Short proofs of normalization for the simply-typed λ-calculus, permutative conversions and Gödel's T: Erratum

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Abstract

The article of the same title by Felix Joachimski and Ralph Matthes that appeared in the Archive for Mathematical Logic [1] contains an imprecision in the description of the adopted induction principle in two lemmas and one wrong elementary step of reasoning. They can be corrected by reorganizing the adopted induction principle.

The crucial lemmas in [1] for establishing that the set SN (the syntax-directed characterization of strongly normalizing terms) is closed under application and substitution are correct in all cases albeit not immediately provable as stated in the cases with permutative conversions. The affected lemmas are thus Lemma 5.7 and Lemma 6.6. In both cases, the statement is proven by main induction on the type ρ and by side induction on $r \in \mathrm{SN}$, where the order of statements is used in that not only the side induction hypothesis is invoked but also one of the statements earlier in that order.

Lemma 6.6

The case (ii)(Var) with $y \equiv x$ is flawed: The term r on which we do side induction is now $x(s', z'.t')$. We already know $s'_x[s], t'_x[s] \in \mathrm{SN}$, and we want to invoke (i) with s in place of r, $s'_x[s]$ in place of s, z' in place of z and $t'_x[s]$ in place of t in order to obtain $s(s'_x[s], z'.t'_x[s]) \in \mathrm{SN}$. This does not fit with the side induction since s may even be more complex than $x(s', z'.t')$.

The very same proof *is* valid if it is perceived as a proof by main induction on ρ that, firstly, for all $r \in \mathrm{SN}$, (i) holds—to be proven by side induction on $r \in \mathrm{SN}$—and, secondly, that for all $r \in \mathrm{SN}$, (ii) holds—again to be proven by side induction on $r \in \mathrm{SN}$.

A typo: In case (i)(Var), we need (Var_π)&(Var), not (Var_π)&(Var_0).

Lemma 5.7

A similar problem occurs in (iii), case $y\vec{r}[R]$ if $x \equiv y$, $\vec{r}$ is empty and $[R]$ is one critical elimination R. The term r on which we do side induction is now xR, and (ii) is needed for s in place of r, which again is not captured by the side induction hypothesis. Again, this can be allowed if the statements (i), (ii) and (iii) are separately proven by induction on $r \in \mathrm{SN}$.

First solution: Prove by main induction on ρ that, firstly, for all $r \in \mathrm{SN}$, (ii) holds—to be proven by side induction on $r \in \mathrm{SN}$—and, secondly, that for all $r \in \mathrm{SN}$, (iii) holds—again to be proven by side induction on $r \in \mathrm{SN}$—and, thirdly, (i) for all $r \in \mathrm{SN}$ by side induction on $r \in \mathrm{SN}$. All the reasoning steps in the published proof may remain unchanged (including the critical case shown above that is incorrect in the published proof structure), with the following exception: In the proof of (i) for $\lambda xr \in \mathrm{SN}$ and $s^\rho \in \mathrm{SN}$, there is no side induction hypothesis for (iii) available but due to our changed order, (iii) is already proven for all terms $r \in \mathrm{SN}$ and $s^\rho \in \mathrm{SN}$ (with that same type ρ).

Second (alternative) solution: One can keep the order by adopting the same trick as in Lemma 6.6 where the type of the main premise in the rule of closure under application is controlled by the main induction. This means that (i) has to be replaced by

(i)' if $r : \rho \equiv \rho_0 \to \rho_1$ and $s \in \mathrm{SN}$ then $rs \in \mathrm{SN}$.

Then one proves by main induction on ρ that, firstly, for all $r \in \mathrm{SN}$, (i)' holds—to be proven by side induction on $r \in \mathrm{SN}$—and, secondly, that for all $r \in \mathrm{SN}$, (ii) holds—again to be proven by side induction on $r \in \mathrm{SN}$—and, thirdly, (iii) for all $r \in \mathrm{SN}$ by side induction on $r \in \mathrm{SN}$. The proof case that had to be changed in the first solution above is also changed in this proposal: In the proof of (i)' for λxr in place of r, with $\lambda xr : \rho_0 \to \rho_1 \equiv \rho$, we know that typability of $(\lambda xr)s$ entails that s has type ρ_0, hence we conclude $r_x[s] \in \mathrm{SN}$ from (iii) by the main induction hypothesis for type ρ_0. (In a sense, this is analogous to the case of an injection in the proof of (ii).) Moreover, in the proof of (iii), case $y\vec{r}[R]$ with $x \equiv y$ and $\vec{r}$ non-empty, we only get $s(r_1[x := s]) \in \mathrm{SN}$ by invoking the already proven (i)' at type ρ and not by main induction hypothesis for (i)'.

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References

[1] Felix Joachimski and Ralph Matthes. Short proofs of normalization for the simply-typed lambda-calculus, permutative conversions and Gödel's T. *Archive for Mathematical Logic*, 42(1):59–87, 2003.