Batch Arrival Processor-Sharing with Application to Multi-Level Processor-Sharing Scheduling

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Abstract

We analyze a Processor-Sharing queue with Batch arrivals. Our analysis is based on the integral equation derived by Kleinrock, Muntz and Rodemich. Using the contraction mapping principle, we demonstrate the existence and uniqueness of a solution to the integral equation. Then we provide asymptotical analysis as well as tight bounds for the expected response time conditioned on the service time. In particular, the asymptotics for large service times depends only on the first moment of the service time distribution and on the first two moments of the batch size distribution. That is, similarly to the Processor-Sharing with single arrivals, in the Processor-Sharing queue with batch arrivals the expected conditional response time is finite even when the service time distribution has infinite second moment. Finally, we show how the present results can be applied to the Multi-Level Processor-Sharing scheduling.

Keywords. $M^X/G/1$, Processor-Sharing, Batch arrivals, Work conservation, Multi-level Processor-Sharing.

I. Introduction and Motivation

The Processor-Sharing queue with batch arrivals (BPS) has not been fully characterized yet. Kleinrock *et al.* [1], [2] showed that the derivative of the expected response time conditioned on the service time satisfies an integral equation. Furthermore, they obtained an analytic solution for service time distributions of the type $\overline{F}(x) = q(x)e^{-\mu x}$ where $q(x)$ is a polynomial. Bansal [3], using Kleinrock's integral equation, obtained the Laplace transform of the expected conditional response time for hyperexponential distributions and more generally for distributions with rational Laplace Transforms. Rege and Sengupta [4] found the distribution of the expected conditional response time for a tagged customer, given the service times of all customer in the system. More recently, Feng and Misra [5] provided bounds for the expected conditional response time. Their bounds depend on the second moment of the service time distribution.

One of the main motivations to study the BPS queue is its application to size-based scheduling. Size-based scheduling has recently received a fairly big attention in connection with the differentiation of Short and Long flows in the Internet [6], [7], [8], [5], [9]. Kleinrock *et al.* [1], [10], [2] introduced a quite general set of size-based scheduling termed as Multi-Level Processor-Sharing (MLPS). In MLPS, jobs are served with a discipline that will depend on their attained amount of service. That is, based on their attained service, jobs are classified into different classes. Jobs within the same class are served either with First-Come-First-Serve (FCFS), Processor-Sharing (PS) or Foreground Background (FB)[1]

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[1]The Foreground-Background policy is also known as Least Attained Service (LAS).

policy. The classes themselves are served according to the FB policy, that is, the class that contains jobs with smallest attained service is served first. It turns out, that when PS is used to serve jobs in any of the classes, the expected conditional response time in this class can be expressed as a function of the expected conditional response in an BPS queue.

The organization of the paper is as follows: First we prove the existence and uniqueness of a solution to Kleinrock's integral equation. Second we show that under natural conditions, the expected conditional response time has an asymptote and we give an analytical expression for the slope and the bias of the asymptote. In particular, this asymptote provides a tight upper bound for the expected conditional response time for large jobs. Yet another upper bound is obtained for small jobs. Combining these two bounds we obtain a very good characterization of the expected conditional response time for all service times. In particular these bounds are insensitive to the service time distribution and depend only on the distribution through the first moment. Finally, as an example of the application of these results, we show that in the case of MLPS schedulers, the expected conditional response time has an asymptote with the same slope as in a PS queue. This result indicates that with an MLPS discipline, very large jobs perceive the same service rate they would perceive in a PS queue.

II. Analysis of the Batch arrival Processor-Sharing queue

A. Model and Notation

Let us denote $T_{BPS}(x)$ the expected conditional response time for a job with service time x in an BPS queue. Let $T'_{BPS}(x)$ be its derivative. We assume the batch inter-arrival time is exponentially distributed with mean arrival rate equal to λ. Kleinrock *et al.* [1], [2] showed that $T'_{BPS}(x)$ is a solution of the following integral equation

$$\begin{aligned} T'_{BPS}(x) &= \lambda E[N]\int_0^\infty T'_{BPS}(y)\overline{F}(x+y)dy \\ &+\lambda E[N]\int_0^x T'_{BPS}(y)\overline{F}(x-y)dy \\ &+b\overline{F}(x)+1, \end{aligned} \tag{1}$$

where λ is the batch arrival rate, $E[N]$ is the average batch size, b is the average number of jobs that arrive in addition to the tagged job, i.e., $b+1 = E[N^2]/E[N]$ and $\overline{F}(x) = 1-F(x)$ is the complementary distribution of the service time. The load in the BPS system is given by $\rho = \lambda E[N]E[X]$.

B. Fixed Point Approach to the Kleinrock's integral equation

In Theorem 1 we show that there exists a unique solution to the integral equation (1).

Theorem 1: Let the service time distribution have a finite mean, the batch size distribution have a finite second moment and $\rho < 1$. Then there exists a unique solution of the integral equation (1).

Proof: We consider the fixed point iterations

$$\begin{aligned} T'_{k+1}(x) &= \lambda E[N]\int_0^\infty T'_k(y)\overline{F}(x+y)dy \\ &+\lambda E[N]\int_0^x T'_k(y)\overline{F}(x-y)dy \\ &+b\overline{F}(x)+1, \quad k=0,1,... \end{aligned} \tag{2}$$

on the complete functional space of continuous bounded non-negative functions $\mathcal{C}[0,\infty)$ with the supremum metric. Let $\|T'\| = \sup_x\{T'(x)\} < \infty$. Define the linear integral operator $\mathcal{A}[\beta(x)]$ as follows:

$$\begin{aligned}\mathcal{A}[\beta(x)] &= \lambda E[N]\int_0^\infty \beta(y)\overline{F}(x+y)dy \\ &\quad +\lambda E[N]\int_0^x \beta(y)\overline{F}(x-y)dy + b\overline{F}(x)+1.\end{aligned}$$

Clearly the operator $\mathcal{A}[\beta(x)]$ maps the space $\mathcal{C}[0,\infty)$ into itself.

If we show that the linear integral operator $\mathcal{A}[\beta(x)]$ is a contraction, then the integral equation (3) has a unique solution in $\mathcal{C}[0,\infty)$. Let us denote as d the distance in the metric space $\mathcal{C}[0,\infty)$, that is, $d(\beta_1,\beta_2) = \sup_x|\beta_1(x)-\beta_2(x)|$. We show now that the linear operator $\mathcal{A}[\beta(x)]$ is indeed a contraction mapping on $\mathcal{C}[0,\infty)$.

$$\begin{aligned}d(\mathcal{A}[\beta_1],\mathcal{A}[\beta_2]) &= \sup_x\{|\mathcal{A}[\beta_1]-\mathcal{A}[\beta_2]|\} \\ &\leq \lambda E[N]\sup_x\{|\beta_1-\beta_2|\} \\ &\quad \sup_x\left(\int_0^\infty \overline{F}(x+y)dy + \int_0^x \overline{F}(x-y)dy\right) \\ &= \lambda E[N]d(\beta_1,\beta_2)E[X] \\ &= \rho d(\beta_1,\beta_2).\end{aligned}$$

Thus, the mapping is a contraction if $\rho < 1$. ■

Theorem 1 implies that we can apply the Fixed Point Iterations (2) for the solution of the integral equation (1). A numerical example will be provided in Section III.

Corollary 1: Let the service time distribution have a finite mean, the batch size distribution have a finite second moment and $\rho < 1$. Then, the average number of jobs in the BPS queue is finite.

Proof: Because of PASTA, the distribution at arrival epochs of batches is equal to the stationary distribution. Hence, when a batch arrives to the system, it finds on average $E[Q_{BPS}]$ jobs in the queue, where Q_{BPS} has the time average distribution of the number of jobs in the queue. Clearly, $T'_k(0)$ is equal to the derivative of the rate at which a job gets served upon arrival to the system. In a BPS system, this value is equal to the total number of jobs present in the queue upon arrival, thus,

$$T'_{BPS}(0) = E[Q_{BPS}] + b + 1.$$

Since from Theorem 1 $T'_{BPS}(0)$ is finite, and by hypothesis b is finite, it follows that the average number of jobs in the queue is finite. ■

Note that by Little's law, Corollary 1 is equivalent to stating that the expected unconditional response time is finite. In Theorem 4 we provide an upper bound of the expected unconditional response time.

For the ensuing analysis, it will be convenient to remove the constant component of the solution of equation (1), hence we note that the solution of the integral equation (1) is equivalent to the solution of the following integral equation

$$\begin{aligned}\delta T'(x) &= \lambda E[N]\int_0^\infty \delta T'(y)\overline{F}(x+y)dy \\ &\quad +\lambda E[N]\int_0^x \delta T'(y)\overline{F}(x-y)dy \\ &\quad +b\overline{F}(x),\end{aligned} \tag{3}$$

where

$$\delta T'(x) := T'_{BPS}(x) - \frac{1}{1-\rho}. \quad (4)$$

C. Asymptotic Analysis

It is known that in a queue, under any work conserving discipline, the total unfinished work in the system does not depend on the particular scheduling policy being used. This fact has been widely exploited since it poses a constraint on the average conditional response time $T(x)$ of the system among the set on non-anticipative scheduling disciplines, i.e., the set of disciplines that do not take advantage of the service time of the jobs when deciding which job(s) will be served.

Lemma 1: [11], [12] In an ergodic queue, under any work conserving and service-time independent scheduling discipline, the expected conditional response time satisfies

$$\lambda \int_0^\infty T(x)\overline{F}(x)dx = \overline{U}, \quad (5)$$

where λ is the job arrival rate and $\overline{U}$ is the time-average unfinished work in the system.

The interest of Lemma 1 lies in the fact that if the expected conditional response time is known for a particular scheduling discipline, then one can compute the average unfinished work in the system $\overline{U}$. Since this quantity is independent of the scheduling discipline, the expected conditional time for any other scheduling discipline must satisfy equation (5).

Let $\overline{U}^B$ be the expected unfinished work in the case of Poisson batch arrival queue. In order to apply Lemma 1 to the Poisson batch arrival system, we note that the job arrival rate is $\lambda E[N]$, thus

$$\lambda E[N] \int_0^\infty T(x)\overline{F}(x)dx = \overline{U}^B. \quad (6)$$

The expected unfinished work $\overline{U}^B$ in a Poisson batch arrival queue can be easily computed [13], [14].The basic step is to consider a FCFS discipline and to define the random variable $Y = \sum_i^N X_i$, where N is the size of the batch and X_i is the size of the i-th job. Then the expected unfinished work can be computed directly by the Pollaczek-Khinchin formula. The expressions given in [13], [14] become more transparent if they are expressed as a function of b, namely,

$$\overline{U}^B = \overline{U}^{Batch-FCFS} = \frac{\lambda E[Y^2]}{2(1-\lambda E[Y])} = \frac{\lambda E[N]E[X^2]}{2(1-\rho)} + \frac{bE[X]\rho}{2(1-\rho)}. \quad (7)$$

Let us illustrate Lemma 1 with one particular example.

Example 1: Let us consider a BPS queue with exponentially distributed service time. This is the only distribution for which there exists an analytical expression for the expected conditional response time. Let $T_{BPS_{exp}}(x)$ be the expected conditional response time in a BPS queue with exponential service time distribution. Then it is known that [1], [2], [4]

$$T_{BPS_{exp}} = \frac{x}{1-\rho} + \frac{b(2-\rho)E[X]}{2(1-\rho)^2}(1 - e^{\frac{-(1-\rho)}{E[X]}x}). \quad (8)$$

Then, calculating equation (6) we obtain

$$\begin{aligned}
\lambda E[N]\int_0^\infty T_{BPS_{exp}}(x)\overline{F}(x)dx &= \frac{\lambda E[N]E[X^2]}{2(1-\rho)} + \lambda E[N]\frac{b(2-\rho)E[X]}{2(1-\rho)^2}E[X] \\
&\quad -\lambda E[N]\frac{b(2-\rho)E[X]}{2(1-\rho)^2}\int_0^\infty e^{\frac{-(2-\rho)}{E[X]}x}dx \\
&= \frac{\lambda E[N]E[X^2]}{2(1-\rho)} + \frac{b(2-\rho)\rho E[X]}{2(1-\rho)^2}\rho - \frac{b(2-\rho)\rho}{2(1-\rho)^2}\frac{E[X]}{(2-\rho)} \\
&= \frac{\lambda E[N]E[X^2]}{2(1-\rho)} + \frac{b\rho E[X](2-\rho-1)}{2(1-\rho)^2} = \overline{U}^B.
\end{aligned}$$

In the following Lemma, we take advantage of equation (6) to obtain a result that is crucial for the ensuing analysis.

Lemma 2: Let $\delta T(x) = T_{BPS}(x) - \frac{x}{1-\rho}$, and let the service time distribution have a finite mean, the batch size distribution have a finite second moment and $\rho < 1$. Then it holds that

$$\lambda E[N]\int_0^\infty \delta T(x)\overline{F}(x)dx = \frac{bE[X]\rho}{2(1-\rho)}. \tag{9}$$

Proof: Let X be a random variable with complementary distribution function $\overline{F}(x)$. The second moment of X is allowed to be infinite. We consider the truncated random variable X_t at t, that is $X_t = \min\{X, t\}$. The complementary distribution of the truncated random variable is

$$\overline{F}_t(x) = \begin{cases} \overline{F}(x), & x \le t, \\ 0, & \text{otherwise}. \end{cases}$$

The mean of the truncated random variable is given by

$$E[X_t] = \int_0^\infty \overline{F}_t(y)dy = \int_0^t \overline{F}(y)dy.$$

We note that the second moment of the truncated random variable is always finite for finite t,

$$E[X_t^2] = \int_0^t 2y\overline{F}(y)dy < \infty.$$

Let $T_{BPS}^t(x)$ be the expected conditional response time in a BPS queue in the case when jobs are distributed according to the random variable X_t and $\rho_t = \lambda E[N]E[X_t]$. Then from equation (7) and Lemma 1 we have

$$\begin{aligned}
\frac{\lambda E[N]E[X_t^2]}{2(1-\rho_t)} + \frac{\rho E[X_t]b}{2(1-\rho_t)} &= \lambda E[N]\int_0^\infty T_{BPS}^t(x)\overline{F_t}(x)dx \\
&= \lambda E[N]\int_0^\infty \left(\frac{x}{1-\rho_t} + \delta T_t(x)\right)\overline{F_t}(x)dx \\
&= \frac{\lambda E[N]E[X_t^2]}{2(1-\rho_t)} + \lambda E[N]\int_0^\infty \delta T_t(x)\overline{F_t}(x)dx.
\end{aligned}$$

Consequently, we have that $\lambda E[N]\int_0^\infty \delta T_t(x)\overline{F_t}(x)dx = \frac{\rho E[X_t]b}{2(1-\rho_t)}$. Taking the limit when $t \to \infty$ and invoking the monotone convergence theorem we get

$$\lambda E[N]\int_0^\infty \delta T(x)\overline{F}(x)dx = \frac{bE[X]\rho}{2(1-\rho)}.$$

∎

Let us now prove another Lemma.

Lemma 3: Let the service time distribution have a finite mean, the batch size distribution have a finite second moment and $\rho < 1$. Then, $\delta T(x) = T_{BPS}(x) - \frac{x}{1-\rho}$ is increasing with respect to x.

Proof: Let us show that $\delta T'(x) = T'_{BPS}(x) - \frac{1}{1-\rho} \geq 0$.

$$\begin{aligned}
\inf_{x\geq 0}\{\delta T'(x)\} &= \inf_{x\geq 0}\{\lambda E[N]\int_0^\infty \delta T'(y)\overline{F}(x+y)dy + \lambda E[N]\int_0^x \delta T'(y)\overline{F}(x-y)dy + b\overline{F}(x)\} \\
&\geq \lambda E[N]\inf_{y\geq 0}\{\delta T'(y)\}\left(\int_0^\infty \overline{F}(x+y)dy + \int_0^x \overline{F}(x-y)dy\right) \\
&= \lambda E[N]\inf_{y\geq 0}\{\delta T'(y)\}E[X] \\
&= \rho \inf\{\delta T'(x).\}
\end{aligned}$$

Hence $\delta T'(x) \geq 0$ and in particular $T'_{BPS}(x) - \frac{1}{1-\rho} \geq 0$. ∎

A direct consequence of Lemma 3 is that $\delta T(x) \geq 0\ \forall x \geq 0$. Next we obtain an upper bound for the expected conditional response time.

Lemma 4: Let the service time distribution have a finite mean, the batch size distribution have a finite second moment and $\rho < 1$. An upper bound for the expected conditional response time is given by

$$T_{BPS}(z) \leq \frac{z}{1-\rho} + \frac{b\left(\rho E[X] + 2E[X_z](1-\rho)\right)}{2(1-\rho)(1-\rho_z)}.$$

Proof: Let us consider the first term on the right hand-side of equation (3). First, integrating by parts we note that

$$\begin{aligned}
\lambda E[N]\int_0^\infty \delta T'(y)\overline{F}(x+y)dy &= \lambda E[N]\left(\delta T(y)\overline{F}(x+y)\right)|_{y=0}^{y=\infty} \\
&\quad +\lambda E[N]\int_0^\infty \delta T(y)dF(x+y)dy. \qquad (10)
\end{aligned}$$

We evaluate the first term on the right hand side. Integrating by parts we get

$$\int_0^\infty x dF(x) = \int_0^\infty \overline{F}(x)dx + \lim_{x\to\infty} x\overline{F}(x),$$

thus, if the service time requirement has a finite mean, $\lim_{x\to\infty} x\overline{F}(x) = 0$. Now we show there exists some $L < \infty$ such that $T_{BPS}(x) \leq Lx$ for all $x \geq 0$. Let us estimate $\sup_{x\geq 0}\{T'_{BPS}(x)\}$.

$$\begin{aligned}
\sup_{x\geq 0}\{T'_{BPS}(x)\} &= \sup_{x\geq 0}\{\lambda E[N]\int_0^\infty T'_{BPS}(y)\overline{F}(x+y)dy \\
&\quad +\lambda E[N]\int_0^x T'_{BPS}(y)\overline{F}(x-y)dy + b\overline{F}(x) + 1\} \\
&\leq \lambda E[N]\sup_{x\geq 0}\{T'_{BPS}(x)\}\left(\int_0^\infty \overline{F}(x+y)dy + \int_0^x \overline{F}(x-y)dy\right) + b + 1 \\
&= \lambda E[N]\sup_{x\geq 0}\{T'_{BPS}(x)\}\int_0^\infty \overline{F}(z)dz + b + 1 \\
&= \lambda E[N]\sup_{x\geq 0}\{T'_{BPS}(x)\}E[X] + b + 1 = \rho\sup_{x\geq 0}\{T'_{BPS}(x)\} + b + 1,
\end{aligned}$$

and hence $\sup\{T'_{BPS}(x)\} \leq \frac{b+1}{1-\rho}$. Noting that $\lim_{x\to 0} T_{BPS}(x) = 0$ in a Processor-Sharing system and integrating between 0 and x we obtain an upper bound

$$T_{BPS}(x) \leq \frac{1+b}{1-\rho}x, \tag{11}$$

and hence,

$$\lim_{x\to\infty} T_{BPS}(x)\overline{F}(x) \leq \lim_{x\to\infty} \frac{1+b}{1-\rho}x\overline{F}(x) = 0.$$

Therefore equation (10) becomes

$$\lambda E[N]\int_0^\infty \delta T'(y)\overline{F}(x+y)dy \;=\; \lambda E[N]\int_0^\infty \delta T(y)dF(x+y).$$

Using the integral equation (3) and the fact that $\delta T(z) = \int_0^z \delta T'(x)dx$, we can write

$$\begin{aligned}
\int_0^z \delta T'(x)dx &= \lambda E[N]\int_0^z\int_0^\infty \delta T'(y)\overline{F}(x+y)dydx \\
&\quad +\lambda E[N]\int_0^z\int_0^x \delta T'(y)\overline{F}(x-y)dydx + b\int_0^z \overline{F}(x)dx \\
&= \lambda E[N]\int_{x=0}^z\int_{y=0}^\infty \delta T(y)dF(x+y)dx \\
&\quad +\lambda E[N]\int_0^z \delta T'(y)\int_y^z \overline{F}(x-y)dxdy + bE[X_z] \\
&= \lambda E[N]\int_{y=0}^\infty \delta T(y)\int_{x=0}^z dF(x+y)dy \\
&\quad +\lambda E[N]\int_0^z \delta T'(y)\int_0^{z-y} \overline{F}(h)dhdy + bE[X_z] \\
&= \lambda E[N]\int_0^\infty \delta T(y)\left(\overline{F}(y) - \overline{F}(y+z)\right)dy \\
&\quad +\lambda E[N]\int_0^z \delta T'(y)\int_0^{z-y} \overline{F}(h)dhdy + bE[X_z]
\end{aligned}$$

Next by Lemma 3 it follows that

$$\begin{aligned}
&\leq \lambda E[N]\int_0^\infty \delta T(y)\overline{F}(y)dy \\
&\quad +\lambda E[N]E[X_z]\int_0^z \delta T'(y)dy + bE[X_z]
\end{aligned}$$

Consequently we get that

$$(1-\lambda E[N]E[X_z])\delta T(z) \;\leq\; \lambda E[N]\int_0^\infty \delta T(y)\overline{F}(y)dy + bE[X_z].$$

Substituting the result obtained in Lemma 2 and taking into account that $\rho_z < 1$, we get

$$\delta T(z) = \int_0^z \delta T'(x)dx \leq \frac{b\left(\rho E[X] + 2E[X_z](1-\rho)\right)}{2(1-\rho)(1-\rho_z)}.$$

Thus, we obtain an upper bound for $T_{BPS}(z)$

$$\begin{aligned}
T_{BPS}(z) &= \frac{z}{1-\rho} + \delta T(z) \\
&\leq \frac{z}{1-\rho} + \frac{b\left(\rho E[X] + 2E[X_z](1-\rho)\right)}{2(1-\rho)(1-\rho_z)}.
\end{aligned}$$

■

In the next Theorem we state the main result of this paper. Namely we show that $T_{BPS}(x)$ has an asymptote. This result will be useful afterwards to provide tight upper bounds on the expected conditional and unconditional response times.

Theorem 2: Let the service time distribution have a finite mean, the batch size distribution have a finite second moment and $\rho < 1$. The conditional response time for the BPS queue has an asymptote with slope $1/(1-\rho)$ and bias

$$\lim_{x\to\infty}\left(T_{BPS}(x) - \frac{x}{1-\rho}\right) = \frac{bE[X](2-\rho)}{2(1-\rho)^2}.$$

Proof: Let us show that there exists an asymptote. From Lemma 4 we know that $T_{BPS}(x) - \frac{x}{1-\rho}$ is upper bounded and from Lemma 3 that $T_{BPS}(x) - \frac{x}{1-\rho}$ is increasing with respect to x. Consequently $\lim_{x\to\infty} T_{BPS}(x) - \frac{x}{1-\rho}$ exists. This justifies the following calculation of the asymptote bias. Proceeding in a similar way as in the proof of Lemma 4, we can write

$$\begin{aligned}
\lim_{x\to\infty}\left(T_{BPS}(x) - \frac{x}{1-\rho}\right) &= \int_0^\infty \delta T'(x)dx = \\
&= \lambda E[N]\int_0^\infty\int_0^\infty \delta T'(y)\overline{F}(x+y)dydx \\
&\quad +\lambda E[N]\int_0^\infty\int_0^x \delta T'(y)\overline{F}(x-y)dydx + b\int_0^\infty \overline{F}(x)dx \\
&= \lambda E[N]\int_{x=0}^\infty\int_{y=0}^\infty \delta T(y)dF(x+y)dx \\
&\quad +\lambda E[N]\int_0^\infty \delta T'(y)\int_y^\infty \overline{F}(x-y)dxdy + bE[X] \\
&= \lambda E[N]\int_{y=0}^\infty \delta T(y)\int_{x=0}^\infty dF(x+y)dy \\
&\quad +\lambda E[N]\int_0^\infty \delta T'(y)\int_0^\infty \overline{F}(h)dhdy + bE[X] \\
&= \lambda E[N]\int_0^\infty \delta T(y)\overline{F}(y)dy \\
&\quad +\lambda E[N]E[X]\int_0^\infty \delta T'(y)dy + bE[X] \\
&= \frac{bE[X]\rho}{2(1-\rho)} + \lambda E[N]E[X]\int_0^\infty \delta T'(y)dy + bE[X].
\end{aligned}$$

Solving the equation for $\int_0^\infty \delta T'(y)dy$, we obtain

$$\lim_{x\to\infty}\left(T_{BPS}(x) - \frac{x}{1-\rho}\right) = \frac{bE[X](2-\rho)}{2(1-\rho)^2}.$$

■

Interestingly, we observe that the value of the bias is insensitive with respect to the service time distribution, that is, it depends on the distribution only through the first moment.

Corollary 2: Let the service time distribution have a finite mean, the batch size distribution have a finite second moment and $\rho < 1$, then the slowdown for large service times in a BPS queue satisfies

$$\lim_{x\to\infty} \frac{T_{BPS}(x)}{x} = \frac{1}{1-\rho}.$$

Proof: The result is a direct consequence of Theorem 2. ■

Corollary 2 shows that in a BPS queue, very large jobs obtain service at the same rate they would in the equivalent PS queue, that is $\lim_{x\to\infty} \frac{T_{BPS}(x)}{x} = \lim_{x\to\infty} \frac{T_{PS}(x)}{x}$.

D. Bounds

In this section, we use the results obtained in the preceding section to obtain tight upper bounds for the expected conditional response time as well as for the expected unconditional response time. We start by providing upper and lower bounds for the expected conditional response time.

Theorem 3: Lower and upper bounds for the expected conditional response time in the BPS queue are given by:

$$\frac{x}{1-\rho} \leq T_{BPS}(x) \leq \min\left\{\frac{b+1}{1-\rho}x, \frac{x}{1-\rho} + \frac{bE[X](2-\rho)}{2(1-\rho)^2}\right\}.$$

The bounds on the right hand part of the inequality intersect at the point $x^* = \frac{E[X](2-\rho)}{2(1-\rho)}$.

Proof: Since $T'_{BPS}(x) - \frac{1}{1-\rho} \geq 0$, $T_{BPS}(x)$ approaches the asymptotic from below. Hence for large service times we obtain a bound that is asymptotically tight. Thus, from Lemma 2 we have:

$$T_{BPS}(z) \leq \frac{z}{1-\rho} + \frac{bE[X](2-\rho)}{2(1-\rho)^2}. \tag{12}$$

Clearly, the upper bound (12) is not appropriate for small service times, since we know that $T_{BPS}(0) = 0$. For small service times we consider the upper bound obtained in equation (11).

Equating the bounds (12) and (11), we find the intersection point

$$\frac{x}{1-\rho} + \frac{bE[X](2-\rho)}{2(1-\rho)^2} = \frac{1+b}{1-\rho}x \Longrightarrow x^* = \frac{E[X](2-\rho)}{2(1-\rho)}. \tag{13}$$

Thus, we have

$$T_{BPS}(x) \leq \min\left\{\frac{b+1}{1-\rho}x, \frac{x}{1-\rho} + \frac{bE[X](2-\rho)}{2(1-\rho)^2}\right\}.$$

The lower bound is a direct consequence of the inequality $T'_{BPS}(x) - \frac{1}{1-\rho} \geq 0$. ■

In Section III we provide numerical examples that indicate that the upper bound of Theorem 3 characterizes quite closely the expected conditional response time for large jobs in the BPS queue. In the next theorem, we use the upper bound of the expected conditional response time to provide an upper bound on the unconditional response time.

Theorem 4: Lower and upper bounds for the expected unconditional response time in a BPS queue are given by:

$$\frac{E[X]}{1-\rho} \leq E[T_{BPS}] \leq \frac{E[X]}{1-\rho} + \frac{b}{1-\rho}E[X_{x^*}].$$

where x^* is the same as in Theorem 3.

Proof: The lower bound is straightforward from the lower bound in Theorem 3. Now we calculate the upper bound.

$$\begin{aligned}E[T_{BPS}] &= \int_0^\infty T_{BPS}(x)dF(x)\\ &\leq \int_0^\infty \frac{x}{1-\rho}dF(x) + \int_0^{x^*} \frac{bx}{1-\rho}dF(x)\\ &\quad + \int_{x^*}^\infty \frac{bE[X](2-\rho)}{2(1-\rho)^2}dF(x)\\ &= \frac{E[X]}{1-\rho} + \frac{b}{1-\rho}E[X_{x^*}] - \frac{b}{1-\rho}\frac{E[X](2-\rho)}{2(1-\rho)}\overline{F}(x^*)\\ &\quad + \frac{bE[X](2-\rho)}{2(1-\rho)^2}\overline{F}(x^*)\\ &= \frac{E[X]}{1-\rho} + \frac{b}{1-\rho}E[X_{x^*}].\end{aligned}$$

■

Interestingly, we note that the upper and lower bound get tight as the value of b decreases. Furthermore, the next corollary shows that when $b \to 0$ (the zero value of b corresponds to the deterministic batch size) both upper and lower bound coincide.

Corollary 3: Let the batch size distribution be deterministic. Then for all $x \geq 0$,

$$T_{BPS}(x) = \frac{x}{1-\rho},$$

and

$$E[T_{BPS}] = \frac{E[X]}{1-\rho},$$

where $\rho = \lambda E[N]E[X]$.

Proof: The result follows directly from Theorems 3 and 4 after noting that the lower and upper bounds become tight as $b \to 0$. ■

The values of the expected conditional and unconditional response time provided in Corollary 3 indicate that a BPS queue with deterministic batch sizes is equivalent to an ordinary PS. Indeed, a BPS queue with batch arrival rate λ, deterministic batch size $E[N]$ and service time distribution $F(x)$, can be related to a PS queue with job arrival rate λ and service time distribution $F(x/E[N])$. We note that in both queues the load will be $\rho = \lambda E[N]E[X]$.

III. Numerical examples

In this section we provide some numerical examples of the results of the preceding sections. In order to compute numerically $T_{BPS}(x)$ (or its derivative) for a general distribution we perform the Fixed Point Iterations (2). Indeed, we have shown in Theorem 1 that the Fixed Point Iterations (2) will converge to the solution of equation (1).

First of all, we consider the case of exponentially distributed service time and we demonstrate a high speed of convergence of the Fixed Point Iterations. Taking the derivative of the expected conditional response time for the exponential distribution case, see equation (8), we obtain

$$\frac{dT_{BPS_{exp}}(x)}{dx} = \frac{1}{1-\rho} + \frac{b(2-\rho)}{2(1-\rho)}e^{\frac{-(1-\rho)}{E[X]}x}. \tag{14}$$

In Figure 1 we depict equation (14) and the Fixed Point Iterations (1st, 6th and 11th) of equation (2). We take $E[N] = 2$, $b = 5$, $E[X] = 20$ and $\rho = 0.7$. By Theorem 1 the rate of convergence of the iterations is in the worst case geometric with parameter ρ. Hence, unless for the highly loaded case, it is expected that the Fixed Point Iterations will converge very rapidly to the analytic solution.

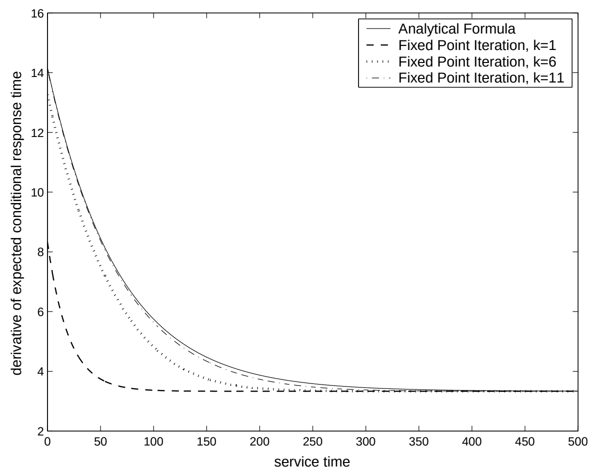

Fig. 1. Convergence of $T'_{BPS}(x)$ for exponential distribution: Analytical formula (14) and Fixed Point Iterations of equation (2)

In Figure 2 we plot the value of $T_{BPS}(x)$ obtained by Fixed Point Iterations for the case of Pareto distribution with infinite variance ($1 < \alpha < 2$). We also plot the upper bound for the conditional response time of Theorem 3. The Pareto distribution is $F(x) = 1 - \frac{k^\alpha}{x^\alpha}$ and the parameters are $k = 10$, $\alpha = 1.5$, $E[N] = 3$, $b = 10$ and $\rho = \{0.3, 0.7\}$. We observe that the upper bound of Theorem 3 provides a good qualitative characterization of the expected conditional response time.

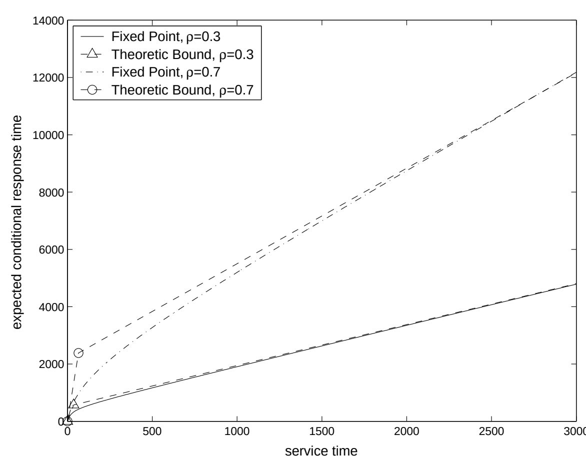

Fig. 2. T_{BPS} for Pareto distribution: Fixed Point iterations and upper bound of Theorem 3

The ratio between the service time and the expected conditional response time ($R(x) = x/T(x)$) is commonly considered as a good measure of the average service rate obtained by a job. We study now the effect of the batch size distribution on $R(x)$. To that extent, we consider a batch size distribution with $E[N] = 2$ and we vary the value of b. In Figure 3 we compare the average service rates of a BPS for different values of b. We note that the load is the same regardless the value of b. The service time distribution is Pareto with $k = 10$, $\alpha = 1.5$ and the load $\rho = 0.7$.

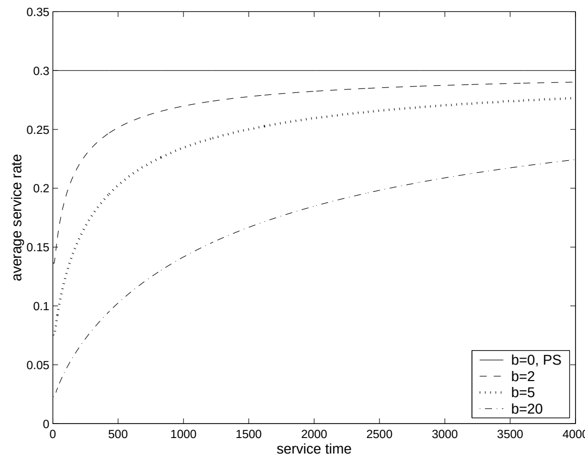

Fig. 3. Average service rate comparison in BPS and PS queues

As a consequence of Lemma 3, it follows that $R_{BPS}(x) \leq R_{PS}(x)$, for all $x \geq 0$. Thus, the fact that jobs arrive grouped has a negative effect on the job's perceived performance. From Figure 3 we conclude that the performance degrades as the variability of the batch size distribution increases.

We consider now the upper bound for the unconditional expected response time obtained in Theorem 4. In the case of a general distribution, we can calculate the tightness of the upper bound of Theorem 4. In Figure 4, we consider a Pareto distribution, and we plot for different loads the relative error between the upper bound provided in Theorem 4 and the numerical calculation of the expected unconditional response time obtained by the Fixed Point Iterations. As in the previous numerical example with Pareto distribution, we take $k = 10$, $\alpha = 1.5$, $E[N] = 3$. We consider different values for b.

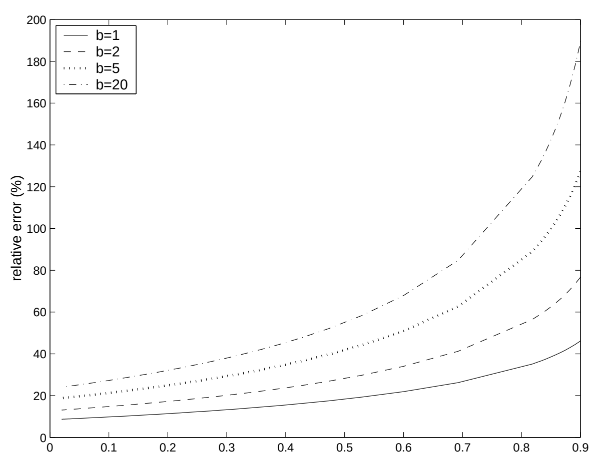

Fig. 4. Expected unconditional response time for Pareto service time distribution: Relative error of the upper bound of Theorem 4

The tightness of the upper bound provided in Theorem 4 depends on the characteristics of the service time distribution. We conclude that Theorem 4 provides a quite tight upper bound for light to medium loads. We emphasize that the bounds provided in Theorems 3 and 4 are useful to characterize in a simple way and with good accuracy the performance of the BPS queue.

We end this section with a conjecture. In view of Figures 1 and 2 we claim that:

Conjecture 1 In a BPS queue, the derivative of the expected conditional response time is decreasing, and thus the expected conditional response time is a concave function.

IV. Application to Multi-Level Processor-Sharing

One of the classical results of queuing theory says that when information on the service time of jobs is available to the server, the *Shortest Remaining Processing Time* (SRPT) scheduling discipline is optimal with respect to the expected unconditional response time of the system [15]. In some scenarios, this information is not available to the scheduler, for example, in the context of computer networks the file size is not known in the core of the network. Hence, scheduling disciplines that only take advantage of the attained service of jobs have drawn significant attention recently. The performance of size-based scheduling depends on the characteristics of the distribution function. For instance, it is known that when the distribution of the service time has a decreasing hazard rate $\mu(x) = dF(x)/\overline{F}(x)$, the FB scheduling discipline minimizes the expected unconditional response time among all non-anticipative disciplines [16], [17]. The set of non-anticipative disciplines is made up by those disciplines that do not take advantage of the total service time of jobs. In few words, we can describe FB as the scheduler who gives full service to the job who has obtained the least amount of service. Thus, it is also referred to as Least Attained Service (LAS).

It is clear that choosing an appropriate scheduling policy may significantly improve the performance of the system. In the current TCP/IP architecture of the Internet the length of a flow is not known in advance. This, coupled with the fact that no job obtains preferential treatment, have led researchers to propose PS as a good mathematical abstraction for the bandwidth allocation that the network provides [18], [19].

It has been widely reported that whereas most of the connections are made up of few packets, most of the data is carried by some few large connections [20]. This type of distributions (Pareto, hyperexponential) have decreasing hazard rate. Therefore, from the theoretic point of view, it seems that giving priority to short flows might improve the overall performance of the system [21].

Even though the apparent desirable properties of FB, its deployment does not seem to be a simple task. Hence, researchers have recently analyzed and proposed different size-based scheduling disciplines that aim to improve the performance that the current network architecture provides [6], [7], [5], [8], [9]. In the next section, we describe these schedulers as particular examples of the set of MLPS disciplines introduced by Kleinrock [1, Section 4.7] and we show how the results presented for the BPS queue can be applied. The main objective of this section is to evaluate how giving preference to small service times affects the expected conditional response time of large service times.

A. Multi-Level Processor-Sharing Scheduling

The framework of MLPS allows us to define a very large class of scheduling disciplines. We assume that jobs arrive to the system according to a Poisson process of rate λ. Let $F(\cdot)$ denote the required service time distribution. The mean required service time is denoted by $E[X]$. We consider the queue is in stable regime, i.e., $\rho = \lambda E[X] < 1$. Let a_i be a set of numbers such that

$$0 = a_0 < a_1 < \ldots < a_N < a_{N+1} = \infty.$$

We consider $N+1$ scheduling disciplines, where π_i is the discipline which is used to serve jobs whose attained service τ belongs to i-th level, that is $a_{i-1} < \tau \leq a_i$. We permit π_i to be either FB or PS. Intervals are served according to a FB discipline with respect to each other, that is, at any instant of time, the processor will give full service to the jobs belonging to the lowest nonempty level. For example, let us consider a two level MLPS scheduler with threshold at a_1, jobs with attained service smaller than a_1 are served with PS and jobs that have attained more service than a_1 with FB. If there are in the system jobs who have attained less service than a_1, those jobs receive full service and they will be served according to a PS discipline. When there are no such jobs, the MLPS scheduler will give full service to those jobs who have attained more service than a_1, in this case following a FB scheduler. As soon as there is a new arrival, the server will interrupt serving jobs with attained service greater than a_1 and start serving the new arrival.

Let x be some value belonging to the i-th interval, i.e. $a_{i-1} < x \leq a_i$ and let $T(x)$ be the expected conditional response time in the MLPS queue. $T(x)$ is made up of two components. The first one corresponds to the difference between the arrival time and the time at which the jobs starts getting served at the i-th interval. The second component is the amount of time spent in the i-th level itself. An important characteristic of MLPS disciplines is that neither of these two components are affected by the scheduling discipline utilized in intervals $j < i$. This is a direct consequence of the fact that the length of the busy-period (of the first j intervals in this case) being independent of the scheduling discipline.

Therefore, in a sample path sense, the time epochs at which the server switches among classes are independent of the disciplines π_i, $i = \{1, \ldots, N+1\}$. In particular, for all $x > a_N$, the expected conditional response time $T(x)$ will depend only on the discipline deployed at the $(N+1)$-st interval.

B. Truncated and Residual random variables

Let us introduce the notation that will be used in the next section. Given a random variable X that takes values in $[0, \infty)$, we consider the truncated random variables $X_t = \min\{t, X\}$ with distribution function

$$F_t(x) = \begin{cases} F(x), & x \leq t, \\ 1, & \text{otherwise}. \end{cases}$$

We consider as well the residual random variables $X_{t,\infty} = \max\{X - t, 0\}$ with distribution function given by $F_{t,\infty}(x) = \frac{F(x+t)-F(t)}{\overline{F}(t)}$ $\forall x \geq 0$. The mean of the residual random variable is given by

$$E[X_{t,\infty}] = \int_0^\infty \frac{\overline{F}(x+t)}{\overline{F}(x)} dx = \frac{E[X] - E[X_{0,t}]}{\overline{F}(t)}.$$

We use the notation $E[T_t^S]$ and $E[T_{t,\infty}^S]$ to denote the average waiting time in a queue $S \in \{PS, FB, BPS\}$ with truncated and residual random variable respectively.

C. *Asymptotic Analysis*

From now on, we concentrate on MLPS disciplines such that $\pi_{N+1} = PS$. We explain first the link between an MLPS discipline and BPS systems. Consider the busy period of the system composed of all intervals but the $(N+1)$-st one. During such a busy period, each time a job gets a_N units of service, it stops being served and joins the $(N+1)$-st interval. Hence, at the end of the busy period, there is a batch of jobs that have arrived to the $(N+1)$-st interval. Then, the MLPS discipline serves the $(N+1)$-st interval until a new job arrives to the queue (or the queue drains completely). Because of the memoryless property of Poisson arrivals, this inter-arrival time is exponentially distributed. Upon the new arrival, the MLPS discipline preempts serving the $(N+1)$-st interval and starts serving the new arrival. Therefore, if we consider the virtual time axis obtained by removing the periods in which the $(N+1)$-st interval is preempted, it turns out that the $(N+1)$-st interval is equivalent to a Batch Processor-Sharing queue with exponentially distributed batch inter-arrival distribution. The required service time distribution of jobs getting to the $(N+1)$-st is $F_{a_N,\infty}(x)$, for all $x \geq 0$.

Kleinrock showed that the parameters $E[N]$ and b of the batch size distribution are given by

$$E[N] = \frac{1 - F(a_N)}{1 - \rho_{a_N}}, \tag{15}$$

and

$$b = 2\lambda(1 - F(a_N))\frac{\overline{W}_{a_N} + a_N}{1 - \rho_{a_N}},$$

where $\overline{W}_{a_N} = \frac{\lambda E[X^2_{a_N}]}{2(1-\rho_{a_N})}$. We note that $\overline{W}_{a_N}$ is always finite, and hence so is b.

Let $T(x)$ denote the expected conditional response time in an MLPS queue with $\pi_{N+1} = PS$ for a job that requires x units of service. Following the arguments in Section IV-A, for all $x > a_N$, $T(x)$ is independent of π_i, for all $i < N+1$. Kleinrock [1, Equation (4.39)] showed that for all $x > a_N$

$$T(x) = \frac{a_N + \overline{W}_{a_N}}{1 - \rho_{a_N}} + \frac{T_{BPS}(x - a_N)}{1 - \rho_{a_N}}. \tag{16}$$

The first term on the right hand side accounts for the expected elapse time between the arrival epoch and the time at which the job is served for the first time at the $(N+1)$-st level. This time can be interpreted as the expected length of the busy period (without the $(N+1)$-st interval) that a job with a service time requirement larger than a_N originates upon arrival. The second term accounts for the time spent in the $(N+1)$-st interval. $T_{BPS}(x - a_N)$ is equal to the elapsed time in the virtual time and the term $1/(1 - \rho_{a_N})$ accounts for the preempted periods of the $(N+1)$-st that were taken out when constructing the virtual time.

There is an undesirable property of FB. In the case of distributions with infinite second moment, there is no asymptote and the expected conditional response time for large jobs deviates from PS [9]. For example, in the case of Pareto distribution with infinite second moment, the asymptotics of FB has the following form [9]

$$T_{FB}(x) = \frac{1}{1-\rho}x + \frac{\lambda k^{\alpha}}{(1-\rho)^2(2-\alpha)}x^{2-\alpha} + \mathsf{o}(x^{2-\alpha}).$$

There is no asymptote in this case, even though the limit $\lim_{x\to\infty}\frac{T_{FB}(x)}{x}$ exists. This implies that the performance of FB deviates increasingly from PS performance with the increase of the service time.

In the next proposition, we show the expected conditional response time in an MLPS queue with $\pi_{N+1} = PS$ has an asymptote with slope $1/(1-\rho)$ even in the case when the service time distribution has an infinite second moment.

Proposition 1: Consider an MLPS queue with $\pi_{N+1} = PS$. Let the service time distribution have a finite mean, let $\rho < 1$, then the response time of the queue has an asymptote with slope $1/(1-\rho)$ and bias

$$\lim_{x\to\infty}\left(T(x) - \frac{x}{1-\rho}\right) = \frac{\overline{W}_{a_N}}{1-\rho_{a_N}} + \frac{a_N(\rho_{a_N}-\rho)}{(1-\rho)(1-\rho_{a_N})} + \frac{bE[X_{a_N,\infty}](2-(\rho+\rho_{a_N}))}{2(1-\rho)^2},$$

where $\rho_{a_N} = \lambda E[X_{a_N}]$.

Proof: From Theorem 2 and equation (16) it follows that as $x \to \infty$

$$T(x) = \frac{a_N + \overline{W}_{a_N}}{1-\rho_{a_N}} + \frac{1}{1-\rho_{a_N}}\frac{x-a_N}{1-\rho_{a_N,\infty}} + \frac{1}{1-\rho_{a_N}}\frac{bE[X_{a_N,\infty}](2-\rho_{a_N,\infty})}{2(1-\rho_{a_N,\infty})^2} + o(1),$$

where $\rho_{a_N,\infty} = \lambda E[N]E[X_{a_N,\infty}]$. In view of equation (15) we have

$$\frac{1}{1-\rho_{a_N,\infty}} = \frac{1}{1-\lambda\frac{\overline{F}(a_N)}{1-\rho_{a_N}}\frac{E[X]-E[X_{a_N}]}{\overline{F}(a_N)}} = \frac{1-\rho_{a_N}}{1-\rho},$$

and similarly $2-\rho_{a_N,\infty} = \frac{2-(\rho+\rho_{a_N})}{1-\rho_{a_N}}$.

Then as $x \to \infty$ we obtain

$$\begin{aligned} T(x) &= \frac{a+\overline{W}_{a_N}}{1-\rho_{a_N}} + \frac{x-a_N}{1-\rho} + \frac{bE[X_{a_N,\infty}](2-\rho_{a_N,\infty})}{2(1-\rho_{a_N,\infty})(1-\rho)} + o(1) \\ &= \frac{x}{1-\rho} + \frac{\overline{W}_{a_N}}{1-\rho_{a_N}} + \frac{a_N(\rho_{a_N}-\rho)}{(1-\rho)(1-\rho_{a_N})} + \frac{bE[X_{a_N,\infty}](2-(\rho+\rho_{a_N}))}{2(1-\rho)^2} + o(1). \end{aligned}$$

■

We conclude with the comparison of the slowdown of an MLPS discipline with $\pi_{N+1} = PS$ and the egalitarian PS discipline.

Corollary 4: Consider an MLPS discipline such that $\pi_{N+1} = PS$. Let $\rho < 1$, then the asymptotic slowdown in the MLPS and PS queues are the same, that is,

$$\lim_{x\to\infty}\frac{T(x)}{x} = \lim_{x\to\infty}\frac{T_{PS}(x)}{x} = \frac{1}{1-\rho}.$$

Proof: The result is a direct consequence of Proposition 1.

■

This result shows that the performance that very large jobs obtain is equivalent under both MLPS and PS disciplines.

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