Weak bisimulations and weak mu-calculus

Quentin Gougeon

Unpublished note

1 Introduction

The modal μ-calculus is a logic that allows definition of operators through fixpoint equations, and praised for its high expressiveness. More precisely, Janin and Walukiewicz [JW96] proved that the μ-calculus is exactly the bisimulation-closed fragment of Monadic Second Order Logic (or MSO): any formula of MSO that can not discriminate bisimilar transitions systems, can be translated in the μ-calculus.

The standard notion of bisimulation admits several variants, including weak bisimulation [Mil89]. We have identified the weak bisimulation-closed fragment of MSO, which appears to be a variant of the μ-calculus that we call the *weak μ-calculus*. In section 2, we introduce notations and recall some standard results. In section 3, we prove that the weak μ-calculus is indeed embedded into MSO, an easy but necessary step. In section 4, we introduce τ-equivalence between states and show that it implies logical equivalence with respect to the weak μ-calculus. In section 5, we define the closure of a transition system, which is similar to the construction used to eliminate ϵ-transitions in automata. This powerful tool allows us to show that the weak μ-calculus is closed under weak bisimulation. In section 6, we prove the final result as follows: given a weak bisimulation-closed class $\mathcal{C}$ of transition systems, we apply Janin and Walukiewicz's theorem, before tweaking the resulting formula of the μ-calculus into a formula of the weak μ-calculus; we then show that this new formula defines $\mathcal{C}$.

2 Background

For everything below we consider A a set of *actions*, $\tau \in \mathsf{A}$ and Prop a set of *atomic propositions*.

2.1 Transition systems

Definition. A transition system is a tuple $\mathcal{M} = (S, s_0, (a^{\mathcal{M}})_{a\in\mathsf{A}}, (p^{\mathcal{M}})_{p\in\mathsf{Prop}})$ where:

- S is a set of *states*;
- $s_0 \in S$ is an *initial state*;
- for every $a \in \mathsf{A}$, $a^{\mathcal{M}}$ is a binary relation over S;
- for every $p \in \mathsf{Prop}$, $p^{\mathcal{M}}$ is a subset of S.

Given two states s and t, $s \xrightarrow{a}_{\mathcal{M}} t$ holds for $(s, t) \in a^{\mathcal{M}}$. This notation can be extended to any regular expression e: we write $s \xrightarrow{e}_{\mathcal{M}} t$ whenever there exists a word $a_1 \dots a_n \in \mathcal{L}(e)$ and some states $(t_i)_{i\ \in [\![0,n]\!]}$ such that $t_0 = t$, $t_n = t$ and for all $i \in [\![1, n-1]\!]$, $t_i \xrightarrow{a_i}_{\mathcal{M}} t_{i+1}$. Intuitively, $s \xrightarrow{a}_{\mathcal{M}} t$ means that action a performs a *transition* from state s to state t, while $s \in p^{\mathcal{M}}$ means that property p is true at state s.

2.2 Bisimulations

The notion of bisimulation between two transition systems is meant to express the property of behaving in the same way.

Definition. A *bisimulation* between two transition systems $\mathcal{M}$ and $\mathcal{M}'$ is a relation $\mathfrak{R} \subseteq S \times S'$ such that for every $(s, s') \in \mathfrak{R}$:

- $\forall p \in \mathsf{Prop}, s \in p^{\mathcal{M}} \iff s' \in p^{\mathcal{M}'}$;
- for all $a \in \mathsf{A}$ and $t \in S$, if $s \xrightarrow{a}_{\mathcal{M}} t$, then there exists $t' \in S'$ such that $(t, t') \in \mathfrak{R}$ and $s' \xrightarrow{a}_{\mathcal{M}'} t'$;
- for all $a \in \mathsf{A}$ and $t' \in S$, if $s' \xrightarrow{a}_{\mathcal{M}'} t'$, then there exists $t \in S$ such that $(t, t') \in \mathfrak{R}$ and $s \xrightarrow{a}_{\mathcal{M}} t$.

If $(s, s') \in \mathfrak{R}$ then we write $\mathcal{M}, s \leftrightarroweq \mathcal{M}', s'$. If $(s_0, s'_0) \in \mathfrak{R}$ then we write $\mathcal{M} \leftrightarroweq \mathcal{M}'$ and in this case $\mathcal{M}$ and $\mathcal{M}'$ are said to be *bisimilar*.

Weak bisimulation is a variant where τ-transitions are considered to be invisible to an external observer.

Definition. A *weak bisimulation* between two transition systems $\mathcal{M}$ and $\mathcal{M}'$ is a relation $\mathfrak{R} \subseteq S \times S'$ such that for every $(s, s') \in \mathfrak{R}$:

- for all $p \in \mathsf{Prop}$ and $t \in p^{\mathcal{M}}$, if $s \xrightarrow{\tau^*}_{\mathcal{M}} t$, then there exists $t' \in p^{\mathcal{M}'}$ such that $s' \xrightarrow{\tau^*}_{\mathcal{M}'} t'$;
- for all $t \in S$, if $s \xrightarrow{\tau^*}_{\mathcal{M}} t$, then there exists $t' \in S'$ such that $(t, t') \in \mathfrak{R}$ and $s' \xrightarrow{\tau^*}_{\mathcal{M}'} t'$;
- for all $a \in \mathsf{A} \setminus \{\tau\}$ and $t \in S$, if $s \xrightarrow{\tau^* a \tau^*}_{\mathcal{M}} t$, then there exists $t' \in S'$ such that $(t, t') \in \mathfrak{R}$ and $s' \xrightarrow{\tau^* a \tau^*}_{\mathcal{M}'} t'$;
- for all $p \in \mathsf{Prop}$ and $t' \in p^{\mathcal{M}'}$, if $s' \xrightarrow{\tau^*}_{\mathcal{M}'} t'$, then there exists $t \in p^{\mathcal{M}}$ such that $s \xrightarrow{\tau^*}_{\mathcal{M}} t$;
- for all $t' \in S'$, if $s' \xrightarrow{\tau^*}_{\mathcal{M}'} t'$, then there exists $t \in S$ such that $(t, t') \in \mathfrak{R}$ and $s \xrightarrow{\tau^*}_{\mathcal{M}} t$;
- for all $a \in \mathsf{A} \setminus \{\tau\}$ and $t' \in S'$, if $s' \xrightarrow{\tau^* a \tau^*}_{\mathcal{M}'} t'$, then there exists $t \in S$ such that $(t, t') \in \mathfrak{R}$ and $s \xrightarrow{\tau^* a \tau^*}_{\mathcal{M}} t$;

If $(s, s') \in \mathfrak{R}$ then we write $\mathcal{M}, s \leftrightarroweq_\tau \mathcal{M}', s'$. If $(s_0, s'_0) \in \mathfrak{R}$ then we write $\mathcal{M} \leftrightarroweq_\tau \mathcal{M}'$ and in this case $\mathcal{M}$ and $\mathcal{M}'$ are said to be *weakly bisimilar*.

Note that $\leftrightarroweq$ and $\leftrightarroweq_\tau$ are both equivalence relations, and that any bisimulation is also a weak bisimulation.

2.3 Logics over transition systems

Here we recall the syntax of MSO and the μ-calculus, but we do not expand on their semantics since it is well known. We also introduce a variant of the μ-calculus named the weak μ-calculus. We fix a countable set Var of *variables*.

Definition. The syntax of MSO is defined as follows:

$$\phi = \mathsf{r}(X) \mid p(X) \mid a(X, Y) \mid X \subseteq Y \mid \neg\phi \mid \phi \wedge \phi \mid \exists X.\phi(X)$$

Definition. The set $\mathcal{L}_\mu$ of the formulas of the μ-calculus is defined as follows:

$$\phi = p \mid X \mid \phi \wedge \phi \mid \neg\phi \mid \langle a \rangle \phi \mid \mu X.\phi$$

where in every formula of the form $\mu X.\phi$, the variable X only appears under an even number a negations.

Given a transition system $\mathcal{M}$ and a valuation $V : \mathsf{Var} \to \mathcal{P}(S)$, we denote by $[\![\phi]\!]^{\mathcal{M},V} \subseteq S$ the extension of ϕ in $\mathcal{M}$. We $\mathcal{M}, V, s \vDash \phi$ whenever $s \in [\![\phi]\!]^{\mathcal{M},V}$. If ϕ has no free variable, we simply write $\mathcal{M}, s \vDash \phi$. In case $s = s_0$ we simply write $\mathcal{M}, V \vDash \phi$. If V is a valuation, $X \in \mathsf{Var}$ and $B \subseteq S$, the valuation $V[B/X]$ is defined by:

- $V[B/X](X) = B$
- $V[B/X](Y) = V(Y)$ for all $Y \neq X$

We also recall well known properties of the μ-calculus [BvBW06]:

Proposition 1. MSO is more expressive than the μ-calculus, i.e. every formula of $\mathcal{L}_\mu$ is equivalent to some formula of MSO.

Proposition 2. The μ-calculus is bisimulation-closed, i.e. for any close formula $\phi \in \mathcal{L}_\mu$, if $\mathcal{M} \vDash \phi$ and $\mathcal{M} \underline{\leftrightarrow} \mathcal{M}'$ then $\mathcal{M}' \vDash \phi$.

Just like a weak bisimulation is a bisimulation where τ-transitions are counted for free, the weak μ-calculus is the μ-calculus without the ability to talk about τ-successors:

Definition. The set $\mathcal{L}_\mu^\tau$ of the formulas of the weak μ-calculus is defined as follows:

$$\psi = p^\tau \mid X^\tau \mid \psi \wedge \psi \mid \neg\psi \mid \langle \tau^* a \tau^* \rangle \psi \mid \langle \tau^* \rangle \psi \mid \mu X.\psi$$

where $a \neq \tau$ and, as above, X appears under an even number a negations in every formula of the form $\mu X.\psi$. To avoid misunderstanding when dealing with those two versions of the μ-calculus, ϕ will always denote a formula of $\mathcal{L}_\mu$, while ψ will always denote a formula of $\mathcal{L}_\mu^\tau$. The semantics for those new formulas is the following:

- $[\![p^\tau]\!]^{\mathcal{M},V} = \{s \in S \mid \exists t \in p^{\mathcal{M}},\ s \xrightarrow{\tau^*}_{\mathcal{M}} t\}$
- $[\![X^\tau]\!]^{\mathcal{M},V} = \{s \in S \mid \exists t \in V(X),\ s \xrightarrow{\tau^*}_{\mathcal{M}} t\}$
- $[\![\langle \tau^* \rangle \psi]\!]^{\mathcal{M},V} = \{s \in S \mid \exists t \in [\![\psi]\!]^{\mathcal{M},V},\ s \xrightarrow{\tau^*}_{\mathcal{M}} t\}$
- $[\![\langle \tau^* a \tau^* \rangle \psi]\!]^{\mathcal{M},V} = \{s \in S \mid \exists t \in [\![\psi]\!]^{\mathcal{M},V},\ s \xrightarrow{\tau^* a \tau^*}_{\mathcal{M}} t\}$

2.4 Classes of models

Let ϕ be a close MSO formula. We denote by $\mathsf{mod}(\phi) = \{\mathcal{M} \text{ transition system} \mid \mathcal{M} \vDash \phi\}$ the class of all models of ϕ. Let $\mathcal{C}$ be a class of transition systems. Given a logic $\mathcal{L}$, $\mathcal{C}$ is said to be *$\mathcal{L}$-definable* if there exists a close formula $\phi \in \mathcal{L}$ such that $\mathsf{mod}(\phi) = \mathcal{C}$. Moreover, $\mathcal{C}$ is said to be *closed under bisimulation* (respectively *closed under weak bisimulation*) if $\mathcal{M} \in \mathcal{C}$ and $\mathcal{M} \underline{\leftrightarrow} \mathcal{M}'$ (respectively $\mathcal{M} \underline{\leftrightarrow}_\tau \mathcal{M}'$) implies $\mathcal{M}' \in \mathcal{C}$. The theorem attributed to Janin and Walukiewicz [JW96] we mentioned in the introduction is the following:

Theorem 1. Every MSO-definable and bisimulation-closed class is $\mathcal{L}_\mu$-definable.

3 Matters of expressivity

The first step is to show that MSO is more expressive than the weak μ-calculus, and that the weak μ-calculus is closed under bisimulation.

Proposition 3. The μ-calculus is more expressive than the weak μ-calculus.

Proof. We define $\mathsf{tr} : \mathcal{L}_\mu^\tau \to \mathcal{L}_\mu$ by induction:

- $\mathsf{tr}(p^\tau) = \mu X.(p \vee \langle \tau \rangle X)$
- $\mathsf{tr}(X^\tau) = \mu Y.(X \vee \langle \tau \rangle Y)$
- $\mathsf{tr}(\langle \tau^* \rangle \psi) = \mu X.(\langle \tau \rangle X \vee \mathsf{tr}(\psi))$
- $\mathsf{tr}(\langle a \rangle^\tau \psi) = \mu X.(\langle \tau \rangle X \vee \langle a \rangle \mu Y.(\mathsf{tr}(\psi) \vee \langle \tau \rangle Y))$
- $\mathsf{tr}(\neg\psi) = \neg \mathsf{tr}(\psi)$

- $\mathsf{tr}(\psi \wedge \psi') = \mathsf{tr}(\psi) \wedge \mathsf{tr}(\psi')$
- $\mathsf{tr}(\mu X.\psi) = \mu X.\mathsf{tr}(\psi)$

It is then clear that $\mathsf{tr}(\psi) \equiv \psi$ for all $\psi \in \mathcal{L}^\tau_\mu$. □

From proposition 1, proposition 2 and proposition 3 we derive the following results:

Corollary 1. Every formula of $\mathcal{L}^\tau_\mu$ is equivalent to some formula of MSO.

Corollary 2. The weak μ-calculus is closed under bisimulation.

4 The relation of τ-equivalence

In this section we define the relation of τ-equivalence between states: two states are τ-equivalent if one can be reached from another through a sequence of τ-transitions. We then prove lemma 1, a crucial result that we will use extensively, and which states that τ-equivalent states satisfy the same formulas of the weak μ-calculus.

Definition. Let $\mathcal{M}$ be a transition system. The equivalence relation $\sim_\tau$ is defined by

$$s \sim_\tau t \iff s \xrightarrow{\tau^*}_{\mathcal{M}} t \ \wedge \ t \xrightarrow{\tau^*}_{\mathcal{M}} s$$

for all $s, t \in S$.

Lemma 1. Assume that $s \sim_\tau t$. Then for all close formula $\psi \in \mathcal{L}^\tau_\mu$, we have $\mathcal{M}, s \vDash \psi$ if and only if $\mathcal{M}, t \vDash \psi$.

Proof. We proceed by induction on ψ. The only non-trivial case is for formulas of the form $\mu X.\psi$. So assume that $H(\psi)$ and let V be a valuation. Assume that $\mathcal{M}, V, s \vDash \mu X.\psi$ and let B^* be the smallest fixpoint of $B \mapsto [\![\psi]\!]^{\mathcal{M},V[B/X]}$. We have $s \in B^* = [\![\psi]\!]^{\mathcal{M},V[B^*/X]}$, so $\mathcal{M}, V[B^*/X], s \vDash \psi$, so $\mathcal{M}, V[B^*/X], t \vDash \psi$ as well according to $H(\psi)$. Consequently $t \in B^*$ and $\mathcal{M}, V, t \vDash \mu X.\psi$. The proof of the other implication is similar. □

5 τ-closures

Here we define the τ-closure of a transition system $\mathcal{M}$. For any $s \in S$, we write $\overline{s} = \{t \in S \mid s \xrightarrow{\tau^*}_{\mathcal{M}} t\}$. For any $B \subseteq S$, we write $\overline{B} = \{\overline{s} \mid s \in S\}$. For any $V : \mathsf{Var} \to \mathcal{P}(S)$, we write $\overline{V} : X \mapsto \overline{V(X)}$.

Definition. The *τ-closure* of a transition system $\mathcal{M}$ is the system $\overline{\mathcal{M}} = (\overline{S}, \overline{s_0}, (a^{\overline{\mathcal{M}}})_{a\in\mathsf{A}}, (p^{\overline{\mathcal{M}}})_{p\in\mathsf{Prop}})$ where:

- for all $a \in \mathsf{A} \setminus \{\tau\}$, $a^{\overline{\mathcal{M}}} = \{(\overline{s}, \overline{t}) \in \overline{S}^2 \mid s \xrightarrow{\tau^* a \tau^*}_{\mathcal{M}} t\}$;
- $\tau^{\overline{\mathcal{M}}} = \{(\overline{s}, \overline{t}) \in \overline{S}^2 \mid s \xrightarrow{\tau^*}_{\mathcal{M}} t\}$
- for all $p \in \mathsf{Prop}$, $p^{\overline{\mathcal{M}}} = \{\overline{s} \in S \mid \exists t \in p^{\mathcal{M}}, s \xrightarrow{\tau^*}_{\mathcal{M}} t\}$.

Note that $a^{\overline{\mathcal{M}}}$ and $p^{\overline{\mathcal{M}}}$ are well defined, as they do not depend on the representative of any element of $\overline{S}$.

Proposition 4. Let $\mathcal{M}$ be a transition system. Then $\mathcal{M} \underline{\leftrightarrow}_\tau \overline{\mathcal{M}}$.

Proof. It suffices to point out that $\mathfrak{R} = \{(s, \overline{s}) \mid s \in S\}$ is a weak bisimulation. □

Proposition 5. If $\mathcal{M} \underline{\leftrightarrow}_\tau \mathcal{M}'$ then $\overline{\mathcal{M}} \underline{\leftrightarrow} \overline{\mathcal{M}'}$.

Proof. Suppose $\mathfrak{R}$ is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$ such that $(s_0, s'_0) \in \mathfrak{R}$. One can see that $\overline{\mathfrak{R}} = \{(\overline{s}, \overline{s'}) \mid (s, s') \in \mathfrak{R}\}$ is a weak bisimulation relating $\overline{s_0}$ to $\overline{s'_0}$. □

Lemma 2. Let $\psi \in \mathcal{L}^\tau_\mu$ and $\mathcal{M}$ a transition system. Then $\mathcal{M} \vDash \psi$ if and and only if $\overline{\mathcal{M}} \vDash \psi$.

Proof. We proceed by induction on ψ. Again, we only consider the case $\mu X.\psi$.

- Assume that $\mathcal{M}, V, s \vDash \mu X.\psi$ and let B^* be the smallest fixpoint of $\overline{F} : B \mapsto [\![\psi]\!]^{\overline{\mathcal{M}},\overline{V}[B^*/X]}$. Writing $B' = \{t \in S \mid \overline{t} \in B^*\}$ it is clear that $B^* = \overline{B'}$, and also that $\overline{V}[B^*/X] = \overline{V}[\overline{B'}/X] = \overline{V[B'/X]}$. We now show that B' is the smallest fixpoint of $F : B \mapsto [\![\psi]\!]^{\mathcal{M},V[B/X]}$.

  - We have

    $$\begin{aligned} \mathcal{M}, V[B'/X], t \vDash \psi &\iff \overline{\mathcal{M}}, \overline{V}[\overline{B'}/X], \overline{t} \vDash \psi && \text{by the induction hypothesis} \\ &\iff \overline{t} \in \overline{B'} && \text{because } \overline{B'} \text{ is a fixpoint of } \overline{F} \\ &\iff t \in B' \end{aligned}$$

    which means that $F(B') = B'$
  - Let B'' bet an other fixpoint of F. To show that $B' \subseteq B''$, we first prove that $\overline{B''}$ is a fixpoint of $\overline{F}$. If $\overline{t} \in \overline{F}(\overline{B''})$ then $t \in F(B'')$ by the induction hypothesis, so $t \in B''$ by assumption, which means that $\overline{t} \in \overline{B''}$.
    Conversely, if $\overline{t} \in \overline{B''}$, then there exists $u \in B''$ such that $\overline{t} = \overline{u}$. Then $u \in F(B'')$, so $t \in F(B'')$ as well thanks to lemma 1. Consequently we have $\overline{t} \in \overline{F}(\overline{B''})$ by the induction hypothesis.
    Since $\overline{B'}$ is the smallest fixpoint of $\overline{F}$, we get $\overline{B'} \subseteq \overline{B''}$ and we now prove that it implies $B' \subseteq B''$. Let $t \in B'$. We have $\overline{t} \in \overline{B''}$ by assumption, so there exists $u \in B''$ such that $\overline{t} = \overline{u}$. Using lemma 1 again, we deduce from $u \in F(B'')$ that $t \in F(B'')$ as well, which means that $t \in B''$.

  Now the conclusion comes easily: by assumption we have $s \in \overline{B'}$, and therefore $\overline{s} \in \overline{B'} = B^*$ which means that $\overline{\mathcal{M}}, \overline{V}, \overline{s} \vDash \mu X.\psi$.

- Conversely, assume $\overline{\mathcal{M}}, \overline{V}, \overline{s} \vDash \mu X.\psi$. We only give the proof scheme which is very similar to the previous one (here again lemma 1 is very useful). Let B^* be the smallest fixpoint of F. We show that $\overline{B^*}$ is the smallest fixpoint of $\overline{F}$.

  - We check that $\overline{F}(\overline{B^*}) = \overline{B^*}$.
  - Given a fixpoint B' of $\overline{F}$, let $B'' = \{s \in S \mid \overline{s} \in B'\}$ so that $B' = \overline{B''}$. Again we show that B'' is a fixpoint of F, which implies $B^* \subseteq B''$. From this we obtain $\overline{B^*} \subseteq \overline{B''} = B'$.

  Consequently $\overline{s} \in \overline{B^*}$ by assumption and lemma 1 yields $s \in B^*$, which means that $\mathcal{M}, V, s \vDash \mu X.\psi$.

□

We can now assert the following:

Theorem 2. $\mathcal{L}_\mu^\tau$ is closed under weak bisimulation.

Proof. Let $\psi \in \mathcal{L}_\mu^\tau$ and assume $\mathcal{M} \vDash \psi$ and $\mathcal{M} \underline{\leftrightarrow}_\tau \mathcal{M}'$. By lemma 2 we get $\overline{\mathcal{M}} \vDash \psi$. By proposition 5 we also have $\overline{\mathcal{M}} \underline{\leftrightarrow} \overline{\mathcal{M}'}$. Therefore corollary 2 yields $\overline{\mathcal{M}'} \vDash \psi$. By lemma 2 again we finally get $\mathcal{M}' \vDash \psi$. □

6 Formula weakening

As mentioned in the introduction, we now introduce a way to weaken a formula of the μ-calculus into a formula of the weak μ-calculus. This operation presents some desirable properties regarding the τ-closure, and will allow us to derive theorem 3 from theorem 1 in a quite straightforward way.

Definition. The application $f : \mathcal{L}_\mu \to \mathcal{L}_\mu^\tau$ is defined as follows:

- $f(p) = p^\tau$
- $f(X) = X^\tau$
- $f(\langle\tau\rangle\phi) = \langle\tau^*\rangle f(\phi)$

- $f(\langle a\rangle\phi) = \langle\tau^* a\tau^*\rangle f(\phi)$ for all $a \in \mathsf{A} \setminus \{\tau\}$
- $f(\neg\phi) = \neg f(\phi)$
- $f(\phi \wedge \phi') = f(\phi) \wedge f(\phi')$
- $f(\mu X.\phi) = \mu X.f(\phi)$

Lemma 3. Let $\mathcal{M}$ be a transition system and $\phi \in \mathcal{L}_\mu$. Then $\overline{\mathcal{M}} \vDash \phi$ if and only if $\overline{\mathcal{M}} \vDash f(\phi)$.

Proof. It is not difficult to prove by induction on ψ that for all valuation $V : \mathsf{Var} \to \mathcal{P}(\overline{S})$, for all $\overline{s} \in \overline{S}$, we have $\overline{\mathcal{M}}, V, \overline{s} \vDash \phi$ if and only if $\overline{\mathcal{M}}, V, \overline{s} \vDash f(\phi)$. □

From lemma 2 and lemma 3 we get the following corollary:

Corollary 3. Let $\mathcal{M}$ be a transition system and $\phi \in \mathcal{L}_\mu$. Then $\overline{\mathcal{M}} \vDash \phi$ if and only if $\mathcal{M} \vDash f(\phi)$.

We are now ready to prove our final theorem:

Theorem 3. Every MSO-definable and weak bisimulation-closed class is $\mathcal{L}_\mu^\tau$-definable.

Proof. Let $\mathcal{C}$ be a MSO-definable, weak bisimulation-closed class. Then $\mathcal{C}$ is bisimulation-closed as well and from theorem 1 there exists $\phi \in \mathcal{L}_\mu$ such that $\mathcal{C} = \mathsf{mod}(\phi)$. We show that $\mathcal{C} = \mathsf{mod}(f(\phi))$:

$$\begin{array}{rcll} \mathcal{M} \vDash f(\phi) & \iff & \overline{\mathcal{M}} \vDash \phi & \text{by corollary 3} \\ & \iff & \overline{\mathcal{M}} \in \mathcal{C} & \text{by assumption on } \phi \\ & \iff & \mathcal{M} \in \mathcal{C} & \text{because proposition 4 yields } \mathcal{M} \underline{\leftrightarrow}_\tau \overline{\mathcal{M}} \end{array}$$

□

References

[BvBW06] Patrick Blackburn, Johan van Benthem, and Frank Wolter. *Handbook of modal logic*, volume 3. Elsevier, 2006.

[JW96] David Janin and Igor Walukiewicz. On the expressive completeness of the propositional mu-calculus with respect to monadic second order logic. In *International Conference on Concurrency Theory*, pages 263–277. Springer, 1996.

[Mil89] Robin Milner. *Communication and concurrency*, volume 84. Prentice hall New York etc., 1989.