The Common Structure of Paradoxes in Aggregation Theory

Umberto Grandi

Institute for Logic, Language and Computation
University of Amsterdam

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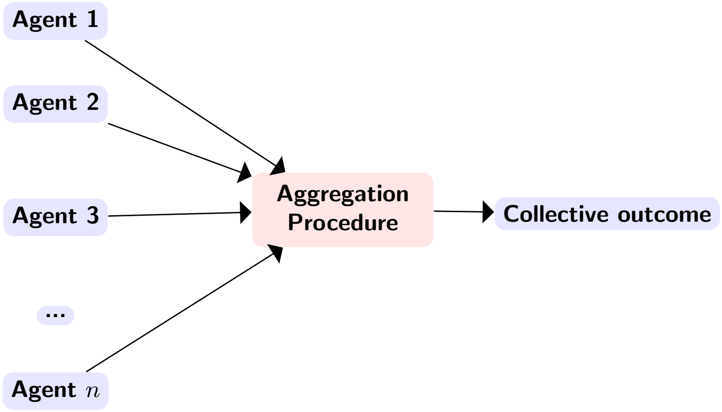
What is an aggregation problem?



What is a paradox (of aggregation)?

Condorcet Paradox

$$\triangle \succ_1 \bigcirc \succ_1 \square$$
$$\square \succ_2 \triangle \succ_2 \bigcirc$$
$$\bigcirc \succ_3 \square \succ_3 \triangle$$

$$\triangle \succ \bigcirc \succ \square \succ \triangle$$

Discursive Dilemma

	α	$\alpha \to \beta$	β
Judge 1	Y	Y	Y
Judge 2	N	Y	N
Judge 3	Y	N	N
Majority	Y	Y	N

To analyse the common structure of paradoxes of aggregation we need:

- A general framework for the study of aggregation problems: binary aggregation/abstract aggregation
- An explicit representation of rationality assumptions: integrity constraints

Outline

1. A general framework for the study of aggregation problems:
   binary aggregation with integrity constraints

2. Paradoxes of aggregation: Condorcet, Discursive, Ostrogorski

3. Observation: paradoxical integrity constraints have a common structure!

4. Characterisation of paradoxes for the majority rule

Binary Aggregation with Integrity Constraints

Ingredients:

- A finite set $\mathcal{N} = \{1, \ldots, n\}$ of individuals
- A finite set $\mathcal{I} = \{1, \ldots, m\}$ of issues
- A boolean combinatorial domain: $\mathcal{D} = \{0, 1\}^m$

Definition

An aggregation procedure is a function $F : \mathcal{D}^{\mathcal{N}} \to \mathcal{D}$ mapping each profile of ballots $\mathbf{B} = (B_1, \ldots, B_n)$ to an element of the domain $\mathcal{D}$.

Wilson (1975), Dokow and Holzman (2008), Nehring and Puppe 2010

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A propositional language $\mathcal{L}$ to express integrity constraints:

- One propositional symbol for every issue: $PS = \{p_1, \dots, p_m\}$
- $\mathcal{L}_{PS}$ closing under connectives $\wedge$, $\vee$, $\neg$, $\to$ the set of atoms PS

Given an integrity constraint $\mathsf{IC} \in \mathcal{L}_{PS}$, *a rational ballot is* $B \in \mathrm{Mod}(\mathsf{IC})$.

Example (paradoxical)

Example: Three agents with sensors

- $\mathcal{N} = \{1, 2, 3\}$
- $\mathcal{I} = \{T_1, T_2, A\}$
- Agents submit ballots in $\mathcal{D} = \{0, 1\}^3$

Perform action A if both sensors T_1 and T_2 are active.
Propositional constraint: $\text{IC} = (T_1 \wedge T_2) \rightarrow A$

	T_1	T_2	A
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0

B_1 satisfies IC ✓
$B_2 \models \text{IC}$ ✓
$B_3 \models \text{IC}$ ✓

IC not satisfied by the outcome of the majority.

Every individual satisfies the same rationality assumption IC...
...what about the collective outcome?

Definition

A paradox is a triple $(F, \mathbf{B}, IC)$, where:

- F *is an aggregation procedure*
- $\mathbf{B} = (B_1, \dots, B_n)$ *a profile*
- $IC \in \mathcal{L}_{PS}$ *an integrity constraint*

such that $B_i \models IC$ *for all* $i \in \mathcal{N}$ *but* $F(\mathbf{B}) \not\models IC$.

Preference Aggregation and Judgment Aggregation

Linear order $<$ over alternatives $\mathcal{X}$ $\Leftrightarrow$ Ballot $B_<$ over issues $\mathcal{I} = \{ab \mid a \neq b \in \mathcal{X}\}$

IC$_<$ encodes the rationality assumption for preferences:

Irreflexivity: $\neg p_{aa}$ for all $a \in \mathcal{X}$

Completeness: $p_{ab} \vee p_{ba}$ for all $a \neq b \in \mathcal{X}$

Transitivity: $p_{ab} \wedge p_{bc} \rightarrow p_{ac}$ for $a, b, c \in \mathcal{X}$ pairwise distinct

Judgment sets J over agenda Φ $\Leftrightarrow$ Ballot B_J over issues $\mathcal{I} = \Phi$

IC$_\Phi$ encodes the rationality assumption for judgments:

Completeness: $p_\alpha \vee p_{\neg\alpha}$ for all $\alpha \in \Phi$

Consistency: $\neg(\bigwedge_{\alpha \in S} p_\alpha)$ for every minimally inconsistent set $S \subseteq \Phi$

Condorcet Paradox Revisited

	$\triangle\bigcirc$	$\bigcirc\square$	$\triangle\square$
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Maj	1	1	0

Our definition of paradox:

- F is issue by issue majority rule
- profile as described in the table
- IC that is violated is $p_{\triangle\bigcirc} \wedge p_{\bigcirc\square} \to p_{\triangle\square}$

Doctrinal Paradox

	α	$\alpha \to \beta$	β
Agent 1	1	1	1
Agent 2	0	1	0
Agent 3	1	0	0
Majority	1	1	0

Our definition of paradox:

- F is issue by issue majority rule
- profile as described in the table
- IC that is violated is $\neg(p_\alpha \wedge p_{\neg\beta} \wedge p_{(\alpha\to\beta)})$

Ostrogorski Paradox

A majority of individuals may disagree with the majoritarian party on a majority of the issues:

	E	V	I	P
Agent 1	0	1	0	0
Agent 2	0	1	0	0
Agent 3	1	0	0	0
Agent 4	1	1	1	1
Agent 5	1	1	0	1
Majority	1	1	0	0

Our definition of paradox:

- F is issue by issue majority rule
- profile as described in table
- IC that is violated is $P \leftrightarrow [(E \wedge V) \vee (V \wedge I) \vee (I \wedge E)]$

The Common Structure of Paradoxes

Integrity constraints formalising classical paradoxes:

Condorcet Paradox: $p_{\triangle\bigcirc} \wedge p_{\bigcirc\square} \rightarrow p_{\triangle\square}$

Discursive Dilemma: $\neg(p_{\alpha} \wedge p_{\neg\beta} \wedge p_{(\alpha\rightarrow\beta)})$

Ostrogorski Paradox: $(P \vee \neg E \vee \neg V) \wedge (P \vee \neg E \vee \neg I) \wedge$
$\wedge (P \vee \neg I \vee \neg V) \wedge (\neg P \vee E \vee V) \wedge (\neg P \vee E \vee I) \wedge (\neg P \vee I \vee V)$

Multiple Election Paradox: can be related to our definition

The Common Structure of Paradoxes

Integrity constraints formalising classical paradoxes:

Condorcet Paradox: $\neg p_{\triangle\bigcirc} \vee \neg p_{\bigcirc\square} \vee p_{\triangle\square}$

Discursive Dilemma: $\neg p_{\alpha} \vee \neg p_{\neg\beta} \vee \neg p_{(\alpha\rightarrow\beta)}$

Ostrogorski Paradox: $(P \vee \neg E \vee \neg V) \wedge (P \vee \neg E \vee \neg I) \wedge$
$\wedge (P \vee \neg I \vee \neg V) \wedge (\neg P \vee E \vee V) \wedge (\neg P \vee E \vee I) \wedge (\neg P \vee I \vee V)$

Multiple Election Paradox: can be related to our definition

Integrity constraints formalising classical paradoxes of aggregation all feature a clause of size 3!!

Theorem

The majority rule (for an odd number of individuals) is collectively rational wrt. IC if and only if IC is equivalent to a conjunction of clauses of size $\leqslant 2$.

$$\mathcal{IC}(\mathit{Maj}) = 2\mathrm{CNF}^{\sim}$$

Indirect proof: from Nehring and Puppe (2002), clauses as *critical fragments*.

Direct proof (sketch)

- A clause $\ell_1 \vee \ell_2$ of size 2 cannot generate a paradox: if ℓ_1 and ℓ_2 are both rejected then there exists an individual which rejects both literals.
- Every formula is equivalent to the conjunction of its *prime implicates*. If there exists a prime implicate of size $\geqslant 3$ then we can devise a paradoxical situation for the majority rule.

Conclusions

1. Classical paradoxes of aggregation share a common structure;
2. The majority rule is very problematic on multiple issues;
3. Computational-friendly framework for aggregation. Propositional integrity constraints enable the use of known complexity results and tools from knowledge representation.

You are kindly invited to the public defence of my PhD thesis:

Binary Aggregation with Integrity Constraints

25 September 2012, University of Amsterdam

Workshop on "Frameworks for Multiagent Aggregation"