Bus schedule for optimal bus bunching and waiting times

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***Abstract*—In a bus transportation system the time gap between two successive buses is called headway. When the headways are small (high-frequency bus routes), any perturbation (e.g., in the number of passengers using the facility, traffic conditions, etc.) makes the system unstable, and the headway variance tends to increase along the route. Eventually, buses end up bunching, i.e, they start travelling together. Bus bunching results in an inefficient and unreliable bus service and is one of the critical problems faced by bus agencies. Another important aspect is the expected time that a typical passenger has to wait before the arrival of its bus. The bunching phenomenon might reduce if one increases the headway, however this can result in unacceptable waiting times for the passengers. We precisely study this inherent trade-off and derive a bus schedule optimal for a joint cost which is a convex combination of the two performance measures. We assume that the passengers arrive according to a fluid process, board at a fluid rate and using gate service, to derive the performance. We derive the stationary as well as the transient performance. Further using Monte-Carlo simulations, we demonstrate that the performance of the system with Poisson arrivals can be well approximated by that of the fluid model.**

We make the following interesting observations regarding the optimal operating frequency of the buses. If the randomness in the traffic (variance in travel times) increases, it is optimal to reduce the bus frequency. More interestingly even with the increase in load (passenger arrival rates), it is optimal to reduce the bus frequency. This is true in the low load regimes, while for high loads it is optimal to increase the frequency with increase in load.

I. Introduction

In daily life, majority of people in a city rely on public transport and benefit from affordable service due to subsidized rates from the government. However, significant percentage of people are shifting towards private transport due to unreliable, inefficient service provided by the public systems. This is leading to undesirable issues like traffic congestion, pollution etc. To keep cities green, government and city planners need to enhance the efficiency and reliability of the public transport.

Typically, most popular bus routes have high frequency of buses and buses keep circulating around the route. It is well-known that such transit routes without any intervention or control are unstable ([5]). Any perturbation, typically, in the number of passengers arriving (demand) at the bus stops or in traffic conditions, can cause bus bunching. That is, two (or sometimes more) buses arrive at a bus stop simultaneously and start travelling together. Such a perturbation and hence bus bunching is inevitable and needs to be controlled using some intervention strategy.

Headway can be defined as the time gap between two consecutive buses. This headway is predetermined at the depot (where the buses start their journey) and hence is known. However the headways at the other bus-stops, encountered on the route, have random fluctuations as discussed. When a bus is delayed for a long period at a particular stop (due to larger demand and or larger transit times from previous stops), it has to cater to the increased number of passengers at the next stop. Hence it gets further delayed. Whereas, the following bus observes less passengers than anticipated, as the time gap between the buses is reduced. Hence it further speeds up due to lesser dwell times (the boarding and de-boarding time). Eventually, the headway becomes zero at a certain point along the route and both the buses travel together. This phenomenon is called 'Bus bunching'. Thus, the headway varies as the buses circulate along their path. As a result, passengers waiting at various bus stops experience large variance in their waiting times leading to an unreliable transport service. Further, this results in an inefficient usage of resources (as bunched buses run empty) even for bus operators. Thus efficient control of bunching of buses is important from the perspective of both the public using the service, as well as the operator.

Bus bunching is a critical issue faced by bus agencies and this problem has been thoroughly investigated over past few decades. However, it is still an active area of research as it is challenging to provide a generic, practical, and effective solution. Existing control strategies are based on ideas like skipping the forthcoming bus stops (e.g., [1], [4], [6]), limited boarding (e.g., [2], [3]), holding buses at specific locations (e.g., [1], [7], [2]) etc. Holding control applied at intermediate bus-stops and or skipping of stops may not be comfortable from the perspective of the passengers travelling in the bus. Hence our focus is on the holding control strategy only at depot. Further in papers like [2], [3], [7] etc., authors discuss and control the eventual variance of the error between the ideal schedule and the schedule considering random fluctuations, when the number of stops and buses converges to infinity. They assume no bunching. However in many scenarios it is not possible to completely avoid bunching. Further when the randomness is very high, it is almost impossible to adhere to the ideal schedules. In such scenarios, it is rather important to reduce the probability of bunching, and we precisely consider this probability. For such highly random scenarios, it is also important to consider the passenger waiting times. Further in all scenarios the number of stops and the number of trips is finite, hence such a modelling more realistic. In [1], authors consider finite number of buses looping continuously in the circular path and covering finite number of stops. However they work with expected value of the squared difference between the actual headway and the supposed headway, and not with the probability of bunching. Further the average passenger waiting times are defined as the expected value of the product of the headway and the number of passengers arrived during the headway. This definition does not consider the influence of passenger arrivals spread over the entire (bus inter-arrival) interval. We consider (customer) average of the waiting times of the passengers, the time gaps between their arrival epoch to the stop and the arrival epoch of the bus that they board. We derive theoretical expression for fluid arrivals, we also demonstrate through simulations that the derived expressions well approximate that corresponding to Poisson arrivals.

The main goal of this work is to derive optimal headway between the buses at depot, that minimizes a convex combination of two costs: a) the average passenger waiting times; and b) the probability of bunching. Using our results one can obtain a bus frequency optimal for stationary performance as well as the one that optimizes the performance for finite number of trips. Both the optimizers are the same once the number of trips is greater than the number of stops.

II. System Model

We consider buses moving on a single route and this route has M number of stops $Q_1, Q_2, \ldots, Q_M$. Each stop has infinite waiting capacity. Any bus starts at the depot (Q_0) and travels along a predefined cyclic path, while boarding/de-boarding passengers at the encountered bus stops. Passengers arrive independently of others in each stop Q_i according to a fluid process with rate λ. We also consider Poisson arrivals and show that the fluid model can well approximate the system with Poisson arrivals, using simulations. The passengers board the bus at 'fluid' rate. That is, the time taken to board x number of passengers equals bx, where b is the boarding time per passenger at any stop. Let S_k^i be the time taken by k-th bus to travel from $(i-1)$- th stop Q_{i-1} to i-th stop Q_i. These random travel times, $\{S_k^i\}_k$ (for each stop Q_i), are independent and identically distributed (IID). The buses start at the depot after fixed headway times (interval between two successive buses) and traverse through the M stops before concluding their trip. The main purpose of this paper is to obtain this depot-headway optimally. We further make the following assumptions to model the problem:

R.1) Surplus number of buses: the next bus can start at any specified headway in the depot, without having to wait for the return of the previous bus.

R.2) Parallel boarding and de-boarding: and the time taken for de-boarding is smaller with probability one.

R.3) Gated service: Only the passengers that arrived before the arrival of the bus can board the bus. Passengers arrived during the boarding process, will wait for the next bus.

R.4) If buses are bunched at any stop then the second bus will wait till the previous bus departs, before it starts boarding its passengers. Thus overtaking of buses does not happen.

R.5) There is no constraint on the capacity of the buses.

In most of the cities the buses have two doors, boarding and de-boarding happens in parallel and hence one can neglect the de-boarding times. Boarding times are negligible compared to bus travel times. Thus we will have negligible number of arrivals during a boarding time, and hence gated service is a reasonable assumption. Assumption **R**.*1 simplifies the model sufficiently and is instrumental in deriving closed-form expressions for performance measures of this kind of a complicated system.* Without this assumption one would have to take care of looping effects. Assumption of availability of few extra buses can easily ensure this assumption is satisfied. Even availability of one extra bus can ensure such a condition is satisfied with sufficiently large probability, and is often a well practised method. The remaining assumptions are mentioned as and when required.

Bunching: Because of variability in demand and travelling times, some buses are delayed with respect to the scheduled times. Delay of the bus results in more number of passengers to be boarded, and hence longer dwell time at the stop. Thus it gets delayed further. This would also imply smaller dwell time for the following bus at the same stop, as it has to board lesser number of passengers. This can continue for the following bus-stops, the bus-stop headway times (time gap between two consequent buses at the same stop) become smaller and can eventually become zero. This phenomenon is called Bus bunching. We define bunching probability as the probability of occurrence of this event. *Bunching probability for the k-th trip at i-th stop, b_k^i, is defined as the probability that the dwell time of $(k-1)$-th bus at stop Q_i is greater than the inter arrival time between $(k-1)$ and k-th buses to the same stop.*

Waiting times: Passengers wait for the bus at every bus stop. When a bus is delayed their waiting time increases. If the delay results in bunching, the waiting times can be longer. Further the waiting times also depend upon the depot-headway time. The larger this headway is, the longer are the waiting times. We define the (passenger) waiting times as the time difference between their arrival instance and the arrival instance of the bus in which they board. Let W_n^i be the waiting time of the n-th passenger that arrived to bus stop Q_i. We define the customer average of these waiting times specific to stop Q_i by:

$$\bar{w}^i := \lim_{N \to \infty} \frac{1}{N} \sum_{n \le N} W_n^i. \tag{1}$$

When the depot-headway increases bunching happens less often. However the passengers have to wait longer. We precisely study this trade-off. We derive the required performance measures, and, obtain headway times that optimize a convex combination of the two performance measures.

III. System Dynamics

Dwell time is the amount of time spent by a bus in the stop. By **R**.1, it equals total boarding time of passengers waiting at the bus stop. Let X_k^i be the number of passengers waiting at stop Q_i, at the arrival instance of k-th bus. Because of gated service and fluid boarding at rate b the dwell time equals:

$$V_k^i = X_k^i b. \tag{2}$$

By our assumption **R**.4, the buses serve one after the other. That is, even in the event of bunching, the trailing bus starts boarding its customers only after the preceding one departs. In such events the dwell times would be bigger than $X_k^i b$. However practically one design systems with small bunching chances (typically less than 10%) and hence we neglect the influence of these extra terms in the dwell times.

A. Bus inter-arrival times

Let $\mathcal{N}^i(I)$ be the number of passengers that arrived in an interval of length I at Q_i for any $i \leq M$. For fluid arrivals this equals $\mathcal{N}^i(I) = \lambda I$, while, for Poisson arrivals $\mathcal{N}^i(I)$ is Poisson distributed with parameter λI.

Recall by definition that X_k^i passengers wait at Q_i and are served by the k-th bus. Also, note that the passengers that arrived during the dwell time ($V_k^i = X_k^i b$) of k-th bus would be served by $(k+1)$-th bus. We begin with stop Q_1. Recall S_k^1 represents the travel time between depot and stop Q_1 for the k-th bus. Further the k-th bus departs the depot after $(k-1)$-th bus, with a time gap equal to the (depot) headway time h. Thus the inter arrival time between $(k-1)$-th bus and k-th bus at Q_1 equals, $I_k^1 = h + S_k^1 - S_{k-1}^1$. Thus the number of passengers served by k-th bus at Q_1 equals, $X_k^1 = \mathcal{N}^1(I_k^1)$.

It is clear that the k-th bus takes, $\sum_{j \leq i} S_k^j + \sum_{j<i} V_k^j$ time, to reach stop Q_i after its departure at depot, and the time gap between departures of k-th and $(k-1)$-th buses at depot equals h. Thus the inter-arrival time between k-th and $k-1$-th buses at stop Q_i,

$$I_k^i = h + \sum_{j \leq i} S_k^j + \sum_{j<i} V_k^j - \sum_{j \leq i} S_{k-1}^j - \sum_{j<i} V_{k-1}^j. \quad (3)$$

Hence the number of passengers waiting at stop Q_i at the arrival instance of k-bus equals, $X_k^i = \mathcal{N}^i(I_k^i)$.

There are random variations in travel times. We model these variations by Gaussian random variables. To be precise we assume the travel time by k-th bus between i and $(i-1)$-th bus stop to be $S_k^i = s^i + N_k^i$, where $\{N_k^i\}_{i,k}$ are IID Gaussian random variables with mean zero and variance ϵ^2 and $\{s^i\}_i$ are constants. For fluid arrivals and Gaussian travel times, the inter arrival times from equation (3) are (with $\rho := \lambda b$):

$$\begin{aligned} I_k^i &= h + \sum_{j \leq i} N_k^j + \sum_{j<i} V_k^j - \sum_{j \leq i} N_{k-1}^j - \sum_{j<i} V_{k-1}^j \\ &= h + \sum_{j \leq i} N_k^j + \rho \sum_{j<i} \left(I_k^j - I_{k-1}^j \right) - \sum_{j \leq i} N_{k-1}^j. \quad (4) \end{aligned}$$

In the above, by notation, we set $N_k^i = 0$ for all $k \leq 0$. Thus even with independent travel times (and passenger arrivals), the bus inter-arrival times and hence the dwell times are correlated. One needs to study these correlations to obtain the performance, and we begin with the following:

Lemma 1. *Define the following Gaussian vectors:*

$$\mathbb{N}_k := [N_k^1, N_k^2, \cdots, N_k^M] \text{ and } \mathbb{N}_l^k := [\mathbb{N}_k, \mathbb{N}_{k-1}, \cdots, \mathbb{N}_l].$$

Then from (4), I_k^i for any i, k can be expressed as:

$$\begin{aligned} I_k^i &= I_k^i(\mathbb{N}_{k-i}^k) = h + N_k^i - N_{k-1}^i + \sum_{s=1}^{i-1} \Bigg[(1+\rho)^s N_k^{i-s} \\ &\quad + \sum_{r=1}^{s+1} (-1)^r N_{k-r}^{i-s} \sum_{l=0}^{i-r} \binom{s}{l+r-1} \binom{l+r}{r} \rho^{l+r-1} \Bigg]. \end{aligned}$$

Thus for any stop Q_i, bus inter-arrival times $\{I_k^i\}$ are Gaussian random variables with common mean h and common variance $\sigma_{a,i}^2$ once[1] $k > i$, with ($\binom{n}{r} = 0$ if $n < r$),

$$\begin{aligned} \sigma_{a,i}^2 &= E[I_k^i - h]^2 = 2\epsilon^2 + \epsilon^2 \sum_{s=1}^{i-1} \Bigg[(1+\rho)^{2s} + \sum_{r=1}^{s+1} \\ &\quad \left(\sum_{l=0}^{i-r} \binom{s}{l+r-1} \binom{l+r}{r} \rho^{l+r-1} \right)^2 \Bigg]. \end{aligned}$$

Proof: is in Appendix A. ■

B. Transience, Stationarity and Time (Trip) averages

The bus inter-arrival times $\{I_k^i\}_k$, for any given stop Q_i, are not independent as seen from equation (4). Nevertheless by Lemma 1 inter-arrival times of a trip, $\mathbb{I}_k := [I_k^1, I_k^2, \cdots, I_k^M]$, depend only upon $\mathbb{N}_{k-M}^k$ and so the sampled inter-arrival times

$$\{I_{j+kl}^i\}_{k \geq 1} = I_{l+j}^1, I_{2l+j}^l \cdots (\text{ for any stop } Q_i),$$

with $l > M+1$ and for any $0 \leq j \leq l-1$ form an IID sequence.

Thus one of the important aspects that is clear from Lemma 1 is that the *system is in transience only for the first M trips.* After the M-th trip, the trips (are not independent, but) are identical. Thus stationarity is reached within M trips. To be precise, any expected performance measure related to a single trip is the same for all trips other than the first M trips. Thus *the transient performance after M trips equals the stationary performance.* Passenger waiting times related to a trip can be one such example performance measure (more details to follow). On the other hand for performance measures like bunching probability, which depend upon two consecutive trips (details to follow), stationarity is reached after $M+1$ trips. Hence again the bunching probability of k-th trip equals stationary bunching probability, for all $k > M+1$.

One can also derive the time (trip) average of any performance measure, using the above 'block' IID characteristics. By Law of large numbers, for any (integrable) performance f that depends (for example) upon one trip (almost surely (a.s.)):

$$\bar{f} := \lim_{K \to \infty} \frac{1}{K} \sum_{k=1}^{K} f(\mathbb{I}_k) \quad (5)$$

$$= \lim_{K \to \infty} \frac{1}{K(M+1)} \sum_{j=1}^{M+1} \sum_{k=1}^{K} f(\mathbb{I}_{j+k(M+1)}) \stackrel{a.s.}{=} E[f(\mathbb{I}_{M+1})].$$

In the above the expectation is with respect to the Gaussian measure of Lemma 1. Thus the time average performance also equals (a.s.) the stationary as well as the transient performance (with trips $> M$). One can derive trip average of the performance measures that depend upon finite number of consecutive trips (e.g., bunching probability) in a similar way.

[1] The variance for smaller k is different but can easily be computed, if required.

IV. Performance and optimization

A. Passenger waiting times

As already mentioned, waiting times are the times for which a typical passenger waits before its bus arrives. We first discuss the trip-wise passenger waiting times. Towards this we first gather together the waiting times of the passengers that arrived during one (bus) inter-arrival time. Recall X_k^i is the number of passengers that arrived during k-th trip, i.e., during the inter arrival time between the k-th and $(k-1)$-th buses at Q_i, and let the sum of the waiting times of the passengers that arrived during this trip be:

$$\bar{W}_k^i \triangleq \sum_{n=1}^{X_k^i} W_n^i.$$

For fluid arrivals, customers are assumed to arrive at regular intervals (of length $1/\lambda$) and if the time duration for this arrivals is large in comparison with $1/\lambda$ then $X_k^i \approx \lambda I_k^i$ and the sum of the waiting times[2], $\bar{W}_k^i \approx \lambda (I_k^i)^2/2$. For Poisson arrivals, due to memoryless property, the conditional expectation (see [8]) $E\left[\bar{W}_k^i \middle| I_k^i\right] = \lambda (I_k^i)^2/2.$

In all, the sum of waiting times of passengers of stop Q_i in the k-trip equals $E[\bar{W}_k^i] = \lambda E[(I_k^i)^2]/2$.

Customer average: This performance is important from the perspective of passengers and hence it is more appropriate to consider the 'passenger' average of the waiting times ($\{\bar{w}^i\}$) defined in (1). For fluid arrivals, the trip-wise sum waiting times equal (in the limit) $\bar{W}_k^i \approx \lambda (I_k^i)^2/2$, and using this we obtain:

Lemma 2. *For any stop Q_i, with $\bar{W}_k^i = \lambda (I_k^i)^2/2$:*

$$\bar{w}^i = \lim_{K\to\infty} \frac{K}{\sum_{k=1}^K X_k^i} \frac{\sum_{k=1}^K \bar{W}_k^i}{K} \stackrel{a.s.}{=} \frac{E[(I_k^i)^2]}{2E[I_k^i]} = \frac{\sigma_{a,i}^2 + h^2}{2h}, \qquad (7)$$

for any $k > M$.

Proof: is in Appendix A. ∎

We would like to give equal importance to passengers of all stops. Hence we consider the following for optimization purposes:

$$\bar{w} = \sum_{i=1}^M \bar{w}^i = \sum_{i=1}^M \frac{\sigma_{a,i}^2 + h^2}{2h}. \qquad (8)$$

B. Bunching probability

Bunching occurs at a stop when two buses meet at the stop, i.e., when the headway time (time gap between buses) of consequent buses becomes zero. Bunching probability, b_k^i, of k-th bus at i-th stop is the probability that the dwell time (equation (2)) of $(k-1)$-th bus is greater than the inter arrival time (equation (4)) between $(k-1)$ and k-th buses. Thus for fluid arrivals:

$$b_k^i = P(\mathcal{N}^i(I_{k-1}^i)b > I_k^i) \; = \; P(I_k^i - \rho I_{k-1}^i < 0). \qquad (9)$$

The above expression is true because of assumption **R**.4. Thus we require the (marginal) distribution of $(I_k^i - \rho I_{k-1}^i)$ for computing the bunching probability at any stop i. Hence we consider the analysis of $I_k^i - \rho I_{k-1}^i$ in the lemma below.

Lemma 3. *The term $I_k^i - \rho I_{k-1}^i$ is Gaussian and can be expressed as:*

$$I_k^i - \rho I_{k-1}^i = h(1-\rho) + N_k^i - (1+\rho)N_{k-1}^i + \rho N_{k-2}^i + \sum_{r=1}^{i-1} \Bigg\{ (1+\rho)^r N_k^{i-r} + \sum_{l=1}^{r+2} (-1)^l N_{k-l}^{i-r} \Bigg[\binom{r+1}{l-1} \rho^{l-1}(1+\rho)^{(r+1)-(l-1)} + \binom{r}{l} \rho^l (1+\rho)^{(r+1)-(l+1)} \Bigg] \Bigg\}.$$

Thus the mean and variance of $I_k^i - \rho I_{k-1}^i$ (with $k > M+1$) are given respectively by $E[I_k^i - \rho I_{k-1}^i] = h(1-\rho)$ and

$$\sigma_{b,i}^2 = \epsilon^2 + \epsilon^2(1+\rho)^2 + \epsilon^2\rho^2 + \epsilon^2 \sum_{r=1}^{i-1} \Bigg\{ (1+\rho)^{2r} + \sum_{l=1}^{r+2} \Bigg[\binom{r+1}{l-1} \rho^{l-1}(1+\rho)^{(r+1)-(l-1)} + \binom{r}{l} \rho^l (1+\rho)^{(r+1)-(l+1)} \Bigg]^2 \Bigg\}.$$

Proof: is in Appendix A. ∎

As already discussed, various trips can be correlated, however the bunching probabilities in different trips remains the same. This is because the bunching probabilities depend only upon $I_k^i - \rho I_{k-1}^i$ and because these are identically distributed for all $k > M+1$. For all such trips the bunching probability of a stop in a trip is the same and equals,

$$b_k^i = 1 - \Phi\left(\frac{h(1-\rho)}{\sigma_{b,i}}\right), \qquad (10)$$

where Φ is the standard Gaussian cumulative distribution:

$$\Phi(x) := \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-t^2}{2}\right) dt.$$

As mentioned already, this also represents *the bunching probability of a trip under stationarity.* Note that the bunching probabilities of initial trips can be different, and these can be computed in a similar way if required.

C. Total cost and optimization

Our aim is to minimize a joint cost that considers both the factors (8) and (10). Towards this we consider a weighted average of the two costs with α and $\{\beta_i\}_i$ representing the weights for various components as below:

$$T = \sum_{i=1}^M \left\{ \frac{\sigma_{a,i}^2 + h^2}{2h} + \alpha\beta_i \left(1 - \Phi\left(\frac{h(1-\rho)}{\sigma_{b,i}}\right)\right) \right\}. \qquad (11)$$

Here α is the trade-off parameter between bunching probability and waiting times, while $\{\beta_i\}_i$ determine the trade-off

[2] The waiting time of first passenger during that period is approximately I_k^i, that of the second passenger is approximately $I_k^i - 1/\lambda$ and so on. As $\lambda \to \infty$, the following Riemann sum converges as below:

$$\frac{\bar{W}_k^i}{\lambda} = \frac{1}{\lambda} \sum_{i=0}^{\lambda I_k^i} \left(I_k^i - \frac{i}{\lambda}\right) \to \int_0^{I_k^i} (I_k^i - x)dx = \frac{(I_k^i)^2}{2}. \qquad (6)$$

In the above the residual passenger waiting time at the bus-arrival epoch is neglected and this also becomes negligible with $\lambda \to \infty$. Thus for large λ, $\bar{W}_k^i \approx \lambda (I_k^i)^2/2$.

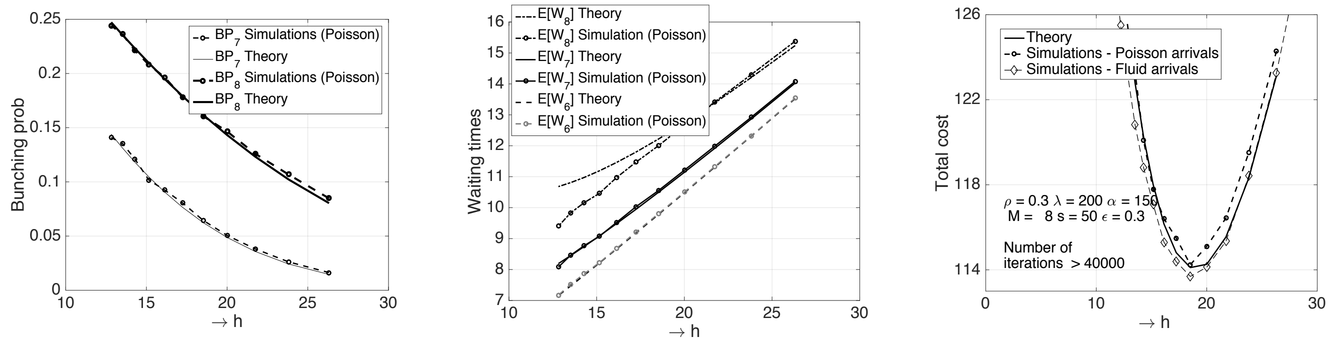

Fig. 1. Bunching probability, Waiting time and Total cost vs headway

for bunching probabilities of various stops[3]. Let h^* be the minimizer for total cost (11). The total cost is a differentiable function and hence h^* is the zero of the following derivative (obtained using Leibniz rule):

$$\frac{dT(h)}{dh} = \frac{M}{2} - \sum_{i=1}^{M} \left(\frac{\sigma_{a,i}^2}{2h^2} + \frac{\alpha\beta_i}{\sqrt{2\pi}} \exp\left(\frac{-h^2(1-\rho)^2}{2\sigma_{b,i}^2} \right) \frac{1-\rho}{\sigma_{b,i}} \right). \quad (12)$$

Also, one can easily verify that $d^2T(h)/dh^2 > 0$ for all h, and hence that h^* *is the unique minimizer.*

We consider the special case with $\beta_M = 1$ and $\beta_m = 0$ for all $m \neq M$ to obtain a good approximation for the above optimizer. Under suitable conditions (e.g., when the depot headway time is large) one can neglect the second term in the derivative to obtain approximate h^* as below:

$$h^* \approx \sqrt{\frac{-\log(C_1)}{C_2}},\ C_1 = \frac{M\sqrt{2\pi}\sigma_{b,M}}{2\alpha(1-\rho)},\ C_2 = \frac{(1-\rho)^2}{2\sigma_{b,M}^2}. \quad (13)$$

Remarks: It is immediately clear that the optimal bus frequency (inverse of h^*) decreases as the number of stops increase. This is in fact true even for the general case, as seen from (12). Similarly, the optimal frequency decreases with increase in traffic variability factor ϵ (see (13)). We compute the optimizers for remaining cases using numerical computations in the next section to derive some more interesting inferences.

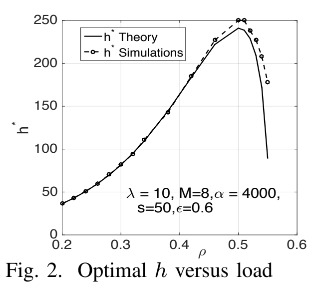

Fig. 2. Optimal h versus load

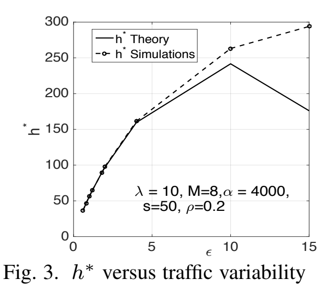

Fig. 3. h^* versus traffic variability

V. Simulations

In this section, we verify the derived performance measures of the proposed model through Monte-Carlo simulations. We emulate the buses travelling on a single route with 8 bus stops, boarding a random number of passengers using gated service and avoiding parallel boarding (into two or more buses simultaneously) of passengers at any stop. The travelling times between stops are perturbed by normally distributed noise with zero mean and variance ϵ^2. Passenger arrivals are either due to fluid arrivals or due to Poisson arrivals, and we consider fluid boarding. We conduct the simulations with $M = 8, \lambda = 200,\ \rho = 0.3, s = 50$ and $\alpha = 150$ in Figure 1.

In Figure 1, we compared the theoretical quantities with the ones estimated through simulations. We plot bunching probability, average passenger waiting times and total cost respectively as a function of headway. We find a good match between theory (curves without markers) and simulations considering Poisson arrivals (curves with circular markers). We also conducted simulations using fluid arrivals. The simulation results with fluid arrivals better match the theoretical counterparts. We included the simulation based results for fluid arrivals (curves with diamond markers) in the third sub-figure, i.e., for total cost. We notice a good match between (both) the simulated quantities and the theoretical expressions in majority of the cases. These observations affirm the theory derived.

From the sub-figures of Figure 1, we observe that the bunching probability improves (decreases) with depot-headway (h), while, the passenger waiting times degrade (increase). This is the *inherent trade-off that needs to be considered to design an efficient system.* We plot optimal depot-headways for some examples in Figures 2-3, which are estimated using numerical simulations (dashed curves with circular markers). We also plot the approximate h^* given by (13) in the same figures (solid curves). We notice that the approximate optimizer well matches the ones estimated using numerical simulations, for small load factors (ρ) and or small traffic variability (ϵ). When variability increases the first factor in waiting times (8) becomes significant and then the theoretical h^* (13) is no more a good approximation (for load factors bigger than 0.5 in Figure 2 and traffic variance $\epsilon > 5$ in Figure 3).

As the load factor increases, or equivalently when the customer arrival rates increase, one would anticipate an increased bus-frequency to be optimal. On the contrary we notice that the optimal headway increases (initially) with increase in load-factor in Figure 2. This is because the passenger waiting times for any given headway h, approximately equal $h/2$ when the variability components ($\{\sigma_{a,i}\}_i$ due to traffic and or load

[3]Note that the bunching probability is low at initial stops and increases with stop number (see Lemma 3) and hence the need for different $\{\beta_i\}_i$.

conditions) are negligible (see equation 8). Thus with increase in load factor, the bunching probabilities increase sharply, while waiting times are less influenced and hence an increase in optimal depot-headway. However as seen in the same figure, when load increases beyond 0.5, the variability components in waiting times also become significant and now we notice that the optimal depot-headways are smaller. To summarize, the optimal frequency of the buses decreases initially with increase in load and for higher range of load factors it increases with load.

On the other hand, with increase in traffic variability factor ϵ, we notice that the optimal depot-headway always increases (see Figure 3). Thus it is optimal to decrease the frequency of buses, when traffic variability increases. These results are true as long as the buses have sufficient capacity (to board all the customers).

CONCLUSIONS

We modelled the bus bunching problem with Gaussian bus travel times and fluid arrivals. We studied the related performance measures. We discussed stationary as well as transient (suitable for finite trip problems) performance measures. Using numerical simulations, we showed that the performance of the system with Poisson arrivals can be well approximated with the derived theoretical expressions, when the arrival rates are large. We obtained the optimal depot headway time, i.e., the optimal bus frequency as a function of parameters like load conditions (passenger arrival rates), number of bus stops, traffic variability conditions (variance of the travel times) etc. We made the following observations using the theoretical as well as numerical study: a) When bus frequency decreases, bunching probability decreases and passenger waiting times increase; b) Optimal bus frequency decreases with increase in traffic variability and load conditions; and c) If the load is significantly larger, then the optimal bus frequency actually increases with load.

These are just initial results and we have many future directions. Previously in literature the focus mostly has been on ensuring that the bus schedules adhere as closely as possible to the ideal schedules. Towards this they consider optimal holding of buses at various stops and or skipping of stops. But the passengers might be uncomfortable with such strategies. Further our focus is on systems where bunching can't be avoided completely. Thus our focus has been on reducing the bunching probability by (static) controlling only the depot-headway times. In future we would like to consider dynamic policies which control the bus frequency based on the state of the system. We would also like to derive the performance for non-stationary (Markov modulated) Poisson arrivals or even for the renewal arrival process.

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APPENDIX A: PROOFS

Proof of Lemma 1: The proof is based on mathematical induction. We can easily verify that it is true for $i = 1$,

$$I_k^1 = h + N_k^1 - N_{k-1}^1.$$

It is also true for $i = 2$,

$$I_k^2 = h + N_k^2 - N_{k-1}^2 + (1+\rho)N_k^1 - N_{k-1}^1(1+2\rho) + \rho N_{k-2}^1.$$

Assuming that the result is true for $i = m$, we prove it for $i = m+1$. We begin by observing a simple relation as below (see equation (4)):

$$I_k^{m+1} \;\; = \;\; N_k^{m+1} - N_{k-1}^{m+1} + (1+\rho)I_k^m - \rho I_{k-1}^m.$$

The rest of the proof is in [8]. ■

Proof of Lemma 2: By law of large numbers and the block IID structure described in section IV-A,

$$\lim_{K\to\infty} \frac{\sum_{k=1}^K \bar{W}_k^i}{K} \to E[\bar{W}_k^i] \text{ and } \lim_{K\to\infty} \frac{X_K^i}{K} \to \lambda E[I_k^i].$$

Hence,

$$\bar{w}^i = \frac{E[(I_k^i)^2]}{2E[I_k^i]}.$$ ■

Proof of Lemma 3: The proof is again based on mathematical induction. We can easily verify that it is true for $i = 1$,

$$I_k^1 - \rho I_{k-1}^1 = h(1-\rho) + N_k^1 - (1+\rho)N_{k-1}^1 + \rho N_{k-2}^1.$$

It is also true for $i = 2$,

$$I_k^1 - \rho I_{k-1}^1 = h(1-\rho) + N_k^2 - (1+\rho)N_{k-1}^2 + \rho N_{k-2}^2 + N_k^1 (1+\rho) - N_{k-1}^1(1+3\rho+\rho^2) + N_{k-2}^1(2\rho+2\rho^2) - N_{k-3}^1\rho^2.$$

Assuming that the result is true for $i = m$, we prove it for $i = m+1$. Again the proof is based on the following observation:

$$I_k^{m+1} - \rho I_{k-1}^{m+1} \;\; = \;\; N_k^{m+1} - (1+\rho)N_{k-1}^{m+1} + \rho N_{k-2}^{m+1} + (1+\rho)(I_k^m - \rho I_{k-1}^m) - \rho(I_{k-1}^m - \rho I_{k-2}^m).$$

The rest of the proof is in [8]. ■