Belief Merging

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Introduction

The situation: several agents holding beliefs about the state of the world

States are described using a finite number of propositional symbols, there is no hierarchy, nor priority, nor any difference in reliability of the sources.

The problem: how to define the beliefs of the group? How to merge the individual belief sets into a collective one?

Example:

\[ K_1 = \{a, b \rightarrow c\} \]
\[ K_2 = \{a, b\} \]
\[ K_3 = \{\neg a\} \]

\[ K = ? \] a candidate set could be: \( \{b \rightarrow c, b, a\} \)
Overview

Plan for today:

- Logic-based belief merging
- Axioms for IC merging operators and representation result
- Examples: Sum operator, Gmax operator, DA^2 operators
- Related work: Strategy-proof operators, belief revision, belief merging in judgment aggregation...

Based on the following survey paper:

Konieczny and Pino Pérez. Logic Based Merging. JPL, 2011.
Basic Definitions

Ingredients:
- A finite set of individuals $\mathcal{N}$
- A propositional language $\mathcal{L}_{PS}$ defined from a finite set of atoms $PS$

A knowledge base is a finite set of formulas $K \subseteq \mathcal{L}_{PS}$

Each individual submits a consistent knowledge base $K_i$

A profile is a multi-set of knowledge bases

$$E = \{K_1, \ldots, K_{|\mathcal{N}|}\}$$

Remark: Given a knowledge base $K$, denote with $\wedge K$ the conjunction of the formulas in $K$. Given a profile $E$ denote with $\wedge E$ the conjunction of all the bases in $E$. A profile $E$ is consistent if $\wedge E$ is consistent.
Belief Merging

Call:
- \( \mathcal{E} \) the set of all profiles
- \( \mathcal{K} \) the set of all knowledge bases
- integrity constraint any formula \( \mu \in \mathcal{L} \)

A merging operator is a function
\[
\Delta : \mathcal{E} \times \mathcal{L} \rightarrow \mathcal{K}
\]

The idea is to start from a profile \( E \) and an integrity constraint \( \mu \) and get as an outcome \( \Delta_\mu(E) \) the closest knowledge base to \( E \) satisfying the constraint \( \mu \).

How to formalise these properties?
The following remarks and notation:

- $K$ and $\wedge K$ will be interchangeable unless it creates ambiguity
- Two profiles are equivalent $E_1 \equiv E_2$ iff $\vdash \wedge E_1 \leftrightarrow \wedge E_2$
- $E_1 \sqcup E_2$ is the multi-set union
- $E^n = E \sqcup \cdots \sqcup E$ n-times
Axioms for IC merging operators

Definition

An operator $\Delta : \mathcal{E} \times \mathcal{L} \to \mathcal{K}$ is said to be an integrity constraint merging operator if it satisfies the following axioms (IC0)-(IC8).

(IC0): $\Delta_\mu(E) \vdash \mu$

(IC1): If $\mu$ is consistent then $\Delta_\mu(E)$ is consistent

(IC2): If $\wedge E$ is consistent with $\mu$ then $\Delta_\mu = \wedge E \wedge \mu$

(IC3): If $E_1 \equiv E_2$ and $\mu_1 \equiv \mu_2$ then $\Delta_{\mu_1}(E_1) \equiv \Delta_{\mu_2}(E_2)$

Axioms II

\((\text{IC4})\): If \(K_1 \vdash \mu\) and \(K_2 \vdash \mu\) then \(\Delta_\mu(\{K_1, K_2\}) \land K_1\) is consistent if and only if \(\Delta_\mu(\{K_1, K_2\}) \land K_2\) is consistent.

This axioms is called \textit{fairness} by Konieczny and Pérez (2011). It resembles a neutrality axiom with respect to consistency: When merging two knowledge bases we shall not give preference to neither of the two.

\((\text{IC5})\): \(\Delta_\mu(E_1) \land \Delta_\mu(E_2) \vdash \Delta_\mu(E_1 \sqcup E_2)\)

\((\text{IC6})\): If \(\Delta_\mu(E_1) \land \Delta_\mu(E_2)\) is consistent, then \(\Delta_\mu(E_1 \sqcup E_2) \vdash \Delta_\mu(E_1) \land \Delta_\mu(E_2)\)

These two axioms together state that if we can subdivide the group in \textit{two subgroups} that separately agree on certain beliefs, then the result is the conjunction of the aggregated beliefs of the two groups.
Axioms III

Last two axioms:

\[(IC7)\]: \( \Delta_{\mu_1}(E) \land \mu_2 \vdash \Delta_{\mu_1 \land \mu_2}(E) \)

\[(IC8)\]: If \( \Delta_{\mu_1}(E) \land \mu_2 \) is consistent, then \( \Delta_{\mu_1 \land \mu_2}(E) \vdash \Delta_{\mu_1}(E) \land \mu_2 \)

These two axioms come from belief revision (see 2nd part of lecture)

The idea behind \((IC7)\) and \((IC8)\) is that when an alternative/belief is chosen in a certain set of feasible ones then if we restrict this set further and this alternative remains available it will still be chosen (strong similarity with properties of social choice functions).
Examples: Merging Operators That DO NOT Satisfy the Axioms

Meet revision:

$$\Delta_\mu(E) = \begin{cases} 
\land E \land \mu & \text{if consistent} \\
\mu & \text{otherwise}
\end{cases}$$

Does not satisfy axiom \((\text{IC6})!!\)

Take the conjunction of the belief bases if consistent, the disjunction otherwise:

$$\Delta_\mu(E) = \begin{cases} 
\land E \land \mu & \text{if consistent} \\
\lor E \land \mu & \text{otherwise, if consistent} \\
\mu & \text{otherwise}
\end{cases}$$

Does not satisfy axiom \((\text{IC6})!!\)
More Axioms...

There are other properties we might want to require for an IC merging operator:

**Majority:** \( \exists n \ \Delta_\mu(E_1 \sqcup E^n_2) \sqsubseteq \Delta_\mu(E_2) \)

Strong similarity with some axioms of Social Choice Theory

**Majority Independence:** \( \forall n \ \Delta_\mu(E_1 \sqcup E^n_2) \leftrightarrow \Delta_\mu(E_1 \sqcup E_2) \)

The result is independent from the “popularity”: impossible!!

The following remarks and notation:

- $K$ and $\wedge K$ will be interchangable unless it creates ambiguity
- Two profiles are equivalent $E_1 \equiv E_2$ iff $\vdash \wedge E_1 \leftrightarrow \wedge E_2$
- $E_1 \sqcup E_2$ is the multi-set union
- $E^n = E \sqcup \cdots \sqcup E$ n-times
**Arbitration**: complicated... cf. Konieczny and Pino Pérez (2011)

**Idea**: if the alternatives/belief preferred by two groups over two different integrity constraints are the same, and if the whole group is indifferent with respect to what satisfies one constraint but not the other, then the result of merging the belief of the two groups together is the initial set of preferred alternatives. (In accordance to one of the first papers on belief merging by Liberatore and Schaerf, 1995.)

Syncretic Assignments

Definition

A **syncretic assignment** is a function mapping each profile $E$

\[ E \rightarrow \leq_E \text{ over } \rho : PS \rightarrow \{0, 1\} \]

to a total preorder $\leq_E$ over assignments, satisfying the following conditions:

- If $\omega \models E$ and $\omega' \models E$ then $\omega \sim_E \omega'$
- If $\omega \models E$ and $\omega' \not\models E$ then $\omega <_E \omega'$
- If $E_1 \equiv E_2$ then $\leq_{E_1} = \leq_{E_2}$
- ...
Syncetic Assignments II

- $\forall \omega \models K_1 \exists \omega' \models K_2$ such that $\omega' \leq_{K_1 \cup K_2} \omega$

A version of “fairness”: no priority given to any knowledge base $K$

- If $\omega \leq_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$ then $\omega \leq_{E_1 \cup E_2} \omega'$
- If $\omega <_{E_1} \omega'$ and $\omega \leq_{E_2} \omega'$ then $\omega <_{E_1 \cup E_2} \omega'$

If an assignment is at least as plausible as another for two profiles, then it continue to do so if we join the two profiles: similarity with reinforcement axiom in social choice theory.
Representation Theorem

**Theorem (Konieczny and Pino Pérez, 2002)**

An operator $\Delta$ is an IC merging operator if and only if there exists a syncretic assignment that maps each profile $E$ to a total preorder $\leq_E$ such that

$$\text{Mod}(\Delta_\mu(E)) = \min_{\text{Mod}(\mu) \leq E}$$

Other properties can be provided to represent majoritarian operators and arbitration.
Example: the Sum Operator

Let $d$ be a counting distance between assignments $d: \mathcal{W} \times \mathcal{W} \to \mathbb{N}$.

If $K$ is a knowledge base, define $d(K, \rho) = \min_{\varphi \in K} d(\varphi, \rho)$.

**Definition**

Call sum operator the following:

$$\text{Mod}(\Delta_{\mu}^{\Sigma}(E)) = \arg \min_{\rho \in \text{Mod}(\mu)} \sum_{K \in E} d(K, \rho)$$

**Proposition**

$\Delta_{\mu}^{\Sigma}(E)$ is a majority IC merging operator.
Example: the GMax Operator

Definition

Let $E = \{K_1, \ldots, K_n\}$ and let $d_j^p = d(K_j, \rho)$. Let $L_E^\rho = (d_{i_1}^p, \ldots, d_{i_n}^p)$ sorted in descending order. Using the lexicographic ordering over vectors of integers we can define the ordering $\leq_{E}^{G\text{max}}$ over assignments and the Gmax operator

$$\text{Mod}(\Delta_{\mu}^{G\text{max}}(E)) = \min_{\text{Mod}(\mu)} \leq_{E}^{G\text{max}}$$

Proposition

$\Delta_{\mu}^{G\text{max}}(E)$ is an arbitration IC merging operator.
A "Concrete" Example

At a meeting of flat co-owners, the chairman proposes the construction of a swimming-pool, a tennis-court and a private-car-park. If two of these three items are build the rent will increase. Integrity Constraint:

$$\mu = ((S \land T) \lor (S \land P) \lor (T \land P)) \rightarrow I$$

**Hamming distance:** cardinality of the symmetric difference. Distance between (1, 0, 0,1) and (1, 1, 0,1) is 1 since they differ on only one issue.

Individual goals:

- $K_1 = K_2 = S \land T \land P$
- $K_3 = \neg S \land \neg T \land \neg P \land \neg I$
- $K_4 = T \land P \land I$

(see some calculation on blackboard...)

The result:

- $\text{Mod}(\Delta^\Sigma_\mu(E)) = \{(1, 1, 1, 1)\}$
- $\text{Mod}(\Delta^\text{Gmax}_\mu(E)) = \{(0, 0, 1, 0), (0, 1, 0, 0)\}$
Model-Based IC Merging Operators

Definition

Given:

- A pseudo-distance \( d : \mathcal{W} \times \mathcal{W} \rightarrow \mathbb{R}^+ \), defining \( d(K, \rho) = \min_{\varphi \in K} d(\varphi, \rho) \)
- An aggregator \( f : \mathbb{R}^n \rightarrow \mathbb{R}^+ \) (satisfying certain basic properties)

We can define the following operator:

\[
\text{Mod}(\Delta_{d,f}^{\mu}(E)) = \arg \min_{\rho \in \text{Mod}(\mu)} f(d(K_1, \rho), \ldots, d(K_n, \rho))
\]

Proposition

\( \Delta_{d,f}^{\mu}(E) \) is an IC merging operator for all \( f \) and \( d \).
DA² and Formula-Based Operators

**Definition**

Given:

- A pseudo-distance \( d : \mathcal{W} \times \mathcal{W} \to \mathbb{R}^+ \)
- Two aggregators \( f, g : \mathbb{R}^n \to \mathbb{R}^+ \) (satisfying certain basic properties)

We can define the following operator:

\[
\text{Mod}(\Delta^d_{\mu}(E)) = \arg \min_{\rho \in \text{Mod}(\mu)} f(d^g(K_1, \rho), \ldots, d^g(K_n, \rho))
\]

Where \( d^g(\rho, K) = g(d(\rho, \varphi_1), \ldots, d(\rho, \varphi_m)) \) with \( \varphi_j \in K \).

Complexity results for various distances can be found in:

A satisfaction index $i$ is a total function $i : \mathcal{L} \times \mathcal{L} \to \mathbb{R}$

**Definition**

- A profile $E$ is **manipulable** by a base $K$ for $i$ given $\Delta$ iff there exists a base $K'$ such that $i(K, \Delta_\mu(E \sqcup \{K'\})) > i(K, \Delta_\mu(E \sqcup \{K\}))$

- $\Delta$ is **strategy-proof** for $i$ iff there is no IC $\mu$ and profile $E$ such that $E$ is manipulable for $i$.

Examples of satisfaction index:

Weak drastic index

$$i_{dw}(K, K\Delta) = \begin{cases} 
1 & K \land K\Delta \text{ is consistent} \\
0 & \text{otherwise}
\end{cases}$$

Probabilistic index

$$i_p(K, K\Delta) = \begin{cases} 
0 & \text{if } |\text{Mod}(K\Delta)| = 0 \\
\frac{|(\text{Mod}(K) \cap \text{Mod}(K\Delta))|}{|\text{Mod}(K\Delta)|} & \text{otherwise}
\end{cases}$$

Proposition (Everaere et Al. 2004)

Let $f$ be an aggregation function, $\Delta^H,\Sigma_\mu$ is strategy proof for $i_p$ if and only if the initial knowledge base $K$ is complete (i.e., it has only one model).
Belief Revision (AGM Approach)

Given a knowledge base $K$ (be it a scientific theory, a set of beliefs, a corpus of norms...), there are three operations that can be performed when confronted with new information:

- **Expansion** $K \oplus \varphi$: adding the new information to the knowledge base (possibly getting an inconsistent outcome).
- **Contraction** $K \ominus \varphi$: deleting a previously kept belief, obtaining something from which $\varphi$ does not follow.
- **Revision** $K \otimes \varphi = (K \ominus \neg \varphi) \oplus \varphi$
  
  Firsts make space for $\varphi$ by deleting everything that contradicts it, and then expand the knowledge base safely in order to maintain consistency.

AGM give axioms to norm the behaviour of these three operators:

Alchourrón, Gärdenfors and Makinson. On The Logic of Theory Change... JSL, 1985.
Belief Revision and Belief Merging

For finite propositional languages these are equivalent axioms for an AGM revision operator (Katsuno and Mendelson, 1991):

R1: $\varphi \circ \mu$ implies $\mu$
R4: If $\varphi_1 \leftrightarrow \varphi_2$ and $\mu_1 \leftrightarrow \mu_2$ then $\varphi_1 \circ \mu_1 \leftrightarrow \varphi_2 \circ \mu_2$
R2: If $\varphi \land \mu$ is consistent then $\varphi \circ \mu \leftrightarrow \varphi \land \mu$
R5: $(\varphi \circ \mu) \land \psi$ implies $\varphi \circ (\mu \land \psi)$
R3: If $\mu$ is consistent then $\varphi \circ \mu$ is consistent
R6: If $(\varphi \circ \mu) \land \psi$ is consistent then $\varphi \circ (\mu \land \psi)$ implies $(\varphi \circ \mu) \land \psi$

**Theorem**

If $\Delta$ is an IC merging operator then the operator $\circ$ defined as $K \circ \mu = \Delta_\mu(K)$ is an AGM revision operator.

The other way round (from belief revision to merging) is not as easy:

Belief Merging and Judgment Aggregation

The two frameworks have many similarities, but at the moment there has not been any systematic study that attempted at combining them:

<table>
<thead>
<tr>
<th>Judgment Aggregation</th>
<th>Belief Merging</th>
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</thead>
<tbody>
<tr>
<td>Finite agenda</td>
<td>Infinite number of formulas $\mathcal{L}_{PS}$ (easy to fix)</td>
</tr>
<tr>
<td>Non-anonymous profiles</td>
<td>Anonymous profiles (multi-sets)</td>
</tr>
<tr>
<td>Complete judgment sets</td>
<td>No assumption on knowledge bases</td>
</tr>
<tr>
<td>No IC (easy to fix)</td>
<td>IC</td>
</tr>
<tr>
<td>Constraints of rationality</td>
<td>Constraints of feasibility (and rationality)</td>
</tr>
<tr>
<td>Focus on existing rules and on consistency</td>
<td>Focus on new rules while consistency is enforced by definition</td>
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Can we import results from one framework to the other?
Belief Merging in JA and SCT

Distance based aggregation procedures were introduced in the literature on judgment aggregation in the following paper:

Pigozzi. Belief merging and the discursive dilemma: an argument-based account to paradoxes of judgment aggregation. Synthese, 2006

Followed by several other papers:


Rules based on distances were already studied in classical SCT (Kemeny rule, distance rationalizability...). But belief merging might play an interesting role if we try to combine beliefs and preferences into a collective action:

Related Work

Syntax-Based merging operators:


Merging of weighted goal/belief bases, belief merging and negotiation...

References in the following survey papers:


Konieczny and Pino Pérez. Logic Based Merging. JPL, 2011.
Summary

During today’s lecture we have seen:

- a model of multi-agents beliefs based on propositional logic
- an axiomatisation for belief merging operators based on this model
- a representation result for this axiomatisation: syncretic assignments
- examples of operators: Sum, Gmax, DA²...

A number of related topics:

- Judgment aggregation: similar but different
- Belief revision: single-agent version of belief merging
- Strategy-proofness: negative results