From Individual to Collective Rationality: a Systematic Study in Binary Aggregation

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Everything Starts From a Paradox

In 1785 Monsieur le Marquis de Condorcet pointed out that:

$$\triangle \succ_{\mathbf{1}} \bigcirc \succ_{\mathbf{1}} \square$$
$$\square \succ_{\mathbf{2}} \triangle \succ_{\mathbf{2}} \bigcirc$$
$$\bigcirc \succ_{\mathbf{3}} \square \succ_{\mathbf{3}} \triangle$$

$$\triangle \succ \bigcirc \succ \square \succ \triangle$$

???

- Why is this a paradox?
- Why does this happen?

Outline

1. What is a paradox?
   - Various notions of individual rationality
   - A propositional language for rationality assumptions
   - Binary aggregation with integrity constraints
   - General definition of paradox
2. Why do paradoxes come about?
   - Languages for integrity constraints
   - Collective rationality and axiomatic properties
   - Characterisation Results
   - An answer to M.Condorcet: the majority rule
3. Conclusions and related work

Part I:
Individual Rationalit<u>ies</u> and Paradoxes

Individual Rationality in Decision Theory

The problem:	Individuals choosing over a set of alternatives $\mathcal{X}$
Rational behaviour:	Maximise a weak order over $\mathcal{X}$ (transitive, complete and reflexive binary relation)

- Linear orders to avoid ties
- Partial orders over large domains
- Acyclic relations defined from choice functions

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Remark: we do not talk about uncertainties.

Many Rationalities?

Judges in a court (cf. judgment aggregation):

"O.J.Simpson is guilty" "O.J.Simpson wore bloody gloves"
"If O.J.Simpson wore the glove then he is guilty"

Rational judges?
Consistent and complete judgment sets

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Rational judges?
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Committee deciding over multiple issues:

"Cut pensions" "Cut the number of MPs"
"Cut funding to local provinces"

Rational members?
No political power to enforce all three austerity measures:
Ballots with at most 2 yes

Binary Aggregation with Integrity Constraints

"Everything is binary"

- Individuals express yes/no ballots over a finite set of issues $\mathcal{I}$
- A propositional language can be interpreted over ballots
- Rationality assumptions/integrity constraints are formulas in this language

Binary Aggregation

Ingredients:

- A finite set N of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- A boolean *combinatorial domain*: $\mathcal{D} = D_1 \times \cdots \times D_m$ with $|D_i| = 2$

Definition

An aggregation procedure is a function $F : \mathcal{D}^N \to \mathcal{D}$ mapping each profile of ballots $\mathbf{B} = (\mathbf{B}_1, \ldots, \mathbf{B}_n)$ to an element of the domain $\mathcal{D}$.

Dokow and Holzman (2005), Grandi and Endriss (AAAI-2010)

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Example: Science Park

- $N = \{1, 2, 3\}$
- $\mathcal{I} = \{\text{University}, \text{Sportcentrum}, \text{Food supply}\}$
- Individuals submit ballots in $\mathcal{D} = \{0, 1\}^3$

$B_1 = (0, 1, 1)$ the first individual wants to have a good meal after the gym.

Integrity Constraints

A propositional language $\mathcal{L}$ to express integrity constraints on $D = \{0,1\}^m$

- One propositional symbol for every issue: $PS = \{p_1, \ldots, p_m\}$
- $\mathcal{L}_{PS}$ closing under connectives $\wedge$, $\vee$,$\neg$, $\rightarrow$ the set of atoms PS

Given an integrity constraint IC $\in \mathcal{L}_{PS}$, a rational ballot is $B \in \mathrm{Mod}(\mathrm{IC})$

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Example: Science Park (the true story)

If there is both a university and a sport center then food supply is necessary
Propositional constraint: IC $= (p_U \wedge p_S) \rightarrow p_F$

Individual 1 submits $B_1 = (1,0,0)$: B_1 satisfies IC ✓
Individual 2 submits $B_2 = (1,1,1)$: $B_2 \models$ IC ✓
Individual 3 submits $B_3 = (0,1,0)$: $B_3 \models$ IC ✓

Majority aggregation outputs $(1,1,0)$: IC not satisfied (as are all employees)

Paradoxes of Aggregation

Every individual satisfies the same rationality assumption IC...
...what about the collective outcome?

Definition

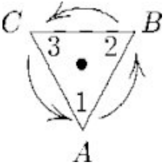
A paradox is a triple $(F, \mathbf{B}, IC)$, where:

- *F is an aggregation procedure*
- *$\mathbf{B} = (B_1, \dots, B_n)$ a profile*
- *$IC \in \mathcal{L}_{PS}$ an integrity constraint*

such that $B_i \models IC$ for all $i \in \mathcal{N}$ but $F(\mathbf{B}) \not\models IC$.

Condorcet Paradox Revisited

	ab	bc	ac
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0



Our definition of paradox:

- F is issue by issue majority rule
- the profile is the one described in the table
- IC that is violated is $p_{ab} \wedge p_{bc} \to p_{ac}$

Doctrinal Paradox

	α	$\alpha \to \beta$	β
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0!!

Our definition of paradox:

- F is issue by issue majority rule
- profile described in the table
- IC that is violated is $\neg(p_\alpha \wedge p_{\neg\beta} \wedge p_{(\alpha\to\beta)})$

Common feature: clauses of size 3

Part II:
Characterisation Results for Collective Rationality

Collective Rationality

Definition

F is collectively rational (CR) for IC $\in \mathcal{L}_{PS}$ if for all profiles $\mathbf{B}$ such that $\mathbf{B}_i \models$ IC for all $i \in N$ then $F(\mathbf{B}) \models$ IC.

F lifts the rationality assumption given by IC
from the individual to the collective level.

$$\mathcal{CR}[\mathcal{L}] = \{F : \mathcal{D}^{\mathcal{N}} \to \mathcal{D} \mid \mathcal{N} \text{ is finite and } F \text{ is CR for all IC} \in \mathcal{L}\}$$

where $\mathcal{L} \subseteq \mathcal{L}_{PS}$ is a sublanguage

Languages for Integrity Constraints

Definition

A language for integrity constraints is a subset $\mathcal{L} \subseteq \mathcal{L}_{PS}$ that is closed under conjunctions, logical equivalence and contains $\top$ and $\bot$.

- if F is CR wrt. φ and to ψ then is CR wrt. $\varphi \wedge \psi$
- if F is CR wrt. φ then it is so for every equivalent formulas
- $\top$ and $\bot$ are trivial requirements for CR

Lemma

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are languages for IC, then if $\mathcal{L}_1 \neq \mathcal{L}_2$ then $\mathcal{CR}[\mathcal{L}_1] \neq \mathcal{CR}[\mathcal{L}_2]$

Languages and Axioms

Several languages for integrity constraints:

- *cubes*: conjunctions
- *k-pclauses*: positive disjunctions of size $\leqslant k$
- *XOR*: conjunctions of $p \leftrightarrow \neg q$
- ...

Several axioms to classify aggregation procedures:

- Unanimity (U): For any profile $\mathbf{B} \in X^N$ and any $x \in \{0, 1\}$, if $\mathbf{B}_{i,j} = x$ for all $i \in N$, then $F(\mathbf{B})_j = x$.
- Independence, Neutrality...

Characterisation Results

Different lists of axioms AX define classes of functions:

$$\mathcal{F}_{\mathcal{L}}[\mathrm{AX}] = \{F : \mathcal{D}^{\mathcal{N}} \to \mathcal{D} \mid F_{\restriction \mathrm{Mod}(\mathsf{IC})^N} \text{ sat. } \mathrm{AX} \text{ for all } \mathsf{IC} \in \mathcal{L}\}$$

Recall that the class of CR procedures for a language $\mathcal{L}$ is:

$$\mathcal{CR}[\mathcal{L}] = \{F : \mathcal{D}^{\mathcal{N}} \to \mathcal{D} \mid F \text{ is CR for all } \mathsf{IC} \in \mathcal{L}\}$$

What we want:

$$\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[\mathrm{AX}]$$

Characterisation Results: Examples

Cubes (conjunctions of literals) are lifted iff the procedure satisfies unanimity:

Proposition

$\mathcal{CR}[cubes] = \mathcal{F}_{cubes}[Unanimity]$.

Similar results can be proven for language of equivalences (issue-neutrality), XOR formulas (domain-neutrality), positive implications (neutral-monotonicity).

Characterisation Results: Examples

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Similar results can be proven for language of equivalences (issue-neutrality), XOR formulas (domain-neutrality), positive implications (neutral-monotonicity).

For the axioms of independence we prove instead a negative result:

Proposition

There is no language $\mathcal{L} \subseteq \mathcal{L}_{PS}$ *such that* $\mathcal{CR}[\mathcal{L}] = \mathcal{F}_{\mathcal{L}}[\mathrm{I}]$.

The same result holds for the axiom of anonymity and monotonicity

Committee and Quota Rules

Interesting classes of procedures:

- Independence $\rightarrow$ Committee rules
  Every issue j a decisive committee $N_j \subseteq \mathcal{N}$

$$\mathcal{F}[\mathrm{I}] = \text{Committee rules}$$

- Anonymity $\rightarrow$ Only size matter
  Committee of same size have same decision power

- Monotonicity $\rightarrow$ Quota rules
  Every issue j a quota q_j

$$\mathcal{F}[\mathrm{I}, \mathrm{A}, \mathrm{M}] = \text{Quota rules}$$

Example: The majority rule has uniform quota $\lceil \frac{n+1}{2} \rceil$

Quota Rules and Languages of Clauses

Interesting results concerning positive/negative clauses, and a general equation:

Proposition

A quota rule is CR with respect to a k-clause IC iff

$$\sum_{j \text{ negative}} q_j + \sum_{j \text{ positive}} (n - q_j + 1) > n(k-1) \tag{1}$$

for issues j that occur positive or negative in IC, or $q_j = 0$ for some issue j that occurs positive in IC, or $q_j = n + 1$ for issue j negative in IC.

Yet no characterisation result for general languages of clauses (for $k > 2$):

$$\mathcal{CR}[\textit{k-clauses}] \cap \mathcal{QR} = \emptyset$$

My Answer to M. Condorcet: The Majority Rule

Proposition

The majority rule does not generate a paradox with respect to IC if and only if IC is equivalent to a conjunction of clauses of size $\leqslant 2$.

$$\mathcal{IC}(\textit{Maj}) = \textit{2-clauses}$$

Common feature of all paradoxes:
clauses of size 3 are not lifted by majority

Part III:
Related Work and Conclusion

Are These Results Useful for Social Choice Theorists?

Yes, we can talk about orders:

Call an aggregation procedure imposed if there are two alternatives x and y such that x is collectively preferred to y in every profile:

Proposition

Any anonymous, independent and monotonic aggregation procedure for more than 3 alternatives and 2 individuals is imposed.

Proof sketch:

- Translate preference aggregation into BA with IC
- Study the syntactic property of the $\text{IC}_<$
- Use a characterisation result!
- Go back to preference aggregation

Are These Results Useful for Computer Scientists?

Yes, we can talk about multi-agent systems:

Systems of automatic agents embedded with preferences or judgments to control and perform actions: need systematic theory of consistent aggregation!

Several applications:

- Preferential dependencies in elections
- Combinatorial vote
- Distance-based procedures

Airiau et al., Aggregating dependency graphs into voting agendas in multi-issue elections, *IJCAI-11*

Are These Results Useful for Philosophers?

Maybe.

A question: can we model a group of rational agents as a rational agent itself?

Proposition

F is CR with respect to all IC in $\mathcal{L}_{PS}$ if and only if F copies the ballot of a (possibly different) individual in every profile.

This class includes:

- Classical dictatorships $F(B_1, \dots, B_n) = B_i$ for $i \in \mathcal{N}$
- Distance-based generalised dictatorship: map $(B_1, \dots, B_n)$ to the ballot B_i that minimises the sum of the Hamming distance to the others (a sort of "median voter"...). An interesting procedure!

Conclusions

Many notions of individual rationality:

- binary issues as a general model of individual expressions;
- rationality assumptions as propositional formulas;
- focus on syntactic properties of the rationality assumption.

Collective rationality:

- Unifying framework for paradoxes;
- Systematic study of collective rationality;
- Application in preference/judgment aggregation & co.

Thanks for your attention!

Grandi and Endriss. Binary Aggregation with Integrity Constraints, *IJCAI-2011*, *LIRA Yearbook-10*.
Grandi and Endriss. Lifting Rationality Assumptions in Binary Aggregation, *AAAI-2010*.