Convergence of trajectories and optimal buffer sizing for MIMD congestion control

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Abstract

We study the interaction between the MIMD (Multiplicative Increase Multiplicative Decrease) congestion control and a bottleneck router with Drop Tail buffer. We consider the problem in the framework of deterministic hybrid models. We study conditions under which the system trajectories converge to limiting cycles with a single jump. Following that, we consider the problem of the optimal buffer sizing in the framework of multi-criteria optimization in which the Lagrange function corresponds to a linear combination of the average throughput and the average delay in the queue. As case studies, we consider the Slow Start phase of TCP New Reno and Scalable TCP for high speed networks.

Keywords: Deterministic hybrid model, Stability, Pareto set, Optimization

1 Introduction

Most traffic in the Internet is governed by TCP/IP (Transmission Control Protocol and Internet Protocol) [1, 15]. TCP protocol tries to adjust the sending rate of a source to match the available bandwidth along the path. The current TCP New Reno uses MIMD congestion control during the initial Slow Start phase and AIMD (Additive Increase Multiplicative Decrease) congestion control during the principal Congestion Avoidance phase. In the AIMD congestion control

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scheme in the absence of congestion signals from the network, TCP enlarges the congestion window linearly in round trip times and, upon the reception of a congestion signal, TCP reduces the congestion window by a multiplicative factor. In the MIMD congestion control scheme in the absence of congestion signals from the network TCP enlarges the congestion window exponentially in round trip times.

A significant increase of link capacities has posed a challenge to the current TCP implementation. The current TCP New Reno version is not able to utilise efficiently high speed links [12]. To mitigate this problem, several new TCP versions (HS-TCP, FAST-TCP, Scalable TCP, H-TCP, CUBIC-TCP, BIC-TCP for example) have been proposed [12, 16, 17, 18, 25, 30]. These algorithms have in common that in the absence of congestion, the sources enlarge the congestion window in a much more aggressive fashion than the standard TCP New Reno does. An extensive overview and comparison of different TCP versions for high capacity links is given in [19]. In the present work we analyze the MIMD congestion control which is a base for Scalable TCP [17].

On the other hand, most of the routers in the Internet are of Drop Tail type. In basic Drop Tail routers, apart from the router capacity, the buffer size is the only parameter to be tuned. In fact, the buffer size is one of the few parameters of the TCP/IP network that can be managed by network operators. This makes the choice of the router buffer size very important in the TCP/IP network design. This choice has recently received considerable attention [3, 4, 5, 6, 7, 11, 13, 22, 23, 24, 26, 27, 28, 29]. (This is far from an exhaustive list of relevant references.) However, most of these works study only the AIMD congestion control algorithm.

In this paper we study the interaction of MIMD congestion control algorithms with Drop Tail buffers. We consider the problem in the framework of deterministic hybrid models, which describe systems with both discrete and continuous behavior. Recently, hybrid models have been successfully applied to the modeling of communication networks [4, 5, 7, 8, 14]. The model in the present paper is a significant extension of the models in [7]. In particular, in [7], the Round Trip Time (RTT) is regarded ignorably small, so that there is no delay between sending data out and receiving the corresponding acknowledgements. This means that as soon as the buffer is filled full, there will be an instantaneous multiplicative reduction (without any delay) on the sending rate. In comparison, in the current work, as will be seen in Section 2, we take accurately into account the time-varying nature of the RTT, resulting in a time-varying delay between sending out data and receiving corresponding acknowledgements. The present more accurate model allows us to provide conditions for the absence of multiple subsequent reductions of the congestion window and estimate more accurately the minimal buffer size for the full link utilization. Furthermore, we recommend the use of the Delayed Ack mechanism [1] and the reduction of the window growth parameter in order to avoid the undesirable regime of subsequent window reductions. Additionally to the analytical expression for the minimal buffer size for the full link utilization, we construct the Pareto set to achieve the trade off between the high link utilization and small queueing delays. In particular, our results suggest that in order to achieve high utilization, one can size the buffer much smaller than the bandwidth-delay product. Our analytical results are confirmed by NS simulations [20].

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2 Mathematical model

Consider a long-lived MIMD TCP connection that sends data through a bottleneck router. Denote by $w(t)$ the instantaneous congestion window of the TCP connection at time $t \in [0, \infty)$. Let $x(t)$ be the amount of data in the bottleneck queue at time t, $B > 0$ be the size of the Drop Tail buffer, and μ be the capacity of the bottleneck router.

If $x(t) < B$, the evolution of $w(t)$ is given by differential equation

$$\frac{dw}{dt} = \frac{mw}{T + x(t)/\mu}. \tag{1}$$

Here T is the two way propagation delay and m being a constant, is some fixed multiplicative factor. Note that $T + x(t)/\mu$ corresponds to the RTT at time moment t.

The sending rate of the window based congestion control is given by

$$\lambda(t) = \frac{w(t)}{T + x(t)/\mu}. \tag{2}$$

We emphasize that the time parameter t corresponds to the local time observed at the router.

When x reaches B at time t^*, i.e. $x(t^*) = B$, the buffer starts to overflow. The overflow of the buffer will be noticed by the sender only after the time delay $\delta = T + B/\mu$. Upon the reception of the congestion signal at time $t^* + \delta$, the congestion window is reduced according to

$$w(t^* + \delta + 0) = \beta^k w(t^* + \delta - 0). \tag{3}$$

Usually, $k = 1$, but sometimes it is necessary to send several congestion signals in order to reduce the sending rate below the transmission capacity of the bottleneck router.

Therefore, between the instantaneous jumps of the congestion window w, we have the dynamical system

$$\dot{x} = \begin{cases} \lambda(t) - \mu, & \text{if } 0 < x(t) < B, \text{ or } x(t) = 0 \text{ and } \lambda(t) \geq \mu, \\ & \text{or } x(t) = B \text{ and } \lambda(t) \leq \mu; \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

where $\lambda(t)$ is given by (2).

Let us discuss particular parameter settings. Curently, the MIMD congestion control mechanism is used in:

(a) Slow Start regime [1] in the standard TCP New Reno;

(b) Scalable TCP [17] for high speed links.

In the Slow Start regime we have $\beta = 0.5$. The value of m depends on whether the Delayed Ack mechanism [1] is enabled or not. If the Delayed Ack mechanism is enabled, $m = 0.5$, and if it is not enabled, $m = 1$.

In Scalable TCP we have $\beta = 0.875$ and $m = 0.01$.

We would like to recall that a similar hybrid model can be used to study the AIMD congestion control [5, 8, 14]. One only needs to change equation (1) to the following equation

$$\frac{dw}{dt} = \frac{M}{T + x(t)/\mu}.$$

The AIMD congestion control is used in the principal Congestion Avoidance regime of TCP New Reno. In this case, we have $\beta = 0.5$, and M is equal to the half packet size if the Delayed Ack mechanism is enabled, and otherwise M is equal to the packet size.

3 Convergence to limiting cycles

Let us first begin with some definitions.

Definition 1 *A cycle is defined as the trajectory starting with the initial state* $w(0) = w_0 = W_0 \in [\beta(\mu T+B), \mu T+B)$, $x(0) = x_0 = B$ *at* $t = 0$, *and reaching the same point for the first time at some moment* T_{cycle}, *called the duration of the cycle. Note that* $T_{cycle} > \delta = T + B/\mu$ *because* $W_0 < B + \mu T$. *A cycle with* $x(t)$ *staying at zero for a positive time interval is called clipped. Otherwise it is unclipped. In particular, a cycle with* $x(t)$ *staying at* 0 *at a single time moment is called critical, and it is referred to as an unclipped cycle. In addition, a cycle, possibly with more than one instantaneous jump though (i.e.* $k > 1$ *in (3)), is called simple, if it has only one loop (one convex time interval containing no jumps (3)). Otherwise, it is called complicated.*

The case of a simple cycle with $k = 1$ is most interesting because in this case we avoid multiple subsequent packet losses. Such a cycle will be called a $1-$cycle or a cycle of order one. In the general case, a simple cycle is called $k-$cycle (a cycle of order k).

Let

$$B^* = \mu T \frac{1 - e^{m+1}\beta^{\frac{m+1}{m}} - (m+1)e^m\left(1 - e\beta^{\frac{1}{m}}\right)\beta e^{ms_1}}{(m+1)e^m\left(1 - e\beta^{\frac{1}{m}}\right)\beta e^{ms_1}}, \tag{5}$$

where

$$s_1 = \frac{1}{m+1}\ln\left(\frac{1 - e^m\beta}{\beta e^m m\left(1 - e\beta^{\frac{1}{m}}\right)}\right). \tag{6}$$

Theorem 1 *(a) For an arbitrary* $B > B^*$, *the system trajectory converges to the limiting unclipped 1-cycle from an arbitrary initial state iff* $\beta < \bar{\beta}$, *where* $\bar{\beta}$ *is the single solution to*

$$(m+1)e^m\bar{\beta}(1 - e\bar{\beta}^{\frac{2}{m}}) + e^{m+1}\bar{\beta}^{2(1+\frac{1}{m})} = 1 \tag{7}$$

in the interval $(0, e^{-m})$.

(b) Suppose $B = B^*$. *Then the limiting cycle is of order one and critical iff* $\beta < e^{-m}$.

(c) Suppose $B < B^*$. *Then the limiting cycle is of order one and clipped iff* $\beta < e^{-m}$.

In cases (b) and (c), the system trajectory also converges to the limiting cycle from an arbitrary initial state.

A simple $1-$*cycle (clipped or unclipped) exists iff* $\beta e^m < 1$.

Remark 1 *According to the proofs given in the Appendix, in case (a), condition* $\beta < \bar{\beta}$ *can be relaxed to* $\beta < e^{-m}$ *sacrificing the convergence from an arbitrary initial state. Specifically, if* $\beta < e^{-m}$, $B > B^*$, *and* $w_0 \in [\beta e^m(B + \mu T), B + \mu T)$, *then the system trajectory converges to the limiting cycle, which is of order one and unclipped.*

Inequality $\beta < \bar{\beta}$ is a sufficient condition for the convergence from an arbitrary initial state in all three cases of Theorem 1.

Suppose $\bar{\beta} \le \beta < e^{-m}$. According to the proof of Theorem 1, in case $B > B^*$ a trajectory does not converge to the limiting unclipped $1-$cycle iff after each series of jumps $w(t^* + \delta + 0) < \beta e^m(B + \mu T)$. In this situation, double jumps always happen, so that one can use the developed theory with β being replaced with β^2. As a result, one can face only the convergence to a simple $2-$cycle which can be unclipped, critical or clipped. Complicated cycles never appear.

In particular, the above theorem implies that the buffer size B^* is the minimal buffer size for the full link utilization. The following asymptotics holds for small values of m

$$B^*(m) = \mu T \frac{(1-\beta + m\ln(m))}{\beta} + \mathrm{o}(m\ln(m)). \tag{8}$$

The asymptotics (8) can be verified by the application of the L'Hôpital's rule. The asymptotics (8) together with the exact expression (5) can be considered as an improvement of the results presented in [7, 10]. In particular, for Scalable TCP the above asymptotics gives $B^* \approx 0.09\mu T$. Thus, a single Scalable TCP connection requires about 10 times less buffer space than a standard TCP New Reno connection, which requires up to μT buffer space [27].

Note that in the Slow Start phase of TCP New Reno without the Delayed Ack mechanism [1] the condition $\beta < e^{-m}$ is violated and it is possible to have subsequent window reductions. However, if the Delayed Ack mechanism is enabled, the value of m reduces from 1 to 0.5 and condition $\beta < e^{-m}$ is satisfied. We note that if $m = 0.5$, condition $\beta < \bar{\beta}$ is not satisfied since in this case $\bar{\beta} = 0.43$. However, even if condition $\beta < \bar{\beta}$ is violated, the system trajectory can still converge to 1-cycle from some initial conditions. To avoid for sure the undesired regime of multiple window reductions, one can reduce the value of m to 0.4 in the Slow Start regime.

In the case of Scalable TCP, the inequality $\beta < \bar{\beta}$ is valid as $\bar{\beta} \approx 0.98$, and the regime of multiple window reductions is not realized in any network conditions and configurations.

4 Pareto set for the buffer sizing

Let us study what effect has the choice of the buffer size on the performance of TCP with MIMD congestion control. In particular, we are interested in the optimal buffer sizing. We have two criteria here, namely the average throughput, defined by

$$\bar{g} = \lim_{t\to\infty} \frac{1}{t}\int_0^t g(t)dt,$$

where

$$g(t) = \begin{cases} \lambda(t) & \text{if } x(t) < B \\ \mu & \text{if } x(t) = B, \end{cases}$$

and the average amount of data in the buffer, defined by

$$\bar{x} = \lim_{t\to\infty} \frac{1}{t}\int_0^t x(t)dt.$$

More precisely, one is interested in maximizing $\bar{g}$ and minimizing $\bar{x}$. Clearly those two objectives are contradictory. This is a typical situation in multi-criteria optimization. A standard approach is to optimize one criterion under constraints on the other one. And the solution providing the optimality gives a point in the Pareto set. As is known, see e.g. [21], it can be obtained by solving the optimization problem

$$\max_B \left\{ \lim_{t\to\infty} \frac{1}{t}\int_0^t (c_1 g(t) - c_2 x(t))dt \right\}$$

with $(c_1, c_2) \in \mathbb{R}_+^2$. Different values of $c_1 > 0$ and $c_2 > 0$ lead to the complete Pareto set which must be closed. Based on the Pareto set, one can make the decision on the parity between the two objectives. Mathematical description of partial orders and connected Pareto sets can be found in [9].

We study the Pareto optimality in the framework of the simple clipped (or critical) $1-$cycle, i.e. we assume that $\beta < e^{-m}$ and $B \leq B^*$. The formulae for $\bar{g}$ and $\bar{x}$ can be written as

$$\bar{g} = \frac{1}{T_{cycle}} \int_0^{T_{cycle}} g(t)dt,$$

and

$$\bar{x} = \frac{1}{T_{cycle}} \int_0^{T_{cycle}} x(t)dt,$$

where T_{cycle} is the duration of the cycle. The following propositions provide expressions for the average sending rate, throughput, and amount of data in the buffer. In particular, the expressions allow us to plot the Pareto set parameterized by the buffer size.

Firstly, consider the case $B \geq B^*$ and suppose the limiting $1-$cycle is realized (see Theorem 1(a)). Then the duration of that cycle equals

$$T_{cycle} = \frac{B + \mu T}{\mu m} \left\{ m + \frac{(1 - e\beta^{\frac{1}{m}})(m+1)(1 - e^m \beta)}{1 - e^{m+1}\beta^{1+\frac{1}{m}}} \right\}, \tag{9}$$

and the following proposition holds.

Proposition 1 *The average sending rate is given by*

$$\bar{\lambda} = \frac{(1-\beta)e^m(1 - e\beta^{1/m})(m+1)\mu}{\left[m(1 - e^{m+1}\beta^{1+1/m}) + (1 - e\beta^{1/m})(m+1)(1 - e^m\beta)\right]}, \tag{10}$$

the average throughput is given by

$$\bar{g} = \mu,$$

and the average amount of data in the buffer is given by

$$\bar{x} = \frac{1}{T_{cycle}} \left\{ TB \int_0^S y(s)ds + \frac{B^2}{\mu} \int_0^S y^2(s)ds + BT + \frac{B^2}{\mu} \right\}, \tag{11}$$

where $S = \frac{1}{m} \ln \frac{1}{\beta} - 1$, *and*

$$y(s) = \frac{\beta(1 + \frac{\mu T}{B})e^m(1 - e\beta^{1/m})}{1 - e^{m+1}\beta^{1+1/m}}(e^{ms} - e^{-s}) - \frac{\mu T}{B} + e^{-s}(1 + \frac{\mu T}{B}).$$

The proofs are presented in the Appendix.

Secondly, consider the case $B < B^*$ and $\beta < e^{-m}$. According to Theorem 1(c), all trajectories converge to the clipped limiting $1-$cycle; the phase portrait is presented in Figure 1.

To calculate the main parameters $\bar{\lambda}$, $\bar{g}$, and $\bar{x}$, we need the following quantities and functions.

- Starting point of the cycle, i.e., the minimal value of w in Figure 1:

$$w_0 = \mu T \beta e^{m(S_{CD}+1)}, \tag{12}$$

where S_{CD} is the single positive solution of

$$\mu T(e^{mS_{CD}} + e^{-S_{CD}}m) - (m+1)(\mu T + B) = 0. \tag{13}$$

-Duration of the cycle:

$$T_{cycle} = \frac{T}{m} \ln \frac{1}{\beta} + \frac{B}{\mu} \left\{ \int_0^{S_{AB}} Y_{AB}(s)ds + \int_0^{S_{CD}} Y_{CD}(s)ds + 1 \right\}, \tag{14}$$

where S_{AB} is the smaller positive solution of

$$0 = w_0 e^{mS_{AB}} - \mu T(m+1) + e^{-S_{AB}} \left[(m+1)(B+\mu T) - w_0\right]; \tag{15}$$

here

$$Y_{AB}(s) = \frac{1}{B(m+1)} \left\{ w_0 e^{ms} - \mu T(m+1) + e^{-s} \left[(B+\mu T)(m+1) - w_0\right] \right\}, \; s \in [0, S_{AB}], \tag{16}$$

$$Y_{CD}(s) = \frac{\mu T}{B(m+1)} e^{ms} - \frac{\mu T}{B} + e^{-s} \left(\frac{\mu T m}{B(m+1)} \right), \; s \in [0, S_{CD}]. \tag{17}$$

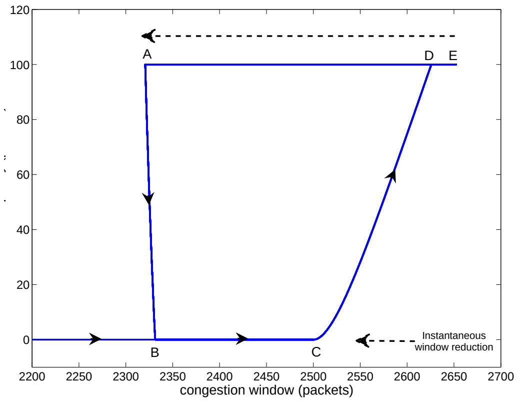

Figure 1: Clipped 1-cycle for Scalable TCP with μ = 1Gbps, T = 10ms, and B = 100pkts. Packet size is 4000 bits.

Proposition 2 *The average sending rate is given by*

$$\bar{\lambda} = \frac{w_0}{T_{cycle} m} \left(\frac{1}{\beta} - 1 \right), \tag{18}$$

the average throughput is given by

$$\bar{g} = \frac{1}{T_{cycle}} \left\{ \frac{w_0}{m} \left(\frac{1}{\beta e^m} - 1 \right) + \mu T + B \right\}, \tag{19}$$

and the average amount of data in the buffer is given by

$$\begin{aligned} \bar{x} &= \frac{1}{T_{cycle}} \left\{ TB \left(\int_0^{S_{AB}} Y_{AB}(s) ds + \int_0^{S_{CD}} Y_{CD}(s) ds \right) \right. \\ &\left. + \frac{B^2}{\mu} \left(\int_0^{S_{AB}} Y_{AB}^2(s) ds + \int_0^{S_{CD}} Y_{CD}^2(s) ds \right) + B \left(T + \frac{B}{\mu} \right) \right\}. \end{aligned} \tag{20}$$

The proofs are presented in the Appendix.

According to Proposition 1, if $B \geq B^*$ and the limiting 1-cycle is realized, then $\bar{\lambda}$ given by (10) is strictly greater than μ and B-independent. Thus, $(\bar{\lambda} - \mu) \not\to 0$ as $B \to \infty$. It means that in the MIMD case the rate of data loss in buffer overflow does not decrease as the buffer size increases. In contrast, in the AIMD case, we have $(\bar{\lambda} - \mu) \to 0$ as $B \to \infty$ [5]. This surprising result has the following explanation. In the MIMD case, when the cycle is unclipped both the amount of data transfered over the cycle and the cycle duration are proportional to $B + \mu T$. For the parameters of the Slow Start phase of TCP New Reno with the Delayed Ack mechanism ($m = 0.5$), the expression (10) gives $(\bar{\lambda} - \mu)/\mu \approx 0.3$. Fortunately, the Slow Start phase switches to the Congestion Avoidance phase after the first loss is detected by triple duplicate acknowledgement [1]. According to (10), Scalable TCP induces as little as 0.1% losses.

According to Proposition 2, as $B \to 0$, we have $\bar{x} \to 0$ and $\bar{g} \to \mu(\beta e^m - 1 - m)/\ln(\beta)$. In particular, in the case of Scalable TCP, we have $\bar{g} \to 0.95\mu$ as $B \to 0$. We recall from [5] that for AIMD, when the packet size is small in comparison with the BDP (Bandwidth Delay Product) μT, we have $\bar{g} \to \mu(1+\beta)/2$ as $B \to 0$. Thus, the Congestion Avoidance phase of TCP New Reno with $\beta = 0.5$ has the worse link utilization of 0.75μ than that of Scalable TCP with $\beta = 0.875$ (0.95μ) when the buffer size is small. It turns out that this difference mostly comes from different values of β. In fact, one can easily check that $\mu(\beta e^m - 1 - m)/\ln(\beta) = \mu(1 + \beta)/2 + \mathrm{o}(1 - \beta)$ and consequently, if one chooses the same value of β close to one for AIMD and MIMD, the link utilization would be the same for the two congestion control mechanisms for small buffer sizes.

5 Simulation results

We perform network simulations with the help of NS-2, the widely used open-source network simulator [20]. We consider the following benchmark example of a TCP/IP network with a single bottleneck link. The topology may for instance represent an access network. The capacity of the bottleneck link is denoted by μ and its propagation delay is denoted by d. We will consider several choices for the values of μ and d. The packet size is 500bytes = 4000bits. When we simulate a scenario with multiple connections, we will assume that each connection is connected to the bottleneck link via its own access link. The capacities of the access links are supposed to be large enough so that they do not hinder the traffic.

We consider the MIMD control strategy with $m = 0.01$ and $\beta = 0.875$, that is, the standard values for Scalable TCP.

5.1 Impact of the buffer size on the link utilization

We first study how the utilization depends on the buffer size. We consider the values $\mu = 1$Gbps = 1 Gigabit per second and $d = 5$ms (thus $T = 2d = 10$ms).

In Figure 2, based on our analytical results, we plot the value of B^* (equation (5)) as a function of m. We observe from Figure 2 that for $m = 0.01$, the value of B^* is approximately 230 packets (the packet size is 4000 bits).

We investigate the impact of the buffer size on the link utilization. From Theorem 1 it follows that according to the fluid model, $B^* = 230$ packets is the minimum buffer size such as the link is utilized at 100%. Note that the BDP for these values is equal to 2500 packets. According to the well known rule of thumb for AIMD connections [27], the minimum buffer size that guarantees 100% utilization is 2500 packets.

Our fluid model predicts that for MIMD, the minimum buffer is much smaller (230 in this example). In Figure 3 we provide the utilization of the link for several values of the buffer size. We note that in the simulation the minimum buffer size where we observe 100% utilization is 450 packets. We note that the utilization when the buffer size is 230 packets is already quite high since it is very close to 99%. Clearly our fluid model predicts a much smaller value, which can

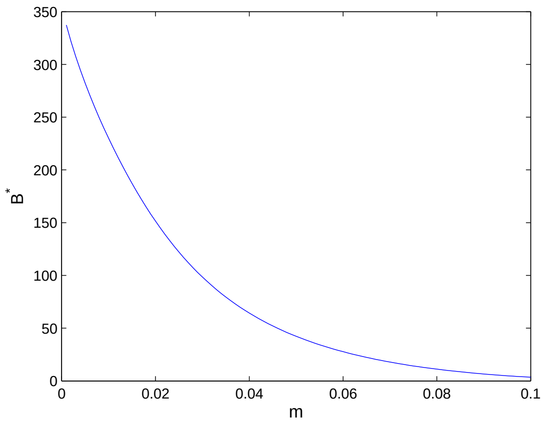

Figure 2: B^* (in packets) as a function of m for Scalable TCP with $\mu = 1$Gbps, $T = 10$ms, and $\beta = 0.875$.

be explained by the fact that the simulated traffic is not as smooth as it is in the fluid model. However we note that the fluid model estimation for B^* is of the same order as the optimal value obtained via simulations when comparing it with the BDP rule-of-thumb for AIMD given in [27].

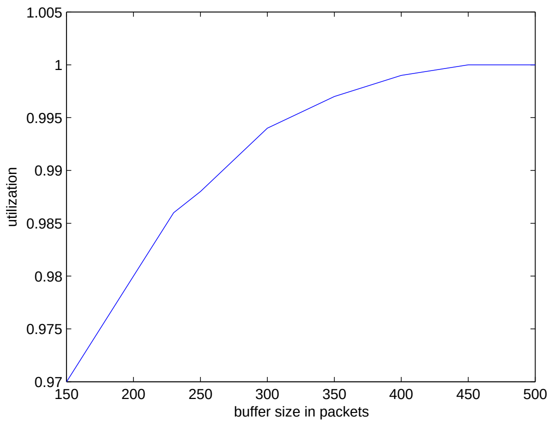

Figure 3: Utilization against buffer size

5.2 Trajectories of the dynamical systems

We simulate now the evolution in time of the congestion window, the buffer occupancy and the sending rate. We consider the same example as above, namely, $\mu = 1$Gbps $= 1$ Gigabit per second and $d = 5$ms (thus $T = 2d = 10$ms). The packet size is 4000 bits. We consider again Scalable TCP, that is, $m = 0.01$ and $\beta = 0.875$.

In Figures 4 and 5 we depict the curves of $x(t)$, $w(t)$ and $\lambda(t)$ for $B = 230$ and $B = 500$, respectively. As predicted by Theorem 1, for $B = 230$, the cycle is critical, and the link is utilized at 100%. For $B = 500$, the cycle is unclipped and the buffer is never empty. For $B = 100$, we plot the phase portrait of a clipped cycle in the plane (w, x) in Figure 1 for illustrative means.

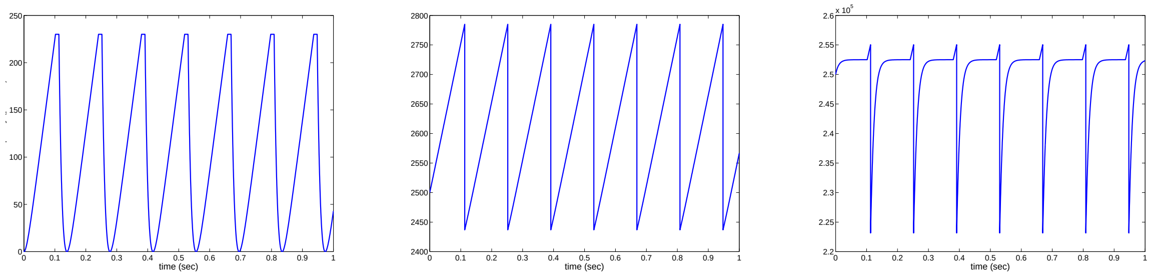

Figure 4: Evolution in time of the buffer occupancy, congestion window and sending rate for Scalable TCP with $\mu = 1$Gbps, $T = 10$ms, and $B = 230$pkts.

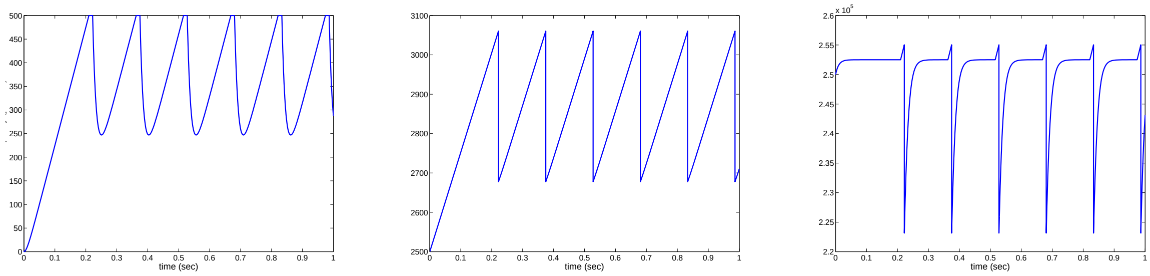

Figure 5: Evolution in time of the buffer occupancy, congestion window and sending rate for Scalable TCP with $\mu = 1$Gbps, $T = 10$ms, and $B = 500$pkts.

The figures for sending rate $\lambda(t)$ might appear a bit odd from the first glance. However, the flat part with the steep increasing part following it can be understood in the following way. Consider the derivative of λ with respect to t (corresponding to the part before x reaches buffer size B). Based on equations (1), (2) and (4), one can easily show that $\frac{d\lambda}{dt} = \frac{\lambda(m+1-\frac{\lambda}{\mu})}{T+\frac{x}{\mu}}$. Now, focusing on the numerator, clearly, $\frac{d\lambda}{dt} = 0$ when $\lambda = \mu(m+1)$, as confirmed also by the figures. Say $\lambda(\hat{t}) = \mu(m + 1)$. After this point $\hat{t}$, we have a sliding mode, since $\lambda > \mu(m + 1) \Rightarrow \frac{d\lambda}{dt} < 0$ and $\lambda < \mu(m + 1) \Rightarrow \frac{d\lambda}{dt} > 0$. This sliding mode explains the flat part. On the other hand, this motion is up to the point when x reaches B. Then as far as x stays there, $\frac{d\lambda}{dt} = \frac{m\lambda}{T+\frac{B}{\mu}}$, explaining the steep increasing part after the flat part.

5.3 Pareto set

Now we compare the numerical Pareto Set with the expressions for $\bar{\lambda}$ and $\bar{g}$ given in Propositions 1 and 2. We consider AIMD (New Reno version [1]) and MIMD connections. In the case of AIMD we will obtain the Pareto Set for several values of number of persistent connections, whereas for MIMD we will only consider one. We recall that several symmetric synchronized MIMD connections are equivalent to a single MIMD connection.

Let N denote the number of persistent connections in the simulation. We will assume that each connection is connected to the bottleneck link via its own access link. The capacities of the N access links leading to the bottleneck link are supposed to be large enough (or the load on each access link is small enough) so that they do not hinder the traffic. For each of these N links, the delay and capacity are $d_i = 1$ms and $\mu_i = 1000$Mbps, respectively. The fact that the delays in the access links are the same implies that the TCP connections will be synchronized.

We consider the following values for the bottleneck link: capacity is $\mu = 100$Mbps, bottleneck link propagation delay $d = 1$ms, the access link capacity and delay are 1000Mbps and 1ms, respectively. Thus $T = 2(d + d_i) = 0.004$ sec.

In Figure 6 we depict the Pareto set for the cases of AIMD with $N = 2$, $N = 5$ and $N = 20$ connections, and MIMD with just one connection. The qualitative shape of the curves agrees with what our model predicts. In particular, MIMD achieves the full link utilization with a much smaller buffer size than in the case of AIMD. We also display the theoretical trade-off

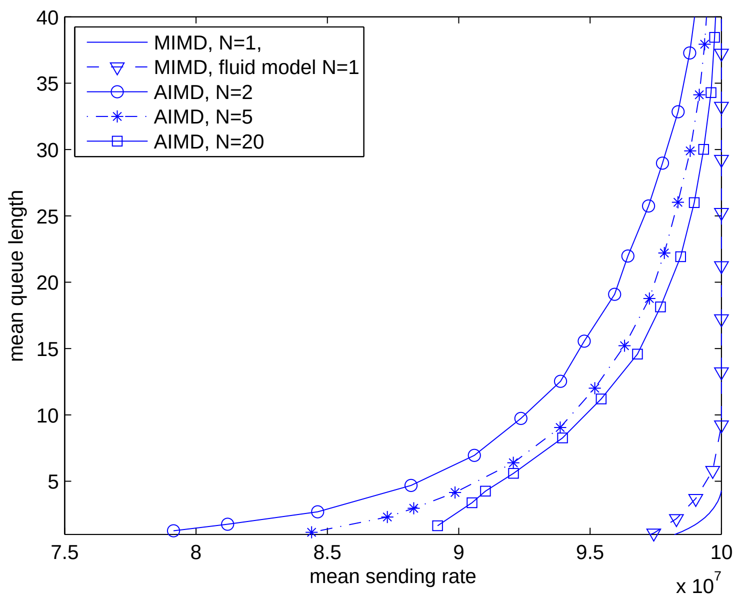

Figure 6: The trade-off curves for AIMD ($N = 2$, $N = 5$, $N = 20$, $M = 1$ packet $= 500$ bytes) and MIMD ($N = 1$, $m = 0.01$), $\beta = 0.875$, $T = 0.004$ sec, $\mu = 1000$ Mbps.

curve for the mathematical fluid model as given in Propositions 1, 2. It turns to be close to the curve coming from simulations. However, when comparing the results obtained from the analytical model and from simulations we have observed some differences. For example, when the buffer size is zero, the simulated average sending rate is smaller than the one obtained with the fluid model. Similarly, in the simulated scenario the minimal buffer size that guarantees the full utilization of the link is larger than the one predicted by the fluid model. These differences can be explained by the fact that the traffic in the simulations is not as smooth as the fluid model that we have used.

6 Conclusions

We have analyzed a hybrid model for the interaction between the MIMD congestion control mechanism and a Drop Tail Internet router buffer. The present hybrid model is a significant extension of the model in [7]. The present model allows us to study the impact of the time-varying Round Trip Times on the system performance. We have obtained conditions for the absence of multiple reductions of the congestion window within one congestion cycle. It turns out that these conditions are violated in the Slow Start phase of TCP New Reno without the Delayed Ack mechanism. Therefore, it is indeed recommended to use the Delayed Ack mechanism in the Slow Start phase. Fortunately, the obtained conditions are satisfied by the parameters of Scalable TCP. For Scalable TCP, we construct the Pareto set that allows us to choose a buffer size which achieves a trade off between high link utilization and small queueing delays.

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Appendix

The Appendix is organized as follows. Firstly, we prove a series of Lemmas and then use them to prove Theorem 1. The proofs of Propositions 1, 2 come at the end.

To make the model more tractable, we change the time scale and the variables as follows.

$$ds = \frac{dt}{T + x(t)/\mu}\ ,\ y = x/B, \text{ and } v = w/B.$$

Then

$$\frac{dv}{ds} = \frac{dv}{dw}\frac{dw}{dt}\frac{dt}{ds} = \frac{mw(s)}{B} = mv(s), \tag{21}$$

and

$$\frac{dy}{ds} = \begin{cases} \frac{dy}{dx}\frac{dx}{dt}\frac{dt}{ds} = v(s) - q - y(s) & \text{if } 0 < y(s) < 1, \text{ or } y(s) = 0 \text{ and } v(s) > q, \\ & \text{or } y(s) = 1, \text{ and } v(s) \le q + 1 \\ 0 & \text{otherwise,} \end{cases} \tag{22}$$

where we have put $q = \frac{\mu T}{B}$, which is a positive constant. Now everything is in the new time scale. Let s^* be the time moment in the new time scale when the state of the system reaches 1. That is, $y(s^*) = 1$. Then the impulsive control (3) can now be written as

$$v(s^* + 1 + 0) = \beta^k v(s^* + 1 - 0), \tag{23}$$

where $k = \min\{i = 1, 2, \ldots : \beta^i v(s^* + 1 - 0) < q + 1\}$, and we notice that the time delay δ has been standardized in the new time scale. With the new variables and time scale, we see when the buffer is filled full, $y(s^*)$ reaches 1, after 1 RTT, the congestion signal is received leading to a multiplicative reduction on $v(s^*)$ with a factor β^k, where k is just defined above. Note, reducing v below $q + 1 = \frac{\mu T}{B} + 1$ exactly corresponds to reducing the instantaneous sending rate defined as $\frac{v(^*)B}{T + \frac{\mu}{B}}$ below the capacity μ.

If we ignore the non-negativity constraint on variable y, then one can solve (21) and (22) for $v(s)$ and $y(s)$ with initial conditions $v(0) = v_0$ and $y(0) = y_0$ respectively, and obtain

$$v(s) = v_0 e^{ms} \tag{24}$$

$$y(s) = \frac{v_0}{m+1}e^{ms} - q + e^{-s}\left(y_0 + q - \frac{v_0}{m+1}\right), \tag{25}$$

and the existence and uniqueness of the above two solutions follow from the initial value problems of ordinary differential equations.

With the new variables in the new time scale, we give a corresponding version of Definition 1 as follows.

Definition 1' *A cycle is defined as the trajectory starting with the initial point (a particular* $V_0 \in [\beta(q+1), q+1)$, $y(0) = y_0 = 1$*) at* $s = 0$*, and reaching the same point for the first time at some* $S + 1 \ge 1$*. And* $S + 1$ *is called the duration of the cycle. A cycle with* $y(s)$ *staying at zero for a positive time interval is called clipped. Otherwise it is unclipped. In particular, a cycle with* $y(s)$ *staying at* 0 *at a single time moment is called critical, and it could be referred* *to as an unclipped cycle. In addition, cycles, possibly with more than one instant jump though, are called simple, if they have only one loop. Otherwise, they are called complicated.*

In what follows, expression "unconstrained case" means that we ignore the non-negativity constraint on variable y. Expression "general case" means that we impose constraint $y \geq 0$. Under "trajectory" we mean the phase portrait $y(s)$ against $v(s)$: see Figure 1.

Lemma 1 *In the unconstrained case, 1-cycle exists iff $\beta e^m < 1$.*

Proof. Consider the unconstrained case. A $1-$cycle exists iff there exists a nonnegative number V_0 such that $V_0 \in [\beta(q+1), q+1)$, and

$$\beta V_0 e^{m(S+1)} = V_0 \tag{26}$$

$$1 = \frac{V_0}{m+1} e^{mS} - q + e^{-S}\left(1 + q - \frac{V_0}{m+1}\right), \tag{27}$$

where we have already put $y_0 = 1$.

Firstly one can check the existence of a solution to equations (26) (27). Indeed from (26) we have

$$S = \frac{1}{m} \ln \frac{1}{\beta} - 1, \tag{28}$$

so that (27) results in

$$V_0 = \frac{(1+q)e^m \beta \left(1 - e\beta^{\frac{1}{m}}\right)(m+1)}{1 - e^{m+1}\beta^{\left(1+\frac{1}{m}\right)}}. \tag{29}$$

Secondly, one can check that V_0 given by equation (29) is in the interval $[\beta(q+1), q+1)$, provided $1 + q > 0$ and $\beta \in (0, e^{-m})$. The latter condition is necessary and sufficient for the presented reasoning to hold. ■

In what follows, it is assumed that $e^m \beta < 1$.

Lemma 2 *Consider the unconstrained case. Starting from an arbitrary initial state $v_0 \in (0, 1+q)$, $y_0 = 1$, component $y(s)$ attains its single minimum at the moment*

$$s_1(v_0) = \frac{1}{m+1} \ln \frac{(1+q)(1+m) - v_0}{m v_0} > 0. \tag{30}$$

The value $y(s_1)$ increases with v_0 and $y(s_1) \to 1$ as $v_0 \to 1+q$.

A $1-$cycle is critical for a single nonnegative value of q given by

$$q^* = \frac{(m+1)e^m \left(1 - e\beta^{\frac{1}{m}}\right) \beta e^{m s_1^*}}{1 - e^{m+1}\beta^{\frac{m+1}{m}} - (m+1)e^m \left(1 - e\beta^{\frac{1}{m}}\right)\beta e^{m s_1^*}}, \tag{31}$$

where $s_1^* = s_1(V_0^*) = \frac{1}{m+1} \ln \left(\frac{1 - e^m \beta}{\beta e^m m \left(1 - e\beta^{\frac{1}{m}}\right)} \right)$, and V_0^* is given by (29) with $q = q^*$.

In the general case (if we impose constraint $y \geq 0$) the $1-$cycle is unclipped iff $q \leq q^*$.

Proof. According to equations (24) (25), we have the following equations satisfied by $s_1(v_0)$:

$$\begin{cases} v(s_1) = v_0 e^{m s_1} \\ y(s_1) = \frac{v_0}{m+1} e^{m s_1} - q + e^{-s_1}\left(1 + q - \frac{v_0}{m+1}\right) \\ y(s_1) = v(s_1) - q. \end{cases}$$

Solving them for s_1 gives (30), and

$$y(s_1(v_0)) = \left\{ \frac{(1+q)(1+m)v_0^{\frac{1}{m}} - v_0^{\frac{1+m}{m}}}{m} \right\}^{\frac{m}{m+1}} - q.$$

One can easily check that $y(s_1(v_0)) \to 1$ as $v_0 \to 1+q$. For $\frac{dy(s_1(v_0))}{dv_0}$, we have

$$\frac{dy(s_1(v_0))}{dv_0} = \frac{m}{m+1} \left\{ \frac{(1+q)(1+m)v_0^{\frac{1}{m}} - v_0^{\frac{1+m}{m}}}{m} \right\}^{\frac{-1}{m+1}} \times \left\{ \frac{1+m}{m^2} v_0^{\frac{1}{m}-1}(1+q-v_0) \right\} > 0.$$

Let us fix an arbitrary $q > 0$ and consider the corresponding simple $1-$cycle with the corresponding value of V_0 defined in (29). Now

$$s_1(V_0) = \frac{1}{m+1} \ln \frac{1-e^m\beta}{me^m\beta(1-e\beta^{1/m})} = s_1^* \tag{32}$$

and according to (22) and (24)

$$y(s_1(V_0)) = v(s_1) - q = V_0 e^{ms_1} - q.$$

Since s_1 is $q-$independent, $y(s_1(V_0))$ is a **linear** function of q. Let us show that it decreases with q.

Indeed, if $q \to 0$ then $y(s_1(V_0))$ has a positive limit. When q increases, $y(s_1(V_0))$ becomes negative. To see this, notice that at the beginning of the cycle, starting from $v(0) = V_0 < 1+q$, $y(0) = 1$, component y decreases. Moreover,

$$\frac{d}{ds}\left(\frac{y(s)}{q}\right)\Big|_{s=0} = \frac{V_0}{q} - 1 - \frac{1}{q} \to \frac{e^m\beta(1-e\beta^{1/m})(m+1)}{1-e^{m+1}\beta^{1+1/m}} - 1$$

as $q \to \infty$. And the latter expression is negative because $e^m\beta(1-e\beta^{1/m})(m+1) - 1 + e^{m+1}\beta^{1+1/m} < 0$ for $\beta \in (0, e^{-m})$. Therefore, $\frac{y(s)}{q}$ decreases with time s, when s is small, at large values of q, starting from initial value $\frac{y_0}{q} = \frac{1}{q}$, meaning that $\frac{y(s)}{q}$ takes negative values if q is sufficiently big, i.e. the minimal value, $y(s_1) < 0$.

Therefore, there exists a single value $q^* > 0$ such that $y(s_1(V_0)) = 0$. Clearly, the last equality holds iff

$$V_0^* e^{ms_1^*} - q^* = \frac{(1+q^*)e^m\beta(1-e\beta^{1/m})(m+1)}{1-e^{m+1}\beta^{1+1/m}} e^{ms_1^*} - q^* = 0.$$

It only remains to solve the equation obtained for q^*.

The last statement is obvious. ∎

Remark 2 *According to (29)*

$$\frac{dV_0}{d\beta} = \frac{(1+q)e^m(m+1)[1 - e(1+\frac{1}{m})\beta^{1/m} - e^{m+1}\beta^{1+1/m} + e^{m+1}(1+\frac{1}{m})\beta^{1+1/m}]}{(1-e^{m+1}\beta^{1+1/m})^2},$$

and the standard analysis of the derivatives shows that the latter function of β is positive if $\beta \in (0, e^{-m})$. Trajectories $(v(s), y(s))$ cannot cross when starting from different initial points $(v(0) = V_0^1, y(0) = 1)$ and $(v(0) = V_0^2, y(0) = 1)$; thus the minimal value $y(s_1(V_0))$ increases with β.

Corollary 1 *In the general case, where constraint $y(s) \geq 0$ is imposed, if a trajectory starting with some $v_0 \in (0, 1+q)$ is clipped, there will be some $\hat{v}_0 \in (v_0, 1+q)$, starting with which the trajectory just touches the horizontal v axis, i.e., $y(s_1(\hat{v}_0)) = 0$. Furthermore, trajectories starting with $v_0 \in [\hat{v}_0, 1+q)$ are unclipped, while those with $v_0 \in (0, \hat{v}_0)$ are clipped. As a result, if $q \leq q^*$, $V_0 \geq \hat{v}_0$, where V_0 is given by (29).*

Proof. Everything follows directly from the first part of Lemma 2, bearing in mind that increasing $y(s_1(v_0))$ is a continuous function of v_0. ■

After the continuous trajectory starting with $v(0) = v_0 < 1+q$ and $y(0) = 1$ finishes, that is, the buffer is filled up and the congestion is noticed after the delay, there will be a reduction on the variable v leading to $v_1 \in [\beta(1+q), 1+q)$. Therefore, as the process proceeds, we have a sequence $\{v_i\}$. If this sequence has a limit, namely v_∞, a limiting cycle exists and will be realized.

According to (24) and (25), we introduce the following denotations (for $v < 1+q$):

$$\varphi(v) = \beta v e^{m(s+1)}, \tag{33}$$

where $s > 0$ solves equation

$$F(v,\, s) = \frac{v}{m+1}\left(e^{ms} - e^{-s}\right) + (1+q)\left(e^{-s} - 1\right) = 0. \tag{34}$$

In the unconstrained case, (or if an actual continuous trajectory is unclipped), if only one jump is sufficient, $v_{i+1} = \varphi(v_i)$. We shall also use the denotation $\psi(v) = \varphi(\varphi(v)) = \varphi^2(v)$ for brevity.

Note that, if the actual continuous trajectory starting from $v(0) = v$, $y(0) = 1$ is clipped then, at the next time moment s^* when $y(s^*) = 1$, $v(s^*) < ve^{ms}$ implying $v(s^* + 1 + 0) < \varphi(v)$ provided only one jump is sufficient in the unconstrained case.

Lemma 3 *In the unconstrained case, starting with an arbitrary $v_0 \in [\beta e^m(1+q), 1+q)$, the limiting simple cycle exists and is of order one.*

Proof. One can check that there exists only one $s > 0$ solving (34) for $v \in (0, 1+q)$. It is convenient to investigate the mapping φ defined on the closed segment $[\beta e^m(1+q),\, 1+q]$: $\varphi(1+q) = \beta e^m(1+q)$. (Equation (34) has only one zero solution for $v = 1+q$ and $\lim_{v \to 1+q-0} \varphi(v) = \beta e^m(1+q)$.)

Our proof will be performed in three steps:

1. $\varphi(v)$ decreases with v. Hence $\psi(v)$ increases with v. This statement holds for all $v \in (0, 1+q)$.

2. $\varphi : [\beta e^m(1+q),\, 1+q] \to [\beta e^m(1+q),\, 1+q]$ and $\psi : [\beta e^m(1+q),\, 1+q] \to [\beta e^m(1+q),\, 1+q]$.

3. $\{\varphi^n(v)\}$ and $\{\psi^n(v)\}$ both converge to $v_\infty \in [\beta e^m(1+q), 1+q)$.

For **item 1**, according to implicit differentiation and partial differention,

$$\frac{d\varphi(v)}{dv} = \frac{\beta e^{-s}(m+1)e^{m(s+1)}(v - (q+1))}{v\left(me^{ms} + e^{-s}\right) - (m+1)(q+1)e^{-s}}, \tag{35}$$

where the numerator of the last expression is smaller than zero for $v < 1+q$.

The denominator of the last expression equals

$$v\left(e^{ms}m + e^{-s}\right) - (m+1)(q+1)e^{-s} = \frac{(1+q)(m+1)e^{-s}}{e^{ms} - e^{-s}} G_1(s),$$

where $G_1(s) = me^{(m+1)s} - me^{ms} + 1 - e^{ms}$. (We have put in $v = \frac{(1+q)(1-e^{-s})(m+1)}{e^{ms}-e^{-s}}$, according to equation (34).) Finally, $G_1(s) > 0$ for $s > 0$.

For **item 2**, we consider $v \in [\beta e^m(1+q), 1+q)$. According to item 1, $\varphi(1+q)$ and $\varphi(\beta e^m(1+q))$ give a lower and an upper bounds for $\varphi(v)$, respectively. We then need to show that $\varphi(1+q) \geq \beta e^m(1+q) > \beta(1+q)$, and $\varphi(\beta e^m(1+q)) < 1+q$. Since $\varphi(1+q) = \beta e^m(1+q)$, it remains to prove that $\varphi(\beta e^m(1+q)) \leq 1+q$.

According to (34), where we put in $v = \beta e^m(1+q)$, we have

$$\frac{(1+q)\beta e^m}{m+1}\left(e^{ms} - e^{-s}\right) + (1+q)\left(e^{-s} - 1\right) = 0 \Leftrightarrow G_2(s, \beta, m) = 0,$$

where $G_2(s, \beta, m) = \beta e^m\left(e^{ms} - e^{-s}\right) + (m+1)\left(e^{-s} - 1\right)$. Function $G_2(s, \beta, m)$ firstly decreases with respect to s from zero and then increases up to ∞ after the single minimum point, resulting in a single positive solution s solving (34) with $v = \beta e^m(1+q)$.

Clearly, $\varphi(\beta e^m(1+q)) = \beta^2(1+q)e^m e^{m(s+1)}$, where s solves (34) with $v = \beta e^m(1+q)$. Define the increasing (with respect to s) auxiliary function $G_3(s) = \beta^2(1+q)e^m e^{m(s+1)}$. We aim to show that, for $\hat{s}$ satisfying $G_3(\hat{s}) = 1+q$, i.e., $\hat{s}(\beta, m) = \frac{2}{m}\ln\frac{1}{\beta} - 2$, $G_2(\hat{s}(\beta, m), \beta, m) > 0$. That would say, $\hat{s}(\beta, m)$ is greater than the solution of (34) with $v = \beta e^m(1+q)$, and $\varphi(\beta e^m(1+q)) < 1+q$.

We have

$$G_2(\hat{s}(\beta, m), \beta, m) = (e^m\beta)^{-1} - \beta e^m\left(e\beta^{\frac{1}{m}}\right)^2 + (m+1)\left(\left(e\beta^{\frac{1}{m}}\right)^2 - 1\right) = \hat{G}_2(\beta, m).$$

Observe that $\hat{G}_2(\beta, m) \to \infty$ as $\beta \to 0$ and $\hat{G}_2(\beta, m) \to 0$ as $\beta \to e^{-m}$.

Furthermore,

$$\frac{\partial \hat{G}_2(\beta, m)}{\partial \beta} = e^{-m}\beta^{-2}\hat{G}_3(\beta, m),$$

where $\hat{G}_3(\beta, m) = -1 - \frac{m+2}{m}e^{2m+2}\beta^{\frac{2m+2}{m}} + \frac{2(m+1)}{m}e^{m+2}\beta^{\frac{2+m}{m}} < 0$ for $\beta \in (0, e^{-m})$. Therefore, $\frac{\partial \hat{G}_2(\beta,m)}{\partial \beta} < 0 \Rightarrow \hat{G}_2(\beta, m) > 0 \Leftrightarrow G_2(\hat{s}(\beta, m), \beta, m) > 0$, as required.

It follows from item 1 that starting with an arbitrary $v \in [\beta e^m(1+q), 1+q)$, $\{\psi^n(v)\}$ is a monotonic sequence. It follows from item 2 that the sequence $\{\psi^n(v)\}$ is bounded in the closed interval $[\beta e^m(1+q), 1+q]$. Hence, $\psi^n(v_0) \to v_\infty = \psi(v_\infty) \in [\beta e^m(1+q), 1+q]$ as $n \to \infty$. It also follows from item 2 that with $e^m\beta < 1$, exactly one jump is enough, starting with $v \in [\beta e^m(1+q), 1+q)$.

For **item 3**, assume $\varphi(v_\infty) = v'_\infty \neq v_\infty$. Let S_2 and S_3 be such that

$$0 = \frac{v'_\infty}{m+1}\left(e^{mS_2} - e^{-S_2}\right) + (1+q)\left(e^{-S_2} - 1\right) \tag{36}$$

$$0 = \frac{v_\infty}{m+1}\left(e^{mS_3} - e^{-S_3}\right) + (1+q)\left(e^{-S_3} - 1\right). \tag{37}$$

Then $S_2 + 1$ and $S_3 + 1$ are the durations of continuous trajectories starting with v'_∞ and v_∞, respectively. (See (34).) Then by the definition of v_∞,

$$v_\infty = \beta\beta v_\infty e^{m(S_3+1)}e^{m(S_2+1)},$$

leading to

$$\beta = e^{-\frac{1}{2}m(S_2+S_3+2)}. \tag{38}$$

By the definition of v'_∞, we have $v'_\infty = \beta v_\infty e^{m(S_3+1)}$. But from (36) we have in parallel $v'_\infty = \frac{(1+q)(m+1)(1-e^{-S_2})}{e^{mS_2}-e^{-S_2}}$. If we substitute expression for v_∞ coming from (37), and use formula (38), we see that

$$\frac{(1-e^{-S_2})e^{\frac{1}{2}mS_2}}{e^{mS_2}-e^{-S_2}} = \frac{(1-e^{-S_3})e^{\frac{1}{2}mS_3}}{e^{mS_3}-e^{-S_3}}. \tag{39}$$

One can show that function $\frac{(1-e^{-z})e^{\frac{1}{2}mz}}{e^{mz}-e^{-z}}$ strictly decreases if $z > 0$. Hence $S_2 = S_3$ and $v'_\infty = v_\infty < 1+q$ because $\varphi(1+q) = \beta e^m(1+q) \neq 1+q$. Therefore, $\varphi^n(v_0) \to v_\infty$ as $n \to \infty$. ∎

Remark 3 *It follows from* ***item 3*** *in the proof of Lemma 3 that in the unconstrained case, starting with an arbitrary* $v_0 \in [\beta e^m(1+q), 1+q)$*, complicated cycles cannot be realized, and* v_∞ *coincides with* V_0 *given by (29).*

Corollary 2 $-1 < \left.\frac{d\varphi(v_0)}{dv_0}\right|_{V_0} < 0$*, so that the mapping* φ *is a contraction in a neighborhood of the stable point* V_0*.*

Proof. By putting in V_0 given by (29) and S given by (28) into (35), we have $\left.\frac{d\varphi(v)}{dv}\right|_{V_0} = \frac{e\beta^{1/m}\left[(m+1)e^m\beta-1-me^{m+1}\beta^{1+1/m}\right]}{(1-e\beta^{1/m})m+e^{m+1}\beta^{1+1/m}-e\beta^{1/m}}$. We already know that $\frac{d\varphi(v)}{dv} < 0$ (see item 1 above). Hence we just need to prove that $\left.\frac{d\varphi(v_0)}{dv_0}\right|_{V_0} > -1 \Leftrightarrow P_1(\beta, m) > 0$, where $P_1(\beta, m) = (m+1)e^m\beta - 2 - me^{m+1}\beta^{\frac{1+m}{m}} + e^m\beta - m + m(e\beta^{\frac{1}{m}})^{-1}$. But the standard analysis of the derivatives shows that function $P_1(\beta, m)$ monotonically decreases from ∞ to 0 for $\beta \in (0, e^{-m})$. ∎

Corollary 3 *In the unconstrained case, let* $v_0 \in [\beta e^m(1+q), 1+q)$*. Then* $\forall i \in \{0, 1, 2, \ldots\}$ $v_i \in [\beta e^m(1+q), 1+q)$*, and* $v_{i+2} \in [\min(v_i, v_{i+1}), \max(v_i, v_{i+1})]$*.*

Proof. The first statement follows from the proof of Lemma 3.

Without loss of generality, we can put $i = 0$. That is, we aim to show that $v_2 \in [\min(v_0, v_1), \max(v_0, v_1)]$. According to **item 2** in the proof of Lemma 3, $v_i = \varphi(v_{i-1})$. Consider the case $v_0 > v_1$. Automatically we have $v_2 > v_1$, since φ is decreasing. Then there are two possibilities about the relationship between v_0, v_1, and v_2:

1. $v_2 > v_0 > v_1$.
2. $v_2 \in [v_1, v_0]$.

Suppose the first possibility is true, that is, $v_2 > v_0 > v_1$. We aim to show by induction that in this case, $v_{2i+2} > v_{2i} > \ldots > v_2 > v_0 > v_1 > \ldots > v_{2i+1}$, $\forall i \in \{0, 1, 2, \ldots\}$. These inequalities hold for $i = 0$. Suppose they hold for some $i \geq 0$. Consider the case $i + 1$. From the induction supposition we have $v_{2i+2} > v_{2i}$. Therefore $v_{2i+1} > v_{2i+3}$, and $v_{2i+4} > v_{2i+2}$. Hence sequence $\{v_i\}$ does not converge which contradicts Lemma 3.

Therefore, the first possibility is false. And $v_2 \in [v_1, v_0]$ holds automatically, as we want. Exactly in the same manner, one can show that in the case $v_0 < v_1$, $v_2 \in [v_0, v_1]$. And the case $v_0 = v_1$ is trivial. Hence, $v_2 \in [\min(v_0, v_1), \max(v_0, v_1)]$, as required. ∎

Remark 4 *According to Corollary 1 and Corollary 3, in the general case, starting with* $v \in [\beta e^m(1+q), 1+q)$*, once two consecutive unclipped trajectories are realized, all the subsequent trajectories will be unclipped.*

Lemma 4 *Under the main assumption of $e^m\beta < 1$, the trajectory starting with $v(0) = v_0 = \beta(1+q)$, $y(0) = 1$ requires no more than two jumps.*

Proof. Suppose the trajectory is unclipped. Starting with $v_0 = \beta(1+q)$, let us examine the value of $\beta\varphi(\beta(1+q)) = \beta^3(1+q)e^{m(\hat{s}+1)}$, where $\hat{s}$ is the single positive solution to $L(s,\beta,m) = 0$, where $L(s,\beta,m) = \beta(e^{ms} - e^{-s}) + (m+1)(e^{-s} - 1)$ according to (34). Hence, $\beta^3(1+q)e^{m(\hat{s}+1)}$ is the value of v_1 after two instant jumps, if starting with $v_0 = \beta(1+q)$. Define the increasing (with respect to s) auxiliary function $C(s) = \beta^3(1+q)e^{m(s+1)}$. One can easily check that the behaviour of $L(s,\beta,m)$ is similar to that of $G_2(s,\beta,m)$ in the proof of Lemma 3, in the sense that it decreases firstly from zero and then increases up to infinity, with respect to s.

Let us show that $L(\tilde{s}(\beta,m),\beta,m) > 0$, where $\tilde{s}(\beta,m)$ is the single positive solution to $C(s) = 1+q$: $\tilde{s}(\beta,m) = -\frac{3}{m}\ln\beta - 1$.

Now

$$L(\tilde{s}(\beta,m),\beta,m) = \beta^{-2}e^{-m} - \beta^{\frac{3+m}{m}}e + (m+1)(\beta^{\frac{3}{m}}e - 1).$$

Immediately $L(\tilde{s}(\beta,m),\beta,m) \to \infty$ as $\beta \to 0$. And as $\beta \to e^{-m}$, $L(\tilde{s}(\beta,m),\beta,m) \to e^m - e^{-2-m} + (m+1)(e^{-2} - 1) > 0$, as can be verified easily. Now one can calculate the partial derivative

$$\frac{\partial L(\tilde{s}(\beta,m),\beta,m)}{\partial\beta} = -2\beta^{-3}e^{-m} - \frac{m+3}{m}\beta^{\frac{3}{m}}e + (m+1)\frac{3}{m}\beta^{\frac{3}{m}-1}e.$$

Immediately $\frac{\partial L(\tilde{s}(\beta,m),\beta,m)}{\partial\beta} \to -\infty$ as $\beta \to 0$, and one can show that

$$\lim_{\beta\to e^{-m}}\left(\frac{\partial L(\tilde{s}(\beta,m),\beta,m)}{\partial\beta}\right) = -2e^{2m} - \frac{3+m}{m}e^{-2} + \frac{3(m+1)}{m}e^{m-2} < 0$$

for any $m > 0$.

Finally, the analysis of the second order derivative implies $\frac{\partial^2 L(\tilde{s}(\beta,m),\beta,m)}{\partial\beta^2} > 0$, so that $L(\tilde{s}(\beta,m),\beta,m) > 0$.

Hence $\hat{s} < \tilde{s}(\beta,m)$ and $C(\hat{s}) < 1+q$ meaning that $\beta\varphi(\beta(1+q)) < 1+q$.

If the trajectory is clipped then the value after the next two instantaneous jumps is even smaller than $\beta\varphi(\beta(1+q))$. ■

Corollary 4 *If $e^m\beta < 1$ then no-one cycle has more than two instantaneous jumps.*

Proof. It is sufficient to notice that, after any instantaneous series of jumps, $v \geq \beta(1+q)$ and $\beta\varphi(v) \leq \beta\varphi(\beta(1+q))$. ■

Lemma 5 *In the unconstrained case, $2-$cycles are absent iff $\beta < \bar{\beta}$, where $\bar{\beta}$ is the single solution in the interval $(0, e^{-m})$ to equation (7).*

Proof. Clearly a $2-$cycle, described by the starting point

$$V_0^{(2)} = \frac{(1+q)e^m\beta^2\left(1 - e\beta^{\frac{2}{m}}\right)(m+1)}{1 - e^{m+1}\beta^{\left(2+\frac{2}{m}\right)}},$$

does not exist iff $\frac{V_0^{(2)}}{\beta} < 1+q$. (Compare with the proof of Lemma 1.) Or equivalently,

$$Q_1(\beta) = (m+1)e^m\beta(1 - e\beta^{\frac{2}{m}}) + e^{m+1}\beta^{2(1+\frac{1}{m})} < 1.$$

The standard analysis of the derivatives implies that $Q_1(\beta)$ increases with β from $Q_1(0) = 0$ and, after a single stationary point, decreases up to $Q_1(e^{-\frac{m}{2}}) = 1$. Therefore, equation (7) has a single solution in interval $(0, e^{-\frac{m}{2}}) \supset (0, e^{-m})$.

One can easily check that

$$\frac{V_0^{(2)}}{\beta}\,|_{\beta=e^{-m}} > 1 + q, \tag{40}$$

which is equivalent to $\bar{\beta} < e^{-m}$. ■

Remark 5 *Suppose $\beta < e^{-m}$. Then, in the general case, $2-$cycles exist for some (big enough) values of B iff $\beta \geq \bar{\beta}$. According to Lemma 1, $1-$cycles also exist. What is actually realized, depends on the initial conditions $v(0) = v_0, y(0) = 1$.*

Lemma 6 *In the unconstrained case, the continuous trajectory starting from $v(0) = v_0 = \beta(1+q), y(0) = 1$ reaches level $y(\hat{s}) = 1$ with such a value of $v(\hat{s})$ that $\beta v(\hat{s}+1) < 1+q$ if and only if $\beta < \bar{\beta}$.*

Proof. Clearly $\beta v(\hat{s} + 1) = \varphi(v_0) = \beta^2(1 + q)e^{m(\hat{s}+1)}$, where $\hat{s} > 0$ solves equation (34) at $v = v_0 = \beta(1 + q)$. Now $\beta v(\hat{s} + 1) < 1 + q \Leftrightarrow \beta^2 e^{m(\hat{s}+1)} < 1$.

Firstly, one can check that equations

$$\beta^2 e^{m(\hat{s}+1)} = 1; \tag{41}$$

$$\beta(e^{m\hat{s}} - e^{-\hat{s}}) + (m + 1)(e^{-\hat{s}} - 1) = 0 \tag{42}$$

hold iff $\beta = \bar{\beta}$. Indeed, substitute expression $\hat{s} = \frac{-2\ln\beta}{m} - 1$ obtained from (41), into (42):

$$\beta(\frac{e^{-m}}{\beta^2} - e\beta^{2/m}) + (m + 1)(e\beta^{2/m} - 1) = 0 \Leftrightarrow \ (7) \Leftrightarrow \beta = \bar{\beta}.$$

Secondly, from (42) we obtain

$$\frac{d\hat{s}}{d\beta} = \frac{e^{m\hat{s}} - e^{-\hat{s}}}{(m + 1)e^{-\hat{s}} - \beta(me^{m\hat{s}} + e^{-\hat{s}})}.$$

The numerator is positive for $\hat{s} > 0$. After we substitute $\beta = \frac{(1-e^{-\hat{s}})(m+1)}{e^{m\hat{s}}-e^{-\hat{s}}}$, obtained from (42), into the denominator, we obtain

$$\frac{(m + 1)e^{-\hat{s}}(e^{m\hat{s}} - e^{-\hat{s}}) - (1 - e^{-\hat{s}})(m + 1)(me^{m\hat{s}} + e^{-\hat{s}})}{e^{m\hat{s}} - e^{-\hat{s}}} < 0$$

because $e^{m\hat{s}-\hat{s}} - me^{m\hat{s}} + me^{m\hat{s}-\hat{s}} - e^{-\hat{s}} < 0$ at any positive $\hat{s}$ and m. (The latter inequality can be established when analysing the lefthand part as function of $m \in (0, \infty)$.) Thus $\frac{d\hat{s}}{d\beta} < 0$.

Finally, we intend to prove that $\lim_{\beta\to 0} \beta^2 e^{m(\hat{s}+1)} = 0$. When $\beta \to 0$, $\hat{s}$ increases, but the limit cannot be finite. (Otherwise, passing to the limit in (42) would imply $(m+1)(e^{-\lim_{\beta\to 0}\hat{s}}-1) = 0$.) Hence $\lim_{\beta\to 0} \hat{s} = \infty$, and from (42) we have $\lim_{\beta\to 0} \beta e^{m\hat{s}} = m + 1 \Rightarrow \lim_{\beta\to 0} \beta^2 e^{m(\hat{s}+1)} = 0$. Therefore, $\beta^2 e^{m(\hat{s}+1)} < 1 \Leftrightarrow \beta < \bar{\beta}$ because $\bar{\beta}$ is the **single** value of β providing $\beta^2 e^{m(\hat{s}+1)} = 1$, and the lefthand side is obvioulsy a continuous function of β. ■

Proof of Theorem 1. Note that $\bar{\beta}$ is the single solution to equation (7) in the interval $(0, e^{-m})$ according to Lemma 5. If $\beta < e^{-m}$ Lemma 4 excludes trajectories with three or more instantaneous jumps (perhaps after one first continuous trajectory is realised).

(a) Suppose $\beta < \bar{\beta}$ and $B \geq B^* \Leftrightarrow q \leq q^*$. Lemma 6 implies that (perhaps after the first one instantaneous series of jumps) multiple reductions of component v never occur and all the further values of v_i belong to $[\beta e^m(1+q), 1+q)$. For the proof of the latter statement, remember the denotations introduced before Lemma 3, and equality $\varphi(1+q) = \beta e^m(1+q)$. Even if a continuous trajectory starting from $v(0) = v_i, y(0) = 1$ is clipped, $v_{i+1} = \varphi(\hat{v}_0) \in [\beta e^m(1+q), 1+q)$, where $\hat{v}_0$ was defined in Corollary 1.

Suppose there exists a clipped continuous trajectory starting from $v(0) = v_i, y(0) = 1$. (Actually, i can equal 1 or 2.) The next trajectory starting from $v(0) = v_{i+1} \in [\beta e^m(1+q), 1+q), y(0) = 1$ cannot be clipped because otherwise we would have obtained a clipped $1-$cycle which contradicts the last statement of Lemma 2. Thus $v_{i+1} \geq \hat{v}_0$ and $v_{i+2} = \varphi(v_{i+1})$. Since $v_{i+1} = \varphi(\hat{v}_0)$, we can use Corollary 3: $v_{i+2} \geq \hat{v}_0$, so that trajectory starting from $v(0) = v_{i+2}, y(0) = 1$ is also unclipped. According to Remark 4, all the subsequent trajectories are unclipped and converge to the limiting unclipped $1-$cycle in accordance with Lemma 3.

If $\beta \geq \bar{\beta}$ then, according to Remark 5, statement (a) is false.

All the presented reasoning holds also if $\bar{\beta} \leq \beta < e^{-m}$ and $v_0 \in [\beta e^m(1+q), 1+q)$: multiple jumps never occur and $\forall i \geq 0 \; v_i \in [\beta e^m(1+q), 1+q)$. (See the proof of Lemma 3.) On the other hand, according to Remark 5, for some initial conditions, a simple $2-$cycle can be realized if B is big enough. This observation justifies Remark 1.

(b) If $B = B^*$, the previous paragraph is correct, but (independently of the initial state) no-one continuous trajectory can have multiple jumps at the end, because it cannot be situated below the curve starting from $v(0) = q, y(0) = 0$ which results in the single jump at the end. Thus, trajectories converge to the $1-$cycle that is critical according to Lemma 2. The necessity of inequality $\beta < e^{-m}$ can be proved similarly to the part (c).

(c) Similarly to case (b), continuous trajectories having multiple jumps at the end cannot be realized if $B < B^* \Leftrightarrow q > q^*$. According to Lemma 2, one cannot meet an unclipped $1-$cycle. Corollary 1 and Lemma 3 imply that $\hat{v}_0 \in [\beta e^m(1+q), 1+q)$. Moreover, $\varphi(\hat{v}_0) < \hat{v}_0$ because otherwise, starting from $v(0) = \hat{v}_0, y(0) = 1$ we would have had two consecutive unclipped trajectories leading to an unclipped limiting $1-$cycle according to Remark 4 and Lemma 3.

Now one of the following two scenarios can take place.

If $v_0 < \hat{v}_0$, then the first continuous trajectory is clipped and $v_1 = \varphi(\hat{v}_0) < \hat{v}_0$, so that the next continuous trajectory is also clipped, and the limiting clipped $1-$cycle is attained after one iteration.

If $v_0 \geq \hat{v}_0$, then the first continuous trajectory is unclipped, but $v_1 = \varphi(v_0) < \hat{v}_0$. (Otherwise we face two consecutive unclipped trajectories leading to the existence of an unclipped $1-$cycle.) Hence v_1 gives a clipped continuous trajectory, and, according to the previous paragraph, we finish with the clipped $1-$cycle attained after two iterations.

As Lemma 1 says, an unclipped $1-$cycle does not exist if $\beta e^m < 1$. One can easily show that inequality $\beta e^m < 1$ is also necessary for the existence of clipped $1-$cycles. Indeed, if $\beta e^m \geq 1$ then formula (28) gives $S \leq 0$, and that formula remains the same for clipped and unclipped cycles because equation (26) is universal.

The very last statement is justified in full by all the previous reasoning. ■

Before proving Proposition 1, let us justify formula (9). Clearly,

$$T_{cycle} = \int_0^{T_{cycle}} dt = \int_0^{S+1} \left\{ T + \frac{By(s)}{\mu} \right\} ds,$$

where S is given by (28), and expression (9) follows.

Proof of Proposition 1. In the case $q \le q^* \Leftrightarrow B \ge B^* = \frac{\mu T}{q^*}$, the cycle is unclipped. The average sending rate can be calculated according to formula $\bar{\lambda} = \frac{\int_0^{S+1} w(s)ds}{T_{cycle}} = \frac{B}{T_{cycle}} \int_0^{S+1} V_0 e^{ms} ds$ (see (24)). The average throughput can be calculated as the following.

$$\bar{g} \;=\; \frac{1}{T_{cycle}} \left\{ \int_0^{T_{cycle}-T-\frac{B}{\mu}} \lambda(t)dt + \mu \left(T + \frac{B}{\mu} \right) \right\} = \frac{1}{T_{cycle}} \left\{ \int_0^{S} w(s)ds + \mu \left(T + \frac{B}{\mu} \right) \right\} = \mu.$$

Also, the average amount of data in the buffer is calculated as below.

$$\bar{x} = \frac{1}{T_{cycle}} \int_0^{T_{cycle}} x(t)dt = \frac{1}{T_{cycle}} \int_0^{S+1} By(s)(T + \frac{By(s)}{\mu})ds.$$

■

In case $B < B^* \Leftrightarrow q > q^*$, the cycle is clipped. As before, we use $t(s)$ for the original (new) time scale. The graph of the cycle in the plane (v, y) looks similarly to Figure 1; one has only to replace "Buffer size (B)" on the $y-$axis by 1. Suppose the cycle starts at $s = S_A = 0$ from point A, reaches point B at time moment S_B and so on. We shall use denotations like S_{BC} for $S_C - S_B$.

Point C has coordinates $y = 0$ and $v = q$, so that, when $s \in [S_C, S_D]$,

$$y(s) = \frac{q}{m+1} e^{m(s-S_C)} - q + e^{-(s-S_C)} (\frac{mq}{m+1})$$

according to (25). Therefore, $S_{CD} = S_D - S_C$ is the single positive solution to equation $A(s) = 0$ where

$$A(s) = qe^{ms} + e^{-s}qm - (m+1)(q+1).$$

(Note that $\lim_{s\to 0} A(s) = -1 - m, \lim_{s\to\infty} A(s) = \infty$ and $\frac{dA}{ds} > 0$.) Equation (13) is proved. Formula $V_0^{clipped} = v(0) = \frac{\mu T}{B} \beta e^{m(S_{CD}+1)}$ at the beginning of the cycle follows from (24), so that expression (12) is justified. According to (25),

$$y(s) = \frac{V_0^{clipped}}{m+1} e^{ms} - q + e^{-s} \left(1 + q - \frac{V_0^{clipped}}{m+1} \right)$$

for $s \in [0, S_B]$, where $S_B = S_{AB}$ is the minimal positive solution of equation $y_{AB}(S_{AB}) = 0$. (The maximal solution is phantom, corresponding to the last moment when component y equals zero in case we ignore the non-negativity constraint, i.e., if we deal with the unconstrained case.) Equation (15) is obtained.

Now

$$T_{cycle} = \int_0^{S+1} \left[T + \frac{By(s)}{\mu} \right] ds = \frac{T}{m} \ln \frac{1}{\beta} + \frac{B}{\mu} \left\{ \int_0^{S_B} y(s)ds + \int_{S_C}^{S_D} y(s)ds + 1 \right\}$$

and formulae (16) (17) (14) are proved where we made the trivial change of the (new) time scale:

$$Y_{CD}(s) = y(S_C + s); Y_{AB}(s) = y(S_A + s) = y(s).$$

Proof of Proposition 2. Similarly to the case $B \ge B^*$, $\bar{\lambda} = \frac{1}{T_{cycle}} \int_0^{S+1} w(s)ds$ leads to formula (18).

For the average throughput we have, using (24):

$$\bar{g} = \frac{1}{T_{cycle}} \left\{ \int_0^{S_{AD}} w(s)ds + \mu(T + \frac{B}{\mu}) \right\} = \frac{1}{T_{cycle}} \left\{ B \int_0^{S} v(s)ds + \mu T + B \right\}$$

$$= \frac{1}{T_{cycle}} \left\{ \frac{BV_0^{clipped}}{m} (\frac{1}{\beta e^m} - 1) + \mu T + B \right\}.$$

Also the average amount of data in the buffer is calculated as follows.

$$\begin{aligned}
\bar{x} &= \frac{1}{T_{cycle}} \int_0^{T_{cycle}} x(t)dt \\
&= \frac{1}{T_{cycle}} \left\{ \int_0^{S_B} By(s) \left(T + \frac{By(s)}{\mu} \right) ds + \int_{S_C}^{S_D} By(s) \left(T + \frac{By(s)}{\mu} \right) ds \right. \\
&\quad \left. + B \left(T + \frac{B}{\mu} \right) \right\} \\
&= \frac{1}{T_{cycle}} \left\{ TB \left(\int_0^{S_{AB}} Y_{AB}(s)ds + \int_0^{S_{CD}} Y_{CD}(s)ds \right) \right. \\
&\quad \left. + \frac{B^2}{\mu} \left(\int_0^{S_{AB}} Y_{AB}^2(s)ds + \int_0^{S_{CD}} Y_{CD}^2(s)ds \right) + B \left(T + \frac{B}{\mu} \right) \right\}.
\end{aligned}$$

■