Fixed Point Logics and Definable Topological Properties

David Fernández-Duque[1] and Quentin Gougeon[2]

[1] Czech Academy of Sciences, Institute of Computer Science, Prague, Czech Republic
Ghent University, Department of Mathematics WE16, Ghent, Belgium
`David.FernandezDuque@UGent.be`
[2] CNRS-INPT-UT3, Toulouse University, Toulouse, France
`quentin.gougeon@irit.fr`

Abstract. Modal logic enjoys topological semantics that may be traced back to McKinsey and Tarski, and the classification of topological spaces via modal axioms is a lively area of research. In the past two decades, there has been interest in extending topological modal logic to the language of the μ-calculus, but previously no class of topological spaces was known to be μ-calculus definable that was not already modally definable. In this paper we show that the full μ-calculus is indeed more expressive than standard modal logic, in the sense that there are classes of topological spaces (and weakly transitive Kripke frames) which are μ-definable, but not modally definable. The classes we exhibit satisfy a modally definable property outside of their perfect core, and thus we dub them *imperfect spaces*. We show that the μ-calculus is sound and complete for these classes. Our examples are minimal in the sense that they use a single instance of a greatest fixed point.

Keywords: Mu-calculus · Expressivity · Topological semantics.

1 Introduction

Topological semantics for modal logic originated with McKinsey and Tarski [17] in the 1940's, but saw a more recent revival due to the work of Esakia [9], Shehtman [20], and others. In what we call the *closure semantics*, the modal $\Diamond$ is interpreted as the topological closure, and $\Box$ as the interior. The logic of all topological spaces in this semantics is **S4**, and we refer to [4] for an overview of topological completeness of modal logics above **S4**. The more expressive [16] *derivational semantics* has gained traction in recent years, but was already considered by McKinsey and Tarski. It is obtained by interpreting the modal $\Diamond$ as the Cantor derivative.[3] Esakia [8,9] showed that the derivative logic of all topological spaces is the modal logic $\mathbf{wK4} = \mathbf{K} + (\Diamond\Diamond p \to p \vee \Diamond p)$. This is also the modal logic of all *weakly transitive* frames, i.e., those for which the reflexive closure of the accessibility relation is transitive. It is well-known that the modal

[3] Recall that the derivative $\mathsf{d}(A)$ of a set A consists of all limit points of A.

logic of transitive frames is **K4** [6,7], which moreover corresponds to a natural class of topological spaces denoted T_d. Many familiar topological spaces are T_d, such as Euclidean spaces.

Even more recently, topological semantics have been extended to the language of the μ-calculus [3,10,11,13]. The relational μ-calculus is notoriously challenging from a theoretical perspective, with difficult completeness [21] and decidability [15] proofs (see also [1,18,19] for more recent work exhibiting various modifications to these results and their proofs). Since a transitive modality is already definable in the basic μ-calculus, Goldblatt and Hodkinson [14] obtained completeness and decidability as a corollary for transitive frames, and thus for T_d spaces. This does not work for weakly transitive frames, but surprisingly, Baltag et al. [3] showed that the combination of the μ-calculus with topological semantics is much more manageable than the original μ-calculus, with natural and transparent proofs of decidability and completeness involving only classical tools from modal logic (albeit intricately combined).

Thus the topological μ-calculus is decidable and complete, potentially placing it as a powerful yet technically manageable framework for reasoning about topologically-defined fixed points. The Achilles' heel of this proposal is that despite the sophisticated machinery, no class of topological spaces was formerly known to be μ-definable, but not modally definable. Our goal is to exhibit such classes of spaces. Here it is convenient to recall the notion of reducibility of formal languages, following Kudinov and Shehtman [16]. If $\mathcal{L}$ and $\mathcal{L}'$ are sub-languages of the μ-calculus, then $\mathcal{L}$ *reduces to* $\mathcal{L}'$ if every class of spaces definable in $\mathcal{L}$ is also definable in $\mathcal{L}'$ (see Section 2). If $\mathcal{L}$ reduces to $\mathcal{L}'$, we may also say that $\mathcal{L}'$ is *at least as expressive as* $\mathcal{L}$, and if moreover $\mathcal{L}'$ does not reduce to $\mathcal{L}$, we say that $\mathcal{L}'$ is *more expressive than* $\mathcal{L}$.[4]

More precisely, we manage to exhibit infinitely many topologically complete logics in the language of the μ-calculus whose classes of spaces are not modally definable. These axioms separate spaces into two parts, a perfect part (i.e., without isolated points), and a complement satisfying some property definable by a modal formula φ; we call these spaces *φ-imperfect spaces.* The perfect part is defined via a greatest fixed point operator.

The paper is structured as follows: in section 2, we present the relevant material regarding derivative spaces, the μ-calculus and axiomatic expressivity. In section 3, we use greatest fixed points to construct classes of spaces that are not modally definable. Completeness results for some of these classes are then laid out in section 4. We end with some concluding remarks in section 5.

[4] Note that a stronger notion of expressivity is also considered in the literature: namely, $\mathcal{L}'$ is at least as expressive as $\mathcal{L}$ if for every $\varphi \in \mathcal{L}$ there is a logically equivalent $\varphi' \in \mathcal{L}'$. To avoid confusion we may call the latter *local expressivity,* and the notion we are concerned with *axiomatic expressivity.* With this terminology in mind, while it was known that μ-calculus is locally more expressive than the basic modal language over topological spaces (see e.g. [10]), here we will show that it is also axiomatically more expressive.

2 Background

In this section we review the syntax and semantics of the topological μ-calculus. Following [3,12], we present our semantics in the general setting of *derivative spaces,* and work in a language with ν (rather than μ) as primitive.

Definition 1. We fix a countable set Prop of *atomic propositions* (also called *variables*). The language $\mathcal{L}_\mu$ of the modal μ-calculus is defined by the following grammar:

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \nu p.\varphi$$

where $p \in \mathsf{Prop}$ and in the construct $\nu p.\varphi$, the formula φ is *positive in* p, that is, every occurrence of p lies under the scope of an even number of negations. The abbreviations $\varphi \vee \psi$, $\Box\varphi$, $\bot$ and $\top$ are defined as usual. We denote by $\varphi[\psi_1, \ldots, \psi_n/p_1, \ldots, p_n]$ the formula φ where each formula ψ_i is substituted for every free occurrence of the variable p_i. We then introduce the abbreviation $\mu p.\varphi := \neg\nu p.\neg\varphi[\neg p/p]$. Finally, the *basic modal language* $\mathcal{L}_\Diamond$ is the fragment of $\mathcal{L}_\mu$ without occurrences of ν.

Definition 2. A *derivative space* is a pair $\mathcal{X} = (X, \mathsf{d})$, where X is a set of *points* and $\mathsf{d}\colon \mathcal{P}(X) \to \mathcal{P}(X)$ is an operator on subsets of X, satisfying for all $A, B \subseteq X$:

- $\mathsf{d}(\varnothing) = \varnothing$,
- $\mathsf{d}(A \cup B) = \mathsf{d}(A) \cup \mathsf{d}(B)$,
- $\mathsf{d}(\mathsf{d}(A)) \subseteq A \cup \mathsf{d}(A)$.

A *derivative model* based on $\mathcal{X}$ is a tuple of the form $\mathfrak{M} = (X, \mathsf{d}, V)$ with $V\colon \mathsf{Prop} \to \mathcal{P}(X)$ a *valuation*. Given $x \in X$ we then call $(\mathfrak{M}, x)$ a *pointed derivative model.* If $p \in \mathsf{Prop}$ and $A \subseteq X$, we define the valuation $V[p := A]$ by

$$V[p := A](q) := \begin{cases} A & \text{if } p = q \\ V(q) & \text{otherwise} \end{cases}.$$

We then write $\mathfrak{M}[p := A] := (X, \mathsf{d}, V[p := A])$.

Definition 3. Given a derivative model $\mathfrak{M} = (X, \mathsf{d}, V)$, we define by induction on a formula $\varphi \in \mathcal{L}_\mu$ the *extension* $[\![\varphi]\!]_\mathfrak{M}$ of φ in $\mathfrak{M}$ as follows:

$$[\![p]\!]_\mathfrak{M} := V(p) \qquad [\![\neg\varphi]\!]_\mathfrak{M} := X \setminus [\![\varphi]\!]_\mathfrak{M}$$
$$[\![\varphi \wedge \psi]\!]_\mathfrak{M} := [\![\varphi]\!]_\mathfrak{M} \cap [\![\psi]\!]_\mathfrak{M} \qquad [\![\Diamond\varphi]\!]_\mathfrak{M} := \mathsf{d}([\![\varphi]\!]_\mathfrak{M})$$
$$[\![\nu p.\varphi]\!]_\mathfrak{M} := \bigcup\{A \subseteq W \mid A \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]}\}$$

We then write $\mathfrak{M}, x \vDash \varphi$ whenever $x \in [\![\varphi]\!]_\mathfrak{M}$ and we say that φ is *true* at the point x. If $\mathfrak{M}$ is based on $\mathcal{X}$ and $\mathfrak{M}, x \vDash \varphi$, we say that φ is *satisfiable* on $\mathfrak{M}$, or on $\mathcal{X}$, or on $\mathcal{X}, x$ (depending on what is deemed relevant).

If $[\![\varphi]\!]_\mathfrak{M} = X$, we write $\mathfrak{M} \vDash \varphi$. If $\mathfrak{M} \vDash \varphi$ for all models $\mathfrak{M}$ based on $\mathcal{X}$ we write $\mathcal{X} \vDash \varphi$ and we say that φ is *valid* on $\mathcal{X}$. We also have a notion of pointwise validity, that is, if $\mathfrak{M}, x \vDash \varphi$ for every model $\mathfrak{M}$ based on $\mathcal{X}$, then we write $\mathcal{X}, x \vDash \varphi$. If $\mathcal{X} \vDash \varphi$ for all derivative spaces $\mathcal{X}$, we write $\vDash \varphi$. Given a class $\mathcal{C}$ of derivative spaces, we write $\mathcal{C} \vDash \varphi$ whenever $\mathcal{X} \vDash \varphi$ for all $\mathcal{X} \in \mathcal{C}$. If Γ is a set of formulas we write $\mathfrak{M}, x \vDash \Gamma$ whenever $\mathfrak{M}, x \vDash \varphi$ for all $\varphi \in \Gamma$, and all of the other notations are adapted accordingly.

In modal logic it is customary to study morphisms that preserve validity. In the context of derivative spaces, these are known as *d-morphisms* (see e.g. [16]).

Definition 4. Let $\mathcal{X} = (X, \mathsf{d})$ and $\mathcal{X}' = (X', \mathsf{d}')$ be two derivative spaces. A map $f \colon X \to X'$ is called a *d-morphism* from $\mathcal{X}$ to $\mathcal{X}'$ if it satisfies $f^{-1}[\mathsf{d}'(A')] = \mathsf{d}(f^{-1}[A'])$ for all $A' \subseteq X'$.

Proposition 5. *Let $\mathcal{X} = (X, \mathsf{d})$ and $\mathcal{X}' = (X', \mathsf{d}')$ be two derivative spaces and $f \colon X \to X'$ a d-morphism. If $\varphi \in \mathcal{L}_\mu$ and $\mathcal{X} \vDash \varphi$, then $\mathcal{X}' \vDash \varphi$.*

Presenting our semantics in terms of derivative spaces is useful, as both weakly transitive Kripke frames and topological spaces (either with the closure or the d operator) can be viewed as special cases of derivative spaces. While our 'intended' semantics is topological, Kripke semantics will be useful in establishing many of our main results.

Definition 6. A Kripke frame is a pair $\mathfrak{F} = (W, R)$, with W a set of *possible worlds* and $R \subseteq W^2$. We denote by $R^+ := R \cup \{(w, w) \mid w \in W\}$ the *reflexive closure* of R. The frame $\mathfrak{F}$ is said to be *rooted* in r if for all $w \in W$ we have rR^+w. We say that $\mathfrak{F}$ is *weakly transitive* if wRu and uRv implies wR^+v. In this case $\mathfrak{F}$ is also called a **wK4** *frame*, and it induces a derivative space (W, d) with d defined by $\mathsf{d}(A) := \{w \mid wRu \text{ and } u \in A\}$.

Slightly abusing terminology, we will identify $\mathfrak{F}$ and (W, d) (since one can be constructed from the other). Then (pointed) derivative models based on **wK4** frames will be called (pointed) *Kripke models*, while d-morphisms between **wK4** frames will be called *bounded morphisms*.

Now we turn our attention to the 'official' semantics of the topological μ-calculus.

Definition 7. Let X be a set of *points*. A *topology* on X is a set $\tau \subseteq \mathcal{P}(X)$ containing $\varnothing$ and X, closed under arbitrary unions, and closed under finite intersections. The pair (X, τ) is then called a *topological space*. The elements of τ are called the *open* sets of X. The complement of an open set is called a *closed* set. If $x \in U \in \tau$ then U is called an *open neighbourhood* of x. Slightly abusing notation, we will often keep τ implicit and let X refer to the space (X, τ).

Definition 8. Let X be a topological space, $A \subseteq X$ and $x \in X$. The point x is said to be a *limit point* of A if for all open neighbourhoods U of x, we have $U \cap A \setminus \{x\} \neq \varnothing$. We denote by $\mathsf{d}(A)$ the set of all limit points of A and call it the *derived set* of A. The dual of d is defined by $\widehat{\mathsf{d}}(A) := X \setminus \mathsf{d}(X \setminus A)$.

Given a topological space X, it is easily observed that the pair (X, d) is a derivative space. Conversely, the topology τ can be recovered from d since for all $A \subseteq X$, the set A is closed if and only if $\mathsf{d}(A) \subseteq A$. For this reason we choose, again, to identify (X, τ) and (X, d). Then (pointed) derivative models based on topological spaces will be called (pointed) *topological models*. Observe that the familiar *closure* and *interior* operators can be defined by $\mathsf{Cl}(A) := A \cup \mathsf{d}(A)$ and $\mathsf{Int}(A) := A \cap \widehat{\mathsf{d}}(A)$. Writing $\Box^+\varphi := \varphi \wedge \Box\varphi$ and $\Diamond^+\varphi := \varphi \vee \Diamond\varphi$, we then have $[\![\Box^+\varphi]\!]_{\mathfrak{M}} = \mathsf{Int}([\![\varphi]\!]_{\mathfrak{M}})$ and $[\![\Diamond^+\varphi]\!]_{\mathfrak{M}} = \mathsf{Cl}([\![\varphi]\!]_{\mathfrak{M}})$ for all topological models $\mathfrak{M}$. We recall some important classes of topological spaces that will be useful throughout the text.

Definition 9. Let X be a topological space. A point $x \in X$ is said to be *isolated* if $\{x\}$ is open. Given $x \in A \subseteq X$ we say that x is *isolated in* A if there exists U open such that $\{x\} = U \cap A$. The space X is called *dense-in-itself* if it contains no isolated point. The space X is called *scattered* if any subspace of X contains an isolated point. We say that X is T_d if every $x \in X$ is isolated in $\mathsf{Cl}(\{x\})$. We say that X is *extremally disconnected* if $\mathsf{Cl}(U)$ is open for every open set U, and *Aleksandroff* if arbitrary intersections of open sets are open.

Aleksandroff spaces are closely connected to Kripke frames, via the following construction.

Definition 10. Let $\mathfrak{F} := (W, R)$ be a **wK4** frame. A set $U \subseteq W$ is called an *upset* if $w \in U$ and wRu implies $u \in U$. The collection τ_R of all upsets over W is then a topology, and (W, τ_R) is called the topological space *induced* by $\mathfrak{F}$. If $\mathfrak{M} = (W, R, V)$ is a Kripke model based on $\mathfrak{F}$, then $((W, \tau_R), V)$ is the topological model *induced* by $\mathfrak{M}$.

It is not hard to check that a space of the form (W, τ_R) is always Aleksandroff (and, indeed, every Aleksandroff space is of this form [2]). In fact we will simply not distinguish a weakly transitive Kripke frame from the topological space induced by it. This is partly motivated by the following proposition.

Proposition 11. *Let $\mathfrak{M} = (W, R, V)$ be an irreflexive and weakly transitive Kripke model, and $\mathfrak{M}' := ((W, \tau_R), V)$ the space induced by it. For all $w \in W$ and $\varphi \in \mathcal{L}_\mu$ we have*

$$\mathfrak{M}, w \vDash \varphi \iff \mathfrak{M}', w \vDash \varphi.$$

The modal logic of all topological spaces is known as **wK4**, and consists of the following induction rules and axioms:

Name	Axiom/inference rule
	All propositional tautologies
Uniform substitution	From φ infer $\varphi[\psi_1, \ldots, \psi_n / p_1, \ldots, p_n]$
K (Distribution)	$\Box(p \to q) \to (\Box p \to \Box q)$
Modus Ponens	From φ and $\varphi \to \psi$ infer ψ
Necessitation	From φ infer $\Box\varphi$
Weak transitivity	$\Diamond\Diamond p \to p \vee \Diamond p$

The axiomatic system **K4** is the extension of **wK4** with the axiom $4 := \Diamond p \to \Diamond\Diamond p$. The axiomatic system μ**wK4** is the extension of **wK4** with the *fixed point axiom* $\nu p.\varphi \to \varphi[\nu p.\varphi/p]$ and the *induction rule*

$$\text{from } \varphi \to \psi[\varphi/p] \text{ infer } \varphi \to \nu p.\psi.$$

Definition 12. Let **L** be a logic in a sub-language of $\mathcal{L}_\mu$. If φ is a formula, the statement $\mathbf{L} \vdash \varphi$ says that φ is derivable in **L**. We say that **L** is *sound and complete* with respect to a class $\mathcal{C}$ of derivative spaces if for all formulas φ we have $\mathbf{L} \vdash \varphi$ iff $\mathcal{C} \vDash \varphi$. We call **L** *Kripke complete* if it is sound and complete with respect to some class of Kripke frames, and *topologically complete* if it is sound and complete with respect to some class of topological spaces.

Theorem 13 ([3]). *The logic* μ**wK4** *is sound and complete with respect to the class of all* **wK4** *frames, with respect to the class of all topological spaces, and with respect to the class of all derivative spaces.*

In order to compare the expressivity of different languages, we need to introduce the notion of definable classes.

Definition 14. Given a formula φ, we let $\mathcal{C}(\varphi)$ be the class of derivative spaces $\mathcal{X}$ such that $\mathcal{X} \vDash \varphi$. Let $\mathcal{C}_0$ be a class of derivative spaces and $\mathcal{L} \subseteq \mathcal{L}_\mu$. We say that $\mathcal{C}$ is $\mathcal{L}$*-definable within* $\mathcal{C}_0$ if there exists $\varphi \in \mathcal{L}$ such that $\mathcal{C}(\varphi) \cap \mathcal{C}_0 = \mathcal{C} \cap \mathcal{C}_0$.

If $\mathcal{L}, \mathcal{L}' \subseteq \mathcal{L}_\mu$, we say that $\mathcal{L}'$ is *at least as expressive as* $\mathcal{L}$ *over* $\mathcal{C}_0$ if every class definable in $\mathcal{L}$ within $\mathcal{C}_0$ is also definable in $\mathcal{L}'$ within $\mathcal{C}_0$. If $\mathcal{L}'$ is at least as expressive as $\mathcal{L}$ but $\mathcal{L}$ is not at least as expressive as $\mathcal{L}'$, we say that $\mathcal{L}'$ is *more expressive than* $\mathcal{L}$ *over* $\mathcal{C}_0$.

In particular, a $\mathcal{L}_\Diamond$-definable class will be called *modally definable*, and a $\mathcal{L}_\mu$-definable class will be called μ*-definable*. As discussed in Footnote 4, this notion of expressivity is also known as *reducibility* or *axiomatic expressivity*. The choice to compare expressivity relatively to a class of derivative spaces is convenient as it allows to derive all kinds of auxiliary results. We will consider the following classes of interest:

$$\begin{array}{lcl}
\mathcal{C}_{\mathsf{all}} & := & \{\mathcal{X} \mid \mathcal{X} \text{ is a derivative space}\} \\
\mathcal{C}_{\mathsf{fin}} & := & \{(X, \mathsf{d}) \in \mathcal{C}_{\mathsf{all}} \mid X \text{ is finite}\} \\
\mathcal{C}_{\mathsf{Kripke}} & := & \{\mathfrak{F} \mid \mathfrak{F} \text{ is a } \mathbf{wK4} \text{ frame}\} \\
\mathcal{C}_{\mathsf{irrefl}} & := & \{\mathfrak{F} \in \mathcal{C}_{\mathsf{Kripke}} \mid \mathfrak{F} \text{ is irreflexive}\} \\
\mathcal{C}_{\mathsf{topo}} & := & \{X \mid X \text{ is a topological space}\} \\
\mathcal{C}_{\mathbf{K4}} & := & \{\mathcal{X} \in \mathcal{C}_{\mathsf{all}} \mid \mathcal{X} \vDash \mathbf{K4}\}
\end{array}$$

It is well established that $\mathcal{C}_{\mathsf{Kripke}} \cap \mathcal{C}_{\mathbf{K4}}$ is the class of transitive Kripke frames [6], while $\mathcal{C}_{\mathsf{topo}} \cap \mathcal{C}_{\mathbf{K4}}$ is the class of T_d spaces [4].

3 Classes defined by greatest fixed points

The goal of this section is to exhibit μ-definable classes that are not modally definable. It turns out that a whole family of formulas of the form $\theta \vee \nu p.\Diamond p$ will yield the desired result. We easily see that given a pointed Kripke model $(\mathfrak{M}, x)$, we have $\mathfrak{M}, x \vDash \nu p.\Diamond p$ if and only if there exists an infinite path starting from x. Topologically, $\nu p.\Diamond p$ holds in the *perfect core* of X, the largest dense-in-itself subset of X. While the existence of an infinite path is not in general modally definable, it is not hard to check that $\mathcal{C}(\nu p.\Diamond p) = \mathcal{C}(\Diamond\top)$, as this is just the class of dense-in-themselves spaces. However, the story becomes more complicated if we only require certain points in the space to satisfy $\nu p.\Diamond p$. In this case, the following can be applied to exhibit many modally undefinable classes of spaces.

Theorem 15. *Let $\theta \in \mathcal{L}_\mu$ and suppose that for all $n \in \mathbb{N}$ there exists a* **wK4** *frame $\mathfrak{F}_n = (W_n, R_n)$ and $r_n \in W_n$ such that:*

1. *$\mathfrak{F}_n$ is rooted in r_n and $\mathfrak{F}_n, r_n \nvDash \theta \vee \nu p.\Diamond p$;*
2. *$\mathfrak{F}_n$ contains a path of length n;*
3. *for all $w \in W_n \setminus \{r_n\}$ we have $\mathfrak{F}_n, w \vDash \theta$.*

Then $\mathcal{C}(\theta \vee \nu p.\Diamond p)$ is not modally definable within $\mathcal{C}_{\mathsf{irrefl}} \cap \mathcal{C}_{\mathbf{K4}}$. If in addition every $\mathfrak{F}_n$ is finite, then $\mathcal{C}(\theta \vee \nu p.\Diamond p)$ is not modally definable within $\mathcal{C}_{\mathsf{irrefl}} \cap \mathcal{C}_{\mathsf{fin}}$ and $\mathcal{C}_{\mathsf{Kripke}} \cap \mathcal{C}_{\mathsf{fin}} \cap \mathcal{C}_{\mathbf{K4}}$.

Remark 16. We recall that both Kripke frames and topological spaces are identified with their respective derivative spaces, so $\mathcal{C}_{\mathsf{irrefl}} \cap \mathcal{C}_{\mathbf{K4}}$ can equivalently be regarded as the class of all T_d Aleksandroff spaces, and $\mathcal{C}_{\mathsf{irrefl}} \cap \mathcal{C}_{\mathsf{fin}}$ as the class of finite topological spaces. Thus Theorem 15 applies to classes of topological spaces, as well as Kripke frames.

Remark 17. It is easily observed that if $\mathcal{C}$ is not modally definable within $\mathcal{C}_0$ and $\mathcal{C}_0 \subseteq \mathcal{C}_1$, then $\mathcal{C}$ is not modally definable within $\mathcal{C}_1$ as well. This allows us to draw interesting consequences from Theorem 15, as $\mathcal{C}_{\mathsf{irrefl}} \cap \mathcal{C}_{\mathbf{K4}}$ is a subclass of $\mathcal{C}_{\mathsf{all}}$, $\mathcal{C}_{\mathsf{Kripke}}$, $\mathcal{C}_{\mathsf{topo}}$, $\mathcal{C}_{\mathsf{topo}} \cap \mathcal{C}_{\mathbf{K4}}$ and many other relevant classes.

From now on, we fix a formula θ and a family of frames $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Theorem 15. For all $n \in \mathbb{N}$, we assume that $W_n \cap \omega = \varnothing$. We start with an elementary observation.

Claim 18. *For all $n \in \mathbb{N}$, the frame $\mathfrak{F}_n$ is irreflexive and transitive.*

Proof. First assume that $\mathfrak{F}_n$ is not irreflexive, so that there is w with wR_nw. Then, $(r_n, w, w \ldots)$ is an infinite path beginning on r_n, contradicting $\mathfrak{F}_n, r_n \nvDash \nu p.\Diamond p$. If instead $\mathfrak{F}_n$ is not transitive, then since $\mathfrak{F}_n$ is weakly transitive, this can only occur if there exist $w, u \in W_n$ such that wR_nu, uR_nw and not wR_nw. Then, $(r_n, w, u, w, u, \ldots)$ is an infinite path beginning on r_n (or else $(w, u, w, u, \ldots)$ in case $w = r_n$). □

Given a world $w \in W_n$, we define the **wK4** frames $\mathfrak{F}^{\mathsf{point}}_{n,w} = (W^0, R^0)$, $\mathfrak{F}^{\mathsf{cycle}}_{n,w} = (W^1, R^1)$ and $\mathfrak{F}^{\mathsf{spine}}_{n,w} = (W^2, R^2)$ by:

$$
\begin{aligned}
W^0 &:= W_n \cup \{0\} \\
R^0 &:= R_n \cup \{(r_n, 0), (0, 0)\} \cup \{(0, u) \mid w R_n^+ u\} \\
W^1 &:= W_n \cup \{0, 1\} \\
R^1 &:= R_n \cup \{(r_n, 0), (r_n, 1), (0, 1), (1, 0)\} \cup \{(k, u) \mid k \in \{0, 1\} \text{ and } w R_n^+ u\} \\
W^2 &:= W_n \cup \omega \\
R^2 &:= R_n \cup \{(r_n, k) \mid k \in \omega\} \cup \{(m, k) \mid m < k < \omega\} \cup \{(k, u) \mid k \in \omega,\, w R_n^+ u\}
\end{aligned}
$$

In words, $\mathfrak{F}^{\mathsf{point}}_{n,w}$ is the frame $\mathfrak{F}_n$ endowed with a reflexive point reachable from the root, and which sees all the successors of w (as well as w itself). The frames $\mathfrak{F}^{\mathsf{cycle}}_{n,w}$ and $\mathfrak{F}^{\mathsf{spine}}_{n,w}$ are constructed similarly, but with respectively a two-element loop and an infinite branch, instead of a reflexive point. The three frames are depicted in Figure 1.

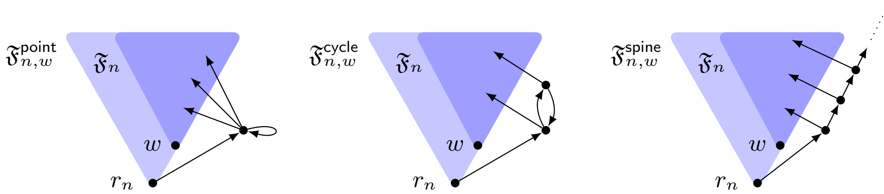

Fig. 1. The frames $\mathfrak{F}^{\mathsf{point}}_{n,w}$, $\mathfrak{F}^{\mathsf{cycle}}_{n,w}$ and $\mathfrak{F}^{\mathsf{spine}}_{n,w}$

If some modal formula ψ defines the same class of spaces as $\theta \vee \nu p.\Diamond p$, then by construction ψ should be refuted at $\mathfrak{F}_n, r_n$ for all n, but not at $\mathfrak{F}^{\mathsf{spine}}_{n,w}, r_n$ or $\mathfrak{F}^{\mathsf{cycle}}_{n,w}, r_n$ or $\mathfrak{F}^{\mathsf{point}}_{n,w}, r_n$ since in all three of them there is an infinite path starting from the root. Yet we will prove that if n is big enough and $\neg\psi$ is satisfiable on $\mathfrak{F}_n, r_n$, then it is also satisfiable on $\mathfrak{F}^{\mathsf{point}}_{n,w}, r_n$ for some w, leading to a contradiction.[5] The proof is rather technical, but we can sketch the main lines of our strategy. First, it is clear that transferring the satisfiability of a diamond formula (i.e., of the form $\Diamond\varphi$) or a Boolean formula from $\mathfrak{F}_n, r_n$ to $\mathfrak{F}^{\mathsf{point}}_{n,w}, r_n$ is immediate, so the challenge really comes from box formulas (of the form $\Box\varphi$). The central argument is that since n may be arbitrarily large, we can select some $\mathfrak{F}_n$ with an arbitrarily long path. By means of a pigeonhole argument, we will then manage to show that on some point w of this path, if $\Box\varphi$ is satisfied, then so is $\Box^+\varphi$ (when $\Box\varphi$ is any subformula of $\neg\psi$). Then, transferring the truth of $\Box\varphi$ to the reflexive point of $\mathfrak{F}^{\mathsf{point}}_{n,w}$ will be straightforward.

First, we recall that the *negative normal form* (or NNF for short) for modal logic is the syntax generated by the following grammar:

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi.$$

It is well known that for all modal formulas, there exists an equivalent formula in NNF. We also introduce a notion of *type* of a possible world.

[5] Later we will see that the same result with $\mathfrak{F}^{\mathsf{spine}}_{n,w}, r_n$ and $\mathfrak{F}^{\mathsf{cycle}}_{n,w}, r_n$ follows for free.

Definition 19. Let φ be a modal formula. We write $\psi \trianglelefteq \varphi$ whenever ψ is a subformula of φ. We also call the *box size* $|\varphi|_\Box$ of φ the number of subformulas of φ of the form $\Box\psi$. If $\mathfrak{M}$ is a derivative model and w a world in $\mathfrak{M}$, we define the *box type* of w relative to φ as the set $t^{\varphi}_{\mathfrak{M}}(w) := \{\Box\psi \mid \Box\psi \trianglelefteq \varphi \text{ and } \mathfrak{M}, w \vDash \Box\psi\}$.

As explained above, the following result allows to transfer the satisfiability of box formulas as soon as the parameter n is large enough.

Claim 20. *Let φ be a modal formula in NNF and $n > 2^{|\varphi|_\Box}$. Suppose that there exists a valuation V over $\mathfrak{F}_n$ such that $\mathfrak{F}_n, V, r_n \vDash \Box\varphi$. Then there exists a world $w \in W_n$ and a valuation V' over $\mathfrak{F}^{\mathsf{point}}_{n,w}$ such that $\mathfrak{F}^{\mathsf{point}}_{n,w}, V', r_n \vDash \Box\varphi$, and V and V' coincide over $\mathfrak{F}_n$.*

Proof. First, we know that $\mathfrak{F}_n$ contains a path $(w_i)_{i\in[1,n]}$ of length n. By construction there are $2^{|\varphi|_\Box}$ different box types relative to φ. Thus, by the pigeonhole principle, there exists $i, j \in \mathbb{N}$ such that $1 \leq i < j \leq n$ and $t^{\varphi}_{\mathfrak{M}}(w_i) = t^{\varphi}_{\mathfrak{M}}(w_j)$. We then define a valuation V' over $\mathfrak{F}^{\mathsf{point}}_{n,w_j}$ by setting, for all $p \in \mathsf{Prop}$:

$$V'(p) := \begin{cases} V(p) \cup \{0\} & \text{if } w_j \in V(p) \\ V(p) & \text{otherwise} \end{cases}.$$

So V and V' coincide over $\mathfrak{F}_n$, and V' is defined over 0 so that this point satisfies the same atomic propositions as w_j. We then prove by induction on $\psi \trianglelefteq \varphi$ that $\mathfrak{F}_n, V, w_j \vDash \psi$ implies $\mathfrak{F}^{\mathsf{point}}_{n,w_j}, V', 0 \vDash \psi$:

- If ψ is of the form $\psi = p$ or $\psi = \neg p$ with $p \in \mathsf{Prop}$ this is just true by construction.
- If ψ is of the form $\psi = \psi_1 \wedge \psi_2$, then $\mathfrak{F}_n, V, w_j \vDash \psi_1 \wedge \psi_2$ implies $\mathfrak{F}_n, V, w_j \vDash \psi_1$ and $\mathfrak{F}_n, V, w_j \vDash \psi_2$ and it suffices to apply the induction hypothesis. If ψ is of the form $\psi = \psi_1 \vee \psi_2$, then $\mathfrak{F}_n, V, w_j \vDash \psi_1 \vee \psi_2$ implies $\mathfrak{F}_n, V, w_j \vDash \psi_1$ or $\mathfrak{F}_n, V, w_j \vDash \psi_2$ and the result follows in the same way.
- Suppose that ψ is of the form $\psi = \Diamond\psi_0$ and $\mathfrak{F}_n, V, w_j \vDash \psi$. Then since V and V' coincide over $\mathfrak{F}_n$, we have $\mathfrak{F}^{\mathsf{point}}_{n,w_j}, V', w_j \vDash \psi$ as well. By transitivity it follows that $\mathfrak{F}^{\mathsf{point}}_{n,w_j}, V', 0 \vDash \psi$.
- Suppose that ψ is of the form $\psi = \Box\psi_0$ and that $\mathfrak{F}_n, V, w_j \vDash \psi$. Then since $t^{\varphi}_{\mathfrak{M}}(w_i) = t^{\varphi}_{\mathfrak{M}}(w_j)$, we have $\mathfrak{F}_n, V, w_i \vDash \psi$ as well. Since $w_i R_n w_j$ it follows $\mathfrak{F}_n, V, w_j \vDash \psi_0$, and then $\mathfrak{F}^{\mathsf{point}}_{n,w_j}, V', 0 \vDash \psi_0$ by the induction hypothesis. Since V and V' coincide over $\mathfrak{F}_n$ we also have $\mathfrak{F}^{\mathsf{point}}_{n,w_j}, V', w_j \vDash \Box^+\psi_0$. All in all we obtain $\mathfrak{F}^{\mathsf{point}}_{n,w_j}, V', 0 \vDash \Box\psi_0$ as desired.

Now observe that since $w_i R_n w_j$ we must have $w_j \neq r_n$, otherwise we would obtain $r_n R_n r_n$ by transitivity. Thus $r_n R_n w_j$ and from $\mathfrak{F}_n, V, r_n \vDash \Box\varphi$ we obtain $\mathfrak{F}_n, V, w_j \vDash \varphi$, and then $\mathfrak{F}^{\mathsf{point}}_{n,w_j}, V', 0 \vDash \varphi$. Since V and V' coincide over $\mathfrak{F}_n$, we conclude that $\mathfrak{F}^{\mathsf{point}}_{n,w_j}, V', r_n \vDash \Box\varphi$. □

We can then extend the result to any modal formula.

Claim 21. *Let φ be a modal formula. There exists $n \in \mathbb{N}$ such that if φ is satisfiable on $\mathfrak{F}_n, r_n$, then there exists a world $w \in W_n$ such that φ is satisfiable on $\mathfrak{F}^{\mathsf{spine}}_{n,w}$ and $\mathfrak{F}^{\mathsf{cycle}}_{n,w}$ and $\mathfrak{F}^{\mathsf{point}}_{n,w}$.*

Proof. Applying the theorem of disjunctive normal form for propositional logic, and using the fact that $\Box$ and $\wedge$ commute, we can assume that φ is of the form $\varphi = \bigvee_{i=1}^{m} \sigma_i$ with $\sigma_i = \rho_i \wedge \Box\psi_i \wedge \bigwedge_{j=1}^{m_i} \Diamond\theta_{i,j}$ for all $i \in [1, m]$, where ρ_i is a propositional formula. Note that since $\Box\top$ is a tautology, we can always assume the presence of $\Box\psi_i$. We also suppose that ψ_i is presented in NNF. We then define

$$n := 1 + \mathsf{max}\ \{2^{|\psi_i|_\Box} \mid 1 \leq i \leq m\}$$

and assume that there exists a valuation V such that $\mathfrak{F}_n, V, r_n \vDash \varphi$. Then there exists $i \in [1, m]$ such that $\mathfrak{F}_n, V, r_n \vDash \sigma_i$. It follows that $\mathfrak{F}_n, V, r_n \vDash \Box\psi_i$ with $n > 2^{|\psi_i|_\Box}$, so by Claim 20 there exists $w \in W_n$ and a valuation V' over $\mathfrak{F}^{\mathsf{point}}_{n,w}$ such that $\mathfrak{F}^{\mathsf{point}}_{n,w}, V', r_n \vDash \Box\psi_i$, and V and V' coincide over $\mathfrak{F}_n$. It is then clear that $\mathfrak{F}^{\mathsf{point}}_{n,w}, V', r_n \vDash \sigma_i$, and thus $\mathfrak{F}^{\mathsf{point}}_{n,w}, V', r_n \vDash \varphi$.

This proves that φ is satisfiable on $\mathfrak{F}^{\mathsf{point}}_{n,w}$. Now consider the function f which maps every $n \in \omega$ (resp. $n \in \{0, 1\}$) to 0, and every $w \in W_n$ to w itself. Then f defines a bounded morphism from $\mathfrak{F}^{\mathsf{spine}}_{n,\mathfrak{F}}$ (resp. $\mathfrak{F}^{\mathsf{cycle}}_{n,w}$) to $\mathfrak{F}^{\mathsf{point}}_{n,w}$, and we conclude that φ is satisfiable on $\mathfrak{F}^{\mathsf{spine}}_{n,w}$ and $\mathfrak{F}^{\mathsf{cycle}}_{n,w}$. □

We are now ready to prove Theorem 15:

Proof. Suppose toward a contradiction that there is a formula $\psi \in \mathcal{L}_\Diamond$ defining the same class as $\theta \vee \nu p.\Diamond p$ within $\mathcal{C}_{\mathsf{irrefl}} \cap \mathcal{C}_{\mathbf{K4}}$. Let n be the integer obtained by applying Claim 21 to $\neg\psi$. By Claim 18, the frame $\mathfrak{F}_n$ is irreflexive and transitive, and we also have $\mathfrak{F}_n \nvDash \theta \vee \nu p.\Diamond p$ by assumption, so $\mathfrak{F}_n \nvDash \psi$ as well.

Thus $\neg\psi$ is satisfiable on $\mathfrak{F}_n, v$ for some $v \in W_n$. If $v \neq r_n$, we denote by $\mathfrak{F}$ the subframe of $\mathfrak{F}_n$ generated by v. Then $\mathfrak{F}$ does not contain r_n, for otherwise we would have $vR_n r_n R_n v$ and thus $vR_n v$, a contradiction. The assumption on $\mathfrak{F}_n$ yields $\mathfrak{F} \vDash \theta$, so $\mathfrak{F} \vDash \theta \vee \nu p.\Diamond p$ and thus $\mathfrak{F} \vDash \psi$. Therefore $\mathfrak{F}_n, v \vDash \psi$, a contradiction. We conclude that $r_n = v$. Then by Claim 21 there exists $w \in W_n$ such that $\neg\psi$ is satisfiable on $\mathfrak{F}^{\mathsf{spine}}_{n,w}$. Yet $\mathfrak{F}^{\mathsf{spine}}_{n,w} \in \mathcal{C}_{\mathsf{irrefl}} \cap \mathcal{C}_{\mathbf{K4}}$ and $\mathfrak{F}^{\mathsf{spine}}_{n,w} \vDash \theta \vee \nu p.\Diamond p$, so $\mathfrak{F}^{\mathsf{spine}}_{n,w} \vDash \psi$, a contradiction.

Now suppose that every $\mathfrak{F}_n$ is finite. By the same reasoning, we can show that $\mathcal{C}(\theta \vee \nu p.\Diamond p)$ is not modally definable within $\mathcal{C}_{\mathsf{irrefl}} \cap \mathcal{C}_{\mathsf{fin}}$ and $\mathcal{C}_{\mathsf{Kripke}} \cap \mathcal{C}_{\mathsf{fin}} \cap \mathcal{C}_{\mathbf{K4}}$. To that end it suffices to replace $\mathfrak{F}^{\mathsf{spine}}_{n,w}$ by respectively $\mathfrak{F}^{\mathsf{cycle}}_{n,w}$, which is irreflexive and finite, and $\mathfrak{F}^{\mathsf{point}}_{n,w}$, which is transitive and finite. □

Theorem 15 remains a very general statement, and it is worth instantiating it with examples. The following result shows the existence of infinitely many non-modally definable classes of spaces.

Proposition 22. *Given $m \in \mathbb{N}$ we define $.2^+_m := (\Diamond^+\Box^+ q \to \Box^+\Diamond^+ q) \vee \Box^m\bot$ and $\mathsf{IP}.2^+_m := .2^+_m \vee \nu p.\Diamond p$. Then the class of topological spaces X such that $X \vDash \mathsf{IP}.2^+_m$ is not modally definable. In addition, whenever $m, k \geq 1$ and $m \neq k$ we have $\mu\mathbf{wK4} + \mathsf{IP}.2^+_m \neq \mu\mathbf{wK4} + \mathsf{IP}.2^+_k$.*

Proof. It suffices to prove that the assumptions of Theorem 15 are satisfied for $\theta := .2^+_m$. In $\Diamond^+\Box^+ q \to \Box^+\Diamond^+ q$ we recognize a variant of the axiom .2 [7], but relative to the reflexive closure R^+; we call it $.2^+$, and this also explains the name $.2^+_m$. Thus, given a frame $\mathfrak{F} = (W, R)$ we have $\mathfrak{F} \vDash \mathsf{IP}.2^+_m$ iff for all $w \in W$ one of the following holds:

- for all $u, v \in W$ such that wR^+u and wR^+v, there exists $t \in W$ such that uR^+t and vR^+t;
- there exists no path of length $m+1$ starting from w;
- there exists an infinite path starting from w.

Consider, for all $n \in \mathbb{N}$, the frame $\mathfrak{F}^m_n := (W^m_n, R^m_n)$ depicted in Figure 2. We can see that the $\mathfrak{F}^m_n$'s fulfil all the conditions of Theorem 15, so we are done (see Remark 17 for why the result applies to topological spaces). Finally, if $1 \leq m < k$ we can see that $\mathfrak{F}^{m-1}_1 \vDash \mathsf{IP}.2^+_m$ whereas $\mathfrak{F}^{m-1}_1 \nvDash \mathsf{IP}.2^+_k$, and this proves that $\mu\mathbf{wK4} + \mathsf{IP}.2^+_m \neq \mu\mathbf{wK4} + \mathsf{IP}.2^+_k$. □

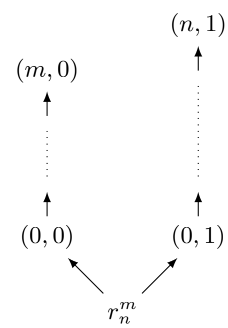

Fig. 2. The fork-like frame $\mathfrak{F}^m_n$

In Section 5 we will analyze these axioms further to see that they are well-behaved, but we find it appropriate to end this section by presenting an intuitive topological interpretation of the axiom $\mathsf{IP}.2^+_0$, which reduces to $.2^+ \vee \nu p.\Diamond p$. Given a formula θ and a space X, we say that X is *θ-imperfect* if there exist two disjoint subspaces Y and Z of X such that $X = Y \cup Z$, $Y \vDash \theta$ and Z is dense-in-itself.

Proposition 23. *Let $\theta \in \mathcal{L}_\mu$ and X a topological space. Then $X \vDash \theta \vee \nu p.\Diamond p$ if and only X is θ-imperfect.*

Proof. From left to right, assume that $X \vDash \theta \vee \nu p.\Diamond p$. We set $Z := \{x \in X \mid X, x \vDash \nu p.\Diamond p\}$ and $Y := X \setminus Z$. The fixed point equation immediately gives $Z = \mathsf{d}(Z)$, so Z is dense-in-itself. From $\mathsf{d}(Z) \subseteq Z$ we also obtain that Z is closed and Y is open. Now, let $x \in Y$ and V be a valuation over Y. We have $X, V, x \vDash \theta \vee \nu p.\Diamond p$ and by construction $X, V, x \nvDash \nu p.\Diamond p$, so $X, V, x \vDash \theta$. Since Y is open we obtain $Y, V, x \vDash \theta$. Therefore $Y \vDash \theta$.

From right to left, suppose that such a decomposition $X = Y \cup Z$ exists. Let $x \in X$ and V be a valuation over X. Suppose that $x \in Z$. Since Z is dense-in-itself we have $Z \subseteq \mathsf{d}(Z) = [\![\Diamond p]\!]_{X,V[p:=Z]}$ so $Z \subseteq [\![\nu p.\Diamond p]\!]_{X,V}$. Therefore $X, V, x \vDash \nu p.\Diamond p$. Otherwise, we have $x \in Y$. If $x \notin \mathsf{Int}(Y)$, then $x \in \mathsf{Cl}(Z)$ and since $x \notin Z$ it follows that $x \in \mathsf{d}(Z)$. We have seen that $X, V, z \vDash \nu p.\Diamond p$ for all $z \in Z$, so $X, V, x \vDash \Diamond \nu p.\Diamond p$, and then the fixed point equation gives $X, V, x \vDash \nu p.\Diamond p$. Otherwise we have $x \in \mathsf{Int}(Y)$. Since $Y \vDash \theta$ and $\mathsf{Int}(Y)$ is open in Y, we have $\mathsf{Int}(Y) \vDash \theta$. Then $\mathsf{Int}(Y), V, x \vDash \theta$ and since $\mathsf{Int}(Y)$ is open, we finally get $X, V, x \vDash \theta$. In all cases we obtain $X, V, x \vDash \theta \vee \nu p.\Diamond p$ as desired. □

Remark 24. By inspection of the proof for the left-to-right implication, we can also assume that Y is scattered and Z is perfect (i.e., closed and dense-in-itself).

In our example, the axiom $.2^+$ is known to define the class of extremally disconnected spaces [4] (see Definition 9). We thus obtain the following result:

Corollary 25. *The class of spaces that can be written as the disjoint union of an extremally disconnected subspace and a perfect subspace is not modally definable.*

4 Completeness for imperfect spaces

We have shown that there are μ-definable classes that are not modally definable, including infinitely many classes of imperfect spaces. We can make these examples even stronger by showing that the logics we have exhibited are complete for these classes. To this end, we construct the canonical model and use the technique of the final model applied by Fine and Zakharyashev to modal logic (see [5,7]) and by Baltag et al. [3] to the μ-calculus. Central will be the notion of cofinal subframe logic.

Definition 26. Let $\mathfrak{F} = (W, R)$ be a Kripke frame. A subframe $\mathfrak{F}' = (W', R')$ of $\mathfrak{F}$ is called a *cofinal subframe* of $\mathfrak{F}$ if $w' \in W'$ and $w'Rw$ implies the existence of $u' \in W'$ such that wR^+u'. Given $\mathfrak{M}$ based on $\mathfrak{F}$ and $\mathfrak{M}'$ a submodel of $\mathfrak{M}$, we call $\mathfrak{M}'$ a *cofinal submodel* of $\mathfrak{M}$ if it is based on a cofinal subframe $\mathfrak{F}'$ of $\mathfrak{F}$.

Definition 27. Let **L** be an extension of **K**. The logic **L** is called *cofinal subframe* if whenever $\mathfrak{F} \vDash \mathbf{L}$ and $\mathfrak{F}'$ is a cofinal subframe of $\mathfrak{F}$, we have $\mathfrak{F}' \vDash \mathbf{L}$.

Definition 28. Let **L** be an extension of **K**. The *canonical model* of **L** is the model $\mathfrak{M} := (\Omega, R, V)$ with:

- Ω the set of maximal **L**-consistent subsets of $\mathcal{L}_\Diamond$;
- $R := \{(\Gamma, \Delta) \mid \Box\varphi \in \Gamma \implies \varphi \in \Delta\}$;
- $V(p) := \{\Gamma \in \Omega \mid p \in \Gamma\}$ for all $p \in \mathsf{Prop}$.

The so-called *Truth Lemma* then establishes an equivalence between truth and membership at the worlds of $\mathfrak{M}$, i.e., $\mathfrak{M}, \Gamma \vDash \varphi$ if and only if $\varphi \in \Gamma$. Combined with the Lindenbaum's Lemma, this yields completeness of **L** with respect to its canonical model [6, sec. 4.2]. If **L** is an extension of μ**wK4**, the canonical model is defined in the same way, but the Truth Lemma then fails to hold. The technique designed in [3] consists in restricting oneself to an appropriate cofinal submodel of $\mathfrak{M}$. First, given a $\mathbf{L}$-consistent formula ψ, one can construct a finite set of formulas Σ containing ψ, closed under subformulas, and closed (up to logical equivalence in $\mathbf{L}$) under negation and $\Diamond^+$. We then define the so-called Σ-final model as follows.

Definition 29. A world $\Gamma \in \Omega$ is called *Σ-final* if there exists $\varphi \in \Sigma \cap \Gamma$ such that whenever $\Gamma R \Delta$ and $\varphi \in \Delta$, we have $\Delta R \Gamma$. The *Σ-final model* is then the submodel $\mathfrak{M}_\Sigma$ of $\mathfrak{M}$ induced by $\Omega_\Sigma := \{\Gamma \in \Omega \mid \Gamma \text{ is } \Sigma\text{-final}\}$.

Under the right assumptions it can be proven that (1) $\mathfrak{M}_\Sigma$ is a cofinal submodel of $\mathfrak{M}$, (2) ψ belongs to some Σ-final world, (3) the Truth Lemma holds in $\mathfrak{M}_\Sigma$ for the formulas in Σ. This yields Kripke completeness of $\mu\mathbf{wK4}$ and, in fact, of any logic of the form $\mu\mathbf{wK4} + \theta$ where $\theta \in \mathcal{L}_\Diamond$ and $\mathbf{wK4} + \theta$ is a canonical and cofinal subframe logic. Note that this result is limited to extensions of $\mu\mathbf{wK4}$ with *basic* modal axioms. This is to be contrasted with the work of the present section, which offers completeness results for axioms with fixed points. First, we need a technical lemma.

Lemma 30. *If $\mu\mathbf{wK4} + \theta \vdash \varphi$, then $\mu\mathbf{wK4} + (\theta \vee \nu p.\Diamond p) \vdash \varphi \vee \nu p.\Diamond p$.*

Proof. We write $\mathbf{L}_0 := \mu\mathbf{wK4} + \theta$ and $\mathbf{L} := \mu\mathbf{wK4} + (\theta \vee \nu p.\Diamond p)$. We proceed by induction on the length of a proof.

- If φ is an axiom of $\mu\mathbf{wK4}$ or θ itself, then this is clear.
- Suppose that this holds for φ, and that $\mathbf{L}_0 \vdash \varphi[\psi_1, \ldots, \psi_n/p_1, \ldots, p_n]$ is obtained from $\mathbf{L}_0 \vdash \varphi$. By the induction hypothesis we have $\mathbf{L} \vdash \varphi \vee \nu p.\Diamond p$ and by substitution it follows that $\mathbf{L} \vdash \varphi[\psi_1, \ldots, \psi_n/p_1, \ldots, p_n] \vee \nu p.\Diamond p$.
- Suppose that this holds for φ and $\varphi \to \psi$, and that $\mathbf{L}_0 \vdash \psi$ is obtained from $\mathbf{L}_0 \vdash \varphi$ and $\mathbf{L}_0 \vdash \varphi \to \psi$. By the induction hypothesis we have $\mathbf{L} \vdash \varphi \vee \nu p.\Diamond p$ and $\mathbf{L} \vdash (\varphi \to \psi) \vee \nu p.\Diamond p$, and we deduce $\mathbf{L} \vdash \psi \vee \nu p.\Diamond p$.
- Suppose that this holds for φ, and that $\mathbf{L}_0 \vdash \Box\varphi$ is obtained from $\mathbf{L}_0 \vdash \varphi$. By the induction hypothesis we have $\mathbf{L} \vdash \varphi \vee \nu p.\Diamond p$, and by necessitation we obtain $\mathbf{L} \vdash \Box(\varphi \vee \nu p.\Diamond p)$ or equivalently $\mathbf{L} \vdash \Box(\neg \nu p.\Diamond p \to \varphi)$. By distribution we obtain $\mathbf{L} \vdash \Box\neg\nu p.\Diamond p \to \Box\varphi$, and the implication $\neg\nu p.\Diamond p \to \Box\neg\nu p.\Diamond p$ can also be derived in $\mu\mathbf{wK4}$. Therefore $\mathbf{L} \vdash \neg\nu p.\Diamond p \to \Box\varphi$, or equivalently $\mathbf{L} \vdash \Box\varphi \vee \nu p.\Diamond p$.
- Suppose that this holds for $\varphi \to \psi[\varphi/p]$ and that $\mathbf{L}_0 \vdash \varphi \to \nu p.\psi$ is obtained from $\mathbf{L}_0 \vdash \varphi \to \psi[\varphi/p]$. By the induction hypothesis we have

  $$\mathbf{L} \vdash \nu p.\Diamond p \vee (\varphi \to \psi[\varphi/p])$$

  and we prove that

  $$\mu\mathbf{wK4} \vdash \psi[\varphi/p] \wedge \neg\nu p.\Diamond p \to \psi[\varphi \wedge \neg\nu p.\Diamond p/p].$$

  Indeed, consider a $\mathbf{wK4}$ frame $\mathfrak{M}$ rooted in w and assume that $\mathfrak{M}, w \vDash \psi[\varphi/p] \wedge \neg\nu p.\Diamond p$. From $\vDash \neg\nu p.\Diamond p \to \Box\neg\nu p.\Diamond p$ we obtain $\mathfrak{M} \vDash \neg\nu p.\Diamond p$, so

$\mathfrak{M} \vDash \varphi \leftrightarrow (\varphi \wedge \neg \nu p.\Diamond p)$ and thus $\mathfrak{M} \vDash \psi[\varphi/p] \leftrightarrow \psi[\varphi \wedge \neg \nu p.\Diamond p/p]$. Therefore $\mathfrak{M}, w \vDash \psi[\varphi \wedge \neg \nu p.\Diamond p/p]$ and the result follows by Theorem 13. We then obtain

$$\mathbf{L} \vdash \varphi \wedge \neg \nu p.\Diamond p \to \psi[\varphi \wedge \neg \nu p.\Diamond p/p]$$

and by the induction rule we derive $\mathbf{L} \vdash \varphi \wedge \neg \nu p.\Diamond p \to \nu p.\psi$, or equivalently $\mathbf{L} \vdash \nu p.\Diamond p \vee (\varphi \to \nu p.\psi)$.

□

Theorem 31. *Let θ be a modal formula such that $\mathbf{wK4} + \theta$ is cofinal subframe and canonical. Then $\mu\mathbf{wK4} + \theta \vee \nu p.\Diamond p$ is Kripke complete.*

Proof. We write $\mathbf{L} := \mu\mathbf{wK4} + \theta \vee \nu p.\Diamond p$ and $\mathbf{L}_0 := \mu\mathbf{wK4} + \theta$. Suppose that $\mathbf{L} \nvdash \neg\psi$ and let Σ be a finite set of formulas containing ψ and $\nu p.\Diamond p$, and with the closure properties enumerated above. We introduce:

- $\mathfrak{M} = (\Omega, R, V)$ the canonical model of $\mathbf{L}$, based on $\mathfrak{F} = (\Omega, R)$;
- $\mathfrak{M}_\Sigma = (\Omega_\Sigma, R_\Sigma, V_\Sigma)$ the Σ-final submodel of $\mathfrak{M}$, based on $\mathfrak{F}_\Sigma = (\Omega_\Sigma, R_\Sigma)$;
- $\mathfrak{M}_0 = (\Omega_0, R_0, V_0)$ the canonical model of $\mathbf{L}_0$, based on $\mathfrak{F}_0 = (\Omega_0, R_0)$.

See Figure 3 for a visual depiction of these frames. We know that $\mathfrak{F}_\Sigma$ is a cofinal subframe of $\mathfrak{F}$. In addition we have $\mathbf{L} \subseteq \mathbf{L}_0$, so for all maximal consistent sets Γ such that $\mathbf{L}_0 \subseteq \Gamma$ we also have $\mathbf{L} \subseteq \Gamma$; it is also clear that R and R_0 coincide over Ω_0, so $\mathfrak{F}_0$ is a subframe of $\mathfrak{F}$. We then introduce

$$\Omega' := \{\Gamma \in \Omega_\Sigma \mid \mathfrak{M}_\Sigma, \Gamma \vDash \neg \nu p.\Diamond p\}$$

which induces a generated subframe $\mathfrak{F}' = (\Omega', R')$ of $\mathfrak{F}$. Indeed, if $\Gamma \in \Omega'$ and $\Gamma R^+_\Sigma \Delta$, then since $\mathfrak{M}_\Sigma, \Gamma \vDash \neg \nu p.\Diamond p$ we have $\mathfrak{M}_\Sigma, \Delta \vDash \neg \nu p.\Diamond p$ too and thus $\Delta \in \Omega'$. Also, given $\Gamma \in \Omega'$ we have $\mathfrak{M}_\Sigma, \Gamma \vDash \neg \nu p.\Diamond p$, and we obtain $\neg \nu p.\Diamond p \in \Gamma$ by the Truth Lemma. If $\mathbf{L}_0 \vdash \varphi$, then $\mathbf{L} \vdash \varphi \vee \nu p.\Diamond p$ by Lemma 30, and from $\varphi \vee \nu p.\Diamond p \in \Gamma$ and $\neg \nu p.\Diamond p \in \Gamma$ we deduce $\varphi \in \Gamma$. Therefore $\mathbf{L}_0 \subseteq \Gamma$, and we obtain $\Gamma \in \Omega_0$. This proves that $\mathfrak{F}'$ is a subframe of $\mathfrak{F}_0$.

Now, suppose $\Gamma \in \Omega'$, $\Delta \in \Omega_0$ and $\Gamma R \Delta$. Since $\mathfrak{F}_\Sigma$ is cofinal in $\mathfrak{F}$, there exists $\Lambda \in \Omega_\Sigma$ such that $\Delta R^+ \Lambda$. By weak transitivity it follows that $\Gamma R^+ \Lambda$, and since $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}_\Sigma$ it follows that $\Lambda \in \Omega'$. Therefore $\mathfrak{F}'$ is a cofinal subframe of $\mathfrak{F}_0$. As observed in [3], that $\mathbf{wK4} + \theta$ is canonical implies that $\mu\mathbf{wK4} + \theta$ is canonical too, so $\mathfrak{F}_0 \vDash \theta$. Since $\mathbf{L}_0$ is cofinal subframe it follows that $\mathfrak{F}' \vDash \theta$ as well.

Now let $V_\bullet$ be a valuation over Ω_Σ and $\Gamma \in \Omega_\Sigma$. If $\Gamma \in \Omega'$, let $(\Omega', R', V'_\bullet)$ be the submodel of $(\Omega_\Sigma, R_\Sigma, V_\bullet)$ induced by Ω'. We know that $(\Omega', R', V'_\bullet), \Gamma \vDash \theta$, and since $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}_\Sigma$, it follows that $(\Omega_\Sigma, R_\Sigma, V_\bullet), \Gamma \vDash \theta$. Otherwise we have $\mathfrak{M}_\Sigma, \Gamma \vDash \nu p.\Diamond p$, but since $\nu p.\Diamond p$ contains no free variable, its truth value does not depend on the valuation V_Σ, and thus $(\Omega_\Sigma, R_\Sigma, V_\bullet), \Gamma \vDash \nu p.\Diamond p$. Therefore $\mathfrak{F}_\Sigma \vDash \theta \vee \nu p.\Diamond p$. As mentioned earlier, ψ is satisfiable on $\mathfrak{M}_\Sigma$ and this proves Kripke completeness. □

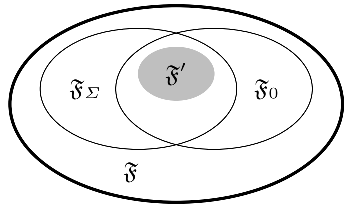

Fig. 3. The canonical frame of **L** and its subframes

In order to prove topological completeness, we apply the technique used in [3] to turn a **wK4** frame into an appropriate topological space. The construction essentially consists of replacing every reflexive point w of a frame by countably many copies of w, and to arrange them all into a dense-in-itself subspace, so as to mimic the reflexivity of w in a topological manner.

Definition 32. Let $\mathfrak{F} = (W, R)$ be a **wK4** frame. We denote by W^r the set of reflexive worlds of $\mathfrak{F}$, and by W^i the set of irreflexive worlds of $\mathfrak{F}$. We then introduce the *unfolding* of $\mathfrak{F}$ as the space $X_\mathfrak{F} := (W^r \times \omega) \cup (W^i \times \{\omega\})$ endowed with the topology $\tau_\mathfrak{F}$ of all sets U such that for all $(w, \alpha) \in U$:

1. there exists $n^U_{w,\alpha} < \omega$ such that for all $(u, \beta) \in X_\mathfrak{F}$, if wRu, uRw and $\beta \geq n^U_{w,\alpha}$ then $(u, \beta) \in U$;
2. if $(u, \beta) \in X_\mathfrak{F}$, wRu and *not* uRw then $(u, \beta) \in U$.

Proposition 33 ([3]). *The pair $(X_\mathfrak{F}, \tau_\mathfrak{F})$ is a topological space and the map $\pi \colon X_\mathfrak{F} \to W$ defined by $\pi(w, \alpha) := w$ is a surjective d-morphism.*

Theorem 34. *Let θ be a modal formula such that* $\mathbf{wK4} + \theta$ *is cofinal subframe and canonical. Then* $\mu\mathbf{wK4} + \theta \vee \nu p.\Diamond p$ *is topologically complete.*

Proof. Suppose that ψ is consistent in $\mu\mathbf{wK4} + \theta \vee \nu p.\Diamond p$. We keep the notations of the proof of Theorem 31. We introduce the spaces $X := X_{\mathfrak{F}_\Sigma}$, $Y := \pi^{-1}[\Omega']$ and $Z := X \setminus Y$. We prove that Y and Z satisfy the conditions of Proposition 23. First, we know that $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}_\Sigma$, so Ω' is open, and thus so is $\pi^{-1}[\Omega'] = Y$. In addition, since $\mathfrak{F}' \vDash \neg\nu p.\Diamond p$, the frame $\mathfrak{F}'$ is irreflexive, so $Y = \Omega' \times \{\omega\}$ and $\pi_{|Y}$ is injective. Since π is a d-morphism, the maps π and π^{-1} are continuous, and since Y is open, so are $\pi_{|Y}$ and ${\pi_{|Y}}^{-1}$. Therefore $\pi_{|Y}$ is a homeomorphism between Y and $\mathfrak{F}'$. From $\mathfrak{F}' \vDash \theta$ and Proposition 11, we conclude that $Y \vDash \theta$.

We then prove that Z is dense-in-itself. Let $(\Gamma, \alpha) \in Z$ and U be an open neighbourhood of (Γ, α). From $(\Gamma, \alpha) \in Z$ we know that $\Gamma \notin \Omega'$, that is, $\mathfrak{M}_\Sigma, \Gamma \vDash \nu p.\Diamond p$. If $\alpha \neq \omega$, then Γ is reflexive. We select some $\beta \geq n^U_{w,\alpha}$ such that $\beta \neq \alpha$, and by definition of $n^U_{w,\alpha}$ we obtain $(\Gamma, \beta) \in U$. We also have $(\Gamma, \beta) \in Z$. Otherwise we have $\alpha = \omega$, and then Γ is irreflexive. From this and $\mathfrak{M}_\Sigma, \Gamma \vDash \nu p.\Diamond p$ we obtain the existence of $\Delta \neq \Gamma$ such that $\Gamma R \Delta$ and $\mathfrak{M}_\Sigma, \Delta \vDash \nu p.\Diamond p$. We set $\beta := n^U_{w,\alpha}$ if Δ is reflexive, and $\beta := \omega$ otherwise; we then have $(\Delta, \beta) \in Z$ by definition. Depending on whether $\Delta R \Gamma$ or not, we apply either item 1 or item 2 of Definition 32, and in both cases we obtain $(\Delta, \beta) \in U$. Both cases bring the existence of some element in $U \cap Z$ different from (Γ, α), and we are done.

It follows that $X \vDash \theta \vee \nu p.\Diamond p$. We know that ψ is satisfiable on $\mathfrak{F}_\Sigma$, and since π is a d-morphism it follows by Proposition 5 that ψ is satisfiable on X as well. This concludes the proof. □

In the following corollary, we finally apply these results to our examples.

Corollary 35. *For all $m \in \mathbb{N}$, the logic $\mu\mathbf{wK4} + \mathsf{IP}.2^+_m$ is Kripke and topologically complete.*

Proof. Since $(\Diamond\Diamond p \to p \vee \Diamond p) \wedge .2^+_m$ is a Sahlqvist formula, the logic $\mathbf{L}_0 := \mathbf{wK4} + .2^+_m$ is canonical [6, sec. 4.3]. In order to apply Theorem 31 and Theorem 34 we prove that $\mathbf{L}_0$ is cofinal subframe. Let $\mathfrak{F} = (W, R)$ be a $\mathbf{wK4}$ frame such that $\mathfrak{F} \vDash \mathbf{L}_0$, and let $\mathfrak{F}' = (W', R')$ be a cofinal subframe of $\mathfrak{F}'$.

Let $w \in W'$. First, suppose that $\mathfrak{F}, w \vDash .2^+$. Then if wR^+u and wR^+v with $u, v \in W'$, we have by assumption uR^+t and vR^+t for some $t \in W$. Then since $\mathfrak{F}'$ is cofinal in $\mathfrak{F}$ we have tR^+t' for some $t' \in W'$, and thus uR^+t' and vR^+t'. This proves that $\mathfrak{F}', w \vDash .2^+$. Otherwise there exists a valuation V such that $\mathfrak{F}, V, w \nvDash .2^+$, and since $\mathfrak{F}, V, w \vDash .2^+_m$ it follows that $\mathfrak{F}, V, w \vDash \Box^m \bot$. From this we deduce $\mathfrak{F}', w \vDash \Box^m \bot$. In both cases we obtain $\mathfrak{F}', w \vDash .2^+_m$. Therefore $\mathfrak{F}' \vDash \mathbf{L}_0$ and this proves the claim. □

5 Conclusion

We have established some fundamental results regarding the expressivity of the topological μ-calculus as opposed to basic modal logic. We have shown that the latter is indeed more expressive axiomatically than the former, a fact that was surprisingly difficult to prove. Accordingly, the examples we have exhibited are optimal in the sense that they involve topologically complete logics, which we have argued correspond to natural classes of spaces. In particular, they are related to the perfect core of a space, equivalent to the unary version of the tangled derivative, perhaps the most fundamental topological fixed point. This suggests that we are only scratching the surface of the jungle of spatial μ-logics, and their classification could be a bold new direction in the study of topological modal logics.

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