On the non-optimality of LAS within the class IMRL

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Scheduling in a M/G/1 Queue

– Poisson arrivals with rate λ. Service requirements are i.i.d. with distribution $F(x)=P[X \leq x]$.

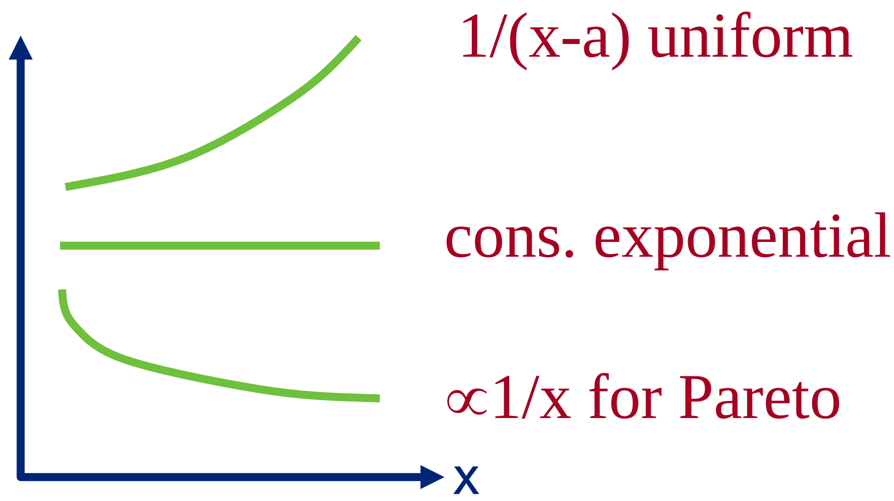
– Attained service is known (total service requirement unknown)

– Optimality criterion: Mean number of jobs in the system

Monotonous Hazard Rate

– Hazard rate of a distribution function: $h(x)dx = P[x < X \leq x + dx \mid X > x]$

$$h(x) = \frac{f(x)}{1 - F(x)}$$



– IHR: Non-preemptive discipline (FCFS etc.)

– Exponential: M/M/1 Mean number of jobs is policy independent

– DHR: Least Attained Service (LAS) is optimal. The job(s) who has attained the least amount of service is served.

Which scheduling when HR not monotonous?

– Mean Residual Life $H(x) = E[X - x \mid X > x] = \dfrac{\int_x^{\infty} (1 - F(u))du}{1 - F(x)}$

– Distribution is IMRL if and only if H(x) is non-decreasing for all $x \geq 0$

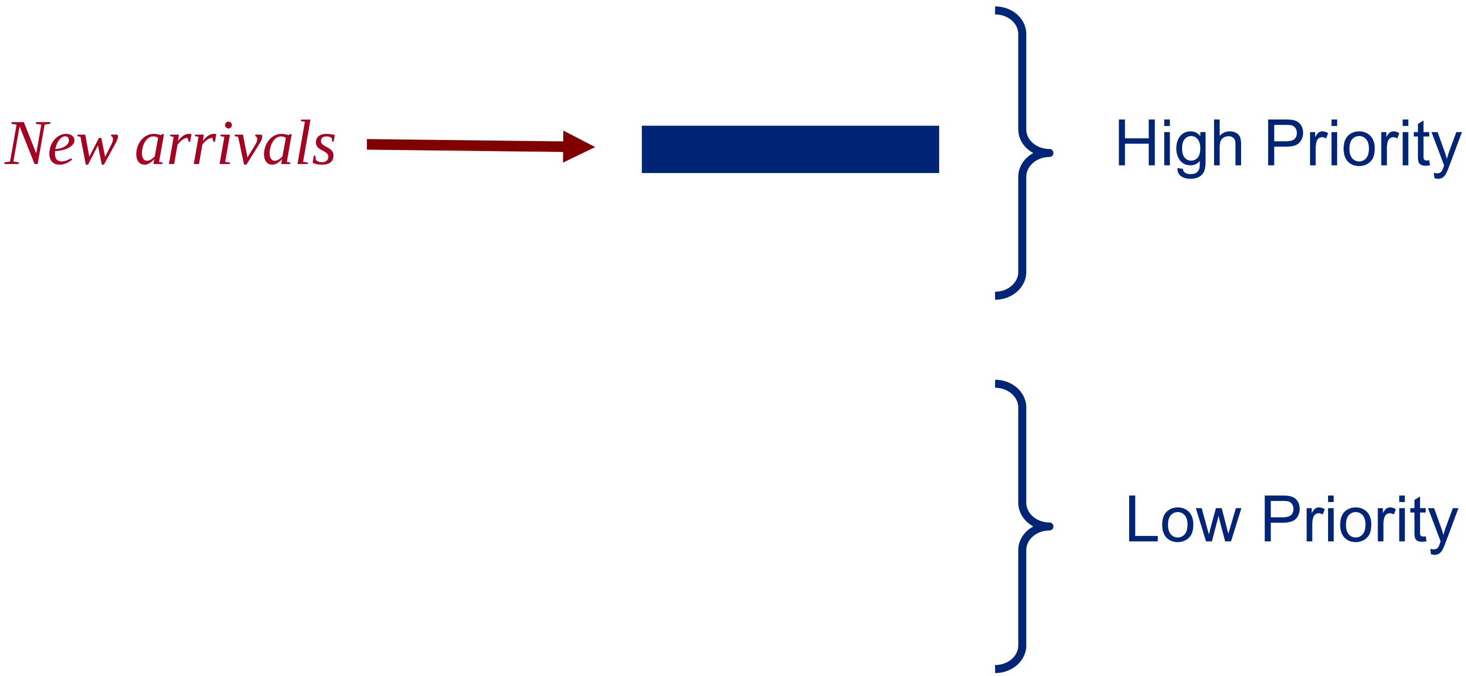
– Larger class of distributions: DFR $\subset$ IMRL

– Claim: LAS is optimal within the class IMRL
  - Known flaw on the proof
  - Does the result hold?
  - What is the optimal policy?

FCFS+LAS(c)

- Classify jobs into two classes depending on the amount of attained service
  - High Priority: Jobs that have obtained less service than c
  - Low Priority: Jobs that have obtained more service than c

- High Priority jobs served according to FCFS and Low Priority with LAS



LAS is not Optimal within IMRL

– Theorem: There exists a service requirement distribution belonging to IMRL and discipline π such that

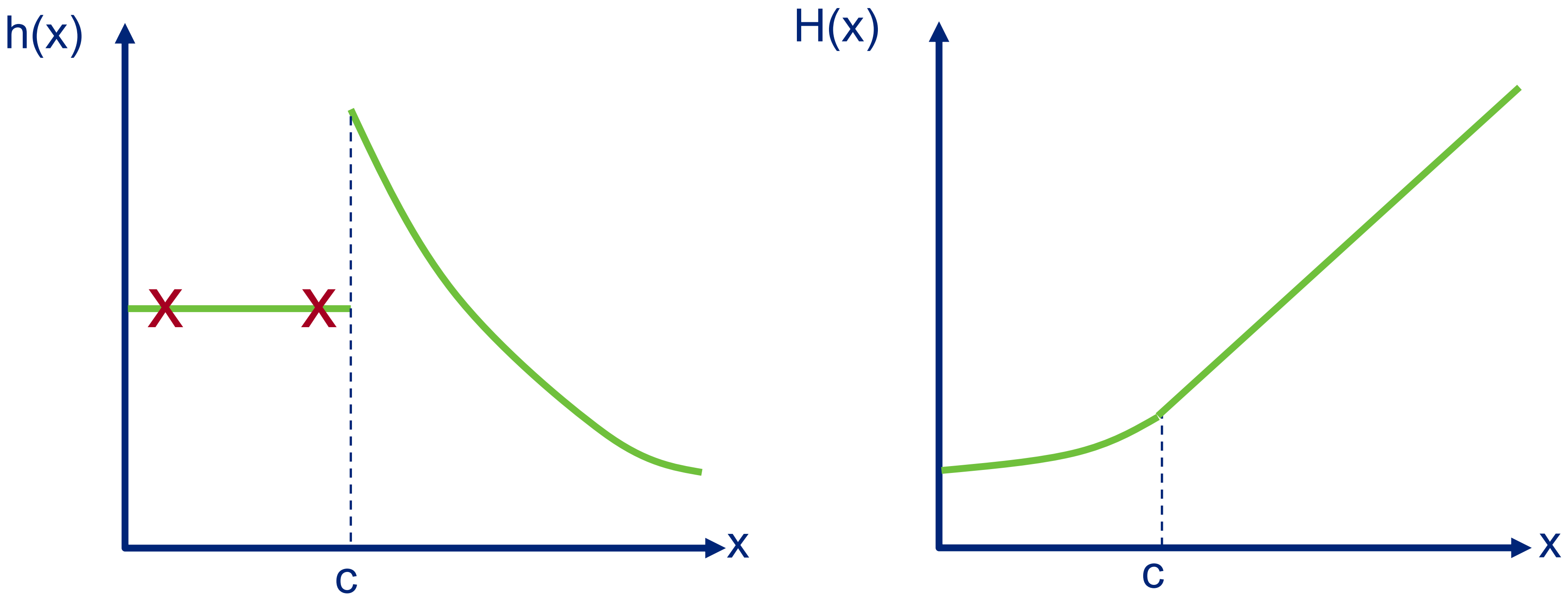
$$N^{\pi} < N^{LAS}$$

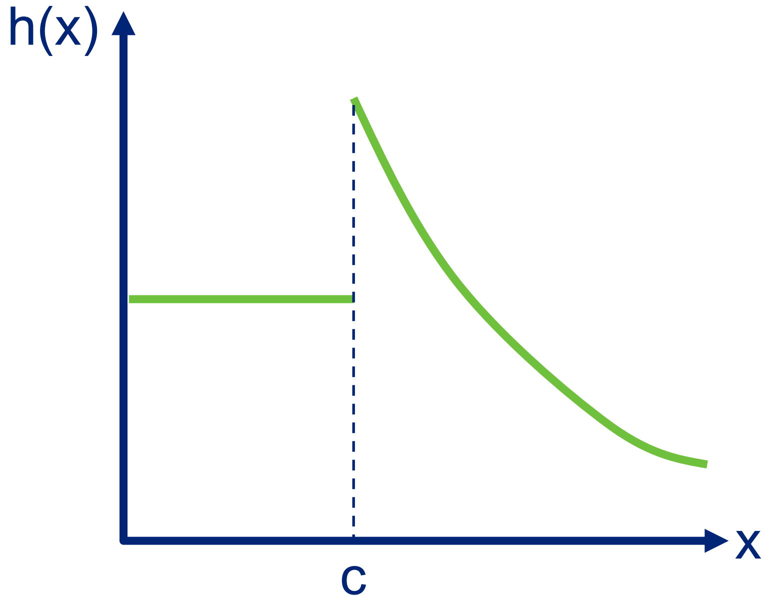
Sketch of the proof: Consider the distribution (CHR+DHR)

$$\overline{F}(x) = \begin{cases} c^{-x}, & 0 < x \le c \\ x^{-c}, & x > c \end{cases}$$

Belongs to IMRL but not to DHR if 1<c<e

Hazard Rate and Mean Residual Life





$$\begin{cases} N^{FCFS+LAS(c)} = N^{LAS} \\ \lim_{\varepsilon \to 0^+} \dfrac{\partial}{\partial \varepsilon} N^{FCFS+LAS(c+\varepsilon)} < 0 \end{cases}$$

$\Rightarrow$There exists $\delta>0$ such that, for any $0<\varepsilon<\delta$

$$N^{FCFS+FB(c+\varepsilon)} < N^{LAS}$$

Optimal discipline for general service requirements

- Gittins' index policy.
  - To each job present in the system, assign an index equal to

$$G(a) = \sup_{\Delta > 0} J(a,\Delta) \quad \text{where} \quad J(a,\Delta) = \frac{\int_0^\Delta f(a+\Delta)}{\int_0^\Delta \overline{F}(a+\Delta)}$$

- Pick the job with highest index value, and assign him a service quota $\Delta^*(a)$

$$\Delta^*(a) = \inf\{\Delta \geq 0 \mid G(a) = J(a,\Delta)\}$$

- Introduced by Sevcik [1974] for static scheduling. Optimality in Stochastic setting by Gittins [1989].

Gittins index policy

- For non-anticipative disciplines, the hazard rate suffices to characterize the optimal scheduling discipline.

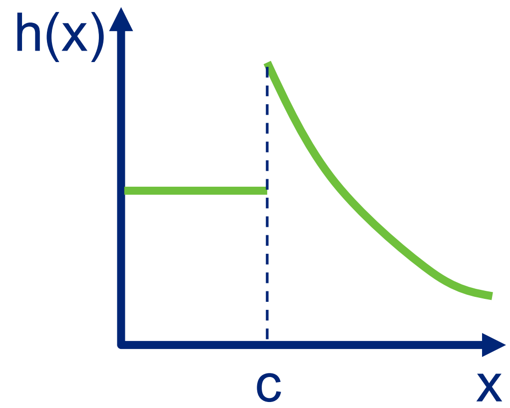
- Theorem: For any attained service a≥0,

$$G(a) = h\left(a + \Delta^*(a)\right)$$

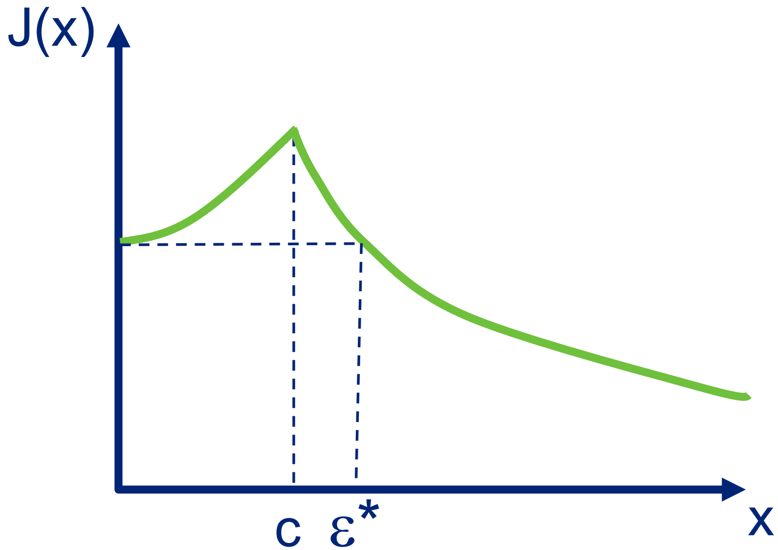
Sketch of the proof:

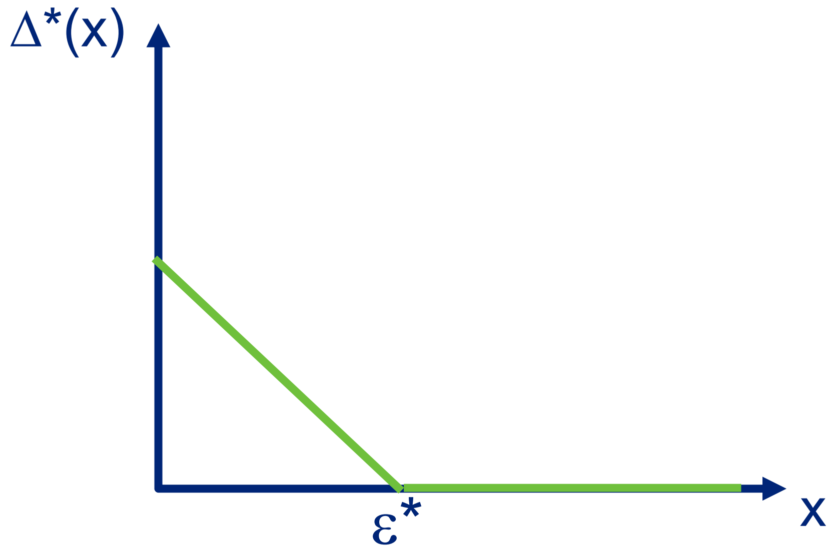
$$\frac{\partial}{\partial \Delta} J(a, \Delta) = 0 \Rightarrow J\left(a, \Delta^*(a)\right) = h\left(a + \Delta^*(a)\right)$$

Gittins index policy



– For a distribution CHR+DHR(k)





– Optimal policy is FCFS+LAS(ε*)

Optimal policy for CHR+DHR distribution

– Theorem: For a CHR+DHR(k) type of distribution, there exists a ε^* <k, such that

$$N^{FCFS+LAS(\varepsilon^*)} < N^{\pi}$$

Conclusion and future research

- In the set of non-anticipative disciplines, the hazard rate characterizes completely the optimal policy.

- Application of index policy for scheduling with multiple nodes?
  - How to cope with non work conserving property of networks?

- Influence of maximum size. Assume that $F(x)=1$ for all $x \geq k$