Graph Aggregation

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[Joint work with Ulle Endriss from University of Amsterdam]

What is this talk about

In this talk I will present a general framework for the aggregation of information coming from different sources in the form of graphs:

- The central problem is that of collective rationality: what graph properties are preserved by the aggregation?
- Axiomatic analysis, focus on Arrovian aggregators
- Two general impossibilities
- Integrity constraints expressed in modal logic: collective rationality at different levels

Results from different papers collected in the following article:

Ulle Endriss and Umberto Grandi. Graph Aggregation. ArXiv:1609.03765, 2016.

Outline

1. Formal definitions of graph aggregation
2. A general impossibility theorem
3. Integrity constraints in modal logic
4. Discussion and potential applications

Graphs and graph properties

Given a finite set of vertices V, a directed graph $G = \langle V, E \rangle$ is defined by a set of edges $E \subseteq V \times V$. A number of graph properties can be considered:

Property	First-Order Condition
Reflexivity	$\forall x.xEx$
Symmetry	$\forall xy.(xEy \rightarrow yEx)$
Right Euclidean	$\forall xyz.[(xEy \wedge xEz) \rightarrow yEz]$
Transitivity	$\forall xyz.[(xEy \wedge yEz) \rightarrow xEz]$
Negative Transitivity	$\forall xyz.[xEy \rightarrow (xEz \vee zEy)]$
Connectedness	$\forall xyz.[(xEy \wedge xEz) \rightarrow (yEz \vee zEy)]$
Completeness	$\forall xy.[x \neq y \rightarrow (xEy \vee yEx)]$
Nontriviality	$\exists xy.xEy$
Seriality	$\forall x.\exists y.xEy$

Aggregation of graphs

Consider the following setting:

- A finite group of agents $\mathcal{N} = \{1, \ldots, n\}$
- Each $i \in \mathcal{N}$ submits a graph on the same set of vertices
- A profile of graphs is $(E_1, ..., E_n)$
- An aggregator is a function that associates a collective graph with a profile

Definition

Given a set of n individuals and a set of vertices V, an **aggregation rule** *is a function $F : (2^{V \times V})^n \to 2^{V \times V}$.*

- The requirement of all graphs with the same set of vertices can be relaxed
- Example 1: accept an edge if $> \frac{n}{2}$ agents do (strict majority)
- Example 2: accept an edge only if all the agents do (intersection)

Potential applications

Elections and preference aggregation: vertices are candidates, and individuals submit reflexive, complete and transitive graphs (= weak orders)

Aggregation of Dung's argumentation graphs: vertices are arguments, and different agents specify an attack relation among them

Epistemology: vertices are possible worlds, each agent has its own accessibility relation. Some well-known aggregators:

- intersection of graphs $\leftrightarrow$ distributed knowledge
- union of graphs $\leftrightarrow$ shared knowledge
- transitive closure of union $\leftrightarrow$ common knowledge

But also: social network analysis (labelled networks), consensus clustering...

We will come back to some of these applications at the end of the talk...

Axiomatic description of aggregators

There is a wide number of possible aggregators:

Quota rules: such as majority, intersection, union...

Successor-approval rules: outgoing edges decided by approvals on successor vertices

Distance-based rules: minimise a distance from a set of individual graphs

We classify aggregators using axiomatic properties:

Nondictatoriality: for no i^* we always have $F(\boldsymbol{E}) = E_{i^*}$

Unanimity: $F(\boldsymbol{E}) \supseteq E_1 \cap \cdots \cap E_n$

Groundedness: $F(\boldsymbol{E}) \subseteq E_1 \cup \cdots \cup E_n$

Neutrality: $N_e^{\boldsymbol{E}} = N_{e'}^{\boldsymbol{E}}$ implies that $e \in F(\boldsymbol{E}) \Leftrightarrow e' \in F(\boldsymbol{E})$

Independence: $N_e^{\boldsymbol{E}} = N_e^{\boldsymbol{E'}}$ implies $e \in F(\boldsymbol{E}) \Leftrightarrow e \in F(\boldsymbol{E'})$

Notation: $N_e^{\boldsymbol{E}}$ is the coalition accepting edge e in profile $\boldsymbol{E}$

NR-axiom means that the axiom only applies to non-reflexive edges

Collective rationality

Which kind of properties are preserved during the aggregation?

Definition

An aggregation rule F is collectively rational (CR) wrt a graph property P if $F(\boldsymbol{E})$ satisfies P whenever all of the individual graphs in $\boldsymbol{E} = (E_1, \ldots, E_n)$ do.

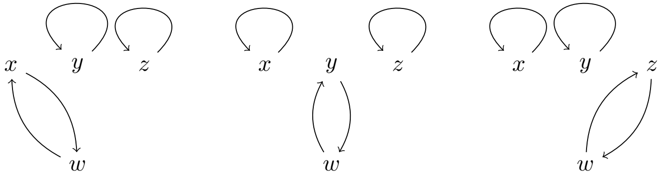
Observation. Same question studied in preference and judgment aggregation:

- A Condorcet paradox shows that the pairwise majority rule is not collectively rational wrt transitivity of preferences
- Judgement aggregation requires the logical consistency of the accepted formulas (a property that is specific to the agenda considered)

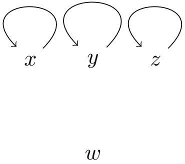
Graph aggregation lies between these two frameworks: preferences can be thought of as graphs over the candidates, and graphs can be represented as sets of propositional statements of the form xEy...

Majority rule and collective rationality

Suppose three agents submit the following graphs:



Aggregated using the majority rule to the following graph:



- •) The majority rule is not collectively rational for seriality
- •) Symmetry is preserved
- •) Reflexivity as well (since individuals violate it)

Part I:
A general impossibility theorem

What is the relation between axioms and collective rationality?

Basic results linking axioms with the preservation of graph properties:

Proposition

Every unanimous aggregator is collectively rational for reflexivity.

Proof. If every individual has edge (x, x) for any vertex x then by unanimity this edge will be present in the collective graph as well.

Proposition

Every grounded aggregator is collectively rational for irreflexivity.

Proposition

Every neutral aggregator is collectively rational for symmetry.

Proof. Suppose each input graph is symmetric: then (x, y) and (y, x) have the same support. By neutrality they are either both accepted or both rejected.

Arrow's Theorem

Can we go further than one-axiom one-property results?

Arrow's Theorem in graph aggregation

For $|V| \geqslant 3$ there exists no NR-nondictatorial, unanimous, grounded and independent aggregator that is CR for transitivity and completeness.

The standard formulation is equivalent:

- weak preference orders are reflexive, transitive and complete graphs
- weak Pareto implies unanimity and groundedness
- standard nondictatoriality is NR-non-dictatoriality on reflexive graphs
- collective rationality for reflexivity is implied by unanimity

Question: For what kind of graph properties does this result go through?

Two general impossibility theorems

We obtain two general results:

Oligarchy theorem

For $|V| \geqslant 3$, any unanimous, grounded, and independent aggregator that is CR for a contagious and implicative graph property must be NR-oligarchic.

Dictatorship theorem

For $|V| \geqslant 3$, any unanimous, grounded, and independent aggregator that is CR for a contagious, implicative and disjunctive property must be NR-dictatorial.

- Implicative property: $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \wedge e_2 \rightarrow e_3]$
- Disjunctive property: $[\bigwedge S^+ \wedge \neg \bigvee S^-] \rightarrow [e_1 \vee e_2]$
- Contagious property: for every accepted edge its "neighbouring edges" must under some conditions be accepted as well

The ultrafilter method

Both theorems are proven using the ultrafilter method:

1. Every Arrovian aggregator is characterised by its winning coalitions $\mathcal{W}$
   - By independence the acceptance of an edge only depends on the coalition of edges accepting it
   - Neutrality Lemma: CR for contagious property implies NR-neutrality
2. CR for an implicative property implies that $\mathcal{W}$ is a filter over $\mathcal{N}$
   - $\mathcal{N}$ is winning by unanimity
   - Closure under intersection ($C_1 \in \mathcal{W} \wedge C_2 \in \mathcal{W} \Rightarrow C_1 \cap C_2 \in \mathcal{W}$) by CR
   - Closure under superset ($C_2 \supset C_1$ and $C_1 \in \mathcal{W} \Rightarrow C_2 \in \mathcal{W}$) by CR
3. CR for a disjunctive property implies that $\mathcal{W}$ is an ultrafilter over $\mathcal{N}$
   - Maximality ($C \in \mathcal{W}$ or $\mathcal{N} \setminus C \in \mathcal{W}$) by CR
4. Conclusion: a filter over a finite set $\mathcal{N}$ is an oligarchy, and an ultrafilter over a finite set is principal, aka dictatorial.

Examples and properties

Arrow's theorem follows since:

- Transitivity is contagious and implicative
- Completeness is disjunctive

Many combination of graph properties have our meta-properties:

Property	Contagious?	Implicative?	Disjunctive?
Transitivity	✓	✓	×
Right Euclidean	✓	✓	×
Negative Transitivity	✓	×	✓
Connectedness	✓	✓	✓
Completeness	×	×	✓
Nontriviality	×	×	✓
Seriality	×	×	✓

Part II:

Integrity constraints in modal logic

Modal formulas and graph properties

Graph aggregation can be used to aggregate Kripke frames:

- V is the set of possible worlds (the same for all agents, eg. the full set of propositional evaluations)
- E_i is agent i's accessibility relation.

Associate modal formulas with graph properties using correspondence theory:

Property	Modal Formula
Reflexivity	$p \to \Diamond p$
Symmetry	$p \to \Box\Diamond p$
Right Euclidean	$\Diamond p \to \Box\Diamond p$
Transitivity	$\Diamond\Diamond p \to \Diamond p$
Connectedness	$\Box(\Box p \to q) \vee \Box(\Box q \to p)$
Seriality	$\Diamond(p \vee \neg p)$

Some properties cannot be expressed so: completenes, negative transitivity...

Three levels for collective rationality

An hierarchy of collective rationality comes naturally from the modal semantics:

Frame collectively rationality for φ if $\langle V, E_i \rangle \models \varphi$ for all $i \in \mathcal{N}$ implies $\langle V, F(\boldsymbol{E}) \rangle \models \varphi$ (corresponds directly to CR wrt to graph properties).

Model collectively rationality for φ if for every valuation $\mathit{Val} : \Phi \to 2^V$ we have $\langle \langle V, E_i \rangle, \mathit{Val} \rangle \models \varphi$ for all $i \in \mathcal{N}$ implying $\langle \langle V, F(\boldsymbol{E}) \rangle, \mathit{Val} \rangle \models \varphi$.

World collectively rationality for φ if for every valuation $\mathit{Val} : \Phi \to 2^V$ and every world $x \in V$ we have $\langle \langle V, E_i \rangle, \mathit{Val} \rangle, x \models \varphi$ for all $i \in \mathcal{N}$ implying $\langle \langle V, F(\boldsymbol{E}) \rangle, \mathit{Val} \rangle, x \models \varphi$.

The following relation of strength is easy to obtain:

Proposition

For all aggregators F and for all modal formulas φ we have that
F is world-CR for φ $\Rightarrow$ F is model-CR for φ $\Rightarrow$ F is frame-CR for φ

Model collective rationality – limitations

The two basic results relating unanimity with reflexivity and neutrality with symmetry at the level of frame collective rationality do not transfer to the level of model collective rationality:

Counterexample for $p \to \Diamond p$ (reflexivity)

Let $V = \{x, y\}$ and let the two submitted graphs be

- $E_1 = \{(x, y), (y, y)\}$
- $E_2 = \{(y, x), (x, x)\}$

Consider the valuation Val$(p) = \{x, y\}$. The formula $p \to \Diamond p$ is globally true in both individual models, but the intersection rule (which is unanimous) returns the empty graph, which falsifies the formula.

Similar example for symmetry using the intersection rule.

World collective rationality – possibilitites

$\Box$-formulas: no occurrence of $\Diamond$ when put in NNF ($\Diamond$-formulas analogously)

Proposition - Box formulas

If F is such that for every profile $\boldsymbol{E}$ there is an individual i^* such that $F(\boldsymbol{E}) \subseteq E_{i^*}$ then it is world-CR for all $\Box$-formulas.

Proposition - Diamond formulas

If F is such that for every profile $\boldsymbol{E}$ there is an individual i^* such that $F(\boldsymbol{E}) \supseteq E_{i^*}$ then it is world-CR for all $\Diamond$-formulas.

Representative voter rules associate with a profile the graph of a (possibly different) individual - examples are dictatorships and average voter selection:

Proposition - Representative voter

Any representative voter rule is world-CR for any modal integrity constraint.

Any interesting interpretation in theories of group agency?

Part I:
Discussion and potential applications

Creating collective agents

Kripke-models are used as models of an individual's knowledge or beliefs:

Graphs $E_1, ..., E_n$ with certain properties (typically validities)

We can treat the aggregation of these graphs $F(E_1, \ldots, E_n)$ as a new agent:

- Study the properties preserved by specific aggregators. Intersection corresponds to D (or R_G?), preserves reflexivity, not completeness...
- Find all the aggregators that preserve given properties: The only Arrovian aggregator for equivalence relation is an oligarchy (a dictatorship if equivalence relations needs to be non-trivial)

Two interesting questions:

- Considerations of computational complexity: non-independent aggregator are typically NP-hard or worse (though transitive closure is polynomial)
- The only F that preserves all properties is a generalised dictatorship: it copies in every profile the ballot of a (possibly different) individual

C. List and P.Pettit. Group Agency: The Possibility, Design, and Status of Corporate Agents. Oxford University Press, 2011.

Incomplete preferences

Bounded rationality in AI consider incomplete preferences:

- Preorders: reflexive and transitive graphs
- MAX-MIN-preorders: preorders with a maximum and a minimum

Consequences of our general impossibility theorem are the following:

Theorem

Let F be aggregator that is collectively rational for preorders over three or more alternatives. Then F is Arrovian (=IND, UN, GR) iff it is oligarchic.

Theorem [Pini Et Al. 2009]

Any Arrovian aggregator that is collectively rational wrt. MAX-MIN-preorders over three or more alternatives must be dictatorial.

M. S. Pini, F. Rossi, K. B. Venable, T. Walsh. Aggregating partially ordered preferences. Journal of Logic and Computation., 2009.

Non-monotonic reasoning and belief merging

A number of different settings in the literature:

1. Belief merging "à la Konieczny and Pino Pèrez" (aggregation of belief bases) is more related to standard judgment aggregation.
2. Belief merging as aggregation of plausibility orders (=preorders): a graph aggregation problem!

Two connections from the literature:

- DW-1991 show that Arrovian aggregators for plausibility orders +one extra axiom need to be dictatorial: a consequence of our general theorem!
- MZL-2003 use preorders with negative transitivity and modify the independence axiom to obtain a possibility result. Our general result shows that the possibility is a consequence only of the latter (not as they claim), since negative transitivity is both contagious and disjunctive.

Doyle and Wellman, Impediments to universal preference-based default theories. *AIJ*, 1991.
Maynard-Zhang and Lehman, Representing and aggregating conflicting beliefs. *JAIR*, 2003.

Multiagent argumentation

Several papers on the aggregation of abstract argumentation frameworks use collective rationality in disguise for a number of properties (such as acyclicity).

Modal logic can be used to define a semantics/labelling, using $\Phi = \{\texttt{in}, \texttt{out}, \texttt{undec}\}$ as variables and the inverse of the attack relation:

$\Diamond\texttt{in}$ means the argument is attacked by an accepted argument.

A complete extension is one that satisfies the following:

- $\texttt{in} \rightarrow \Box\texttt{out}$ (an argument can only be "in", if all of its attackers are "out")
- $\Box\texttt{out} \rightarrow \texttt{in}$
  (if all of an argument's attackers are "out", then it should be "in")
- $\texttt{out} \rightarrow \Diamond\texttt{in}$
  (an argument should only be "out", if one of its attackers is "in")
- $\Diamond\texttt{in} \rightarrow \texttt{out}$ (an argument that has an attacker that is "in" must be "out")

Some are $\Box$ formulas, some other are $\Diamond$ formulas: we can use our results!

Conclusions

In this paper we proposed the framework of graph aggregation:

- Versatile setting given the ubiquity of graphs
- Central problem: collective rationality wrt graph properties
- Two general impossibility theorems for Arrovian aggregators:
  - Arrovian aggregators are oligarchic if CR for contagious and implicative
  - Arrovia aggregators are dictatorial if CR for contagious, implicative and disjunctive properties
- Integrity constraints can be expressed in modal logic

Many potential applications in AI:

- Bounded rationality in preference aggregation
- Belief merging as the aggregation of plausibility orders
- Modal integrity constraints specifying properties of extensions in the aggregation of abstract argumentation graphs

Ulle Endriss and Umberto Grandi. Graph Aggregation. ArXiv:1609.03765, 2016.