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Impact of access rates on the performance of networks

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Introduction

- Previous models, infinite access-rates: DPS,PS, α-fair allocation
- Single bottleneck and Linear network
- Fluid and Diffusion scaling. Diffusion scaling allows to explicitly compute the covariance matrix of the steady-state number of flows

Single bottleneck

- DPS model. M classes. Poisson arrivals with rate λ_i and job sizes exponentially distributed with mean $1/\mu_i$
- Each class-i user is access-link limited at r_i.
- Total capacity allocated to class-i user:

$$R_i(t) = \frac{1}{N_i(t)} \min\{\frac{g_i N_i(t)}{\sum_{j=1}^{M} g_j N_j(t)}, r_i N_i(t)\}$$

Single bottleneck (cont.)

- The process

$$\left(\overrightarrow{N}(t)\right)_{t\geq 0} = \left(N_1(t), \ldots, N_M(t)\right)_{t\geq 0}$$

Is Markovian with transition rates

$$\begin{cases} \left(\overrightarrow{N}(t)\right) \to \left(\overrightarrow{N}(t)\right) + \vec{e}_i : & \lambda_i \\ \left(\overrightarrow{N}(t)\right) \to \left(\overrightarrow{N}(t)\right) - \vec{e}_i : & \mu_i N_i(t) R_i(t) \end{cases}$$

Fluid limit

- Let $\left(\overrightarrow{N}^{(L)}(t)\right)_{t\geq 0}$ denote the process where arrival rates are replaced by $L\lambda_i$, and service rate by L.
- Normalized process $\left(L^{-1}\overrightarrow{N}^{(L)}(t)\right)_{t\geq 0}$ converges to a deterministic limit

$$n_i^{'}(t) = \lambda_i - \phi_i\left(\vec{n}(t)\right)$$

where $\phi_i\left(\vec{n}\right) = \mu_i \min\{\frac{g_i n_i(t)}{\sum_{j=1}^{M} g_j n_j(t)}, r_i n_i(t)\}$

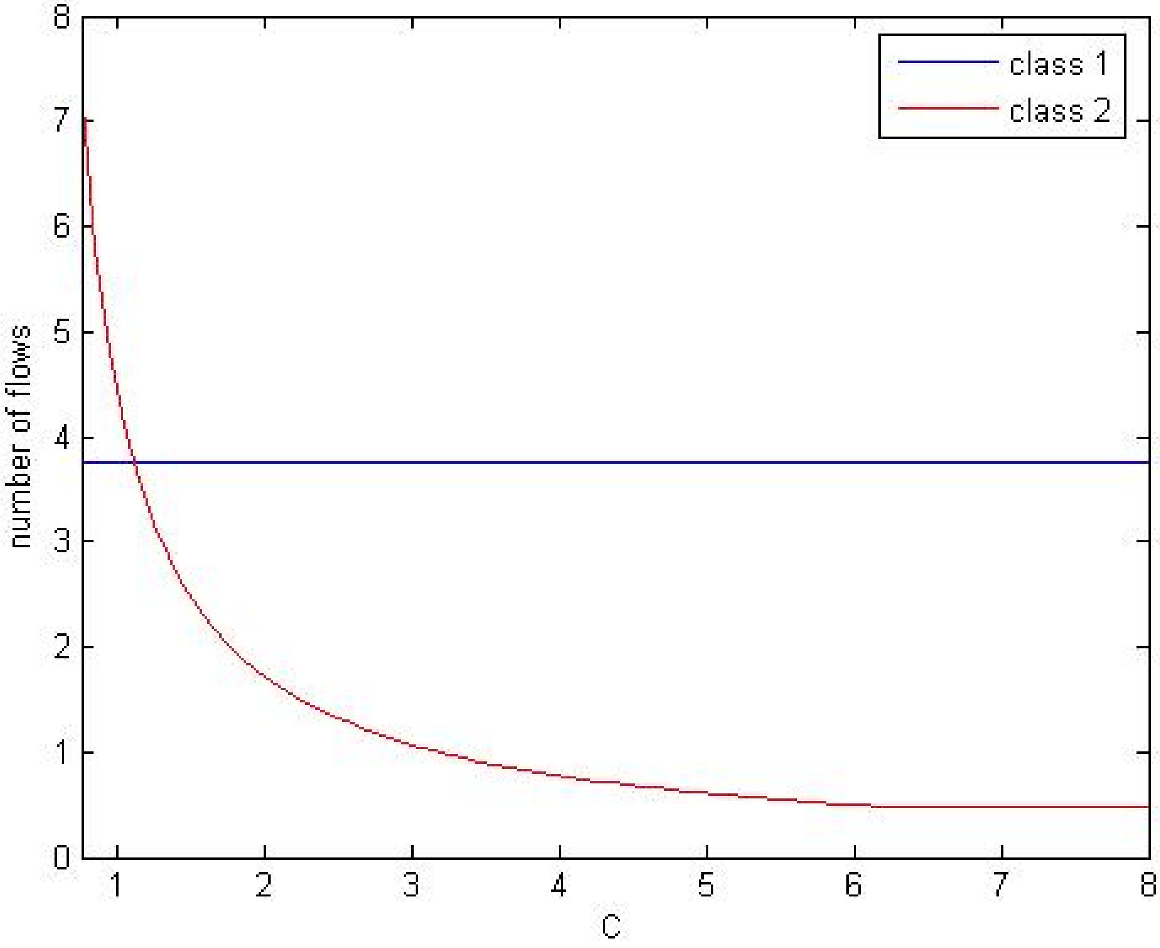
Equilibrium point

- Let $\gamma_i = g_i/r_i$. Relabel the classes such that $\gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_M$.
- Proposition: There exists a unique s={1,…,M} such that S={1,…,s}. The classes belonging to S are access-rate limited. For $i \in S$

$$n_i^*(s) = \frac{\lambda_i}{\mu_i r_i}$$

and for $i \in S^c$ $$n_i^*(s) = \frac{\rho_i}{g_i}\left(\frac{1}{1-\sum_{j=s+1}^{M}\rho_j}\right)\left(\sum_{j=1}^{s}\frac{g_i\lambda_i}{\mu_i r_i}\right)$$

Numerical example: r_1=0.1,r_2=0.8



Linearized system

- Consider the M-dimensional vector

$$\overrightarrow{m}(t) = \overrightarrow{n}(t) - \overrightarrow{n}*$$

- We determine a matrix $P \equiv (p_{ij})_{i,j=1}^{M}$ such that

$$\overrightarrow{m}'(t) = -P\overrightarrow{m}(t)$$

- Proposition: All eigenvalues of P are positive
  – Stability of linearized system

Diffusion scaling

- Let us introduce the perturbation process

$$\vec{Y}(t) = \frac{1}{\sqrt{L}}\left(\vec{N}^{(L)}(t) - L\vec{n}*\right)$$

- It satisfies the stochastic differential equation

$$d\vec{Y}(t) = -P\vec{Y}(t)dt + d\vec{W}(t)$$

where $\vec{W}(t) = A\vec{B}(t)$ with $\vec{B}(t)$ an M-dimensional vector of independent Brownian motions, and A a diagonal matrix with $A_{ii} = \sqrt{2\lambda_i}$

Covariance matrix

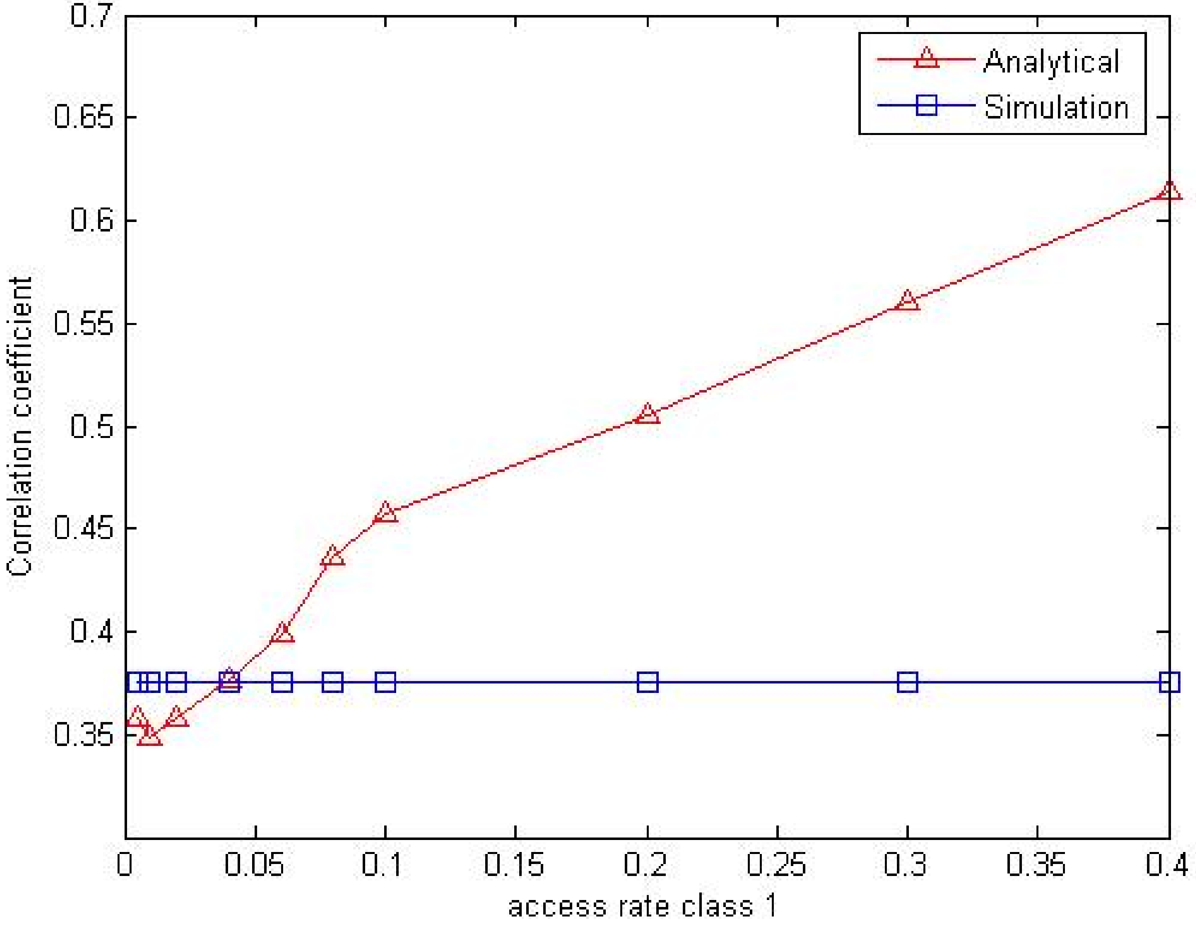
$$E\left[\vec{Y}\vec{Y}^T\right] = \Sigma = \int_0^\infty e^{-Pt} A A^T e^{-P^T t} dt$$

- For M>2 cumbersome. For M=2

$$\Sigma = \begin{pmatrix} \frac{\lambda_1}{r_1 \mu_1} & -\lambda_1 \frac{p_{21}}{p_{11}(p_{11}+p_{22})} \\ -\lambda_1 \frac{p_{21}}{p_{11}(p_{11}+p_{22})} & \lambda_1 \frac{p_{21}^2}{p_{11} p_{22}(p_{11}+p_{22})} + \frac{\lambda_1}{p_{22}} \end{pmatrix}$$

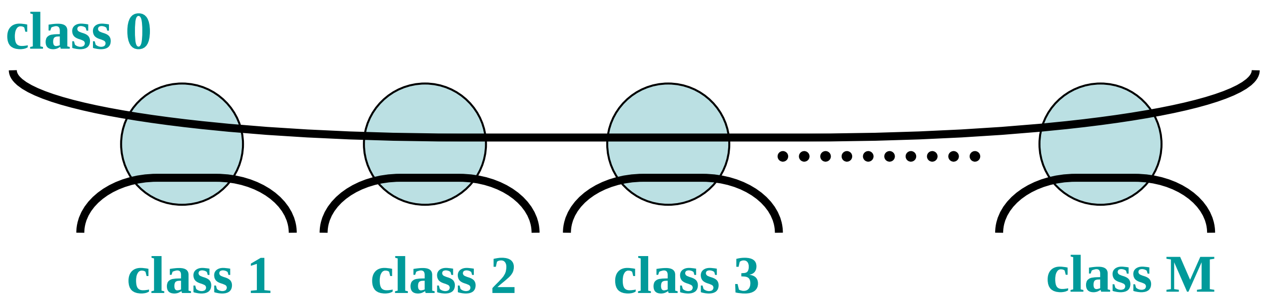
- As expected positive correlation

Covariance: $C=1, r_2=0.8$



Linear network

- M nodes of capacity C and M+1 classes
- Poisson arrival processes with rates λ_i
- Exponentially distributed job sizes with mean $1/\mu_i$



- Assume stability condition $\rho_i+\rho_0<1$, $\forall i$, with $\rho_i=\lambda_i/(\mu_i C)$

Linear network with access-rate limitation

- α-fair allocation $\max_{R_i,\forall i}\{\sum_{k=1}^{M} g_i N_i \frac{R_i^{1-\alpha}}{1-\alpha}\}$

  subject to $N_0 R_0 + N_i R_i \le C$
- Approximately the allocations are

$$R_i(t) = \frac{1}{N_i(t)} \min\{\frac{S_\alpha(\overrightarrow{N}(t))C}{g_0 N_0(t) + S_\alpha(\overrightarrow{N}(t))}, r_0 N_i(t)\}$$

$$R_0(t) = \frac{1}{N_0(t)} \min\{\frac{g_0 N_0(t)C}{g_0 N_0(t) + S_\alpha(\overrightarrow{N}(t))}, r_0 N_0(t)\}$$

- Give rise to an ergodic continuous time Markov chain

Fluid limit

- Equilibrium point is the solution of

$$\rho_0 = \min\{\frac{g_0 n_0}{g_0 n_0 + S_\alpha(\vec{n})}, \frac{r_0 n_0}{C}\}$$

$$\rho_i = \min\{\frac{S_\alpha\left(\vec{n}\right)}{g_0 n_0 + S_\alpha\left(\vec{n}\right)}, \frac{r_i n_i}{C}\}$$

- Assume $\rho_i \neq \rho_j$, for all i,j. Either a crossing class (S^C ={i*}) or the common class (S^C ={0}) is binding.
  - If S^C ={i*}, then i*=argmax{ρ_i}

Equilibrium point

- Proposition: Assume $\rho_i \neq \rho_j$, for all i,j.
  - Crossing class is binding: $S^C = \{i^*\}$. For all $i \in S$

$$n_i^* = \frac{\lambda_i}{\mu_i r_i}$$

and

$$n_{i^*}^* = \left(\frac{\rho_{i^*}}{(1-\rho_{i^*}) g_{i^*}} \left(\frac{g_0 \lambda_0}{r_0 \mu_0} \right)^{\alpha} - \frac{1}{g_{i^*}} \sum_{i=1, i \neq i^*}^{M} \frac{g_i \lambda_i}{r_i \mu_i} \right)^{1/\alpha}$$

  - Common class is binding: $S^C = \{0\}$. Then for all i=1,…,M

$$n_i^*(s) = \frac{\lambda_i}{\mu_i r_i}$$

and

$$n_{i^*}^* = \frac{\rho_0}{(1-\rho_0) g_0} \left(\sum_{i=1}^{M} g_i \left(\frac{\lambda_i}{r_i \mu_i} \right)^{\alpha} \right)^{1/\alpha}$$

Diffusion scaling

- Construct Linearized system (matrix P)
  - P positive eigenvalues
- Diffusion scaling

$$\vec{Y}(t) = \frac{1}{\sqrt{L}}\left(\overrightarrow{N}^{(L)}(t) - L\vec{n}*\right)$$

- Steady state covariance matrix

$$E\left[\vec{Y}\vec{Y}^T\right] = \Sigma = \int_0^\infty e^{-Pt} A A^T e^{-P^T t} dt$$

Steady-state covariance

- Proposition:
  - Crossing class is binding: $S^C = \{i^*\}$.

    For $i \neq 0, i^*$:

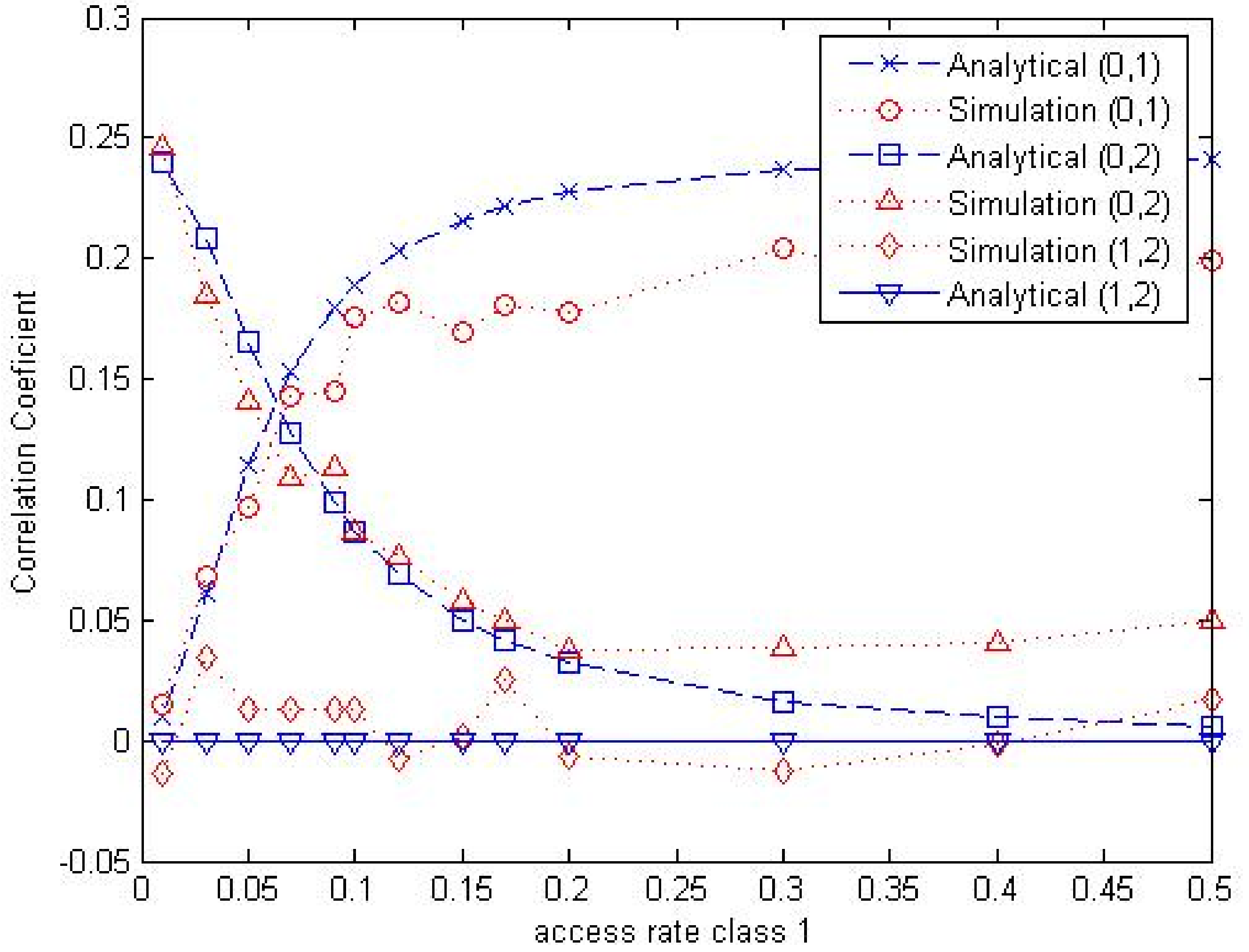
$$\Sigma_{i^*i} = -\lambda_i \frac{p_{i^*i}}{p_{ii}} \frac{1}{p_{ii} + p_{i^*i^*}} < 0$$

and

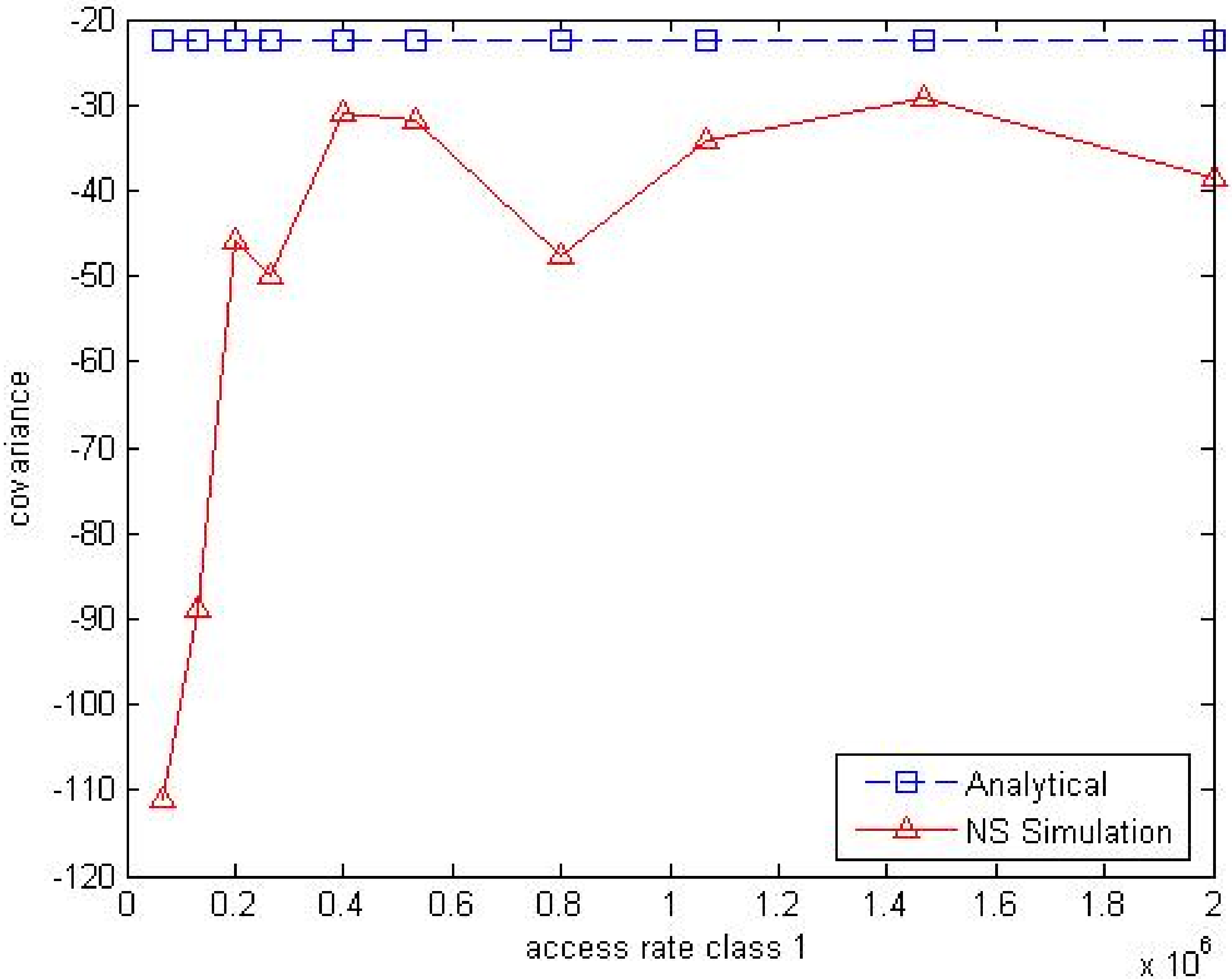
$$\Sigma_{i^*0} = -\lambda_0 \frac{p_{i^*0}}{p_{00}} \frac{1}{p_{00} + p_{i^*i^*}} > 0$$

  - Common class is binding
    - All covariances are positive

Common class binding: r_0=1, r_2=0.05



NS simulations: cross class binding (only class 1 access rate limited)



Conclusions and future work

- Impact of access link: Analytical expressions for the steady-state covariance
- Other networks: Trees, star topology
- Load sharing in order to reduce the steady-state covariance