Basing Belief on Quasi-Factive Evidence

Author version*

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Abstract

Topological semantics have proved to be a very fruitful approach in formal epistemology, two noticeable representatives being the interior semantics and topological evidence models. In this paper, we introduce the concept of *quasi-factive evidence* as a way to account for untruthful evidence in the interior semantics. This allows us to import concepts from topological evidence models, thereby connecting the two frameworks in spite of their apparent disparities. This approach sheds light on the interpretation of belief in the interior semantics, and gives meaning to concepts that used to be essentially technical: the closure-interior semantics can be interpreted as the condition of existence of a quasi-factive justification, while the extremally disconnected spaces are now characterized as those where the available information is always consistent. But our most important result is the equivalence between the interior-closure-interior semantics and what we call the *strengthening condition*, along with a sound and complete axiomatization. Finally, we build on this strengthening condition to introduce a notion of relative plausibility.

1 Introduction

The very beginning of epistemology can be traced back to Plato's dialogue Theaetetus, where the first conception of knowledge as *true justified belief* was proposed, though deemed not fully satisfying. One of the most famous criticisms of this definition was formulated by Gettier in 1963 [Get63]. At the same time, Hintikka had the idea to use modal logic for the purpose of describing epistemic statements [Hin62]. This opened a fruitful philosophical and logical inquiry into the foundations of knowledge.

Two approaches can be distinguished: one is *belief-first* and is focused on identifying a condition on true justified belief that turns it into knowledge; the other is *knowledge-first* and attempts to derive a good notion of belief by weakening knowledge. In particular, the work of Stalnaker [Sta06] stands in the latter trend: in order to investigate the interactions between knowledge and belief, he proposed to address the two notions at once via the bimodal logical system **KB**. From his axioms, he then derived the axiom .2 for knowledge as well as a characterization of belief in term of knowledge, namely $B\varphi \leftrightarrow \hat{K}K\varphi$, which reads "the agent believes φ if and only if it is epistemically possible that they know φ".

The introduction of topology in the elaboration of semantics for modal logics can be attributed to McKinsey and Tarski [MT44], the idea being to interpret a modality as the interior operator of some topological space. In an epistemic framework, this semantics naturally allows for an evidence-based interpretation: an open set is seen as a piece of evidence, and knowing P is then equivalent to *possessing some factive piece of evidence supporting* P [Vic96]. In 2013, Baltag et al. [BBÖS13] (see also [BBÖS19]) built on Stalnaker's logic and, from the equation $B\varphi \leftrightarrow \hat{K}K\varphi$ and the interior semantics for knowledge, deduced a topological semantics for belief. They associated the modality B to the closure of the interior operator, giving birth to the *closure-interior semantics*, or Cl-Int semantics for short. Its interpretation is quite insightful: it means

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that P is believed if and only if the agent can not distinguish the actual situation from a situation where they know P. However, it presents the disadvantage of being restricted to the class of *extremally disconnected* spaces. Besides being little intuitive, the property of extremal disconnectedness is not hereditary[1], which causes some trouble when it comes to deal with public announcements. A solution proposed in [BBÖS15] is to consider a more general semantics: the *interior-closure-interior semantics*, or Int-Cl-Int semantics for short. Despite its good properties, it remains rather unintuitive, as there is limited meaning to be found in a stack of three topological operators. A different route taken in [BÖ20] consists in distinguishing what is known from what is *knowable*, thus introducing a third modality and weakening Stalnaker's system accordingly.

Another approach which has to be mentioned is that of *evidence models*. Those were first introduced by Van Benthem and Pacuit in [vBP11] and further studied in [BBÖS22, vBFDP14]. Contrary to the interior semantics, this framework takes into account false evidence, i.e. pieces of evidence that are available but not factive. The semantics for knowledge and belief proposed in [BBÖS22] is based on the notion of *justification*: a justification is a piece of evidence which is consistent with all of the others. In this paper, we aim at making interior-related semantics more meaningful by importing relevant concepts from evidence models. To this end we introduce the notion of *quasi-factive evidence* to express the property of a piece of evidence being "seemingly" factive at some world. Basically, this is a way to bypass the absence of false evidence, an inherent limitation of the interior semantics. We then give a definition of a local justification and study semantics for knowledge and belief, while drawing analogies with the case of evidence models.

The article is structured as follows: in Section 2, we begin with some preliminaries about modal logics, axiomatic systems, topological semantics and evidence models. In Section 3, we introduce quasi-factive evidence and local justifications, and establish a number of useful characterizations. In Section 4, we define a strengthening condition inspired from evidence models, and prove that it coincides with belief in the Int-Cl-Int semantics – suggesting that this semantics is more relevant than it seemed. In Section 5, we provide a sound and complete axiomatization of knowledge and belief in the Int-Cl-Int semantics. In Section 6 we introduce a plausibility preorder between propositions – in fact the natural generalization of the strengthening condition – and investigate its properties.

2 Background

In Section 2.1, we recall some basics about epistemic logics, the topological interior semantics and a few results of soundness and completeness – for more details we refer to [vBB07, BRV01, vDvdHK07]. In Section 2.2, we present Stalnaker's approach to introduce belief in the picture via a bimodal axiom logic, and we show how this naturally induces the Cl-Int semantics for belief, followed by its close kin, the Int-Cl-Int semantics. In Section 2.3, we summarily introduce evidence models, along with the conceptions of knowledge and belief designed in [BBÖS22].

2.1 Epistemic logics

We first introduce the language of epistemic logic. Let **Prop** be a countable set of atomic propositions. The language $\mathcal{L}_K$ of epistemic formulas is generated by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi$$

where $p \in$ **Prop**. The abbreviations $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined as usual. Given an arbitrary element p of **Prop**; we define the *falsum* $\bot := p \wedge \neg p$ and the *verum* $\top := p \vee \neg p$. Given $\varphi, \psi_1, \ldots, \psi_n \in \mathcal{L}_K$ and $p_1, \ldots, p_n$, we denote by $\varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$ the formula φ where the ψ_i's have been uniformly substituted for the p_i's. A formula of the form $K\varphi$ reads "the agent knows φ", where "the agent" refers to some fictitious individual. The dual of K is defined by $\hat{K}\varphi := \neg K \neg\varphi$ and reads "φ is compatible with the agent's knowledge" or "the agent sees φ as possible".

[1]That is, a subspace of an extremally disconnected space may not be extremally disconnected itself.

We recall in Table 1 the axioms of basic modal logic **K**, along with some additional axioms of particular interest to us. In an epistemic context, T is called the axiom of *truthfulness of knowledge*, while $\mathsf{4}$ and $\mathsf{5}$ are known as the axioms of *positive* and *negative introspection*, respectively. The following logics are all standard:

$$\begin{aligned}
\mathbf{KT} &:= \mathbf{K} + \mathsf{T}, \\
\mathbf{S4} &:= \mathbf{K} + \mathsf{T} + \mathsf{4}, \\
\mathbf{S4.2} &:= \mathbf{K} + \mathsf{T} + \mathsf{4} + \mathsf{.2}, \\
\mathbf{S4.3} &:= \mathbf{K} + \mathsf{T} + \mathsf{4} + \mathsf{.3}, \\
\mathbf{S5} &:= \mathbf{K} + \mathsf{T} + \mathsf{4} + \mathsf{5}.
\end{aligned}$$

While **S5** is criticized for its consequences deemed unrealistic [Sta06], even for our idealized thinkers, **S4** is less controversial. We will deal further with **S4.2** and **S4.3** in Section 2.2. For now we turn our attention

Table 1: Axioms of **K** and other usual axioms

Name	Axiom/inference rule
	all propositional tautologies
K (Distribution)	$K(p \to q) \to (Kp \to Kq)$
Uniform Substitution	from φ infer $\varphi[\psi_1/p_1, \ldots, \psi_n/p_n]$
Modus Ponens	from φ and $\varphi \to \psi$ infer ψ
Necessitation	from φ infer $K\varphi$

Name	Axiom
T	$Kp \to p$
$\mathsf{4}$	$Kp \to KKp$
$\mathsf{.2}$	$\hat{K}Kp \to K\hat{K}p$
$\mathsf{.3}$	$K(Kp \to q) \vee K(Kq \to p)$
$\mathsf{5}$	$\hat{K}p \to K\hat{K}p$

to the topological semantics for modal logic, and assume that the reader is familiar with general topology. If needed one can refer to appendix A for a short reminder about notations and elementary results. For a proper and complete introduction see e.g. [Rys89].

Definition 2.1. Let $\mathcal{X} = (X, \tau)$ be a topological space. A *topological model* based on $\mathcal{X}$ is a tuple (X, τ, ν) where $\nu : \mathbf{Prop} \to \mathcal{P}(X)$ is a *valuation.*

Definition 2.2. Let $\mathfrak{M} = (X, \tau, \nu)$ be a topological model. Given a formula φ, the *extension* $[\![\varphi]\!]^{\mathfrak{M}}$ of φ in $\mathfrak{M}$ is defined inductively as follows:

$$\begin{aligned}
[\![p]\!]^{\mathfrak{M}} &:= \nu(p) && \text{for all } p \in \mathbf{Prop}, \\
[\![\neg\varphi]\!]^{\mathfrak{M}} &:= ([\![\varphi]\!]^{\mathfrak{M}})^c, \\
[\![\varphi \wedge \psi]\!]^{\mathfrak{M}} &:= [\![\varphi]\!]^{\mathfrak{M}} \cap [\![\psi]\!]^{\mathfrak{M}}, \\
[\![K\varphi]\!]^{\mathfrak{M}} &:= \mathrm{Int}([\![\varphi]\!]^{\mathfrak{M}}).
\end{aligned}$$

Informally speaking, $[\![\varphi]\!]^{\mathfrak{M}}$ is meant to denote the set of worlds at which φ is true. Given $x \in X$, we write $\mathfrak{M}, x \vDash \varphi$ whenever $x \in [\![\varphi]\!]^{\mathfrak{M}}$. If $[\![\varphi]\!]^{\mathfrak{M}} = X$, we write $\mathfrak{M} \vDash \varphi$. If $(X, \tau, \nu) \vDash \varphi$ for every valuation ν, then we write $(X, \tau) \vDash \varphi$. Finally, if $\mathcal{X} \vDash \varphi$ for every topological space $\mathcal{X}$, we write $\vDash \varphi$.

In formal epistemology, an open set U is customarily seen as a *piece of evidence* in favour of the worlds lying in U. We say that U is *factive* at world x if $x \in U$. At some world x, the agent is aware of all the pieces of evidence that are factive at x, and only these. Hence, the available evidence is always *truthful.* Note that the smaller a piece of evidence, the more information it carries. Thus, the inclusion relation orders open sets by decreasing precision and, accordingly, if U and V are two pieces of evidence, we will call U *stronger* than V whenever $U \subseteq V$. In a way, the fact that the topology is closed under finite intersection captures the ability of the agent to combine the available information to obtain stronger evidence.

Let $P \subseteq X$ be a proposition (here identified to a set of worlds). We say that a piece of evidence U *supports* P, or is an *argument* for P, if $U \subseteq P$. Note that this defines a notion of *conclusive* support: if one has a argument for P, then one knows P for certain. Looking at the definition of interior, it then appears that P is known at world x if and only if there exists a piece of evidence supporting P and factive at x. In this framework, knowledge is thus seen as the *possession of some factive argument*. Take also note that for all φ, we have $[\![\hat{K}\varphi]\!]^{\mathfrak{M}} = \mathrm{Cl}([\![\varphi]\!]^{\mathfrak{M}})$, which means that $P := [\![\varphi]\!]^{\mathfrak{M}}$ is epistemically possible at world x if and only if P is not falsified by any piece of evidence factive at x.

To close this subsection, let us mention three important soundness and completeness results in the interior semantics.

Theorem 2.1 ([MT44]). *In the interior semantics,* **S4** *is sound and complete for the class of all topological spaces.*

The other two results regard the so-called (hereditarily) extremally disconnected spaces.

Definition 2.3. Let $\mathcal{X}$ be a topological space. We say that $\mathcal{X}$ is *extremally disconnected* (or ED for short) if the closure of every open set is open. We say that $\mathcal{X}$ is *hereditarily extremally disconnected* (or HED for short) if all of its subspaces are extremally disconnected.

Theorem 2.2 ([vBB07]). *In the interior semantics,* **S4.2** *is sound and complete for the class of ED spaces.*

Theorem 2.3 ([BBLBvM15]). *In the interior semantics,* **S4.3** *is sound and complete for the class of HED spaces.*

2.2 Doxastic logics

We now move to the logic of belief, but we first stick to the unimodal framework for a moment. The language $\mathcal{L}_B$ is just like $\mathcal{L}_K$, but the modal operator is written B instead of K. It is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid B\varphi.$$

The formula $B\varphi$ thus reads "the agent believes φ". Since one can have false beliefs, axiom T is not relevant in this context. It is thus replaced by the axiom D, which states that the agent cannot hold inconsistent beliefs:

$$\mathsf{D} := Bp \rightarrow \neg B\neg p.$$

Note that in **K**, it is equivalent to the more concise formula $\neg B\bot$. In addition, the axiom $\mathsf{5}$ is more acceptable when it comes to beliefs, and consequently the logic $\mathbf{KD45} := \mathbf{K} + \mathsf{D} + \mathsf{4} + \mathsf{5}$ has established itself as the standard axiomatic system for belief [vDvdHK07]. On the semantic side, the interior semantics obviously does not fit such axioms. The first attempt to develop a proper topological semantics for belief can be attributed to Steinvold, with his co-derived semantics [Ste07]. It presents however some flaws that have been pointed out in [BBÖS13], so we do not expand on it. Instead we move to Stalnaker's logic **KB** [Sta06], which addresses both knowledge and belief at once. Here we consider the bimodal language $\mathcal{L}_{KB}$ generated by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid B\varphi.$$

The logic **KB** is the logic **S4** for knowledge extended with the axioms of Table 2. Axiom D has already been commented upon. Axioms SPI and SNI state that the agent has full access to their beliefs. Axiom KB means that all that is known is also believed. Axiom FB means that belief is subjectively indistinguishable from knowledge. This last principle is called *strong belief* by Stalnaker, but the term *full belief* is favoured in [BBÖS19]. In this logic, belief then appears to be equivalent to the *epistemic possibility of knowledge*.

Proposition 2.1 ([Sta06]). *The formula* $Bp \leftrightarrow \hat{K}Kp$ *is a theorem of* **KB**.

As a consequence, one can rid the syntax of the operator B and get back to the unimodal language $\mathcal{L}_K$. More precisely, the epistemic fragment of **KB** boils down to **S4.2**.

Proposition 2.2 ([Sta06]). *For all* $\varphi \in \mathcal{L}_K$*, we have* $\mathbf{KB} \vdash \varphi$ *if and only if* $\mathbf{S4.2} \vdash \varphi$.

Table 2: Axioms for belief in **KB**

Name	Axiom	Description
D	$Bp \to \neg B \neg p$	Consistency of belief
SPI	$Bp \to KBp$	Strong positive introspection for belief
SNI	$\neg Bp \to K \neg Bp$	Strong negative introspection for belief
KB	$Kp \to Bp$	Knowledge implies belief
FB	$Bp \to BKp$	Subjective certainty

Alternatively, we can look at the doxastic fragment of **KB**, i.e. its consequences in the language $\mathcal{L}_B$ only. It turns out to be exactly captured by **KD45**, as proven in [BBÖS19].

Theorem 2.4. *For all $\varphi \in \mathcal{L}_B$, we have* $\mathbf{KB} \vdash \varphi$ *if and only if* $\mathbf{KD45} \vdash \varphi$.

On the semantic side, if we start from the interior semantics and accept the axioms of **KB**, then from Proposition 2.1 we immediately obtain a topological semantics for belief: the closure-interior semantics, first introduced in [BBÖS13].

Definition 2.4. Let $\mathfrak{M} = (X, \tau, \nu)$ be a model based on an ED space. Given a formula $\varphi \in \mathcal{L}_{KB}$, the extension $[\![\varphi]\!]^{\mathfrak{M}}$ of φ in $\mathfrak{M}$ is defined by

$$[\![K\varphi]\!]^{\mathfrak{M}} := \mathrm{Int}([\![\varphi]\!]^{\mathfrak{M}}),$$
$$[\![B\varphi]\!]^{\mathfrak{M}} := \mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathfrak{M}})),$$

and the propositional cases are as in Definition 2.2.

One can check, from the properties of the closure and interior operators, that we have $[\![B\varphi]\!]^{\mathfrak{M}} = [\![\hat{K}K\varphi]\!]^{\mathfrak{M}}$ for all $\varphi \in \mathcal{L}_{KB}$. Therefore, Proposition 2.2 and Theorem 2.2 together entail the following:

Theorem 2.5. *In the Cl-Int semantics,* **KB** *is sound and complete for the class of extremally disconnected spaces.*

In words, the interpretation of the operator B reads "$B\varphi$ holds at x if and only if x is arbitrarily close to the extension of $K\varphi$", and this can be seen a spatial analogue of the axiom of full belief which states that "$B\varphi$ is indistinguishable from $K\varphi$". This semantics is thus quite intuitive. On the syntactic side, the logic **KB** has very readable axioms, and its doxastic fragment **KD45** is quite standard.

The downside, however, is that we have to restrict ourselves to ED spaces[2]. Not only they are quite unfamiliar – most usual spaces such as $\mathbb{Q}$ or $\mathbb{R}$ do not have this property – and lack a clear epistemic interpretation, they are also a source of trouble when it comes to dynamic updates (i.e. events that affect the beliefs of the agent). Indeed, if X is a space and $P \subseteq X$ a proposition, the public announcement of P is modelled by restricting everything to the topological subspace induced by P, which might no longer be ED. One solution to make announcements compatible with this semantics is, as proposed in [BBÖS19], to restrict oneself to the class of HED spaces. By construction, this class is contained in the class of ED spaces while being *hereditary*, i.e. closed under taking subspaces. By Theorem 2.3, the logic of knowledge for this class is then **S4.3** – which is stronger than **S4.2**. This fixes the problem, but makes things worse in some respect: we end up with an even smaller class of spaces, and a new axiom (.3) whose epistemic meaning is unclear, and which lacks an independent justification.

Another way to bypass the problems arising from the Cl-Int semantics was presented in [BBÖS15], and consists in designing a variant of it, viable in all topological spaces: the interior-closure-interior semantics.

[2]It technically makes sense to interpret formulas in other spaces, but then the Cl-Int semantics no longer has the intended properties (it does not even define a normal modal operator).

Note that in [BBÖS15], the Cl-Int and Int-Cl-Int semantics are respectively referred to as the "topological semantics" and "weak topological semantics" for belief, but we find our own terms less ambiguous and more evocative.

Definition 2.5. Let $\mathfrak{M} = (X, \tau, \nu)$ be a topological model. Given a formula $\varphi \in \mathcal{L}_{KB}$, the extension $[\![\varphi]\!]^{\mathfrak{M}}$ of φ in $\mathfrak{M}$ is defined by

$$[\![K\varphi]\!]^{\mathfrak{M}} := \mathrm{Int}([\![\varphi]\!]^{\mathfrak{M}}),$$
$$[\![B\varphi]\!]^{\mathfrak{M}} := \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}([\![\varphi]\!]^{\mathfrak{M}}))),$$

and the propositional cases are as in Definition 2.2.

To avoid any ambiguity, we shall make clear, whenever needed, whether we evaluate formulas in the Cl-Int semantics or in the Int-Cl-Int semantics. The crucial result to keep in mind is that the two semantics coincide on ED spaces.

Proposition 2.3. *If (X, τ) is ED, then for all $P \subseteq X$ we have*

$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(P))) = \mathrm{Cl}(\mathrm{Int}(P)).$$

Proof. For all $P \subseteq X$ we know that $\mathrm{Int}(P)$ is open, so by assumption $\mathrm{Cl}(\mathrm{Int}(P))$ is open too and the result follows. □

On the logical side, the Int-Cl-Int semantics *restricted to the doxastic language* $\mathcal{L}_B$ appears to be axiomatized by the logic

$$\mathbf{wKD45} := \mathbf{K} + \mathsf{D} + \mathsf{4} + \mathsf{w5}$$

where $\mathsf{w5}$ is a weak variant of negative introspection:

$$\mathsf{w5} := \neg B\varphi \to \hat{B}B\neg B\varphi.$$

The name **wKD45** stands for "weak **KD45**", since all of the consequences of **wKD45** are also derivable in **KD45** (the converse being false).

Theorem 2.6 ([BBÖS15])**.** *In the Int-Cl-Int semantics,* **wKD45** *is sound and complete for the class of all topological spaces.*

The main benefit of the Int-Cl-Int semantics is that it immediately removes the obstacles to dynamic updates: since it is well defined in all topological spaces, taking subspaces is harmless and no problem of heredity is to bother us. Undeniably, this provides a more harmonious solution than the *ad hoc* restriction to HED spaces, but the price to pay is a loss of conceptual meaning: the Int-Cl-Int semantics appeals little to intuition, unlike the Cl-Int semantics; on the logical side too, the axiom $\mathsf{w5}$ lacks a clear interpretation, as pointed out in [Özg13].

Now, one may wonder which semantics is *the* true semantics of belief (if there is one), but this is not the best phrasing. As stated by Proposition 2.3, the two semantics are essentially the same, except that one is only defined for ED spaces. So what the question really asks is whether we can restrict ourselves to this class of spaces. Since the purpose of topological spaces is to depict epistemic situations, this can be reformulated as follows.

Question 1. Can any epistemic situation be represented by an extremally disconnected space?

If the answer is yes, then the Cl-Int semantics alone is sufficient to address our needs. If the answer is no, then there exist some spaces that deserve our attention while not being ED, and wherein formulas can be interpreted in the Int-Cl-Int semantics only. In logical terms, a negative answer would also mean that **KD45** does not faithfully capture the notion of belief, that is, the axiom $\mathsf{5}$ is too strong and only $\mathsf{w5}$ is acceptable in general. However, as mentioned before, it is not clear what the condition of extremal disconnectedness mean in epistemic terms. Without any intuitive understanding of it, we are unlikely to make any progress toward the resolution of Question 1. It seems that further conceptualization is needed to escape this dead end, and this is why we now turn our attention to evidence models, and see what inspiration can be drawn from them.

2.3 Evidence models

Evidence models are a framework introduced by Van Benthem and Pacuit [vBP11] to deal with situations in which some pieces of evidence are unreliable, as opposed to the interior semantics, in which every available piece of evidence is factive. Formally, an evidence model assigns to any world x a collection $E(x)$ of pieces of evidence (those available at this world), resulting in a kind of structure more complex than topological spaces. Yet most of the existing work is concerned with *uniform evidence models*, namely those where the function $x \mapsto E(x)$ is constant, meaning that every piece of evidence is available at all possible worlds. As a result, the agent is fully aware of which evidence they have. We will not discuss here whether this assumption is justified or not, since our goal is only to take a look at the semantics. In [BBÖS22] it is shown that uniform evidence models can alternatively be presented as topological models[3], though we no longer look at them with the point of view of the interior semantics – as all pieces of evidence are now available everywhere. At this point it may seem strange to keep speaking of "evidence models" since we are ultimately dealing with the same models as before, but the term remains appropriate to refer to the overall framework.

Here we fix a topological model (X, τ, ν) and focus our attention on the work conducted in [BBÖS22], which resorts to an approach based on justifications. Following what they see as a *coherentist* perspective, the authors define a justification as an argument which can not be defeated by the available evidence.

Definition 2.6. Two pieces of evidence U and V are said to be *consistent* (with each other) if $U \cap V \neq \varnothing$. Let $P \subseteq X$. An *argument* for P is a non-empty piece of evidence $U \subseteq P$. A *justification* for P is an argument U for P such that any non-empty piece of evidence is consistent with U.

Here the notion of argument is similar to that of Section 2.1. In topological terms, we see that a justification is merely a *dense argument* for P. To define belief, the authors build on the definition proposed in [vBP11] and propose a variant designed to avoid some of its shortcomings. They state that a proposition is believed if it is entailed by *sufficiently strong evidence*.

Definition 2.7. Let $P \subseteq X$. The set BP is defined by

$$x \in BP \iff \forall U \in \tau \setminus \{\varnothing\}, \exists V \in \tau \setminus \{\varnothing\}, V \subseteq U \cap P.$$

Note that the condition on the right-hand side does not depend on x, which means that BP is either the empty set or the whole space. In words, P is believed if every non-empty piece of evidence can be strengthened into an argument for P. We find it appropriate to call this condition the *strengthening condition*, and to stress its "interactive" flavour: for imagine we do not have a single agent pondering on the evidence alone, but two of them engaged in a dialogue and discussing their respective beliefs. Then the strengthening condition reads "for every argument you may oppose to me, I can find a stronger argument supporting my belief", an intuitive formulation that naturally captures the course of a debate. Then, we can prove that this definition is equivalent to the *existence of a justification* (factive or not).

Proposition 2.4. *Let $P \subseteq X$. We have $x \in BP$ iff $\exists U \in \tau, U \subseteq P$ and U is dense.*

This leads the authors to define knowledge as the *possession of some factive justification.*

Definition 2.8. Let $P \subseteq X$. The set KP is defined by

$$x \in KP \iff \exists U \in \tau, x \in U, U \subseteq P \text{ and } U \text{ is dense.}$$

So from this perspective, knowledge is seen as *correctly justified belief*, since the justification is required to be true. This definition is also motivated by arguments related to the Defeasibility Theory that we do not expand on here.

To summarize, we can see that this framework lends itself to natural and insightful conceptions of belief and knowledge, and what makes this possible is the ability to model false evidence. To understand why

[3] Actually the structures they consider are *topological evidence models*, namely topological models equipped with a subbase whose elements are called *basic* pieces of evidence, but this is of little interest here.

false evidence is so central, it suffices to observe what happens when we attempt to adapt justifications to the interior semantics. At a world x, the available pieces of evidence are exactly those factive at x, so they are all pairwise consistent, since they all intersect at x. Thus, *any* available piece of evidence would be a justification – making the concept of justification ultimately vacuous. Have we reached a hard limitation of the interior semantics? In the next section we will argue that this obstacle can be overcome, at the price of a change of perspective.

3 The notion of quasi-factive evidence

For the rest of the paper, we fix a topological space $\mathcal{X} = (X, \tau)$. In Section 2, we have defined the meaning of "the piece of evidence U is *available* at world x" in the interior semantics, before eventually finding it inadequate. The problem is that availability and factivity coincide exactly, an identity that was already dubious in the first place. Indeed, if U is a piece of evidence, the availability of U is a statement about the mind of the agent; whereas the factivity of U is a statement about the current state of affairs. There is, apparently, no reason to equate both, and this is leads us to design a new conception of availability that disentangles these two notions.

That said, availability and factivity should remain related to some extent, for surely the agent should see U as available only if they *think* that U is factive – as no one bases their beliefs on false evidence on purpose. Interestingly, our framework can handle situations in which U *looks* factive, without being factive. Those situations are exactly the elements of $\mathrm{Cl}(U)$, which are the worlds that are *epistemically indistinguishable* from U. This leads us to introduce the notion of *quasi-factive* evidence, the core concept of this paper.

Definition 3.1. A piece of evidence U is *quasi-factive* at world x if $x \in \mathrm{Cl}(U)$.

By the previous observation, we then see that a piece of evidence is quasi-factive at world x if and only if it is indistinguishable from a situation where it is factive at x, whence the term "quasi-factive". With this notion in hand, we now argue for a new point of view: we claim that the available evidence should be the *quasi-factive* one, so that it is possible to model false evidence that looks true. This allows us to bypass the difficulties mentioned at the end of Section 2.3, and to define a notion of justification relative to some world x, as an argument that is consistent with all quasi-factive pieces of evidence at x.

Definition 3.2. Let $P \subseteq X$. A *local justification* for P at world x is an argument U for P such that for every open set V, if $x \in \mathrm{Cl}(V)$ then $U \cap V \neq \varnothing$.

A first immediate observation is that knowledge is equivalent to the *possession of some factive local justification*, a condition analogous to the one we have in evidence models.

Proposition 3.1. *Let $P \subseteq X$. We have $x \in \mathrm{Int}(P)$ if and only if there exists an open set U such that $x \in U$ and U is a local justification for P at x.*

Proof. From left to right, take $U := \mathrm{Int}(P)$. By definition, if $x \in \mathrm{Cl}(V)$ then U intersects V, since U is an open neighbourhood of x. From right to left, $x \in U \subseteq P$ with U open yields $x \in \mathrm{Int}(P)$. □

Likewise, we propose a characterization of belief in the Cl-Int semantics in terms of quasi-factive evidence.

Proposition 3.2. *Let $P \subseteq X$. For all $x \in X$ we have*

$$x \in \mathrm{Cl}(\mathrm{Int}(P)) \iff \exists U \in \tau, x \in \mathrm{Cl}(U) \text{ and } U \subseteq P.$$

Proof. From left to right, simply take $U := \mathrm{Int}(P)$. From right to left, assume that $x \in \mathrm{Cl}(U)$ and $U \subseteq P$ for some open U. This entails $U \subseteq \mathrm{Int}(P)$ and thus $\mathrm{Cl}(U) \subseteq \mathrm{Cl}(\mathrm{Int}(P))$. Therefore $x \in \mathrm{Cl}(\mathrm{Int}(P))$. □

We have seen that knowing P can be interpreted as having some factive argument for P. Similarly, Proposition 3.2 tells us that *in the closure-interior semantics, the agent believes P if and only if they have some quasi-factive argument for P*. Moreover, it is now possible to give a meaningful description of ED spaces, which have been quite unintuitive so far. The following theorem states that they are exactly the spaces wherein at any world, the quasi-factive pieces of evidence are finite-wise consistent. Interestingly, it is in fact sufficient that they are *pairwise* consistent. In other words, ED spaces are the spaces in which the agent *never has conflicting information.*

Lemma 3.1. *Let $A \subseteq X$ and U be open. Then $U \cap \mathrm{Cl}(A) \subseteq \mathrm{Cl}(U \cap A)$.*

Proof. Let $x \in U \cap \mathrm{Cl}(A)$. Assume that $x \in V$ with V open. Then $U \cap V$ is open as well, and contains x, so $(U \cap V) \cap A \neq \varnothing$ by assumption. It follows that $V \cap (U \cap A) \neq \varnothing$, and thus $x \in \mathrm{Cl}(U \cap A)$. □

Theorem 3.1. *The following conditions are equivalent:*

1. *$\mathcal{X}$ is extremally disconnected,*
2. *For all $x \in X$ and U, V open such that $x \in \mathrm{Cl}(U) \cap \mathrm{Cl}(V)$, we have $U \cap V \neq \varnothing$,*
3. *For all $x \in X$ and $U_1, \ldots, U_n$ open such that $x \in \bigcap_{i=1}^n \mathrm{Cl}(U_i)$, we have $\bigcap_{i=1}^n U_i \neq \varnothing$.*

Proof. The direction from 3 to 2 is trivial. From 2 to 1, assume that for all $x \in X$ and U, V open, $x \in \mathrm{Cl}(U) \cap \mathrm{Cl}(V)$ implies $U \cap V \neq \varnothing$. Let U be an open set and suppose that there exists $x \in \mathrm{Cl}(U) \setminus \mathrm{Int}(\mathrm{Cl}(U))$. This entails $x \in \mathrm{Cl}(\mathrm{Int}(U^c))$, so by assumption we have $U \cap \mathrm{Int}(U^c) \neq \varnothing$ and in particular $U \cap U^c \neq \varnothing$, a contradiction. Therefore $\mathrm{Cl}(U) \subseteq \mathrm{Int}(\mathrm{Cl}(U))$, which means that $\mathrm{Cl}(U)$ is open.

From 1 to 3, assume that $\mathcal{X}$ is ED. We prove by induction on $n \in \mathbb{N}$ that $U_1, \ldots, U_n \in \tau$ and $x \in \bigcap_{i=1}^n \mathrm{Cl}(U_i)$ implies $\bigcap_{i=1}^n U_i \neq \varnothing$. For $n = 0$ this is just $x \in X \implies X \neq \varnothing$, which is obvious. Suppose that it holds for n and let $U_1, \ldots, U_{n+1} \in \tau$ be such that $x \in \bigcap_{i=1}^{n+1} \mathrm{Cl}(U_i)$. We know that $\mathrm{Cl}(U_n)$ and $\mathrm{Cl}(U_{n+1})$ are open, so applying Lemma 3.1 twice we obtain

$$x \in \mathrm{Cl}(U_n) \cap \mathrm{Cl}(U_{n+1}) \subseteq \mathrm{Cl}(\mathrm{Cl}(U_n) \cap U_{n+1}) \subseteq \mathrm{Cl}(\mathrm{Cl}(U_n \cap U_{n+1})) = \mathrm{Cl}(U_n \cap U_{n+1}).$$

Then we write $V := U_n \cap U_{n+1}$, apply the induction hypothesis to $U_1, \ldots, U_{n-1}, V$ and conclude. □

Using this theorem, we then prove an insightful characterization of belief: just as Proposition 3.1 describes knowledge as the possession of some factive local justification, we can describe belief in ED spaces as the *possession of some quasi-factive local justification.* This, again, is closely related to the definition of belief in evidence models (see Proposition 2.4).

Proposition 3.3. *Assume that $\mathcal{X}$ is ED and let $P \subseteq X$. We have $x \in \mathrm{Cl}(\mathrm{Int}(P))$ if and only if there exists an open set U such that $x \in \mathrm{Cl}(U)$ and U is a local justification for P at x.*

Proof. From left to right, assume that $x \in \mathrm{Cl}(\mathrm{Int}(P))$. We write $U := \mathrm{Int}(P)$ which is open. If $V \in \tau$ is such that $x \in \mathrm{Cl}(V)$, we have $U \cap V \neq \varnothing$ by Theorem 3.1. Therefore U is a local justification for P at x. The converse is trivial. □

4 The strengthening condition

Now that we have adapted the notion of justification and explored its properties, it remains to do the same with the strengthening condition. Interestingly, this condition will turn out to be an alternative formulation of the Int-Cl-Int semantics.

Definition 4.1. For all $P \subseteq X$, we define

$$\mathbf{B}(P) := \{x \in X \mid \forall U \in \tau, x \in \mathrm{Cl}(U) \implies \exists V \in \tau, x \in \mathrm{Cl}(V) \text{ and } V \subseteq U \cap P\}.$$

The elements of $\mathbf{B}P$ are the worlds at which *every quasi-factive piece of evidence can be strengthened into a quasi-factive argument for P.* The dual of $\mathbf{B}$ also deserves to be made explicit:

$$\hat{\mathbf{B}}(P) := (\mathbf{B}(P^c))^c = \{x \in X \mid \exists U \in \tau, x \in \mathrm{Cl}(U) \text{ and } \forall V \in \tau, (x \in \mathrm{Cl}(V) \text{ and } V \subseteq U) \implies V \cap P \neq \varnothing\}.$$

The elements of $\mathbf{B}P$ are thus the worlds at which *there exists some quasi-factive piece of evidence such that every quasi-factive stronger piece of evidence is consistent with P.*

Theorem 4.1. *For all* $P \subseteq X$ *we have* $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(P))) = \mathbf{B}P$.

Proof. Let $x \in \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(P)))$. Then there exists an open set U_0 such that $x \in U_0 \subseteq \mathrm{Cl}(\mathrm{Int}(P))$. Suppose that $x \in \mathrm{Cl}(U)$ for some open U, and define $V := U \cap \mathrm{Int}(P)$. Clearly V is open and $V \subseteq U \cap P$, and we also prove that $x \in \mathrm{Cl}(V)$. For consider an open neighbourhood W of x. Then $x \in W \cap U_0$ and $x \in \mathrm{Cl}(U)$, so there exists $y \in U \cap W \cap U_0$. Then since $U_0 \subseteq \mathrm{Cl}(\mathrm{Int}(P))$ we obtain $(U \cap W) \cap \mathrm{Int}(P) \neq \varnothing$, that is, $W \cap V \neq \varnothing$, as desired.

Conversely, assume that $x \in \mathbf{B}P$. Suppose that $x \notin \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(P)))$. Then $x \in \mathrm{Cl}(U)$ with $U := \mathrm{Int}(\mathrm{Int}(Q)^c)$ open. Then there exists an open set V such that $V \subseteq U \cap P$ and $x \in \mathrm{Cl}(V)$. Since V is open we have in fact $V \subseteq U \cap \mathrm{Int}(P)$. But then $U \subseteq \mathrm{Int}(P)^c$ yields $V = \varnothing$, contradicting $x \in \mathrm{Cl}(V)$. ☐

This result is intriguing, as it shows that two completely unrelated approaches ultimately lead to the same thing. The Int-Cl-Int semantics was introduced as a technical variant of the Cl-Int semantics, while the strengthening condition results from an attempt to import meaningful concepts from evidence models. Yet these two expressions turn out to be a single one, which thus inherits the best of each framework: it is a generalization of the Cl-Int semantics to all spaces, but also admits an intuitive formulation in terms of evidence. However, the mathematical side of it remains quite intricate, so we present two examples to illustrate the dynamics of strengthening.

Example 1. We propose here a variant of the broken clock of Russell. David is looking at not one but *two* clocks, and he has good reasons to believe that they are functional, out of faith in these devices in general. In fact, when one clock is working, it makes some typical noise, so David definitely knows that it is reliable. However he has never seen these clocks before, so if one is broken he is just going to assume that it is silent. Of course, if the two disagree then David immediately notices that something is wrong and remains uncertain of the current time. But there is also an extreme case of bad (or good) luck where the two are broken while showing the same *correct* time; then David has a true justified belief, but does not *know* the time. How do we model all these subtleties in our framework?

We construct the space of states as follows: first we introduce $H := \{n \in \mathbb{N} \mid 0 \leq n \leq 23\}$ the set of possible times (for convenience we abstract away from minutes); then a state is a tuple (t_1, c_1, t_2, c_2, t) with:

- $t_1 \in H$ the time shown by the first clock,
- $c_1 \in \{0, 1\}$ indicating whether the first clock is functional (0 means broken, 1 means not broken),
- $t_2 \in H$ the time shown by the second clock,
- $c_2 \in \{0, 1\}$ indicating whether the second clock is functional,
- $t \in H$ the actual time.

Of course, the meaning of c_1 (resp. c_2) imposes that whenever it has value 1, the values of t_1 (resp. t_2) and t agree. So the space is

$$X := \{(t_1, c_1, t_2, c_2, t) \in (H \times \{0,1\})^2 \times H \mid (c_1 = 1 \implies t_1 = t) \text{ and } (c_2 = 1 \implies t_2 = t)\}.$$

The pieces of evidence are then of two kinds. First, whatever time is shown by the clocks is known to David by direct observation, so for all $(n, m) \in H^2$ we introduce the piece of evidence

$$U_{n,m} := \{(t_1, c_1, t_2, c_2, t) \in X \mid t_1 = n \text{ and } t_2 = m\}.$$

Second, we must take into account the trust David has toward the clocks, and the confirmation given by the *tick tock* when they are working. This is represented by the pieces of evidence

$$U_1 := \{(t_1, c_1, t_2, c_2, t) \in X \mid c_1 = 1\}$$

and

$$U_2 := \{(t_1, c_1, t_2, c_2, t) \in X \mid c_2 = 1\}.$$

As intended, we see that when the first clock is functional, David knows it thanks to U_1, and that otherwise U_1 can still be quasi-factive and mislead him (the same applies to the second clock).

Now let τ be the topology generated by U_1, U_2 and all of of the $U_{n,m}$'s. We also introduce, for all $t \in H$, the atomic proposition p_t whose interpretation is given by $\nu(p_t) := \{(t_1, c_1, t_2, c_2, t') \in X \mid t' = t\}$, so that p_t reads "the actual time is t". Our model is then $\mathfrak{M} := (X, \tau, \nu)$. We fix a world $x = (t_1, c_1, t_2, c_2, t) \in X$ and consider several cases.

1. Suppose that one of the two clocks is not broken; without loss of generality we take $c_1 = 1$. Then U_1 and U_{t_1,t_2} are factive at x. Let $U := U_1 \cap U_{t_1,t_2}$ which is open. Any element of U is of the form $(t_1, 1, t_2, c_2', t')$, and by construction we then have $t' = t_1 = t$. Hence $U \subseteq [\![p_t]\!]^{\mathfrak{M}}$, and this proves that $\mathfrak{M}, x \vDash Kp_t$. In words, though David may notice a problem if $t_1 \neq t_2$, he stills knows that one clock is working and which one it is, and thus knows the time.
2. Otherwise both clocks are broken, that is, $c_1 = c_2 = 0$, but we suppose that $t_1 = t_2$. We show that $\mathfrak{M}, x \vDash Bp_{t_1}$. First we note that the only neighbourhoods of x are those including U_{t_1,t_2} – as by assumption U_1, U_2 and all of the other $U_{n,m}$'s do not contain x, and thus neither does any intersection of them. Now consider a quasi-factive piece of evidence U at x. We define

$$V := U_{t_1,t_2} \cap U_1 \cap U_2.$$

   In fact we see that V contains only $(t_1, 1, t_2, 1, t_1)$ – a valid state since we assume that $t_1 = t_2$ – and as a result $V \cap U_{t_1,t_2} \neq \varnothing$. Hence V is quasi-factive at x. We also see that $V \subseteq [\![p_t]\!]^{\mathfrak{M}}$, and it remains to show that $V \subseteq U$. By construction U is an union of intersections of sets among the $U_{n,m}$'s, U_1 and U_2. Since we have $U_{t_1,t_2} \cap U_{n,m} = \varnothing$ whenever $(n, m) \neq (t_1, t_2)$, we deduce that U includes an intersection of sets among U_{t_1,t_2}, U_1 and U_2. Hence $V \subseteq U$.
   This proves that David believes that the actual time is t_1, but even if it is true he does not know it: indeed, U_{t_1,t_2} also contains $(t_1, 0, t_2, 0, t')$ for any $t' \neq t$, so there is no factive argument for $[\![p_t]\!]^{\mathfrak{M}}$ at x.
3. Finally suppose that $c_1 = c_2 = 0$ and $t_1 \neq t_2$. We show that David holds no belief regarding the current time, that is, we prove $\mathfrak{M}, x \nvDash Bp_n$ for all $n \in H$. First, assume that $n \neq t_1$. Given $V_1 := U_1 \cap U_{t_1,t_2}$, we observe that $(t_1, 1, t_2, 0, t_1) \in V_1 \cap U_{t_1,t_2}$ and it follows that $x \in \mathrm{Cl}(V_1)$ (recall our observation regarding the neighbourhoods of x). We also have $V_1 \subseteq [\![p_{t_1}]\!]^{\mathfrak{M}}$, and so $V_1 \cap [\![p_n]\!]^{\mathfrak{M}} = \varnothing$. Therefore, V_1 can not be strengthened into a quasi-factive argument for p_n at x. If $n = t_1$, then $n \neq t_2$ by assumption so we can simply apply the same reasoning to $V_2 := U_2 \cap U_{t_1,t_2}$. This proves the claim.
   It is interesting to note that David has no knowledge of this disbelief. Indeed, the set U_{t_1,t_2} contains the world $y = (t_1, 1, t_2, 0, t_1)$, which falls under the first case we have covered: there David knows (and also believes) that the time is t_1. It follows that $\mathfrak{M}, x \vDash \neg K(\bigwedge_{n \in H} \neg Bp_n)$. This is an example of strong negative introspection failing to hold in the Int-Cl-Int semantics.
   We also observe that V_1 and V_2 are inconsistent with each other while both being quasi-factive at x. By Theorem 3.1, this implies that the space is *not* extremally disconnected.

The interested reader may also ponder the following variant.

Exercise 1. Now suppose that a working clock still makes some noise, but it is impossible for David to locate the source nor to estimate the intensity of it. Thus, if at least one of the clocks is functional, then David knows it, but he can not tell whether this is the first one, the second one or both. Make the appropriate changes to the previous model to depict this new situation, and determine David's epistemic attitude in each relevant case.

Example 2. King Angus of Epistemia is in trouble: words have reached him that his close relatives and counselors are conspiring to end his reign. The rumours spreading rapidly, his entire court is soon aware of the conspiracy, and an atmosphere of worry starts to settle. Resolved to expose the members of this outrageous machination, the king sends his dedicated agents to investigate. His suspicions are directed toward:

- Prince Hamish, his son and heir,
- Queen Iona, his wife,
- Chancellor Tancred,
- Bishop Morgan.

King Angus first visits Bishop Morgan and asks for his opinion on these matters. The cleric has not heard of anything suspicious, but warmly ensures the innocence of Prince Hamish. Being his mentor, he has indeed a close eye on his activities. Hamish, he claims, could not have concealed such a plot from him.

Reassured, the king decides to meet with his son who, upon learning the news, shares his mistrust of Chancellor Tancred. "Do not let appearances deceive you: this scoundrel has business with questionable people," the prince reveals with hostility. "I know everything about his scheming. If anyone must be involved in this plot, this is him!" Intrigued, King Angus summons the chancellor, who denies the accusation and in turn blames the ambition of the prince. "Your Highness, I have watched your son for some time and his plans are now clear as day. He wants the crown for himself!"

In the meantime, one of the king's agents has managed to sneak into Chancellor Tancred's office. He reports having found compromising documents proving the involvement of the chancellor. In addition, a letter from him and addressed to another co-conspirator[4] contains the words "make sure that the queen does not notice anything", implying that Iona does not participate in their plan. King Angus is now confident in Tancred's guilt and has a good reason to think that Queen Iona is innocent, but what he ignores is that all the evidence is made up and that the spy had actually been paid by Hamish. The prince wanted to have Tancred charged for the plot, but also to protect his mother from suspicion.

Shortly after, a second agent reports terrible news: he discreetly listened to a discussion between Queen Iona and Chancellor Tancred, and not only they appear to be secretly lovers, they are also together part of the conspiracy. But upon revealing his discovery, his triumphant face fades out before King Angus' incredulous look: the monarch now has conflicting evidence about his wife! The spy has, in fact, jumped to his conclusion a bit too fast: while the adulterous affair is true, he is mistaken about the intention of the lovers. They were just, out of caution, planning to avoid each other until the situation gets resolved, but their whispers have apparently been misheard.

From the beginning, the actual conspirators were Prince Hamish and Bishop Morgan. But what is the king's state of mind? To answer this question, we denote by $\mathbf{C} := \{H, I, M, T\}$ the set of characters. The epistemic situation is represented by the space $X := \mathcal{P}(\mathbf{C})$. Every world $x \in X$ is defined by a possible set of characters involved in the conspiracy. For each character $C \in \mathbf{C}$, we introduce an atomic proposition p_C and we set $\nu(p_C) := \{x \in X \mid C \in x\}$, so that p_C reads "character C is a conspirator". The actual world is $\{H, M\}$ – written HM for short – and the topology τ is generated by four pieces of evidence:

- After his discussion with the bishop, King Angus knows that if Morgan is innocent, then his statement is reliable, and thus Hamish is innocent too. This means that $U_1 := [\![\neg p_M \to \neg p_H]\!]$ is open.
- Because Hamish and Tancred are accusing each other[5], King Angus knows that one of them is guilty, but not both, and this provides the open set $U_2 := [\![p_H \leftrightarrow \neg p_T]\!]$.
- The documents found in Tancred's affairs are represented by the open set $U_3 := [\![p_T \wedge \neg p_I]\!]$.
- The testimony of the second agent is represented by the open set $U_4 := [\![p_T \wedge p_I]\!]$.

The space is pictured in Figure 1. Observe that U_1 and U_2 are factive at world HM, while U_3 and U_4 are only quasi-factive. Take also note that U_3 and U_4 are inconsistent with each other, so once again the space is not extremally disconnected. We write $\mathfrak{M} := (X, \tau, \nu)$ and we examine the opinion of King Angus about each of the p_C's:

H. We have $\mathfrak{M}, HM \vDash B\neg p_H$. Indeed, assume that $HM \in \mathrm{Cl}(U)$ with U open. Writing $V := \{T, MT\}$ and $W := \{IT, IMT\}$, we can check that in any case, we have either $V \subseteq U$ or $W \subseteq U$. In addition V and W are open and included in $[\![\neg p_H]\!]^{\mathfrak{M}}$. We also have $HM \in \mathrm{Cl}(V)$ and $HM \in \mathrm{Cl}(W)$. Therefore $HM \in \mathbf{B}[\![\neg p_H]\!]^{\mathfrak{M}}$.

T. We have $\mathfrak{M}, HM \vDash Bp_T$ and the reasoning is similar: if $HM \in \mathrm{Cl}(U)$ with U open, either $V :=$ $\{IT, IMT\}$ or $W := \{IT, IMT\}$ satisfies the conditions and thus $HM \in \mathbf{B}[\![p_T]\!]^{\mathfrak{M}}$.

M. We have $\mathfrak{M}, HM \vDash K(\hat{B}p_M) \wedge K(\hat{B}\neg p_M)$.
Indeed, we have $x \in U_1$ and we show that $U_1 \subseteq \hat{\mathbf{B}}[\![p_M]\!]^{\mathfrak{M}}$ (resp. $U_1 \subseteq \hat{\mathbf{B}}[\![\neg p_M]\!]^{\mathfrak{M}}$). Let $y \in U_1$. One can see that y belongs to $\mathrm{Cl}(U)$ with either $U := \{T, MT\}$ or $U := \{IT, IMT\}$. In both cases U is

[4] For simplicity the recipient of the letter is none of the other primary characters.

[5] We assume that their accusations are either correct or made up – no honest mistake – and that conspirators do not betray each other.

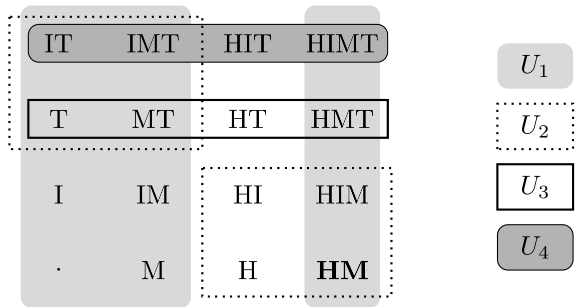

Figure 1: The model of the situation: all possible worlds with four subsets highlighted, corresponding to the four basic pieces of evidence.

open and any open subset $V \subseteq U$ is U itself, which intersects $[\![p_M]\!]^{\mathfrak{M}}$ (resp. $[\![\neg p_M]\!]^{\mathfrak{M}}$).

I. We have $\mathfrak{M}, HM \vDash \hat{B}p_I \wedge \neg K\hat{B}p_I \wedge \hat{B}\neg p_I \wedge \neg K\hat{B}\neg p_I$. Here we only prove $\mathfrak{M}, HM \vDash \hat{B}p_I \wedge \neg K\hat{B}p_I$ and the other two conjuncts are addressed similarly.
We first show that $\mathfrak{M}, HM \vDash \hat{B}p_I$. For consider $U := \{IT, IMT\}$. Then U is open and $HM \in \mathrm{Cl}(U)$. Again, any open subset $V \subseteq U$ is U itself and intersects $[\![p_I]\!]^{\mathfrak{M}}$, so $HM \in \hat{\mathbf{B}}[\![p_I]\!]^{\mathfrak{M}}$.
Now, assume toward a contradiction that there exists an open set W such that $HM \in W$ and $W \subseteq \hat{\mathbf{B}}[\![p_I]\!]^{\mathfrak{M}}$. The reader can check that in any case, $T \in W$, and that $\mathfrak{M}, T \vDash B\neg p_I$, whence $\mathfrak{M}, T \nvDash \hat{B}p_I$. This contradicts $W \subseteq \hat{\mathbf{B}}[\![p_I]\!]^{\mathfrak{M}}$. Therefore $\mathfrak{M}, HM \vDash \neg K\hat{B}p_I$.

To sum everything up: the king believes that Tancred is a plotter due to the fake documents and the report of the spy. Therefore, Prince Hamish was apparently sincere when he denounced the chancellor, and his father now has no doubt about his loyalty. This makes initial Morgan's claim consistent, but does not prove his innocence: it is plausible for King Angus that he is guilty, but also that he is not guilty, and the monarch knows that. The case of his wife is more complex, and makes an other interesting counter-example to strong negative introspection: the available information makes plausible both her guilt and her innocence, but due to the conflicting evidence the king somehow does not know that. In any case, notice how brilliantly Prince Hamish has managed to deceive his father.

5 The logic wKB

The results of the previous section give support to the Int-Cl-Int semantics as the right semantics for belief. But now that we have made this case, there remains to provide an axiomatization of it. Prior to this work, we only had the logic **wKD45**, which is limited to the language of belief only. Of course, one could always extend the logic **S4** for knowledge with the axiom $Bp \leftrightarrow K\hat{K}Kp$, but that would be of little interest. Instead, we want a logic in which this equivalence can be derived from first principles that are intuitive, or that can be compared to other existing axioms.

In this section, we thus present our logic **wKB**, whose name stands for "weak **KB**", and show that it axiomatizes the Int-Cl-Int semantics. The logic **wKB** relates to **KB** just as **wKD45** relates to **KD45**: one is a variant of the other with a weaker version of negative introspection. Formally, **wKB** is the extension of the logic **S4** for knowledge with the axioms presented in Table 3. We see that the axiom SNI, which is known to fail in the Int-Cl-Int semantics, is weakened into the axiom WNI, which is actually very similar to w5. Further, the axiom D is replaced by a stronger version SD, but this is merely a compensation for the previous weakening, since SD could already be derived in **KB** from the axioms D and SNI. Finally, the axiom SPI has disappeared but as we will see in a moment, it can be inferred from the others. First, let us show how these axioms can be derived in **KB**.

Proposition 5.1. *The axioms and inference rules of* **wKB** *are derivable in* **KB**.

Table 3: Axioms for belief in **wKB**

Name	Axiom	Description
SD	$Bp \to K\neg B\neg p$	Strong consistency of belief
WNI	$\neg Bp \to \hat{B}K\neg Bp$	Weak negative introspection for belief
KB	$Kp \to Bp$	Knowledge implies belief
FB	$Bp \to BKp$	Subjective certainty

Proof. The axiom SD is an immediate consequence of D and SNI. The axiom WNI is easily obtained as follows.

(i) $\neg Bp \to K\neg Bp$ — SNI
(ii) $K\neg Bp \to KK\neg Bp$ — by 4
(iii) $KK\neg Bp \to BK\neg Bp$ — by KB
(iv) $BK\neg Bp \to \hat{B}K\neg Bp$ — by D
(v) $\neg Bp \to \hat{B}K\neg Bp$ — by (i),(ii),(iii),(iv) and the properties of $\to$

□

We then prove our main claim.

Proposition 5.2. *The equivalence $Bp \leftrightarrow K\hat{K}Kp$ is derivable in* **wKB**.

Proof. From left to right:

(i) $Bp \to BKp$ — FB
(ii) $BKp \to K\hat{B}Kp$ — by SD
(iii) $\hat{B}Kp \to \hat{K}Kp$ — by contraposition of KB
(iv) $K(\hat{B}Kp \to \hat{K}Kp)$ — by (iii) and necessitation
(v) $K\hat{B}Kp \to K\hat{K}Kp$ — by (iv) and K
(vi) $Bp \to K\hat{K}Kp$ — by (i), (ii), (v) and the properties of $\to$

From right to left:

(i) $\neg Bp \to \neg Kp$ — by contraposition of KB
(ii) $K(\neg Bp \to \neg Kp)$ — by (i) and necessitation
(iii) $K(\neg Bp) \to K(\neg Kp)$ — by (ii) and K
(iv) $\hat{K}Kp \to \hat{K}Bp$ — by contraposition of (iii)
(v) $K(\hat{K}Kp \to \hat{K}Bp)$ — by (iv) and necessitation
(vi) $K\hat{K}Kp \to K\hat{K}Bp$ — by (v) and K
(vii) $K\hat{K}Bp \to B\hat{K}Bp$ — by FB
(viii) $B\hat{K}Bp \to Bp$ — by contraposition of WNI
(ix) $K\hat{K}Kp \to Bp$ — by (vi), (vii), (viii) and the properties of $\to$

□

This result immediately makes the axiom SPI derivable in **wKB**, since from axiom 4 we have **wKB** $\vdash K\hat{K}Kp \to KK\hat{K}Kp$. It also yields the desired theorem of soundness and completeness.

Theorem 5.1. *In the Int-Cl-Int semantics,* **wKB** *is sound and complete for the class of all topological spaces.*

Proof. Follows from Theorem 2.1 and Proposition 5.2. □

With this result, we can then easily prove that the doxastic fragment of **wKB** is **wKD45**, just as **KD45** is the doxastic fragment of **KB** (see Theorem 2.4).

Theorem 5.2. *For all $\varphi \in \mathcal{L}_B$, we have* **wKB** $\vdash \varphi$ *if and only if* **wKD45** $\vdash \varphi$.

Proof. Let $\varphi \in \mathcal{L}_B$. In the Int-Cl-Int semantics we have $\mathbf{wKB} \vdash \varphi$ iff $\vDash \varphi$ by Theorem 5.1, and $\mathbf{wKD45} \vdash \varphi$ iff $\vDash \varphi$ by Theorem 2.6. This proves the claim. □

6 Comparing plausibility

The belief operator $\mathbf{B}$ can be seen as a unary predicate over propositions: at a world x, it classifies P as a belief or a non-belief according to whether $x \in \mathbf{B}P$ or $x \notin \mathbf{B}P$. Yet the epistemic landscape is richer than that: not all propositions are equally disbelieved, and one typically sorts them by *relative plausibility*, that is, via statements of the form "P is more likely than Q". Amazingly, our framework is already equipped to capture this notion. In fact, it naturally emerges from formal considerations only, and so independently of any philosophical motivation. To see why, let us give a closer look at the strengthening condition. Given $Q \subseteq X$, we have:

$$x \in \mathbf{B}Q \ \text{ iff } \ \forall U \in \tau, (x \in \mathrm{Cl}(U) \implies \exists V \in \tau, V \subseteq U, x \in \mathrm{Cl}(V) \text{ and } V \subseteq Q).$$

One can notice a lack of symmetry between the antecedent and the consequent of the implication: the left-hand side says "U is quasi-factive", while the right-hand side says "V is quasi-factive and is an argument for Q". A natural variant would be a version where U is an argument for some proposition as well. This observation thus gives birth to a binary relation parameterized by x and defined by:

$$P \preceq_x Q \ \text{ iff } \ \forall U \in \tau, (x \in \mathrm{Cl}(U) \text{ and } U \subseteq P \implies \exists V \in \tau, V \subseteq U, x \in \mathrm{Cl}(V) \text{ and } V \subseteq Q).$$

This reads "at world x, any quasi-factive argument for P can be strengthened into a quasi-factive argument for Q" or, in plain English, "for every reason to believe P, there is a stronger reason to believe Q" – which really just states that Q is at least as plausible as P. Accordingly, we dub $\preceq_x$ the *plausibility preorder* associated to x. Just as $\mathbf{B}$ coincides with the Int-Cl-Int operator, we can prove that $\preceq_x$ is modally definable. From now on we fix a world $x \in X$, and given $P, Q \subseteq X$ we write $P \Rightarrow Q := P^c \cup Q$.

Theorem 6.1. *For all $P, Q \subseteq X$ we have $P \preceq_x Q$ iff $x \in \mathrm{Int}(\mathrm{Int}(P) \Rightarrow \mathrm{Cl}(\mathrm{Int}(Q)))$.*

Proof. Suppose that $x \in \mathrm{Int}(\mathrm{Int}(P) \Rightarrow \mathrm{Cl}(\mathrm{Int}(Q)))$. Then there exists and open set U_0 such that $x \in U_0 \subseteq \mathrm{Int}(P) \Rightarrow \mathrm{Cl}(\mathrm{Int}(Q))$. Suppose that $x \in \mathrm{Cl}(U)$ for some open $U \subseteq P$, and define $V := U \cap \mathrm{Int}(Q)$. Clearly V is open and $V \subseteq U \cap Q$, and we also prove that $x \in \mathrm{Cl}(V)$. For consider an open neighbourhood W of x. Then $x \in W \cap U_0$ and $x \in \mathrm{Cl}(U)$, so there exists $y \in U \cap W \cap U_0$. Then since $U \subseteq \mathrm{Int}(P)$ we have $U \cap U_0 \subseteq \mathrm{Cl}(\mathrm{Int}(Q))$, and it follows that $(U \cap W) \cap \mathrm{Int}(Q) \neq \varnothing$, that is, $W \cap V \neq \varnothing$, as desired.

Conversely, assume that $P \preceq_x Q$. Suppose that $x \notin \mathrm{Int}(\mathrm{Int}(P) \Rightarrow \mathrm{Cl}(\mathrm{Int}(Q)))$. Then $x \in \mathrm{Cl}(U)$ with $U := \mathrm{Int}(P) \cap \mathrm{Int}(\mathrm{Int}(Q)^c)$ open, and we also have $U \subseteq P$. Then there exists an open set V such that $V \subseteq U \cap Q$ and $x \in \mathrm{Cl}(V)$. Since V is open we have in fact $V \subseteq U \cap \mathrm{Int}(Q)$. But then $U \subseteq \mathrm{Int}(Q)^c$ yields $V = \varnothing$, contradicting $x \in \mathrm{Cl}(V)$. This concludes the proof. □

As we can see, the expression $\mathrm{Int}(\mathrm{Int}(P) \Rightarrow \mathrm{Cl}(\mathrm{Int}(Q)))$ only involves the closure and interior operators, which means that it can be translated in the modal language. It follows that, given $\varphi, \psi \in \mathcal{L}_{KB}$, the formula $K(K\varphi \to \hat{K}K\psi)$ expresses the assertion "ψ is at least as plausible as φ".

Next, we explore further the properties of $\preceq_x$.

Proposition 6.1. *Let $P, Q \subseteq X$. Then:*

1. *$\preceq_x$ is a preorder,*
2. *$\varnothing \preceq_x Q$,*
3. *$P \preceq_x X$,*
4. *$\mathrm{Int}(P) \preceq_x P$ and $P \preceq_x \mathrm{Int}(P)$,*
5. *if $P \subseteq Q$ then $P \preceq_x Q$,*
6. *$x \in \mathbf{B}P$ if and only if $X \preceq_x P$,*
7. *if $x \in \mathbf{B}(P \Rightarrow Q)$ then $P \preceq_x Q$ (but the converse is not true in general),*

8. *$P \cap Q$ is a $\preceq_x$-infimum of P and Q,*
9. *$\mathrm{Int}(P) \cup \mathrm{Int}(Q)$ is a $\preceq_x$-supremum of P and Q.*

Proof.

1. Reflexivity is trivial. For transitivity, assume that $P \preceq_x Q$ and $Q \preceq_x R$. Let U be open and assume that $x \in \mathrm{Cl}(U)$ and $U \subseteq P$. Then there exists V open such that $x \in \mathrm{Cl}(V)$ and $V \subseteq U \cap Q$. Then there exists W open such that $x \in \mathrm{Cl}(W)$ and $W \subseteq V \cap R$. All in all we have $x \in \mathrm{Cl}(W)$ and $W \subseteq U \cap R$. Therefore $P \preceq_x R$.
2. Trivial.
3. Trivial.
4. Stems from the simple fact that if U is open we have $U \subseteq P \iff U \subseteq \mathrm{Int}(P)$.
5. Trivial.
6. Trivial.
7. Assume that $x \in \mathbf{B}(P \Rightarrow Q)$. Suppose that U is open and that $x \in \mathrm{Cl}(U)$ and $U \subseteq P$. By assumption there exists V open such that $x \in \mathrm{Cl}(V)$ and $V \subseteq U \cap (P \Rightarrow Q)$, whence $V \subseteq Q$. All in all we have $x \in \mathrm{Cl}(V)$ and $V \subseteq U \cap Q$. Therefore $P \preceq_x Q$.
   To prove that the converse fails, consider $X := \{0, 1, 2\}$ with $\tau := \{\varnothing, \{1, 2\}, X\}$, $x := 0$, $P := \{1\}$ and $Q := \varnothing$, and check that $P \preceq_x Q$ but $x \notin \mathbf{B}(P \to Q)$.
8. Since $P \cap Q \subseteq P$ we have $P \cap Q \preceq_x P$ by Item 5, and likewise $P \cap Q \preceq_x Q$.
   Now let $R \subseteq X$ be such that $R \preceq_x P$ and $R \preceq_x Q$. We show that $R \preceq_x P \cap Q$. Let U be open and assume that $x \in \mathrm{Cl}(U)$ and $U \subseteq R$. Then there exists V open such that $x \in \mathrm{Cl}(V)$ and $V \subseteq U \cap P$. Since $V \subseteq U \subseteq R$ there also exists W open such that $W \subseteq V \cap Q$. All in all we have $x \in \mathrm{Cl}(W)$ and $W \subseteq U \cap (P \cap Q)$.
9. First $P \preceq_x \mathrm{Int}(P)$ by Item 4. In addition $\mathrm{Int}(P) \subseteq \mathrm{Int}(P) \cup \mathrm{Int}(Q)$, so $\mathrm{Int}(P) \preceq_x \mathrm{Int}(P) \cup \mathrm{Int}(Q)$ by Item 5. Thus, by Item 1, we have $P \preceq_x \mathrm{Int}(P) \cup \mathrm{Int}(Q)$. Similarly we prove that $Q \preceq_x \mathrm{Int}(P) \cup \mathrm{Int}(Q)$.
   Now let $R \subseteq X$ be such that $P \preceq_x R$ and $Q \preceq_x R$. We show that $\mathrm{Int}(P) \cup \mathrm{Int}(Q) \preceq_x R$. Let U be open and assume that $x \in \mathrm{Cl}(U)$ and $U \subseteq \mathrm{Int}(P) \cup \mathrm{Int}(Q)$. We define $U_1 := U \cap \mathrm{Int}(P)$ and $U_2 := U \cap \mathrm{Int}(Q)$ which are open. We have $U_1 \cup U_2 = U$ and thus $\mathrm{Cl}(U_1) \cup \mathrm{Cl}(U_2) = \mathrm{Cl}(U)$, and thus either $x \in \mathrm{Cl}(U_1)$ or $x \in \mathrm{Cl}(U_2)$.
   If $x \in \mathrm{Cl}(U_1)$, then since $U_1 \subseteq P$ and $P \preceq_x R$ there exists an open set V such that $x \in \mathrm{Cl}(V)$, and $V \subseteq U_1 \cap R$. In particular we have $V \subseteq U$. If $x \in \mathrm{Cl}(U_2)$ we apply the same reasoning, using the assumption that $Q \preceq_x R$.

□

Item 1 shows that $\preceq_x$ is indeed a preorder. Item 6 shows that $\mathbf{B}$ can be easily defined from $\preceq_x$. Item 7 gives a sufficient condition for $P \preceq_x Q$ involving $\mathbf{B}$. Since this condition is not necessary, we still lack a characterization of relative plausibility in terms of belief.

We can then lift the preorder $\preceq_x$ to an order in a straightforward way. If $P \preceq_x Q$ and $Q \preceq_x P$, we say that P and Q are *equiplausible* at x and we write $P \sim_x Q$. We denote by $[P]_x$ the equivalence class of P with respect to $\sim_x$, and by $\mathbf{L}_x := \mathcal{P}(X)_{/\sim_x}$ the induced quotient set. We then define the binary relation $\leq_x$ over $\mathbf{L}_x$ by

$$[P]_x \leq_x [Q]_x \text{ iff } P \preceq_x Q$$

and as it is well-known, $\leq_x$ is well-defined and is a partial order. We call it the *plausibility order* at x. The following lemma shows that $\sim_x$ is a congruence with respect to well-chosen operations.

Lemma 6.1. *Suppose that $P \sim_x P'$ and $Q \sim_x Q'$. Then $P \cap Q \sim_x P' \cap Q'$ and $\mathrm{Int}(P) \cup \mathrm{Int}(Q) \sim_x \mathrm{Int}(P') \cup \mathrm{Int}(Q')$.*

Proof. We know that $P \cap Q \preceq_x P$ and $P \cap Q \preceq_x Q$, whence $P \cap Q \preceq_x P'$ and $P \cap Q \preceq_x Q'$. By Proposition 6.1, Item 8, $P' \cap Q'$ is an infimum of P' and Q', so we obtain $P \cap Q \preceq_x P' \cap Q'$. Likewise we prove that $P' \cap Q' \preceq_x P \cap Q$. Therefore $P \cap Q \sim_x P' \cap Q'$. The same reasoning applies to show that $\mathrm{Int}(P) \cup \mathrm{Int}(Q) \sim_x \mathrm{Int}(P') \cup \mathrm{Int}(Q')$, using the items 4 and 9 of Proposition 6.1, □

Lemma 6.1 allows us to define the operations $\sqcup$ and $\sqcap$ by

$$\begin{aligned} [P]_x \sqcap [Q]_x &:= [P \cap Q]_x, \\ [P]_x \sqcup [Q]_x &:= [\mathrm{Int}(P) \cup \mathrm{Int}(Q)]_x. \end{aligned}$$

We also introduce the constants $0 := [\varnothing]_x$ and $1 := [X]_x$. Notice that, by Proposition 6.1, items 3 and 6, 1 is exactly the set of all beliefs at world x. We can then prove that $\mathbf{L}_x$ is actually a lattice.

Theorem 6.2. *The partially ordered set* $(\mathbf{L}_x, \leq_x)$ *induces a bounded distributive lattice* $(\mathbf{L}_x, \sqcap, \sqcup, 0, 1)$.

Proof. Most of the sub-claims of this theorem are straightforward consequences of Proposition 6.1 and Lemma 6.1. Here we only check distributivity. For distributivity of $\sqcup$ over $\sqcap$, we have

$$\begin{aligned} [P]_x \sqcup ([Q]_x \sqcap [R]_x) &= [P]_x \sqcup [Q \cap R]_x \\ &= [\mathrm{Int}(P) \cup (\mathrm{Int}(Q) \cap \mathrm{Int}(R))]_x \\ &= [(\mathrm{Int}(P) \cup \mathrm{Int}(Q)) \cap (\mathrm{Int}(P) \cup \mathrm{Int}(R))]_x \\ &= [\mathrm{Int}(P) \cup \mathrm{Int}(Q)]_x \sqcap [\mathrm{Int}(P) \cup \mathrm{Int}(R)]_x \\ &= ([P]_x \sqcup [Q]_x) \sqcap ([P]_x \sqcup [R]_x). \end{aligned}$$

For distributivity of $\sqcap$ over $\sqcup$, Proposition 6.1, Item 4 gives us $[P]_x = [\mathrm{Int}(P)]_x$, whence:

$$\begin{aligned} [P]_x \sqcap ([Q]_x \sqcup [R]_x) &= [\mathrm{Int}(P)]_x \sqcap ([Q]_x \sqcup [R]_x) \\ &= [\mathrm{Int}(P)]_x \sqcap [\mathrm{Int}(Q) \cup \mathrm{Int}(R)]_x \\ &= [\mathrm{Int}(P) \cap (\mathrm{Int}(Q) \cup \mathrm{Int}(R))]_x \\ &= [(\mathrm{Int}(P) \cap \mathrm{Int}(Q)) \cup (\mathrm{Int}(P) \cap \mathrm{Int}(R))]_x \\ &= [\mathrm{Int}(P \cap Q) \cup \mathrm{Int}(P \cap R)]_x \\ &= [P \cap Q]_x \sqcup [P \cap R]_x \\ &= ([P]_x \sqcap [Q]_x) \sqcup ([P]_x \sqcap [R]_x). \end{aligned}$$

□

7 Conclusion

We have introduced the notion of quasi-factivity in order to properly address false evidence in the interior semantics, a notion already present in evidence models. Doing so allowed us to compare the semantics for knowledge and belief of both frameworks, and to exhibit deep similarities that could not be suspected at a first glance. In Table 4 and Table 5 we summarize the existing semantics and their intuitive characterization in terms of pieces of evidence. We find it convenient to merge the two into a single tabular (Table 6) in order to highlight the connections we have mentioned.

Table 4: Semantics in evidence models

Description	**Semantics**	**Reference**
Possession of a *factive* justification	Knowledge	Definition 2.8
Possession of a justification	Belief	Proposition 2.4
Strengthening condition		Definition 2.7

Our main result is certainly the equivalence between the Int-Cl-Int semantics and the strengthening condition, which corresponds to a notion of belief as an *undefeatable claim in a dialogue*: believing P means being able to reply to any argument with a stronger argument supporting P. This brings a fresh perspective on the Int-Cl-Int semantics and shows that it succeeds at capturing belief in an intuitive manner. We hope that this work will allow it to stand as a legitimate semantics for belief – as opposed to a mere curiosity.

Table 5: Interior-related semantics

Description	**Semantics**	**Reference**
Possession of a *factive* local justification	Knowledge	Proposition 3.1
Possession of a *quasi-factive* local justification	Belief (in extremally disconnected spaces)	Proposition 3.3
Strengthening condition	Belief	Theorem 4.1

Table 6: The semantics of the two frameworks compared

Description	**Interior-related semantics**	**Evidence models**
Possession of a *factive* (local) justification	Knowledge	Knowledge
Possession of a (*quasi-factive* local) justification	Belief (in extremally disconnected spaces)	Belief
Strengthening condition	Belief	

More generally, our work shows that various semantics are ultimately characterized by the same definition sketches. The essential differences only lie in the definition of availability: once this information is provided, local justifications and the strengthening condition immediately arise. These observations suggest that there is something "universal" about these notions, which may motivate further attempts to adapt them to existing or future frameworks.

Another important contribution is Theorem 3.1, which offers a meaningful description of extremally disconnected spaces as those where the available evidence is consistent. In addition to clarifying the difference between the Cl-Int and Int-Cl-Int semantics, this result lifts the conceptual fog surrounding Question 1, which can now be rephrased as "Do all epistemic situations involve consistent evidence?". Experience clearly does not suggest that, and we have seen in Section 4 two fairly "real-life-like" situations depicted by non-extremally disconnected spaces. A negative answer thus seems to impose itself.

Furthermore, we have designed the logic **wKB** for the Int-Cl-Int semantics, under the form of a variant of **KB** with weaker negative introspection. Recall that **wKB** is sound and complete for the class of all spaces, and that **KB** is sound and complete for the class of ED spaces. This means that the frontier separating strong and weak negative introspection is crossed exactly when evidence becomes (potentially) inconsistent. This fact may help to better understand the axiom WNI, which for the moment remains rather obscure. To this end, examples 1 and 2 may serve as "basic material" for the discussion, as they both feature a failure of strong negative introspection.

Finally, we have generalized the strengthening condition to a plausibility preorder between propositions. We have shown that this preorder is definable in terms of closure and interior, and that it induces a lattice, but there is much more to investigate. Degrees of certainty are found naturally in quantitative approaches (e.g. Bayesian epistemology), but have received surprisingly little attention in qualitative approaches. While relative plausibility has been addressed by a few proposals [GdJ13], the state of the art still makes it appear as a side question, and not as a central aspect of epistemic reasoning. In addition, the existing work mainly relies on *ad hoc* structures like orderings of possible worlds. By contrast, our approach lets relative plausibility emerge 'on its own' in topological models – which are already well understood – and thus opens a compelling direction for future work.

A Appendix: general topology

Definition A.1. Let X be a set. A *topology* on a X is a collection of sets $\tau \subseteq \mathcal{P}(X)$ such that:

- τ contains $\varnothing$ and X,
- τ is closed under arbitrary unions, i.e. if $(U_i)_{i\in I} \in \tau^I$, then $\bigcup_{i\in I} U_i \in \tau$,
- τ is closed under finite intersections, i.e. for all $U \in \tau$ and $V \in \tau$, we have $U \cap V \in \tau$.

The pair (X, τ) is then said to be a *topological space*, and elements of τ are said to be *open* in X.

Alternatively, a topology can be presented by a generating family, also called a *subbase*:

Definition A.2. Let $B \subseteq \mathcal{P}(X)$ be a collection of subsets of X. The topology *generated* by B is

$$\tau := \left\{ \bigcup_{i\in I} \bigcap_{j=1}^{n_i} U_j^i \;\middle|\; I \text{ is a set and for all } i \in I, U_1^i, \ldots, U_{n_i}^i \in B \right\}.$$

This is the smallest topology containing the elements of B.

A subset is called *closed* if its complement is open. A subset $A \subseteq X$ is called a *neighbourhood* of a point $x \in X$ if there exists an open set U such that $x \in U$ and $U \subseteq A$.

Definition A.3. Let $Y \subseteq X$. The *subspace topology* on Y is defined by $\tau_Y := \{U \cap Y \mid U \in \tau\}$. The pair (Y, τ_Y) is a topological space and is called a *subspace* of (X, τ).

Two crucial topological notions are the interior and closure operators.

Definition A.4. Let $A \subseteq X$. The *interior* $\mathrm{Int}(A)$ of A is the set of all points of which A is a neighbourhood:

$$\mathrm{Int}(A) := \{x \in X \mid \exists U \in \tau,\ x \in U \subseteq A\}.$$

Definition A.5. Let $A \subseteq X$. The *closure* $\mathrm{Cl}(A)$ of A is the set of all points whose every open neighbourhood intersects A:

$$\mathrm{Cl}(A) := \{x \in X \mid \forall U \in \tau,\ x \in U \implies U \cap A \neq \varnothing\}.$$

We summarize in Table 7 a collection of standard properties of these operators, that we extensively use in this article.

Table 7: Classical properties of the interior and closure operators

Interior	Closure
$\mathrm{Int}(A)$ is the largest open set contained in A	$\mathrm{Cl}(A)$ is the smallest closed set containing A
A is open if and only if $\mathrm{Int}(A) = A$	A is closed if and only if $\mathrm{Cl}(A) = A$
$\mathrm{Int}(\varnothing) = \varnothing$	$\mathrm{Cl}(\varnothing) = \varnothing$
$\mathrm{Int}(X) = X$	$\mathrm{Cl}(X) = X$
$\mathrm{Int}(\mathrm{Int}(A)) = \mathrm{Int}(A)$	$\mathrm{Cl}(\mathrm{Cl}(A)) = \mathrm{Cl}(A)$
$A \subseteq B$ implies $\mathrm{Int}(A) \subseteq \mathrm{Int}(B)$	$A \subseteq B$ implies $\mathrm{Cl}(A) \subseteq \mathrm{Cl}(B)$
$\mathrm{Int}(A) \cap \mathrm{Int}(B) = \mathrm{Int}(A \cap B)$	$\mathrm{Cl}(A) \cup \mathrm{Cl}(B) = \mathrm{Cl}(A \cup B)$
$\mathrm{Int}(A) \cup \mathrm{Int}(B) \subseteq \mathrm{Int}(A \cup B)$	$\mathrm{Cl}(A \cap B) \subseteq \mathrm{Cl}(A) \cap \mathrm{Cl}(B)$
$\mathrm{Int}(A) = \mathrm{Cl}(A^c)^c$	$\mathrm{Cl}(A) = \mathrm{Int}(A^c)^c$

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