

# Inhabitation in Simply-Typed Lambda-Calculus through a Lambda-Calculus for Proof Search

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## Abstract

A new, comprehensive approach to inhabitation problems in simply-typed lambda-calculus is shown, dealing with both decision and counting problems. This approach works by exploiting a representation of the search space generated by a given inhabitation problem, which is in terms of a lambda-calculus for proof search that the authors developed recently. The representation may be seen as extending the Curry-Howard representation of proofs by lambda-terms, staying within the methods of lambda-calculus and type systems. Our methodology reveals inductive descriptions of the decision problems, driven by the syntax of the proof-search expressions, and the end products are simple, recursive decision procedures and counting functions.

## 1 Introduction

In this paper we study inhabitation problems in the simply-typed  $\lambda$ -calculus, by which we mean both decision problems, like “does type  $A$  have an inhabitant?”, and related questions like counting or listing the inhabitants of a type known to have finitely many of them [Hin97]. We propose a new approach based on a  $\lambda$ -calculus for proof search that the authors developed recently [EMP13, EMP16]. This is a  $\lambda$ -calculus with fixed-points and formal sums, here named  $\lambda_{\Sigma}^{\text{fp}}$ , able to represent as a single term the entire space generated by the search for inhabitants for a given type.

Our previous work showed the correctness of this representation. We could add that such representation has a special status: it was derived as an inductive, finitary counterpart to the coinductive characterization of the search process; and the latter is a rather natural (we might say canonical) mathematical definition of the process which, in addition, may be seen as extending the Curry-Howard paradigm of representation, from proofs to runs of search processes (all this will be recalled in Section 2). Furthermore, the finitary representation stays within the methods of  $\lambda$ -calculus and type systems, which dispenses us from importing and adapting methods from other areas, like automata and language theory or games [TAH96, BS11, SDB15], or from creating new representations like in the proof-tree method [BD05, AB15].

Despite these formal merits, the applicability of the finitary representation remains to be illustrated. This is the purpose of the present paper. Consider a decision problem  $D$  and let  $A$  be a type (of simply-typed  $\lambda$ -calculus). Our previous work allowed us (i) to express  $D(A)$  as  $P(S_A)$ , where  $S_A$  is the coinductive description of the search for inhabitants of  $A$ , and  $P$  is some coinductive predicate, and then (ii) to convert to the equivalent  $P'(F_A)$ , where  $F_A$  is the finitary description of  $S_A$  and  $P'$  is still a predicate defined by reference to coinductive structures. The form  $P'(F_A)$  does not yet profit from the finitary description. This is what we achieve in the present paper: one obtains the equivalent  $P''(F_A)$ , where  $P''$  is inductive, actually directed by the syntax of the finitary description and, for this reason, immediately decidable (however, this might

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call, at the leaves of the inductive structure, another decidable predicate, whose decidability has been established before with the same method).

We illustrate in Section 3 the methodology with two decision problems: the problem exemplified above and also “does type  $A$  have finitely many inhabitants?”. Next, in Section 4, we study types  $A$  with finitely many inhabitants to show how their number can be calculated from the finitary description  $F_A$  as a maybe amazingly simple recursive function.

To sum up: we can base a new methodology to study inhabitation problems on the finitary representation offered by  $\lambda_{\Sigma}^{\text{gfp}}$  which (i) is aligned with the Curry-Howard isomorphism; (ii) enjoys economy of means, as it finds resources in the area of  $\lambda$ -calculus; (iii) is modular, as it separates the problems of representing the search space (that exploits the subformula property of the object  $\lambda$ -calculus) from the problem of analyzing it (where types play a minor role beyond being part of the annotations of the deployed  $\lambda$ -calculus for the analysis); (iv) produces algorithms and functions of high simplicity and even beauty.

## 2 Background

This section has four subsections. First we fix our presentation of the simply-typed  $\lambda$ -calculus, next we recall our two representations of proof search, developed before in [EMP13, EMP16], and recast here as search for inhabitants of a given type. Finally, we start introducing new notions needed in this paper.

### 2.1 Simply-typed $\lambda$ -calculus

We lay out a presentation of the simply-typed  $\lambda$ -calculus, a system we often refer to by  $\lambda$ .

Simple types (or simply, types) are given by the grammar:

$$(types) \quad A, B, C ::= p \mid A \supset B$$

where  $p, q, r$  range over *atoms*. We thus do not distinguish types from propositional implicational formulas. We will write  $A_1 \supset A_2 \supset \dots \supset A_k \supset p$ , with  $k \geq 0$ , in vectorial notation as  $\vec{A} \supset p$ . For example, if the vector  $\vec{A}$  is empty the notation means simply  $p$ .

Normal (*i.e.*,  $\beta$ -normal)  $\lambda$ -terms are given by:

$$(terms) \quad t, u ::= \lambda x^A.t \mid x \langle t_1, \dots, t_k \rangle$$

where a countably infinite set of variables, ranged over by letters  $x, y, w, z$ , is assumed. Note that in  $\lambda$ -abstractions we adopt a *domain-full* presentation (a. k. a. Church-style syntax), annotating the bound variable with a formula. As is common-place with lambda-calculi, we will throughout identify terms up to  $\alpha$ -equivalence.

As always, we permanently need access to the head variable of a non-abstraction. To this end, we are using an informal notation, with *vectors* written  $\langle t_1, \dots, t_k \rangle$  (meaning  $\langle \rangle$  if  $k = 0$ ), abbreviated  $\langle t_i \rangle_i$  if there is no ambiguity on the range of indices<sup>1</sup>. The term constructor  $x \langle t_1, \dots, t_k \rangle$  is usually called *application*. When  $n = 0$  we may simply write the variable  $x$ .

We will view contexts  $\Gamma$  as finite sets of declarations  $x : A$ , where no variable  $x$  occurs twice. The context  $\Gamma, x : A$  is obtained from  $\Gamma$  by adding the declaration  $x : A$ , and will only be written if  $x$  is not declared in  $\Gamma$ . Context union is written as concatenation  $\Gamma, \Delta$  for contexts  $\Gamma$  and  $\Delta$  if  $\Gamma \cap \Delta = \emptyset$ . The letters  $\Gamma, \Delta, \Theta$  are used to range over contexts, and the notation  $\text{dom}(\Gamma)$  stands for the set of variables declared in  $\Gamma$ . We will write  $\Gamma(x)$  for the type associated with  $x$  for  $x \in \text{dom}(\Gamma)$ , hence viewing  $\Gamma$  as a function on  $\text{dom}(\Gamma)$ . Context inclusion  $\Gamma \subseteq \Delta$  is just set inclusion.

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<sup>1</sup>If we formalized vectors as a separate syntactic class, with a nil vector and a vector constructor, we would get  $\bar{\lambda}$ -terms [Her95] and would fall, logically, in a sequent calculus format, as in [Her95, EMP16]. But even in [EMP16], despite the concern with proof search in the sequent calculus, the formalization of vectors was of little importance.

Figure 1: Typing rules of  $\lambda$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x^A. t : A \supset B} \text{RIntro} \quad \frac{(x : \vec{B} \supset p) \in \Gamma \quad \forall i, \Gamma \vdash t_i : B_i}{\Gamma \vdash x\langle t_i \rangle_i : p} \text{LVecIntro}$$

The typing rules are in Fig. 1 and derive sequent  $\Gamma \vdash t : A$ . *LVecIntro* presupposes that the indices for the  $t_i$  range over  $1, \dots, k$  and that  $\vec{B} = B_1, \dots, B_k$ , for some  $k \geq 0$ . Such obvious constraints for finite vectors will not be spelt out in the rest of the paper. In the particular case of  $k = 0$ , in which  $(x : p) \in \Gamma$  is the only hypothesis of *LVecIntro*, we type variables (with atoms). Note that the conclusion of the *LVecIntro* rule is an atomic sequent—hence a typable term will always be in  $\eta$ -long form.

## 2.2 Search for inhabitants, coinductively

We are concerned with a specific kind of search problems: given  $\Gamma$  and  $A$ , to find  $t$  such that  $\Gamma \vdash t : A$ , that is, to find an *inhabitant* of type  $A$  in context  $\Gamma$ . Under the Curry-Howard correspondence, a pair  $\Gamma, A$  may be seen as a *logical sequent*  $\Gamma \Rightarrow A$ , and searching for an inhabitant of  $A$  in context  $\Gamma$  is the same as searching for a proof of that sequent<sup>2</sup>.

Following [EMP13, EMP16], we model this search process through the coinductive  $\lambda$ -calculus, denoted  $\lambda^{co}$ . The terms of  $\lambda^{co}$ , also called *coterms* or *Böhm trees*, are given by

$$M, N ::=_{co} \lambda x^A. N \mid x\langle N_1, \dots, N_k \rangle .$$

This is exactly the previous grammar for  $\lambda$ -terms, but read coinductively, as indicated by the index *co* (still with finite tuples  $\langle N_i \rangle_i$ ). The natural notion of equality between coterms is bisimilarity modulo  $\alpha$ -equivalence. Following mathematical practice, this is still written as plain equality.

In  $\lambda^{co}$ , also the typing rules of Fig. 1 have to be interpreted coinductively—but the formulas/types stay inductive and the contexts finite. Following common practice, we will symbolize the coinductive reading of an inference (rule) by the double horizontal line, but we refrain from displaying Fig. 1 again with double lines—a figure where the two inference rules would be called *RIntro<sub>co</sub>* and *LVecIntro<sub>co</sub>*. Such system defines when  $\Gamma \vdash N : A$  holds for a *finite* context  $\Gamma$ , a Böhm tree  $N$  and a type  $A$ .

Suppose  $\Gamma \vdash N : A$  holds. Then this sequent has a derivation which is a (possibly infinite) tree of sequents, generated by applying the inference rules bottom-up; and  $N$  is a (possibly infinite) coterms, which we call a *solution* of  $\sigma$ , with  $\sigma = (\Gamma \Rightarrow A)$ . Therefore, such derivations are the structures generated by the search process which does not fail, even if it runs forever, and so they subsume proofs; likewise solutions subsume typable terms (so we may refer to the latter as *finite solutions*<sup>3</sup>)—with solutions still representing derivations, even if infinite, following the Curry-Howard paradigm.

The next step is to extend even further the paradigm, representing also the choice points of the search process. To this end, we extend  $\lambda^{co}$  to  $\lambda_{\Sigma}^{co}$ , whose syntax is this:

$$\begin{array}{ll} \text{(terms)} & M, N ::=_{co} \lambda x^A. N \mid E_1 + \dots + E_n \\ \text{(elimination alternatives)} & E ::=_{co} x\langle N_1, \dots, N_k \rangle \end{array}$$

where both  $n, k \geq 0$  are arbitrary.  $T$  ranges over both terms and elimination alternatives. Note that summands cannot be lambda-abstractions. We will often use  $\sum_i E_i$  instead of  $E_1 + \dots + E_n$ —in generic situations or if the dependency of  $E_i$  on  $i$  is clear, as well as the number of elements (if this number is 0, we write the sum as  $\mathbb{0}$ ).

<sup>2</sup>To be precise, a proof in natural deduction, which is equivalent to a cut-free, sequent-calculus proof in the system *LJT* [Her95].

<sup>3</sup>Solutions subsume finite solutions conservatively. In fact, it is easy to prove that, given a  $\lambda$ -term  $t$ ,  $\Gamma \vdash t : A$  in  $\lambda$  iff  $\Gamma \vdash t : A$  in  $\lambda^{co}$ .

Figure 2: Membership relations

$$\frac{\text{mem}(M, N)}{\text{mem}(\lambda x^A.M, \lambda x^A.N)} \quad \frac{\forall i, \text{mem}(M_i, N_i)}{\text{mem}(x\langle M_i \rangle_i, x\langle N_i \rangle_i)} \quad \frac{\text{mem}(M, E_j)}{\text{mem}(M, \sum_i E_i)}$$

Figure 3: Extra typing rule of  $\lambda_{\Sigma}^{co}$  w. r. t.  $\lambda^{co}$

$$\frac{\forall i, \Gamma \vdash E_i : p}{\Gamma \vdash \sum_i E_i : p} \text{ Alts}$$

The most natural notion of equality of terms in  $\lambda_{\Sigma}^{co}$  is again bisimilarity modulo  $\alpha$ -equivalence, but the notation  $\sum_i E_i$  already hints that we consider  $+$  to be associative (with  $\mathbb{O}$  as its neutral element). We even want to neglect the precise order of the summands and their (finite) multiplicity. We thus consider the sums of elimination alternatives as if they were sets of alternatives, i. e., we further assume that  $+$  is symmetric and idempotent. As for  $\lambda^{co}$ , we just use mathematical equality for this notion of bisimilarity on expressions of  $\lambda_{\Sigma}^{co}$ , and so the sums of elimination alternatives can plainly be treated as if they were finite sets of elimination alternatives (given by finitely many elimination alternatives of which several might be identified through bisimilarity).

The expressions of  $\lambda_{\Sigma}^{co}$  are also called *Böhm forests*—and a Böhm tree  $M$  is a member of a Böhm forest  $N$  when the relation  $\text{mem}(M, N)$  defined coinductively in Fig. 2 holds.

In the typing system for  $\lambda_{\Sigma}^{co}$ , one derives sequents  $\Gamma \vdash N : A$  and  $\Gamma \vdash E : p$ . The coinductive typing rules are the ones of  $\lambda^{co}$ , together with the rule given in Fig. 3.

A typing derivation of  $\lambda_{\Sigma}^{co}$  is a possibly infinite tree of sequents, generated by the bottom-up application of the inference rules, with “multiplicative” branching (logically: “and” branching) caused by the list of arguments in elimination alternatives, and “additive” branching (logically: “or” branching) caused by sums—the latter being able to express the alternatives found in the search process when an atom  $p$  can be proved by picking different head variables with their appropriate arguments. So, it is no surprise that, with this infrastructure, we can express, as a single Böhm forest, the entire *solution space* generated by the search process when applied to given  $\Gamma$  and  $A$ . That Böhm forest can be defined as a function  $\mathcal{S}$  of  $\Gamma \Rightarrow A$  defined by corecursion as follows:

**Definition 1 (Solution spaces)**

$$\mathcal{S}(\Gamma \Rightarrow \vec{A} \supset p) := \lambda \vec{x} : \vec{A}. \sum_{(y:\vec{B} \supset p) \in \Delta} y \langle \mathcal{S}(\Delta \Rightarrow B_j) \rangle_j \quad \text{with } \Delta := \Gamma, \vec{x} : \vec{A}$$

The following properties witness the robustness of the definition [EMP13, EMP16].

**Proposition 2 (Properties of solution spaces)** *The following properties hold.*

1. Given  $\Gamma$  and  $A$ , the typing  $\Gamma \vdash \mathcal{S}(\Gamma \Rightarrow A) : A$  holds in  $\lambda_{\Sigma}^{co}$ .
2. For  $N \in \lambda^{co}$ ,  $\text{mem}(N, \mathcal{S}(\Gamma \Rightarrow A))$  iff  $\Gamma \vdash N : A$  in  $\lambda^{co}$ .
3. For  $t \in \lambda$ ,  $\text{mem}(t, \mathcal{S}(\Gamma \Rightarrow A))$  iff  $\Gamma \vdash t : A$  in  $\lambda$ .

### 2.3 Search for inhabitants, inductively

Unfortunately, algorithms cannot in general receive Böhm forests as input, so the next step is to find an alternative, equivalent, effective representation that works at least for solution spaces. To

this end, an extension  $\lambda_{\Sigma}^{\text{gfp}}$  of  $\lambda$  is introduced, whose syntax is given by the following grammar (read inductively):

$$\begin{array}{ll} \text{(terms)} & N ::= \lambda x^A.N \mid \mathbf{gfp} X^\sigma.E_1 + \cdots + E_n \mid X^\sigma \\ \text{(elimination alternatives)} & E ::= x\langle N_1, \dots, N_k \rangle \end{array}$$

where  $X$  is assumed to range over a countably infinite set of *fixpoint variables* (also letters  $Y, Z$  will range over them), and where, as for  $\lambda_{\Sigma}^{\text{co}}$ , both  $n, k \geq 0$  are arbitrary. We extend our practice established for  $\lambda_{\Sigma}^{\text{co}}$  of writing the sums  $E_1 + \cdots + E_n$  in the form  $\sum_i E_i$  for  $n \geq 0$ . Also the tuples continue to be communicated as  $\langle N_i \rangle_i$ . As for  $\lambda_{\Sigma}^{\text{co}}$ , we will identify expressions modulo associativity, symmetry and idempotence of  $+$ , thus treating sums of elimination alternatives as if they were the set of those elimination alternatives. Again, we will write  $T$  for expressions of  $\lambda_{\Sigma}^{\text{gfp}}$ , i. e., for terms and elimination alternatives.

In the term formation rules,  $\sigma$  in  $X^\sigma$  is required to be *atomic*, i. e., of the form  $\Gamma \Rightarrow p$ . Let  $FPV(T)$  denote the set of free occurrences of typed fixed-point variables in  $T$ . Perhaps unexpectedly, in  $\mathbf{gfp} X^\sigma.\sum_i E_i$  the fixed-point construction  $\mathbf{gfp}$  binds *all* free occurrences of  $X^{\sigma'}$  in the elimination alternatives  $E_i$ , not just  $X^\sigma$ . But we only want this to happen when  $\sigma \leq \sigma'$ —which means: the context of  $\sigma'$  has more declarations than that of  $\sigma$ , but not with new types. Formally:  $\sigma = (\Gamma \Rightarrow p)$ ,  $\sigma' = (\Gamma' \Rightarrow p)$  and  $\Gamma \leq \Gamma'$ , with the latter meaning  $\Gamma \subseteq \Gamma'$  but  $|\Gamma| = |\Gamma'|$ , and  $|\Delta|$  denoting  $\{A \mid \exists x, (x : A) \in \Delta\}$  for arbitrary context  $\Delta$ .

In the sequel, when we refer to *finitary terms* we have in mind the expressions of  $\lambda_{\Sigma}^{\text{gfp}}$ . The fixed-point operator is called  $\mathbf{gfp}$  (“greatest fixed point”) to indicate that its semantics is—see below—defined in terms of the *infinitary* syntax  $\lambda_{\Sigma}^{\text{co}}$ , but there, fixed points are unique. Hence, the reader may just read this as “the fixed point”.

We now move to the interpretation of expressions of  $\lambda_{\Sigma}^{\text{gfp}}$  in terms of the coinductive syntax of  $\lambda_{\Sigma}^{\text{co}}$  (using the  $\nu$  operation on the meta-level to designate unique fixed points). It is done with the help of *environments*  $\xi$ , which are partial functions from typed fixed-point variables  $X^\sigma$  to (co)terms of  $\lambda_{\Sigma}^{\text{co}}$ , with domain  $\text{dom}(\xi)$  a finite set of typed fixpoint variables *without duplicates*, which means:  $X^{\sigma_1}, X^{\sigma_2} \in \text{dom}(\xi) \Rightarrow \sigma_1 = \sigma_2$ .

Some technicalities are needed before giving the interpretation. We say an environment  $\xi$  is admissible for an expression  $T$  of  $\lambda_{\Sigma}^{\text{gfp}}$  if, for every  $X^{\sigma'} \in FPV(T)$ , there is an  $X^\sigma \in \text{dom}(\xi)$  such that  $\sigma \leq \sigma'$ . It is easy to see that  $T$  admits an environment iff it is *regular* in the following sense: if  $X$  occurs free in  $T$ , there is a sequent  $\sigma$  that is the minimum of all  $\sigma'$  such that  $X^{\sigma'} \in FPV(T)$ . Finally, the interpretation is only given for well-bound expressions, where  $T \in \lambda_{\Sigma}^{\text{gfp}}$  is *well-bound* if, for any of its subterms  $\mathbf{gfp} X^\sigma.\sum_i E_i$  and any (free) occurrence of  $X^{\sigma'}$  in the  $E_i$ 's,  $\sigma \leq \sigma'$ .

**Definition 3 (Interpretation of finitary terms as Böhm forests)** *For a well-bound expression  $T$  of  $\lambda_{\Sigma}^{\text{gfp}}$ , the interpretation  $\llbracket T \rrbracket_{\xi}$  for an environment  $\xi$  that is admissible for  $T$  is given by structural recursion on  $T$ :*

$$\begin{aligned} \llbracket X^{\sigma'} \rrbracket_{\xi} &= [\sigma'/\sigma]\xi(X^\sigma) \quad \text{for the unique } \sigma \leq \sigma' \text{ with } X^\sigma \in \text{dom}(\xi) \\ \llbracket \mathbf{gfp} X^\sigma.\sum_i E_i \rrbracket_{\xi} &= \nu N.\sum_i \llbracket E_i \rrbracket_{\xi \cup [X^\sigma \mapsto N]} \\ \llbracket \lambda x^A.N \rrbracket_{\xi} &= \lambda x^A.\llbracket N \rrbracket_{\xi} \\ \llbracket x\langle N_i \rangle_i \rrbracket_{\xi} &= x\langle \llbracket N_i \rrbracket_{\xi} \rangle_i \end{aligned}$$

If  $T$  is closed, i. e.,  $FPV(T) = \emptyset$ , then the empty function is an admissible environment for  $T$ , and we write  $\llbracket T \rrbracket$ .

The clause for fixpoint variables in the preceding definition has to cope with  $\sigma \leq \sigma'$ . This is done by adjusting the value  $N = \xi(X^\sigma)$  looked up in the environment with an operation on Böhm forests which will add elimination alternatives to the sums in  $N$ , in order to match the new declarations in  $\sigma'$ . If  $\sigma = (\Gamma \Rightarrow p)$  and  $\sigma' = (\Gamma' \Rightarrow p)$ , then  $[\sigma'/\sigma]N$  is defined to be  $[\Gamma'/\Gamma]N$ , with the latter given as follows:

**Definition 4 (Co-contraction)** Let  $\Gamma \leq \Gamma'$ . For  $T$  an expression of  $\lambda_{\Sigma}^{\text{co}}$ , we define  $[\Gamma'/\Gamma]T$  by corecursion as follows:

$$\begin{aligned} [\Gamma'/\Gamma](\lambda x^A.N) &= \lambda x^A.[\Gamma'/\Gamma]N \\ [\Gamma'/\Gamma]\sum E_i &= \sum [\Gamma'/\Gamma]E_i \\ [\Gamma'/\Gamma](z\langle N_i \rangle_i) &= z\langle [\Gamma'/\Gamma]N_i \rangle_i \quad \text{if } z \notin \text{dom}(\Gamma) \\ [\Gamma'/\Gamma](z\langle N_i \rangle_i) &= \sum_{(w:A) \in \Delta_z} w\langle [\Gamma'/\Gamma]N_i \rangle_i \quad \text{if } z \in \text{dom}(\Gamma) \end{aligned}$$

where, in the last clause,  $A := \Gamma(z)$  and  $\Delta_z := \{(z : A)\} \cup (\Gamma' \setminus \Gamma)$ .

Co-contraction captures the extension of the solution space when going from  $\sigma$  to some  $\sigma'$  with  $\sigma \leq \sigma'$ :

**Lemma 5 (Solution spaces and co-contraction)** Let  $\sigma \leq \sigma'$ . Then  $\mathcal{S}(\sigma') = [\sigma'/\sigma]\mathcal{S}(\sigma)$ .

With the finitary calculus and its semantics in place, we can provide an alternative representation  $\mathcal{F}(\sigma)$  of the search space generated by a sequent  $\sigma$ .

**Definition 6 (Finitary solution space)** Let  $\Xi := \overrightarrow{X : \Theta \Rightarrow q}$  be a vector of  $m \geq 0$  declarations ( $X_i : \Theta_i \Rightarrow q_i$ ) where no fixpoint variable name and no sequent occurs twice. The specification of  $\mathcal{F}(\Gamma \Rightarrow \vec{A} \supset p; \Xi)$  is as follows:

If, for some  $1 \leq i \leq m$ ,  $p = q_i$  and  $\Theta_i \subseteq \Gamma$  and  $|\Theta_i| = |\Gamma| \cup \{A_1, \dots, A_n\}$ , then

$$\mathcal{F}(\Gamma \Rightarrow \vec{A} \supset p; \Xi) = \lambda z_1^{A_1} \dots z_n^{A_n}. X_i^\sigma,$$

where  $i$  is taken to be the biggest such index. Otherwise,

$$\mathcal{F}(\Gamma \Rightarrow \vec{A} \supset p; \Xi) = \lambda z_1^{A_1} \dots z_n^{A_n}. \text{gfp } Y^\sigma. \sum_{(y:\vec{B} \supset p) \in \Delta} y\langle \mathcal{F}(\Delta \Rightarrow B_j; \Xi, Y : \sigma) \rangle_j$$

where, in both cases,  $\Delta := \Gamma, z_1 : A_1, \dots, z_n : A_n$  and  $\sigma := \Delta \Rightarrow p$ .

$\mathcal{F}(\sigma)$  denotes  $\mathcal{F}(\sigma; \Xi)$  with empty  $\Xi$ . It can be proved that: (i)  $\mathcal{F}(\sigma)$  is well-defined (the above recursive definition terminates); (ii)  $\mathcal{F}(\sigma)$  is a closed well-bound term.

The semantics into  $\lambda_{\Sigma}^{\text{co}}$  of the finitary representation coincides with  $\mathcal{S}(\sigma)$  [EMP13, EMP16].

**Theorem 7 (Equivalence)** For any sequent  $\sigma$ ,  $\llbracket \mathcal{F}(\sigma) \rrbracket = \mathcal{S}(\sigma)$ .

## 2.4 The finite extension

We now introduce some notions pertaining to the present paper, given its focus on finite inhabitants (i.e.,  $\lambda$ -terms).

For  $T \in \lambda_{\Sigma}^{\text{co}}$ , we call *finite extension* of  $T$ , which we denote by  $\mathcal{E}_{\text{fin}}(T)$ , the set of the finite members of  $T$ , i.e.,  $\mathcal{E}_{\text{fin}}(T) = \{t \in \lambda \mid \text{mem}(t, T)\}$ . We will be mainly interested in the following predicates on Böhm forests concerning the finite extension:

- $\text{exfinext}(T)$  is defined to hold iff  $\mathcal{E}_{\text{fin}}(T)$  is nonempty.
- $\text{nofinext}(T)$  is defined to hold iff  $\mathcal{E}_{\text{fin}}(T)$  is empty.
- $\text{finfinext}(T)$  is defined to hold iff  $\mathcal{E}_{\text{fin}}(T)$  is finite.
- $\text{inffinext}(T)$  is defined to hold iff  $\mathcal{E}_{\text{fin}}(T)$  is infinite.

The predicates  $\text{exfinext}$  and  $\text{finfinext}$  will be characterized inductively in Sect. 3.2 and in Sect. 3.3, respectively, together with coinductive characterizations of  $\text{nofinext}$  and  $\text{inffinext}$  by the generic De Morgan's law relating least and greatest fixed points.

### 3 The inhabitation problems

We will study two decision problems in simply-typed  $\lambda$ -calculus: the inhabitation problem and the type finiteness problem. First, we lay down the common approach we will adopt.

Given  $\Gamma$  and  $A$ , we will write  $\mathcal{I}(\Gamma, A)$  for the set of inhabitants of  $A$  relative to context  $\Gamma$  in  $\lambda$ , i. e., for the set  $\{t \in \lambda \mid \Gamma \vdash t : A \text{ in } \lambda\}$ . Recall that this describes the set of  $\eta$ -long  $\beta$ -normal terms of ordinary simply-typed  $\lambda$ -calculus receiving type  $A$  in context  $\Gamma$ .

The inhabitation problem in simply-typed  $\lambda$ -calculus is the problem “given  $\Gamma$  and  $A$ , is the set  $\mathcal{I}(\Gamma, A)$  nonempty?”, called **INHAB** in this paper. Its negation is called the “emptiness problem” (as is well-known, the answer to this question does not depend on whether all  $\lambda$ -terms are considered or only the  $\beta$ -normal ones or even the  $\eta$ -long  $\beta$ -normal terms). Decidability of the inhabitation problem in simply-typed  $\lambda$ -calculus is a well-known result (see, e. g., [Sta79]).

**Lemma 8 (Characterization of existence of inhabitants in  $\lambda$ )** *There is a  $t \in \lambda$  such that  $\Gamma \vdash t : A$  in  $\lambda$  iff  $\text{exfinext}(\llbracket \mathcal{F}(\Gamma \Rightarrow A) \rrbracket)$ .*

**Proof**

- $\exists t \in \lambda$  s. t.  $\Gamma \vdash t : A$  in  $\lambda$
- iff  $\exists t \in \lambda$  s. t.  $\text{mem}(t, \mathcal{S}(\Gamma \Rightarrow A))$  (Prop. 2.3)
- iff  $\exists t \in \lambda$  s. t.  $\text{mem}(t, \llbracket \mathcal{F}(\Gamma \Rightarrow A) \rrbracket)$  (Theorem 7)
- iff  $\text{exfinext}(\llbracket \mathcal{F}(\Gamma \Rightarrow A) \rrbracket)$  (by definition of  $\text{exfinext}$ ) □

As seen above, the function  $\mathcal{F}$  is effectively computable, and it yields closed well-bound finitary terms. The missing link to deciding **INHAB** is thus the decision of the problem “given a closed well-bound term  $T$ , does  $\text{exfinext}(\llbracket T \rrbracket)$  hold?”. Of course, one cannot deal with closed finitary terms  $T$  in isolation and needs to address fixpoint variables properly. Neither the interpretation function  $\llbracket \cdot \rrbracket$  nor the predicate  $\text{exfinext}$  are effective, but we will define in Section 3.2 a syntax-directed predicate **EF** (more precisely, it will be a predicate  $\text{EF}_P$  parameterized over a decidable predicate  $P$ ) on finitary terms that is equivalent to the composition  $\text{exfinext} \circ \llbracket \cdot \rrbracket$ , for at least those closed well-bound terms that arise as  $\mathcal{F}(\sigma)$  for some sequent  $\sigma$  (technically, the restriction will be to proper terms, as defined in Section 3.1). Syntax-directedness immediately entails that the predicate is decidable.

The appeal of our approach is that, once the finitary representation of the corresponding sequent has been built as  $\mathcal{F}(\sigma)$ , the decision of inhabitation is achieved through a simple recursive function over the structure of  $\lambda_{\Sigma}^{\text{gfp}}$ -terms, corresponding to an inductive predicate adequately characterizing non-emptiness of types.

Using the same methodology, we can also reprove a more difficult and not so well-known result of inhabitation for simply-typed  $\lambda$ -calculus, namely, that the problem “given  $\Gamma$  and  $A$ , is the set  $\mathcal{I}(\Gamma, A)$  finite?” is decidable (see, e. g. [Hir98]). This problem—henceforth called **FINHAB**—depends on studying only  $\beta$ -normal terms; to recall, the inhabitants of our system  $\lambda$  are  $\eta$ -long  $\beta$ -normal simply-typed  $\lambda$ -terms, for which the problem is studied in the literature [Hin97] (there, in particular, the algorithm by Ben-Yelles [BY79]).

**Lemma 9 (Characterization of type finiteness in  $\lambda$ )** *The set of inhabitants  $\mathcal{I}(\Gamma, A)$  is finite iff  $\text{finfinext}(\llbracket \mathcal{F}(\Gamma \Rightarrow A) \rrbracket)$ .*

**Proof** Analogous to the proof of Lemma 8, following from Prop. 2.3, Theorem 7 and definition of  $\text{finfinext}$ . □

Analogously to the emptiness problem, our method for establishing decidability of **FINHAB** is to define a recursive predicate on finitary terms that is equivalent to the composition  $\text{finfinext} \circ \llbracket \cdot \rrbracket$ , for at least those closed well-bound terms that arise as  $\mathcal{F}(\sigma)$  for some sequent  $\sigma$  (with the same technical condition as for the emptiness problem). This will be the predicate **FF** (again, rather a parameterized predicate  $\text{FF}_P$ ), studied in Sect. 3.3. Again, the appeal of our approach is that, after building the finitary representation of the corresponding sequent through the  $\mathcal{F}(\sigma)$  function, **FINHAB** is decided by a simple function given recursively over the structure of  $\lambda_{\Sigma}^{\text{gfp}}$ -terms, which, however, additionally uses the previously established decision algorithm for **INHAB**.

In both cases, the problem is of the form  $P \circ \mathcal{S}$  on sequents, and thanks to  $\mathcal{S} = [\![\cdot]\!] \circ \mathcal{F}$  and associativity, we have to decide  $(P \circ [\![\cdot]\!]) \circ \mathcal{F}$ , where  $\mathcal{F}$  is already computable. The solution is by proposing a recursive predicate  $P'$  that can step in for  $P \circ [\![\cdot]\!]$ , as far the image of  $\mathcal{F}$  is concerned (specifically, those terms are well-bound, have no free fixpoint variables and are proper in the sense of Definition 11 below). Finally, the decision is done by deciding  $P' \circ \mathcal{F}$ .

We will carry out the two instances of this programme, but for this, it will prove useful to simplify our semantics of finitary terms.

### 3.1 A simplified semantics

We introduce a simplified interpretation of expressions of  $\lambda_{\Sigma}^{\text{gfp}}$  in terms of the coinductive syntax of  $\lambda_{\Sigma}^{\text{co}}$ . We now dispense with environments and adopt a simpler and even possibly “wrong” interpretation, which, however, for  $\lambda_{\Sigma}^{\text{gfp}}$ -terms representing solution spaces will be seen to be equivalent.

**Definition 10 (Simplified interpretation of finitary terms as Böhm forests)** *For an expression  $T$  of  $\lambda_{\Sigma}^{\text{gfp}}$ , the simplified interpretation  $\llbracket T \rrbracket^s$  is given by structural recursion on  $T$ :*

$$\begin{aligned} \llbracket X^{\sigma} \rrbracket^s &= \mathcal{S}(\sigma) \\ \llbracket \text{gfp } X^{\sigma} . \sum_i E_i \rrbracket^s &= \sum_i \llbracket E_i \rrbracket^s \\ \llbracket \lambda x^A . N \rrbracket^s &= \lambda x^A . \llbracket N \rrbracket^s \\ \llbracket x \langle N_i \rangle_i \rrbracket^s &= x \langle \llbracket N_i \rrbracket^s \rangle_i \end{aligned}$$

Note that the base case now profits from the sequent annotation at fixpoint variables, and the interpretation of the **gfp**-constructor dispenses with the use of the  $\nu$  operation on the meta-level to designate unique fixed points on  $\lambda_{\Sigma}^{\text{co}}$ -expressions. Of course, this may be “wrong” according to our understanding of a greatest fixed point.

Below, we will be specially interested in the finitary terms which guarantee that a **gfp** $X^{\sigma}$  construction represents the solution space of  $\sigma$ .

**Definition 11 (Proper expressions)** *An expression  $T \in \lambda_{\Sigma}^{\text{gfp}}$  is proper if for any of its subterms  $T'$  of the form  $\text{gfp } X^{\sigma} . \sum_i E_i$ , it holds that  $\llbracket T' \rrbracket^s = \mathcal{S}(\sigma)$ .*

This means that an expression  $T$  is considered proper if, despite having used the simplified definition of semantics for the embedded fixed points, those subterms have the “proper” semantics, and this is only expressed with respect to our main question of representing solution spaces, hence where for the fixed-point variables, the reference semantics of solution spaces is assumed, and this is possible since the fixed-point variables carry the sequent whose solution space they are intended to represent.

For proper expressions, the simplified semantics agrees with the semantics we studied before. Of course, this can only make sense for expressions which have that previous semantics, in other words for well-bound and regular expressions.

**Lemma 12** *Let  $T$  be well-bound and  $\xi$  be an admissible environment for  $T$  such that for all  $X^{\sigma} \in \text{dom}(\xi)$ :  $\xi(X^{\sigma}) = \mathcal{S}(\sigma)$ . If  $T$  is proper, then  $\llbracket T \rrbracket_{\xi} = \llbracket T \rrbracket^s$ .*

We remark that for any regular  $T$ , there is exactly one such environment  $\xi$ . The case of a closed expression  $T$  merits stating a corollary.

**Corollary 13** *For well-bound, closed and proper  $T$ ,  $\llbracket T \rrbracket = \llbracket T \rrbracket^s$ .*

**Proof** (of Lemma 12) By induction on expressions  $T$ . The variable case needs Lemma 5, lambda-abstraction and tuples are fine by the induction hypothesis. For the **gfp** case, it has to be shown that  $\llbracket T \rrbracket^s$  fulfills the fixed-point equation defining  $\llbracket T \rrbracket_{\xi}$ , which suffices by uniqueness of the solution. The induction hypothesis can be applied to the elimination alternatives since the extended environment in which they have to be interpreted is of the required form, just by  $T$  being proper.  $\square$

Figure 4:  $\text{exfin}$  predicate and  $\text{nofin}$  predicate

$$\begin{array}{ccc}
\frac{\text{exfin}(N)}{\text{exfin}(\lambda x^A.N)} & \frac{\text{exfin}(E_j)}{\text{exfin}(\sum_i E_i)} & \frac{\forall i, \text{exfin}(N_i)}{\text{exfin}(x\langle N_i \rangle_i)} \\
\frac{\text{nofin}(N)}{\text{nofin}(\lambda x^A.N)} & \frac{\forall i, \text{nofin}(E_i)}{\text{nofin}(\sum_i E_i)} & \frac{\text{nofin}(N_j)}{\text{nofin}(x\langle N_i \rangle_i)}
\end{array}$$

The corollary is sufficient for our purposes since  $\mathcal{F}(\sigma)$  is not only well-bound and closed, but also proper, as will be seen shortly.

**Theorem 14 (Equivalence for simplified semantics)** *Let  $\sigma$  be a sequent and  $\Xi$  as in Def. 6 so that  $\mathcal{F}(\sigma; \Xi)$  exists (in particular, this holds for empty  $\Xi$ ).*

1.  $\mathcal{F}(\sigma; \Xi)$  is proper.
2.  $\llbracket \mathcal{F}(\sigma; \Xi) \rrbracket^s = \mathcal{S}(\sigma)$ .

**Proof** Both items together by structural induction on the term  $\mathcal{F}(\sigma; \Xi)$ . This all goes by unfolding the definitions and use of the induction hypothesis (the main case in the proof of 1 needs 2 for the subterms, so 1 cannot be proven separately before 2, and the main case of 2 immediately follows from the main case of 1, so it is better to prove both together, although 2 could be proven separately before 1).  $\square$

We remark that the proof is a simplification of the proof for Theorem 7 given previously [EMP16].

### 3.2 Deciding type emptiness

We introduce predicate  $\text{nofin}(T)$ , for  $T$  an expression of  $\lambda_\Sigma^{\text{co}}$  (Böhm forest), which holds iff  $\text{nofinext}(T)$ , i. e., if the finite extension of  $T$  is empty, but it is defined co-inductively in Fig. 4, together (but independently) with the inductive definition of the predicate  $\text{exfin}(T)$  that is supposed to mean the negation of  $\text{nofin}(T)$ , but which is expressed positively as existence of a finite member (i. e., that the finite extension is non-empty—that  $\text{exfinext}(T)$  holds).

**Lemma 15** *Given a Böhm forest  $T$ ,  $\text{exfin}(T)$  iff  $\text{nofin}(T)$  does not hold.*

**Proof** See the appendix.  $\square$

The following lemma shows that the predicate  $\text{nofin}$  corresponds to the intended meaning in terms of the finite extension. Additionally, the lemma shows that the negation of  $\text{nofin}$  holds exactly for the Böhm forests which have finite members.

**Lemma 16 (Coinductive characterization)** *Given a Böhm forest  $T$ . Then,  $\text{nofin}(T)$  iff  $\mathcal{E}_{\text{fin}}(T)$  is empty, i. e.,  $\text{nofin} = \text{nofinext}$  as sets of Böhm forests.*

**Proof** First, let  $\text{inf}(M)$  be defined coinductively, as belonging to the greatest predicate  $\text{inf}$  satisfying

$$\text{inf}(\lambda x^A.M) \Leftrightarrow \text{inf}(M) \quad \text{and} \quad \text{inf}(x\langle M_i \rangle_i) \Leftrightarrow \exists j, \text{inf}(M_j)$$

This is a characterization of infinity: for a Böhm tree  $M$ ,  $M$  is a  $\lambda$ -term iff  $\text{inf}(M)$  does not hold. Now, the statement of the lemma is equivalent to:  $\text{nofin}(T)$  iff  $\text{inf}(M)$  for all  $M$  s. t.  $\text{mem}(M, T)$ . The “only if” is equivalently to: if  $\text{nofin}(T)$  and  $\text{mem}(M, T)$  then  $\text{inf}(M)$ . This is provable by coinduction on  $\text{inf}$ , using the obvious  $\text{mem}(M, M)$  for  $M$  in  $\lambda^{\text{co}}$ . The “if” implication is suitable for coinduction on  $\text{nofin}$ , and this works smoothly.  $\square$

Figure 5:  $\text{EF}_P$  predicate and  $\text{EF}_P$  predicate

$$\begin{array}{cccc}
\frac{P(\sigma)}{\text{EF}_P(X^\sigma)} & \frac{\text{EF}_P(N)}{\text{EF}_P(\lambda x^A.N)} & \frac{\text{EF}_P(E_j)}{\text{EF}_P(\text{gfp } X^\sigma . \sum_i E_i)} & \frac{\forall i, \text{EF}_P(N_i)}{\text{EF}_P(x \langle N_i \rangle_i)} \\
\frac{\neg P(\sigma)}{\text{EF}_P(X^\sigma)} & \frac{\text{EF}_P(N)}{\text{EF}_P(\lambda x^A.N)} & \frac{\forall i, \text{EF}_P(E_i)}{\text{EF}_P(\text{gfp } X^\sigma . \sum_i E_i)} & \frac{\text{EF}_P(N_j)}{\text{EF}_P(x \langle N_i \rangle_i)}
\end{array}$$

Thus, we are authorized to work with `nofin` and `exfin` in place of their “extensional variants” `nofinext` and `exfinext`.

Next we turn to finitary representation of solution spaces and consider the predicate  $\text{EF}(T)$ , for  $T$  an expression in  $\lambda_{\Sigma}^{\text{gfp}}$ , which should hold when there is a finite solution. It is not obvious from the outset if free fixpoint variables should be considered as contributing to these finite solutions. If one already knows that  $\text{exfin}(\mathcal{S}(\sigma))$  holds, then it would be reasonable to put  $X^\sigma$  into the predicate  $\text{EF}$ . However, since our aim is to prove  $\text{exfin} \circ \mathcal{S}$  decidable through decidability of  $\text{EF}$ , we cannot base rules for  $\text{EF}$  on a decision concerning  $\text{exfin} \circ \mathcal{S}$ . Still, once we established decidability of  $\text{exfin} \circ \mathcal{S}$ , we could profit from a definition of  $\text{EF}$  that is sharp in the sense of containing variables  $X^\sigma$  by definition if and only if  $\text{exfin}(\mathcal{S}(\sigma))$ . And this we will do in Section 3.3, building more complex predicates from  $\text{EF}$ .

We therefore consider a parameterized notion  $\text{EF}_P$  with  $P$  a predicate on sequents and instantiate it twice, with

- once  $P := \emptyset$ , the empty predicate which is trivially decidable, and,
- once  $\text{exfin} \circ \mathcal{S}$  is proven decidable, with  $P := \text{exfin} \circ \mathcal{S}$ .

The general proviso on  $P$  is decidability of  $P$  and that, for all sequents  $\sigma$ ,  $P(\sigma)$  implies  $\text{exfin}(\mathcal{S}(\sigma))$ , i. e.,  $P \subseteq \text{exfin} \circ \mathcal{S}$ . This proviso is trivially satisfied in both instantiations.<sup>4</sup> There are still other meaningful parameter settings, e. g., with  $P(\sigma)$  iff  $\sigma = (\Gamma \Rightarrow p)$  is instance of an axiom, i. e.,  $p \in |\Gamma|$ .

The definition of this (parameterized) predicate  $\text{EF}_P$  is inductive and presented in the first line of Fig. 5, although it is clear that it could equivalently be given by a definition by recursion over the term structure. Therefore, the predicate  $\text{EF}_P$  is decidable.

The inductive characterization of the negation of the predicate  $\text{EF}_P$  is easy, as all the rules of  $\text{EF}_P$  are “invertible”, and is given in the second line of Fig. 5.

**Lemma 17** *For all  $T \in \lambda_{\Sigma}^{\text{gfp}}$ ,  $\text{EF}_P(T)$  iff  $\text{EF}_P(T)$  does not hold.*

**Proposition 18 (Finitary characterization)** *Let  $P$  satisfy  $P \subseteq \text{exfin} \circ \mathcal{S}$  (this is part of the general proviso on  $P$ ).*

1. *If  $\text{EF}_P(T)$  then  $\text{exfin}(\llbracket T \rrbracket^s)$ .*
2. *Let  $T \in \lambda_{\Sigma}^{\text{gfp}}$  be well-bound and proper. If  $\text{EF}_P(T)$  and for all  $X^\sigma \in \text{FPV}(T)$ ,  $\text{exfin}(\mathcal{S}(\sigma))$  implies  $P(\sigma)$ , then  $\text{nofin}(\llbracket T \rrbracket^s)$ .*

### Proof

1. is proved by induction on the predicate  $\text{EF}_P$  (or, equivalently, on  $T$ ). The base case for fixed-point variables needs the proviso on  $P$ , and all other cases are immediate by the induction hypothesis.

<sup>4</sup>In a previous version of this paper,  $P$  was accidentally set to the always true predicate, in order to solve a problem of extensionality of a predicate that was used to deal with `FINHAB`. That was an error and led to incorrect proofs. We found this out by ourselves, but we also received a counterexample from Michał Ziobro in January 2017 which we gratefully acknowledge.

Figure 6: finfin predicate and inffin predicate

$$\begin{array}{ccccc}
\frac{\text{nofin}(N)}{\text{finfin}(\lambda x^A.N)} & \frac{\text{finfin}(N)}{\text{finfin}(\lambda x^A.N)} & \frac{\forall i, \text{finfin}(E_i)}{\text{finfin}(\sum_i E_i)} & \frac{\text{nofin}(N_j)}{\text{finfin}(x\langle N_i \rangle_i)} & \frac{\forall i, \text{finfin}(N_i)}{\text{finfin}(x\langle N_i \rangle_i)} \\
\\
\frac{\text{exfin}(N) \quad \text{inffin}(N)}{\text{inffin}(\lambda x^A.N)} & \frac{\text{inffin}(E_j)}{\text{inffin}(\sum_i E_i)} & \frac{\forall i, \text{exfin}(N_i) \quad \text{inffin}(N_j)}{\text{inffin}(x\langle N_i \rangle_i)}
\end{array}$$

2. is proved by induction on the predicate  $\exists F_P$  (or, equivalently, on  $T$ )—the case relative to fixpoints is based on  $T$  being proper and needs an inner co-induction and also the fact that  $\text{exfin}$  is invariant under co-contraction. For details, see the appendix.  $\square$

**Theorem 19 (Decidability of existence of inhabitants in  $\lambda$ )**

1. Let  $P$  satisfy  $P \subseteq \text{exfin} \circ \mathcal{S}$  (this is part of the general proviso on  $P$ ). For any  $T \in \lambda_{\Sigma}^{\text{gfp}}$  well-bound, proper and closed,  $\text{EF}_P(T)$  iff  $\text{exfin}(\llbracket T \rrbracket^s)$ .
2.  $\text{exfin}(\mathcal{S}(\sigma))$  is decidable, by deciding  $\text{EF}_{\emptyset}(\mathcal{F}(\sigma))$ .
3. In other words, **INHAB** is decidable.

**Proof** 1. Follows from both parts of Prop. 18, Lemmas 15 and 17, and the fact that, trivially, the extra condition in Prop. 18.2 is satisfied for closed terms.

2. Apply 1. with both parts of Theorem 14.

3. Analogously to the proof of Lemma 8, apply Prop. 2.3 (and  $\text{exfinext} = \text{exfin}$ ).  $\square$

**Definition 20** Let the predicates  $\text{EF}_{\star}$  and  $\exists F_{\star}$  on  $\lambda_{\Sigma}^{\text{gfp}}$  be defined by  $\text{EF}_{\star} := \text{EF}_P$  and  $\exists F_{\star} := \exists F_P$  for  $P := \text{exfin} \circ \mathcal{S}$ , which satisfies the proviso by Theorem 19.2. In particular,  $\text{EF}_{\star}$  and  $\exists F_{\star}$  are decidable.

Prop. 18.2 gives that  $\exists F_{\star}(T)$  implies  $\text{nofin}(\llbracket T \rrbracket^s)$  for all well-bound and proper expressions  $T$ . However, an inspection of the proof of that lemma even shows that the latter two properties are not needed:

**Lemma 21 (Sharp finitary characterization)** For all  $T \in \lambda_{\Sigma}^{\text{gfp}}$ ,  $\text{EF}_{\star}(T)$  iff  $\text{exfin}(\llbracket T \rrbracket^s)$ .

**Proof** See the appendix.  $\square$

In particular,  $\text{exfin}(\llbracket T \rrbracket^s)$  is decidable, by deciding  $\text{EF}_{\star}(T)$ .

### 3.3 Deciding type finiteness

Now a second and more difficult instance of the programme laid out in the beginning of Section 3.

We will now characterize the predicate  $\text{finfinext}$  by an inductively defined predicate  $\text{finfin}$ . Generically, we will obtain a characterization of its negation  $\text{inffinext}$  by the coinductively defined dual  $\text{inffin}$  of  $\text{finfin}$ . The inductive definition of  $\text{finfin}$  is given in the first line of Fig. 6. Notice that, while  $\text{finfin}$  is inductively defined and has only finitely many premisses in each clause, there is absolutely no claim on decidability since the coinductively defined predicate  $\text{nofin}$  enters the premisses.

By inversion (decomposing the summands into tuples) on  $\text{nofin}$ , one can show that  $\text{nofin} \subseteq \text{finfin}$  (which corresponds semantically to the trivial  $\text{nofinext} \subseteq \text{finfinext}$ ). Thus, in particular, no clause pertaining to  $\text{nofin}$  is necessary for the definition of  $\text{finfin}(\sum_i E_i)$ . We now show that  $\text{finfin}$  is sound and complete in terms of membership.

**Lemma 22 (Coinductive characterization)** *Given a Böhm forest  $T$ . Then,  $\text{finfin}(T)$  iff  $\mathcal{E}_{\text{fin}}(T)$  is finite, i. e.,  $\text{finfin} = \text{finfinext}$  as sets of Böhm forests.*

**Proof** The direction from left to right (“soundness”) is immediate by induction on  $\text{finfin}$ , using Lemma 16. From right to left, we do induction on the sum of the term heights of all finite members, which is a finite measure. The first and fourth rule of  $\text{finfin}$  are necessary to capture the cases when one passes from  $\lambda$ -abstractions to their bodies resp. from tuples to their components—thus when the individual heights decrease—but when there is just no element whose height decreases. The case of sums of elimination alternatives needs a further decomposition into tuples, in order to be able to apply the inductive hypothesis.  $\square$

Combined with Lemma 16, this gives an alternative proof of  $\text{nofin} \subseteq \text{finfin}$ .

The announced coinductive definition  $\text{inffin}$  that is meant to characterize  $\text{inffinext}$  is found in the second line of Fig. 6.

**Lemma 23** *Given a Böhm forest  $T$ ,  $\text{finfin}(T)$  iff  $\text{inffin}(T)$  does not hold.*

**Proof** See the appendix.  $\square$

As a corollary, we obtain  $\text{inffin} = \text{inffinext}$  as sets of Böhm forests.

Now we introduce two predicates on expressions of  $\lambda_{\Sigma}^{\text{gfp}}$  which will allow to characterize type finiteness, with the following intuitive meanings:

1.  $\text{FF}_P(T)$ : there are only finitely many finite members of  $T$  (the case of no finite members is included in this formulation);
2.  $\text{FF}_P(T)$ : there are infinitely many finite members of  $T$ .

Here, the predicate  $P$  on sequents controls the case of fixpoint variables, as before for  $\text{EF}_P$  and  $\text{EF}_P$ . The general proviso on  $P$  is that it is decidable and that for all sequents  $\sigma$ ,  $P(\sigma)$  implies  $\text{finfin}(\mathcal{S}(\sigma))$ , i. e.,  $P \subseteq \text{finfin} \circ \mathcal{S}$ . For our main result, it will be sufficient to take  $P := \emptyset$ . In view of the decidability result of the previous section, another possibility of choosing the predicate would be with  $P := \text{nofin} \circ \mathcal{S}$ , i. e., with the negation of the predicate underlying the definition of  $\text{EF}_{\star}$  and  $\text{EF}_{\star}$ .<sup>5</sup>

The definitions of these predicates are inductive, and they are presented in Fig. 7. Analogously to the predicates  $\text{EF}_P$  and  $\text{EF}_P$ , it is clear that they could equivalently be defined recursively over the term structure, thus ensuring their decidability, thanks to decidability of  $\text{EF}_{\star}$ .

**Lemma 24** *For all  $T \in \lambda_{\Sigma}^{\text{gfp}}$ ,  $\text{FF}_P(T)$  iff  $\text{FF}_P(T)$  does not hold.*

**Proof** Routine induction on  $T$ , using Lemma 17.  $\square$

**Proposition 25 (Finitary characterization)** *Let  $P$  satisfy  $P \subseteq \text{finfin} \circ \mathcal{S}$  (this is part of the general proviso on  $P$ ).*

1. *If  $\text{FF}_P(T)$  then  $\text{finfin}(\llbracket T \rrbracket^s)$ .*
2. *Let  $T \in \lambda_{\Sigma}^{\text{gfp}}$  be well-bound and proper. If  $\text{FF}_P(T)$  and for all  $X^{\sigma} \in \text{FPV}(T)$ ,  $\text{finfin}(\mathcal{S}(\sigma))$  implies  $P(\sigma)$ , then  $\text{inffin}(\llbracket T \rrbracket^s)$ .*

**Proof** Both statements are proven by induction on  $T$  (or, equivalently, by induction on the respective predicate in the premiss). While 1. is straightforward, for 2. the case relative to fixpoints is based on  $T$  being proper and needs an inner co-induction and also the fact that  $\text{finfin}$  is invariant under co-contraction. For details (on both parts), see the appendix.  $\square$

<sup>5</sup>For this specific setting of  $P$ , we could easily establish  $\text{FF}_P \subseteq \text{EF}_{\star}$  or, equivalently,  $\text{EF}_{\star} \subseteq \text{FF}_P$ , by induction. This would allow to remove the condition  $\text{EF}_{\star}(N_j)$  from the tuple rule for  $\text{FF}_P$ .

Figure 7:  $\text{FF}_P$  predicate and  $\text{FF}_P$  predicate

$$\begin{array}{c}
\frac{P(\sigma)}{\text{FF}_P(X^\sigma)} \quad \frac{\text{FF}_P(N)}{\text{FF}_P(\lambda x^A.N)} \quad \frac{\forall i, \text{FF}_P(E_i)}{\text{FF}_P(\text{gfp } X^\sigma. \sum_i E_i)} \\
\frac{\forall i, \text{FF}_P(N_i)}{\text{FF}_P(x(N_i)_i)} \quad \frac{\text{EF}_*(N_j)}{\text{FF}_P(x(N_i)_i)} \quad \text{and} \quad \frac{\text{FF}_P(N_j) \quad \forall i, \text{EF}_*(N_i)}{\text{FF}_P(x(N_i)_i)} \\
\frac{\neg P(\sigma)}{\text{FF}_P(X^\sigma)} \quad \frac{\text{FF}_P(N)}{\text{FF}_P(\lambda x^A.N)} \quad \frac{\text{FF}_P(E_j)}{\text{FF}_P(\text{gfp } X^\sigma. \sum_i E_i)}
\end{array}$$

With these preparations in place, the problem FINHAB can be solved in the same way as INHAB.

**Theorem 26 (Decidability of type finiteness in  $\lambda$ )**

1. Let  $P$  satisfy  $P \subseteq \text{finfin} \circ \mathcal{S}$  (this is part of the general proviso on  $P$ ). For any  $T \in \lambda_\Sigma^{\text{gfp}}$  well-bound, proper and closed,  $\text{FF}_P(T)$  iff  $\text{finfin}(\llbracket T \rrbracket^s)$ .
2.  $\text{finfin}(\mathcal{S}(\sigma))$  is decidable, by deciding  $\text{FF}_0(\mathcal{F}(\sigma))$ .
3. In other words, FINHAB is decidable.

**Proof** 1. Follows from both parts of Prop. 25, Lemmas 23 and 24, and the fact that, trivially, the extra condition in Prop. 25.2 is satisfied for closed terms.

2. Apply 1. with both parts of Theorem 14.

3. Analogously to the proof of Lemma 9, apply Prop. 2.3 (and  $\text{finfinext} = \text{finfin}$ ). □

**Definition 27** Let the predicates  $\text{FF}_*$  and  $\text{FF}_*$  on  $\lambda_\Sigma^{\text{gfp}}$  be defined by  $\text{FF}_* := \text{FF}_P$  and  $\text{FF}_* := \text{FF}_P$  for  $P := \text{finfin} \circ \mathcal{S}$ , which satisfies the proviso by Theorem 26.2. In particular,  $\text{FF}_*$  and  $\text{FF}_*$  are decidable.

Prop. 25.2 gives that  $\text{FF}_*(T)$  implies  $\text{inffin}(\llbracket T \rrbracket^s)$  for all well-bound and proper expressions  $T$ . Again (as for Lemma 21), an inspection of the proof of that proposition even shows that the latter two properties are not needed:

**Lemma 28 (Sharp finitary characterization)** For all  $T \in \lambda_\Sigma^{\text{gfp}}$ ,  $\text{FF}_*(T)$  iff  $\text{finfin}(\llbracket T \rrbracket^s)$ .

In particular,  $\text{finfin}(\llbracket T \rrbracket^s)$  is decidable, by deciding  $\text{FF}_*(T)$ .

## 4 Counting normal inhabitants

The method of the preceding section is not confined to the mere decision problems. In particular, instead of only deciding FINHAB, the finitely many inhabitants can be effectively obtained. We will illustrate this with some detail for the somehow more basic question of determining their number. The function for obtaining the set of inhabitants then follows the same pattern.

We have considered Böhm forests throughout the paper modulo idempotence of the summation operation (among other identifications). This does not hinder us from counting the number of finite members in case it is finite. The finite members themselves are “concrete”, and the only identification that is not expressed in the grammar of  $\lambda$  is  $\alpha$ -equivalence. However, we would prefer counting summand-wise and thus need to be sure that finite members do not belong to more than one summand in a sum, and this by taking into account that occurrences are identified up to bisimulation. Technically, this desideratum is achieved by considering a subset of Böhm

forests that we call *head-variable controlled*. The set  $\text{H}\lambda_{\Sigma}^{\text{co}}$  of head-variable controlled Böhm forests is obtained by the same grammar of terms and elimination alternatives as  $\lambda_{\Sigma}^{\text{co}}$ , but with the restriction for the formation of  $\sum_i E_i$  with  $E_i = x_i \langle N_j^i \rangle_j$  that the  $x_i$  are pairwise different, i. e., no variable is head of two summands in one sum, and this recursively throughout the Böhm forest. If we consider this restriction in our view of sums as sets of elimination alternatives, this only means that a given head variable cannot appear with two distinct tuples of arguments but still can appear multiply. So, in order to profit from the extra property of Böhm forests in  $\text{H}\lambda_{\Sigma}^{\text{co}}$ , we regard sums as functions from a finite set of (head) variables  $x$  into finite tuples of Böhm forest headed by  $x$  and use the associated notion of bisimilarity (modulo  $\alpha$ -equivalence). This means, when we speak about head-variable controlled Böhm forests, we not only consider Böhm forests satisfying this extra property, but also their presentation in this form that takes profit from it. This change of view does not change the notion of bisimilarity. Notice that  $\mathcal{S}(\sigma)$  and  $\mathcal{F}(\sigma)$  always yield head-variable controlled terms, in the respective term systems.

We define the counting function  $\#$  for head-variable controlled Böhm forests in  $\text{finfin}$  only, by recursion on  $\text{finfin}$ .

**Definition 29 (Infinitary counting function  $\# : \text{H}\lambda_{\Sigma}^{\text{co}} \cap \text{finfin} \rightarrow \mathbb{N}$ )**

$$\begin{aligned} \#(\lambda x^A . N) &:= \begin{cases} 0 & \text{if } \text{nofin}(N) \\ \#(N) & \text{else} \end{cases} \\ \#(\sum_i E_i) &:= \sum_i \#(E_i) \\ \#(x \langle N_i \rangle_i) &:= \begin{cases} 0 & \text{if } \exists j, \text{nofin}(N_j) \\ \prod_i \#(N_i) & \text{else} \end{cases} \end{aligned}$$

**Lemma 30** *Let  $T \in \text{H}\lambda_{\Sigma}^{\text{co}}$ . If  $\text{nofin}(T)$  (in particular,  $\text{finfin}(T)$ ) then  $\#(T) = 0$ .*

**Proof** Neither induction on  $T$  nor on  $\text{nofin}$  are available. The proof is by case analysis, where one has to use that elimination alternatives are tuples.  $\square$

While this lemma might allow to remove the case distinction in the  $\lambda$ -abstraction case, the second branch of the tuple case would replace the first one only with a very non-strict reading of the product that would have to be defined and be of value 0 as soon as one of the factors is 0.

The following lemma can be considered a refinement of the soundness part of Lemma 22.

**Lemma 31** *Let  $T$  be a head-variable controlled Böhm forest such that  $\text{finfin}(T)$ . Then,  $\#(T)$  is a well-defined natural number, and it is the cardinality of  $\mathcal{E}_{\text{fn}}(T)$ .*

**Proof** Notice that the clause for sums of elimination alternatives is subject to the presentation we convened for elements of  $\text{H}\lambda_{\Sigma}^{\text{co}}$ , and thus the value is invariant under our identifications. The recursive calls to  $\#$  occur only with Böhm forests that enter  $\text{finfin}$  “earlier”. Being the correct number depends on Lemma 16.  $\square$

Since we have also considered the elements of  $\lambda_{\Sigma}^{\text{gfp}}$  throughout the paper modulo idempotence of the summation operation, we will analogously introduce the set  $\text{H}\lambda_{\Sigma}^{\text{gfp}}$  of head-variable controlled elements. Again, this is not only a subset but comes with a different presentation of sums as functions from a finite set of (head) variables  $x$  into finite tuples of finitary terms headed by  $x$ .

**Definition 32 (Finitary counting function  $\# : \text{H}\lambda_{\Sigma}^{\text{gfp}} \rightarrow \mathbb{N}$ )** *Define by recursion over the term structure*

$$\begin{aligned} \#(X^{\sigma}) &:= 0 \\ \#(\lambda x^A . N) &:= \#(N) \\ \#(\text{gfp } X^{\sigma} . \sum_i E_i) &:= \sum_i \#(E_i) \\ \#(x \langle N_i \rangle_i) &:= \prod_i \#(N_i) \end{aligned}$$

**Lemma 33** *Let  $T \in \text{H}\lambda_{\Sigma}^{\text{gfp}} \cap \mathcal{E}F_{\star}$ . Then  $\#(T) = 0$ .*

**Proof** Obvious induction, see the appendix.  $\square$

**Proposition 34** *Let  $P \subseteq \text{nofin} \circ \mathcal{S}$  and  $T \in \text{H}\lambda_{\Sigma}^{\text{gfp}} \cap \text{FF}_P$ . Then  $\#(T) = \#(\llbracket T \rrbracket^s)$ .*

**Proof** The proof is by induction on  $T$  (equivalently, by induction on  $\text{FF}_P$ ), using Lemma 33 for the last rule of  $\text{FF}_P$ , see the appendix.  $\square$

**Theorem 35 (Counting theorem)** *Let  $P \subseteq \text{nofin} \circ \mathcal{S}$  (e. g.,  $P = \emptyset$ ). If  $\text{FF}_P(\mathcal{F}(\sigma))$  then  $\#(\mathcal{F}(\sigma))$  is the cardinality of  $\mathcal{E}_{\text{fin}}(\mathcal{S}(\sigma))$ .*

**Proof**  $\mathcal{F}(\sigma) \in \text{H}\lambda_{\Sigma}^{\text{gfp}}$ . By the preceding proposition, using the assumption that  $\text{FF}_P(\mathcal{F}(\sigma))$ , we obtain  $\#(\mathcal{F}(\sigma)) = \#(\llbracket \mathcal{F}(\sigma) \rrbracket^s)$ , which is  $\#(\mathcal{S}(\sigma))$  by Theorem 14. Thanks to Proposition 25.1,  $\text{finfin}(\mathcal{S}(\sigma))$ , hence, by Lemma 31,  $\#(\mathcal{S}(\sigma))$  is the cardinality of  $\mathcal{E}_{\text{fin}}(\mathcal{S}(\sigma))$ .  $\square$

Notice that when  $\text{FF}_P(\mathcal{F}(\sigma))$  does not hold, then  $\#(\mathcal{F}(\sigma))$  is meaningless, but  $\text{FF}_P(\mathcal{F}(\sigma))$  holds, and thus,  $\text{inffin}(\mathcal{S}(\sigma))$ , which ensures an infinite number of finite solutions of  $\sigma$ .

Without any extra effort, we can give an effective definition of the associated set of finite inhabitants through a function  $\mathcal{C} : \text{H}\lambda_{\Sigma}^{\text{gfp}} \rightarrow \mathcal{P}_{\text{fin}}(\lambda)$  by

$$\begin{aligned} \mathcal{C}(X^\sigma) &:= \emptyset \\ \mathcal{C}(\lambda x^A.N) &:= \{\lambda x^A.t \mid t \in \mathcal{C}(N)\} \\ \mathcal{C}(\text{gfp } X^\sigma . \sum_i E_i) &:= \cup_i \mathcal{C}(E_i) \\ \mathcal{C}(x\langle N_i \rangle_i) &:= \{x\langle t_i \rangle_i \mid \forall i, t_i \in \mathcal{C}(N_i)\} \end{aligned}$$

Then, for  $T \in \text{H}\lambda_{\Sigma}^{\text{gfp}}$ ,  $\#(T)$  is the cardinality of  $\mathcal{C}(T)$  (notice that the set union in the  $\text{gfp}$  case is always a disjoint union), and if  $\text{FF}_\emptyset(\mathcal{F}(\Gamma \Rightarrow A))$  then  $\mathcal{I}(\Gamma, A) = \mathcal{C}(\mathcal{F}(\Gamma \Rightarrow A))$ . If not,  $\mathcal{I}(\Gamma, A)$  is infinite.

## 5 Final Remarks

This paper illustrates a methodology to address decidability problems in the simply-typed  $\lambda$ -calculus which starts by computing a  $\lambda$ -term (through function  $\mathcal{F}$ ) representing the full set of inhabitants of a given type (using an extension of  $\lambda$ -calculus designed previously by the authors), and then uses that  $\lambda$ -term to decide the problem at hand.

To carry out this program, we had to introduce our simplified semantics that is loose in the sense that it does not guarantee that the interpretation of formal fixed-point constructs indeed denotes a fixed point. This loose semantics can be analyzed very smoothly, and we also identified the notion of a proper expression where the simplified semantics agrees on formal fixed-point constructs with the intended semantics in terms of solution spaces. Our finitary representation function generates proper expressions, and so we can apply the simplified semantics to solve the original problems.

The predicates with which we analyze the finitary expressions representing sets of inhabitants are parameterized by a predicate on sequents for the case of fixpoint variables. The interesting point about our use of this parameter is that, in order to establish decidability, we choose it very simply (as the empty set), but once we obtained decidability, we can in turn use that predicate as parameter when building further definitions. In the end, we only need two instances, but we consider it important—not only in the interest of succinctness—to have identified this abstraction.

We do not claim that our method can confirm sharp complexity results, namely  $\text{PSPACE}$ -completeness of  $\text{INHAB}$  [Sta79] and  $\text{FINHAB}$  [Hir98]. We are rather interested in having a simple representation of the *full* sets of inhabitants, which may have multiple uses, as illustrated by our counting functions. By “full” we mean in particular that we capture all  $\eta$ -long and  $\beta$ -normal terms. The restriction to  $\eta$ -long terms is very convenient for a concise description and does not do much harm to the usability of the results. The concept of *co-contraction* (Def. 4) is crucial for completeness of our method in this respect, and as shows our paper, it is not intrusive in practice, i. e., for the analysis carried out in this paper, its presence is hardly noticed in the proofs.

Note that other approaches dealing with a full set of inhabitants also face questions. For example, in [TAH96], although (finite) context-free grammars suffice to capture inhabitants obeying the *total discharge convention*, an infinite grammar is used to capture the full set of  $\beta$ -normal forms. In [BD05] (Sect. 4.3) a method is presented to produce a context-free grammar to generate the long normal forms of a type, but the produced grammars seem again to be unable to stay within the abovementioned optimal complexity. Also, in [SDB15], as a goal is to achieve machines capable of enumerating all normal inhabitants, and for this, storing a fixed finite number of bound variable names is not sufficient, automata with a non-standard form of register are used.

We believe our compositional methodology of first building a  $\lambda$ -term (more precisely, a closed well-bound term in  $\lambda_{\Sigma}^{\text{gfp}}$ ) representing the full set of inhabitants of interest, and then traversing that  $\lambda$ -term to decide whether a given property of that set holds, can be transferred to other contexts. For example, it would be interesting to know if in the presence of a connective like disjunction, our methodology produces a (simple) decision function for the INHAB problem.

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## A Proofs

**Lemma 15.** *Given a Böhm forest  $T$ ,  $\text{exfin}(T)$  iff  $\text{nofin}(T)$  does not hold.*

**Proof** This is plainly an instance of the generic result in the style of De Morgan’s laws that presents inductive predicates as complements of coinductive predicates, by a dualization operation on the underlying clauses. The principle is recalled with details now.

Assume a set  $U$  (the “universe”) and a function  $F : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$  that is monotone, i. e., for  $\mathcal{M} \subseteq \mathcal{N} \subseteq U$ , one has  $F(\mathcal{M}) \subseteq F(\mathcal{N})$ . Then, by Tarski’s fixed-point theorem, there exist the least fixed-point  $\mu F$  and the greatest fixed-point  $\nu F$  of  $F$ , with respect to set inclusion. Moreover,  $\mu F$  is the intersection of all pre-fixed points  $\mathcal{M} \subseteq U$  of  $F$ , i. e., with  $F(\mathcal{M}) \subseteq \mathcal{M}$ , and  $\nu F$  is the union of all post-fixed points  $\mathcal{M} \subseteq U$  of  $F$ , i. e., with  $\mathcal{M} \subseteq F(\mathcal{M})$ . This lattice-theoretic duality allows to relate both concepts through complements, with  $\mathcal{M}^{\complement} := U \setminus \mathcal{M}$ . Given  $F$  as before, define a monotone function  $F^\dagger : \mathcal{P}(U) \rightarrow \mathcal{P}(U)$  by setting  $F^\dagger(\mathcal{M}) := (F(\mathcal{M}^{\complement}))^{\complement}$ . Then,

$$\mu F = (\nu(F^\dagger))^{\complement} .$$

This formula (written in logical terms with negation in place of set complement) is often used to *define*  $\mu F$ , e. g., in  $\mu$ -calculus. For a proof, it suffices to consider the inclusion from left to right (the other direction is obtained by duality, using  $(F^\dagger)^\dagger = F$ ). Since the left-hand side is included in every pre-fixed point of  $F$ , it suffices to show that the right-hand side is such a pre-fixed point, i. e.,  $F((\nu(F^\dagger))^{\complement}) \subseteq (\nu(F^\dagger))^{\complement}$ . We show the contrapositive  $\nu(F^\dagger) \subseteq F^\dagger(\nu(F^\dagger))$  (using  $F^\dagger$  as abbreviation): but  $\nu(F^\dagger)$  is a post-fixed point itself (it is even a fixed point).  $\square$

**Lemma 36** *Let  $P \in \{\text{exfin}, \text{nofin}\}$  and  $\sigma \leq \sigma'$ . Then for all Böhm forests  $T$ , we have  $P(T)$  iff  $P([\sigma'/\sigma]T)$ .*

**Lemma 17.** *For all  $T \in \lambda_{\Sigma}^{\text{gfp}}$ ,  $\exists F(T)$  iff  $\text{EF}(T)$  does not hold.*

**Proof** Routine induction on  $T$ . In terms of the equivalent recursive definitions of the predicates, this would have been just an application of De Morgan’s laws.  $\square$

**Proposition 18 (Finitary characterization).** *Let  $P$  satisfy  $P \subseteq \text{exfin} \circ \mathcal{S}$  (this is part of the general proviso on  $P$ ).*

1. *If  $\text{EF}_P(T)$  then  $\text{exfin}(\llbracket T \rrbracket^s)$ .*
2. *Let  $T \in \lambda_{\Sigma}^{\text{gfp}}$  be well-bound and proper. If  $\exists F_P(T)$  and for all  $X^\sigma \in \text{FPV}(T)$ ,  $\text{exfin}(\mathcal{S}(\sigma))$  implies  $P(\sigma)$ , then  $\text{nofin}(\llbracket T \rrbracket^s)$ .*

**Proof** 1. is sketched in the main part of the paper.

2. is proved by induction on the predicate  $\exists F_P$  (which can also be seen as a proof by induction on  $T$ ).

Case  $T = X^\sigma$ . Then  $\neg P(\sigma)$ , hence, since  $X^\sigma \in \text{FPV}(T)$ , by contraposition and Lemma 15, we get  $\text{nofin}(\mathcal{S}(\sigma))$ .

Case  $T = \text{gfp } X^\sigma . \sum_i E_i$ . Let  $N := \llbracket T \rrbracket^s = \sum_i \llbracket E_i \rrbracket^s$ . As  $T$  is proper,  $N = \mathcal{S}(\sigma)$ . We hence have to show  $\text{nofin}(\mathcal{S}(\sigma))$ , which we do by an embedded coinduction for the coinductively defined predicate  $\text{nofin}$ . We have  $\exists F_P(E_i)$  for all  $i$  and want to use the induction hypothesis, which would give us  $\text{nofin}(\llbracket E_i \rrbracket^s)$  and thus  $\text{nofin}(\sum_i \llbracket E_i \rrbracket^s)$ , which was our goal. Fix an  $i$ . Of course,  $E_i$  is also well-bound and proper. We have to consider all  $Y^{\sigma'} \in \text{FPV}(E_i)$ . Either  $Y^{\sigma'} \in \text{FPV}(T)$ , and we are fine by hypothesis, or  $Y = X$  and, since  $T$  is well-bound,  $\sigma \leq \sigma'$ . We just show that  $\text{exfin}(\mathcal{S}(\sigma'))$  does not hold: from our coinductive hypothesis  $\text{nofin}(\mathcal{S}(\sigma))$ , we get through Lemma 5 and Lemma 36 even  $\text{nofin}(\mathcal{S}(\sigma'))$ , and this is the negation of  $\text{exfin}(\mathcal{S}(\sigma'))$ . This is a proper application of the coinductive hypothesis since it enters a lemma on  $\text{nofin}$  that does not change needed observation depths and then goes into an elimination alternative, where the occurrences of free fixpoint variables are at least “guarded” by an ordinary variable of a tuple.

The other cases are simple applications of the induction hypothesis.  $\square$

**Lemma 21 (Sharp finitary characterization).** *For all  $T \in \lambda_{\Sigma}^{\text{gfp}}$ ,  $\text{EF}_{\star}(T)$  iff  $\text{exfin}(\llbracket T \rrbracket^s)$ .*

**Proof** In view of the previous proposition, we only need to consider the direction from right to left, and we prove its contraposition  $\neg \text{EF}_{\star}(T)$  implies  $\text{nofin}(\llbracket T \rrbracket^s)$  by induction on the predicate  $\text{EF}_{\star}$ .

Case  $T = X^{\sigma}$ . Then  $\neg \text{exfin}(\mathcal{S}(\sigma))$  by hypothesis of this case, and this is  $\text{nofin}(\llbracket X^{\sigma} \rrbracket^s)$ .

Case  $T = \text{gfp} X^{\sigma}. \sum_i E_i$ . Then  $\llbracket T \rrbracket^s = \sum_i \llbracket E_i \rrbracket^s$ . We have  $\text{EF}(E_i)$  for all  $i$  and we use the induction hypothesis, which gives us  $\text{nofin}(\llbracket E_i \rrbracket^s)$  for all  $i$  and thus  $\text{nofin}(\sum_i \llbracket E_i \rrbracket^s)$ , which was our goal. Notice that this reasoning does not need further properties of  $T$ .

The other cases are likewise simple applications of the induction hypothesis.  $\square$

**Lemma 23.** *Given a Böhm forest  $T$ ,  $\text{finfin}(T)$  iff  $\text{inffin}(T)$  does not hold.*

**Proof**  $\text{inffin}$  is defined from  $\text{finfin}$  by the De Morgan's law (as recalled in the proof of Lemma 15). In the first clause for  $\text{inffin}$ , the proviso  $\text{exfin}(N)$  is necessary for soundness, and as well the proviso  $\text{exfin}(N_j)$  (with  $i = j$ ) in the last clause. Only through these guards we can ensure that  $\text{inffin} \subseteq \text{exfin}$ , which is a minimum requirement given what they say in terms of finite membership. Otherwise, the first clause would allow to derive  $\text{inffin}(N)$  for the infinite  $\lambda$ -abstraction, satisfying the equation  $N = \lambda x^A. N$  for any choice of  $A$  and without any relevance of the variable  $x$ . Similarly for the third clause with  $\nu N. x \langle N \rangle$ .  $\square$

**Lemma 37** *Let  $P \in \{\text{finfin}, \text{inffin}\}$  and  $\Gamma \leq \Gamma'$ . Then for all Böhm forests  $T$ , we have  $P(T)$  iff  $P(\llbracket \Gamma' / \Gamma \rrbracket T)$ .*

**Proposition 25. (Finitary characterization)** *Let  $P$  satisfy  $P \subseteq \text{finfin} \circ \mathcal{S}$  (this is part of the general proviso on  $P$ ).*

1. *If  $\text{FF}_P(T)$  then  $\text{finfin}(\llbracket T \rrbracket^s)$ .*
2. *Let  $T \in \lambda_{\Sigma}^{\text{gfp}}$  be well-bound and proper. If  $\text{FP}_P(T)$  and for all  $X^{\sigma} \in \text{FPV}(T)$ ,  $\text{finfin}(\mathcal{S}(\sigma))$  implies  $P(\sigma)$ , then  $\text{inffin}(\llbracket T \rrbracket^s)$ .*

**Proof** 1. By induction on  $\text{FF}_P$  (or equivalently by structural induction on  $T$ ). We only show the tuple cases with  $T = x \langle N_i \rangle_i$ . The other cases are equally simple.

Case for some  $j$ ,  $\text{EF}_{\star}(N_j)$ . By Lemma 21,  $\text{nofin}(\llbracket N_j \rrbracket^s)$ , hence  $\text{finfin}(x \langle \llbracket N_i \rrbracket^s \rangle_i)$ , which is  $\text{finfin}(\llbracket T \rrbracket^s)$ .

Case for all  $i$ ,  $\text{FF}_P(N_i)$ . By induction hypothesis,  $\text{finfin}(\llbracket N_i \rrbracket^s)$  for all  $i$ , hence  $\text{finfin}(x \langle \llbracket N_i \rrbracket^s \rangle_i)$ .

2. By induction on  $\text{FF}_P$  (or equivalently by structural induction on  $T$ ).

Case  $T = X^{\sigma}$ . Then  $\neg P(\sigma)$ , hence, since  $X^{\sigma} \in \text{FPV}(T)$ , by contraposition and Lemma 23, we get  $\text{inffin}(\mathcal{S}(\sigma))$ .

Case  $T = x \langle N_i \rangle_i$ . For some  $j$ ,  $\text{FP}_P(N_j)$  and, for all  $i$ ,  $\text{EF}_{\star}(N_i)$ . The induction hypothesis is applicable for  $N_j$  since  $\text{FPV}(N_j) \subseteq \text{FPV}(T)$ . Therefore, we have  $\text{inffin}(\llbracket N_j \rrbracket^s)$ . By Lemma 21,  $\text{exfin}(\llbracket N_i \rrbracket^s)$ , for all  $i$ , hence, we are done by definition of  $\text{inffin}$ .

Case  $T = \text{gfp} X^{\sigma}. \sum_i E_i$ . For some  $j$ ,  $\text{FP}_P(E_j)$ . Let  $N := \llbracket T \rrbracket^s = \sum_i \llbracket E_i \rrbracket^s$ . As  $T$  is proper,  $N = \mathcal{S}(\sigma)$ . We hence have to show  $\text{inffin}(\mathcal{S}(\sigma))$ , which we do by an embedded coinduction for the coinductively defined predicate  $\text{inffin}$ . We want to use the induction hypothesis for  $E_j$ , which would give us  $\text{inffin}(\llbracket E_j \rrbracket^s)$  and thus  $\text{inffin}(\sum_i \llbracket E_i \rrbracket^s)$ , which was our goal. Of course,  $E_j$  is also well-bound and proper. We have to consider all  $Y^{\sigma'} \in \text{FPV}(E_j)$ . Either  $Y^{\sigma'} \in \text{FPV}(T)$ , and we are fine by hypothesis, or  $Y = X$  and, since  $T$  is well-bound,  $\sigma \leq \sigma'$ . We just show that  $\text{finfin}(\mathcal{S}(\sigma'))$  does not hold: from our coinductive hypothesis  $\text{inffin}(\mathcal{S}(\sigma))$ , we get through Lemma 5 and Lemma 37 even  $\text{inffin}(\mathcal{S}(\sigma'))$ , and this is the negation of  $\text{finfin}(\mathcal{S}(\sigma'))$ . This is a proper application of the coinductive hypothesis since it enters a lemma on  $\text{inffin}$  that does not change needed observation depths and then goes into an elimination alternative, where the occurrences of free fixpoint variables are at least “guarded” by an ordinary variable of a tuple.

The case of  $\lambda$ -abstractions is a simple application of the induction hypothesis.  $\square$

We remark that the proposition and its proof are rather analogous to Prop. 18 than dual to it, although the logical structure of the predicates is rather dual: to enter a fixed point into  $\mathbf{FF}_P$ , all of the elimination alternatives have to be there already, while for  $\mathbf{EF}_P$ , only one of the elimination alternatives is required. However, this duality is broken for the tuples: while for  $\mathbf{EF}_P$ , all arguments are required to be in the same predicate,  $\mathbf{FF}_P$  has a rule that asks only about one argument, but for a different predicate, and there is even a second possibility. Anyway, the proof structure needs to be analogous since  $\mathbf{exfin}$  and  $\mathbf{finfin}$  are both inductively defined and therefore do not admit reasoning by coinduction.

**Lemma 33.** *Let  $T \in \mathbf{H}\lambda_{\Sigma}^{\mathbf{gfp}} \cap \mathbf{EF}_{\star}$ . Then  $\#(T) = 0$ .*

**Proof** Induction over  $\mathbf{EF}_{\star}$  (or, equivalently, over  $T$ ).

Case  $T = X^{\sigma}$ . Trivial.

Case  $T = \lambda x^A.N$ . Trivial by induction hypothesis.

Case  $T = x\langle N_i \rangle_i$ . By induction hypothesis, one of the factors is 0.

Case  $T = \mathbf{gfp} X^{\sigma} . \sum_i E_i$ . By induction hypothesis, all summands are 0.  $\square$

**Proposition 34.** *Let  $P \subseteq \mathbf{nofin} \circ \mathcal{S}$  and  $T \in \mathbf{H}\lambda_{\Sigma}^{\mathbf{gfp}} \cap \mathbf{FF}_P$ . Then  $\#(T) = \#(\llbracket T \rrbracket^s)$ .*

**Proof** We will write  $L$  and  $R$  for left-hand side and right-hand side of the equation to prove. The proof is by induction on  $T$  (or, equivalently, by induction on  $\mathbf{FF}_P$ ).

Case  $T = X^{\sigma}$ . Then  $\mathbf{nofin}(\mathcal{S}(\sigma))$ , hence  $\#(\mathcal{S}(\sigma)) = 0$  by Lemma 30. Hence,  $R = 0 = L$ .

Case  $T = \lambda x^A.N$ . Then  $\mathbf{FF}_P(N)$ .  $L = \#(N)$ .  $R = \#(\lambda x^A.\llbracket N \rrbracket^s)$ . According to the definition of  $R$ , we have to distinguish if  $\mathbf{nofin}(\llbracket N \rrbracket^s)$  or not. In the first case, by Lemma 30, we have  $\#(\llbracket N \rrbracket^s) = 0$ . Thus, in both case, this gives  $R = \#(\llbracket N \rrbracket^s)$ , while  $L = \#(N)$ . Done by induction hypothesis.

Case  $T = x\langle N_i \rangle_i$ . Subcase  $\mathbf{EF}_{\star}(N_j)$  for some  $j$ . By Lemma 21,  $\mathbf{nofin}(\llbracket N_j \rrbracket^s)$ . Hence,  $R = 0$ . By Lemma 33,  $\#(N_j) = 0$ , hence also  $L = 0$  (since one factor is 0).

Subcase  $\mathbf{FF}_P(N_i)$  for all  $i$ . We may assume that we are not in the first subcase that has already been treated, hence  $\mathbf{EF}_{\star}(N_i)$  for all  $i$ . By Lemma 21,  $\neg \mathbf{nofin}(\llbracket N_i \rrbracket^s)$  for all  $i$ . Therefore,  $R = \prod_i \#(\llbracket N_i \rrbracket^s)$ , while  $L = \prod_i \#(N_i)$ . Done by induction hypothesis for all  $i$ .

Case  $T = \mathbf{gfp} X^{\sigma} . \sum_i E_i$ . Then  $\mathbf{FF}_P(E_i)$  for all  $i$ . Just apply the induction hypothesis to all the summands and sum up. (Notice how this case becomes the simplest one in our setting with simplified semantics.)  $\square$