Euclidean Structure from Confocal Conics: Theory and Application to Camera Calibration

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Abstract

Plane-based calibration is now a very popular procedure because of its flexibility. One key step consists in detecting a set of coplanar features, from which the Euclidean structure of the corresponding 3D plane has to be computed. We suggest to use confocal conics as calibration targets, as they offer undeniable advantages over other ones (e.g., points or lines) in terms of detection and estimation, especially in the presence of partial occlusion. We introduce important projective and Euclidean properties of the linear family of conics (i.e., the confocal conic range), spanned by two confocal conics. In particular, we rely on the fact that the circular point-envelope – a rank-2 conic that encodes the 2D Euclidean structure – is a degenerate member of any confocal conic range. This allows us to give closed-form solutions in three cases: one conic with known foci, two confocal conics with known product of ratios of semi axes, and two unknown confocal conics. In addition to experiments with synthetic data, a video sequence is processed, showing off the interest of using confocal conics as calibration targets, for augmented reality purposes.

1. Introduction

From the early photogrammetric techniques [14] to more recent flexible ones [10][15][19][20], camera calibration [2][3][7] has been widely investigated during the past decades. Not only closed-form solutions exist for calibration with 3D, 2D and even 1D patterns, but the singularities causing the algorithms to fail are now identified.

Plane-based (or 2D) calibration [10][15][19] is a very popular procedure because of its flexibility, with efficient implementations through publicly available toolboxes. A 2D calibration software carries out three basic tasks: (i) detection and matching of coplanar features; (ii) recovery of induced 2D Euclidean structures; (iii) camera calibration and pose estimation i.e., recovery of the 3D Euclidean structure. Regarding intrinsic calibration, albeit three views are sufficient to get a solution, in practice, much more are used. Relating this remark to the first task, in order to make calibration a totally automatic procedure, a “good” planar pattern must be easily detectable, by exhibiting some kind of discriminant information in the views. Numerous types of planar calibration patterns have been devised (see [8]), the most widespread pattern being a point-based checkerboard, whose detection and matching may require some user interaction. The interest of using a circle-pair as 2D pattern has been recently reported [8][9][18] as it offers undeniable advantages over other features (i.e., point- or line-based). Because of intrinsic geometric properties: (i) such a pattern can be easily detected and estimated, even if partially occluded; (ii) it does not require to be matched with some 2D model; (iii) it naturally encodes the plane’s Euclidean structure. On this matter, concentric circles exhibit additional geometric properties which reveals to be ideally suited for feature detection [8] or Euclidean structure recovery [9]. In our opinion, even it is easily admitted that research about calibration has lived its time, these recent papers show that the flexibility and accuracy of calibration can even be more increased, tending towards completely automatic methods.

In this work, we suggest to use a set of \( N \geq 2 \) confocal conics as calibration pattern, offering the same advantages, mentioned above, as a set of \( N \geq 2 \) concentric circles, without its key drawback. Indeed, in multiple views, it is not possible to attach the same XY-frame to a plane supporting concentric circles because of the 1-parameter ambiguity corresponding to the d.o.f. in the 2D rotation around their common centre\(^1\). However, if concentric circles have an infinite number of possible 2D poses, this is not the case for

\(^1\)Practically speaking, using concentric circles, it would have been impossible to integrate the virtual teapot in the video [1], cf. Fig.5.
confocal conics, for which this number reduces to two (like any symmetric pattern e.g., by fixing the axes of the XY-frame, w.r.t. all views, to coincide with the major and minor axes of the confocal conics. In the context of video processing, note that the “wrong pose” can be easily discarded using two consecutive images (see video in [1]), by assuming “small” orientation changes. Eventually, it is worth of note that designing calibration patterns is more flexible when using a set of confocal conics than a set of concentric circles since the former set has two d.o.f. (the semi-major and semi-minor axes) instead of one (the radius).

The key problem is that of recovering the 2D Euclidean structure of a plane supporting two confocal central conics i.e., hyperbolas or ellipses with same foci (e.g., see Fig. 2), in one view. To solve it, all we have at our disposal are their images, in pixel representation. We make the same assumptions as in [18], the world-to-image homography is a quasi-affine transformation [7] i.e., all the observed conics lie in front of the camera. Throughout §3 to §5, we will investigate algebraic and geometric constraints on images of two confocal central conics. Remind that, even if a conic is the former set has two d.o.f. (the semi-major and semi-minor axes) instead of one (the radius).

2. Plane-Based Calibration

Since plane-based calibration and its implementation has been widely outlined in [15][19], we will not give much de-tails here. Let us just remind that the affine structure of a plane is given by the line at infinity as $l_{\infty} = (0, 0, 1)^T$ in every affine representation. Its Euclidean structure is given by two complex conjugate points $I, J$ of $l_{\infty}$, called circular points [13], common to all circles including the absolute conic [7]. They have the canonical forms $I = (1, i, 0)^T$, $J = (1, -i, 0)^T$, in every Euclidean representation. As a direct result of Triggs’ seminal work [17], the plane-based calibration problem can be stated as that of fitting the image of the absolute conic to, at least, three images of circular points, providing a closed-form solution for the intrinsic parameters [15][19]. Once these are estimated, providing a 2D frame can be “fixed” in the plane, the relative poses between planes and cameras (i.e., the external parameters of the camera) can be recovered [16][19].

3. Geometry of Conics and Conic Ranges

Notation-wise, $i$ will always denote $\sqrt{-1}$. Lower-case letters in bold e.g., $f$, denote the homogeneous 3-vectors of 2D points or lines. In uppercase, they denote complex vectors e.g., $F$. Upper-case letters in sans serif font denote (real) $3 \times 3$-matrices e.g., $C$.

In this work, we restrict the term conics to only refer coplanar conics.

3.1. Projective Geometry of Conics

The conic envelope is the dual of a conic locus, as the envelope of lines is the dual of the locus of points. Conic loci and their envelopes are represented by symmetric order-3 matrices, denoted by $C$ and $C^*$ respectively, where $C^*$ is the adjugate2 matrix of $C$. Hence, $C^*$ is the envelope of the set of lines $t \in C^3$ tangent to $C$, satisfying $t^T C^* t = 0$.

A conic envelope $C^*$ is proper or degenerate, whether $C^*$ is or not full rank; a degenerate conic envelope $C^*$ consists of two points $p, q$ such that $C^* \sim pq^T + qp^T$. The line through $p$ and $q$ satisfies $p \times q \sim \text{null} C^*$.

In the sequel, in order to distinguish proper from degenerate envelopes, we will write $pq^T + qp^T \equiv [pq]$.

3.2. Euclidean Geometry of Conics

The center and focus of a conic $C$ are, respectively, affine and Euclidean characterizations of $C$. The center $c$ is defined as the pole of the line at infinity $l_{\infty}$ w.r.t. the envelope $C^*$ i.e., $c \sim C^* l_{\infty}$. Conics whose center is finite are known as central conics: these are ellipses or hyperbolas. A point $m$ is said to be a focus of $C$ if the lines $m \times I$ and $m \times J$ are tangent to $C$. A conic has four foci, two of which are real and two conjugate complex. It can be readily seen that if $f$,
3.3. Projective Geometry of Conic Ranges

Two distinct conic envelopes \( C_1, C_2 \) span the conic range \( C^*(\lambda) \) i.e., the linear family of conic envelopes of the form:

\[
C^*(\lambda) \equiv C_1^* - \lambda C_2^*,
\]

\( \lambda \in \mathbb{C} \) is called parameter and \( C_1^*, C_2^* \) are called bases.

All envelopes in (3) touch the four tangents, either real or conjugate complex, common to \( C_1 \) and \( C_2 \). A conic range includes degenerate conic envelopes as members. These are three – consisting of point-pairs through the four common tangents: from the six ways of combining two tangents out of four – providing six intersection points – there are three ways of combining two points out of these six. Note that substituting any member of \( C^*(\lambda) \), including degenerate ones, for \( C_1^* \) or \( C_2^* \) in (3), we get the same conic range.

Algebraically, the problem of determining the degenerate members of \( C^*(\lambda) \) is that of solving \( \det C^*(\lambda) = 0 \) for \( \lambda \) i.e., of computing the generalized eigenvalues \( \lambda_k \in \mathbb{C} \), \( k \in \{1...3\} \), of the pair \((C_1^*, C_2^*)\). The vectors \( z_k \in \mathbb{C}^3 \) – representing lines – that satisfy \((C_1^* - \lambda_k C_2^*)z_k = 0_3 \), are associated generalized eigenvectors [6].

Projective invariants for pairs of conics have been widely investigated in the early 90's [5][11]. Some of them may involve generalized eigenvalues as follows. Let \( C_1^*, C_2^* \) be the images of \( C_1, C_2 \) under some non-singular homography \( H \in \mathbb{R}^{3\times3} \), satisfying:

\[
\tilde{C}_j^* = \tilde{\mu}_j HC_j^*H^T, \quad \tilde{\mu}_j \neq 0.
\]

where \( j \in \{1, 2\} \). Without loss of generality, assume:

\[
\det C_j^* = \det \tilde{C}_j^* = 1.
\]

so \( \det(C_1^* \tilde{C}_2^*) = \tilde{\mu}_1/\tilde{\mu}_2 \det(C_1^* C_2^*) = 1 \) implies \( \tilde{\mu}_1 = \tilde{\mu}_2 \). The generalized eigenvalues of the matrix-pair \((C_1^*, C_2^*)\) are also the eigenvalues of the matrix \( C_1^* C_2^* \), which are well-known to be preserved under the transformation that maps \( C_1 C_2 \) to \( HC_1^* C_2^* H^{-1} \) [12]. Since \( HC_1^* C_2^* H^{-1} = \tilde{C}_1^* \tilde{C}_2^* \), this entails that the generalized eigenvalues of \((C_1^*, C_2^*)\) are invariant – as a set – under transformations (4) of bases.

3.4. Euclidean Geometry of Conic Ranges

Let \( C_1^*, C_2^* \in \mathbb{R}^{3\times3} \) be two central conic envelopes.

**Proposition 1** A necessary and sufficient condition for (3) to be a confocal conic range is to include \([IJ]\) as member.

**Proof.** (\( \Rightarrow \)) Assume \( C_1^*, C_2^* \) to be confocal. Using the canonical forms of (2) with \( \mu = 1 \), we have \( C_1 - C_2 = (a_1^2 - a_2^2)[IJ] \). (\( \Leftarrow \)) Assume there exists \( \lambda_0 \neq 0 \) such that \([IJ] = C_1^* - \lambda_0 C_2^* \). Write \( C_2^* \) in the form of (2) with \( c = 1 \).

Hence, \( C_1^* = \lambda_0 \operatorname{diag}(\gamma, \gamma - 1, -1) \), where \( \gamma \equiv a_2^2 + 1/\lambda_0 \). The case \( \gamma \leq 0 \) is impossible, otherwise \( C_1^* \) would have
eigenvalues of same signs and so would be the envelope of a set of complex lines, cf. [7, p. 60]. Thus \( \gamma > 0 \) so \( C_1 \) has the form (2), with foci (1) i.e., is confocal with \( C_2 \). ■

It may be derived that if (3) is a confocal conic range i.e., if \( C_1^*, C_2^* \) are confocal bases, then it also includes \([ \tilde{g} \ g] \) and \([ \tilde{F} \ G] \) as degenerate members, in addition to \([ \tilde{I} \ J] \). Indeed, if \( C_1^* \) and \( C_2^* \) are confocal, then the two bases \( C_1^* \) and \([ \tilde{I} \ J] \) span the same conic range than the two bases \( C_2^* \) and \([ \tilde{I} \ J] \).

By intersecting \([ \tilde{I} \ J] \) with \( C_1^* \), we obtain the four isotropic lines \((\tilde{f} \times \tilde{I}), (\tilde{F} \times \tilde{J}), (\tilde{g} \times \tilde{I}) \) and \((\tilde{G} \times \tilde{J})\), shown in Fig. 1, as the four common tangents of the range (by definition of a focus). By picking two points out of the six intersection points obtained by combining two lines out of these four tangents, we obtain \([ \tilde{f} \ g], [ \tilde{F} \ G] \) or \([ \tilde{I} \ J] \).

![Figure 2. A pencil of confocal ellipses and hyperbolas.](image)

**4. Recovering the 2D Euclidean Structure from Images of Central Con focal Conics**

Let \( \tilde{C}_0 \) be the image of a central conic \( C \) under \( H \) and denote by \( \tilde{f}, \tilde{g}, \tilde{F}, \tilde{G} \) the images of its foci \( f, g, F, G \). Our concern is to recover the 2D Euclidean structure of the supporting plane, by computing the image \([ \tilde{I} \tilde{J}] \) of its circular-point envelope \([ IJ] \). To that end, denote by \( C^*(\lambda) \) the range of conics confocal with \( C_0 \) and by \( \tilde{C}^*(\lambda) \) the conic range that is homographically related to \( C^*(\lambda) \) via \( H \) i.e., whose degenerate members are \([ \tilde{g} \ g], [ \tilde{F} \ G] \) and \([ \tilde{I} \ J] \).

**4.1. One Conic with Known Foci**

First we discuss the case in which the images \( \tilde{f}, \tilde{g} \) are known. Our idea is to treat \( \tilde{f}, \tilde{g} \) as a degenerate envelope \([ \tilde{f} \tilde{g}] \) so as to form, along with \( C_0 \), a basis-pair for \( \tilde{C}^*(\lambda) \).

**Proposition 2** Given the images of a central conic and its foci, the image of the circular-point envelope is:

\[
[ \tilde{I} \tilde{J}] \sim [ \tilde{f} \tilde{g}] - \lambda \tilde{C}_0,
\]

where \( \lambda \equiv \tilde{f}^\top \tilde{C}_0 \tilde{g} + \sqrt{(\tilde{f}^\top \tilde{C}_0 \tilde{f})(\tilde{g}^\top \tilde{C}_0 \tilde{g})} \).

**Proof.** We know, as said in §3.3, that \( \lambda \) is a generalized eigenvalue of the matrix-pair \(([ \tilde{f} \ g], [ \tilde{C}_0])\). It can be shown (quite easily using MAPLE) that these are all real, with both algebraic and geometric multiplicity one, and can be written, by introducing bilinear and quadratic forms\(^3\), as:

\[
\lambda_\pm = \tilde{f}^\top \tilde{C}_0 \tilde{g} \pm \sqrt{(\tilde{f}^\top \tilde{C}_0 \tilde{f})(\tilde{g}^\top \tilde{C}_0 \tilde{g})}, \quad \lambda_3 = 0.
\]

The parameter of \([ \tilde{f} \tilde{g}] \) is clearly \( \lambda_3 = 0 \) so we only have to identify \( \lambda \in \{ \lambda_- \lambda_+ \} \). Assume \( \lambda = \lambda_- \), which equals 0 if \( C_0 \) is a circle, since foci then coincide. Hence, 0 can have geometric multiplicity two and so \( \text{Nullity}([ \tilde{I} \tilde{J}]) = 2 \). This is impossible due to the constraint \( \text{Rank}([ \tilde{I} \tilde{J}]) = 2 \). ■

In other words, if \( \tilde{f} \) and \( \tilde{g} \) are known, then (6) is a closed-form solution for \([ \tilde{I} \tilde{J}] \). Let us point out that \( \tilde{f} \) and \( \tilde{g} \) can be given up to arbitrary scale factors\(^4\). We show now that, when \( \tilde{f} \) and \( \tilde{g} \) are unknown, our problem can be solved by replacing the basis envelope \([ \tilde{f} \tilde{g}] \) by the envelope of the image of any central conic confocal with \( C_0 \).

**4.2. Two Confocal Conics with Known Ratio of Semi-axis Product**

Let \( \tilde{C}_1, \tilde{C}_2 \) be the images of two central conics \( C_1, C_2 \) confocal with \( C_0 \). Due to lack of space here, we require these conics to be either two ellipses or two hyperbolas. Furthermore, we assume that \( a_1 > a_2 \) so \( C_1 \) (resp. \( C_2 \)) is the outer (resp. inner) conic.

Let us take \( \tilde{C}_1, \tilde{C}_2 \) as bases for \( C^*(\lambda) \) i.e., let us define:

\[
\tilde{C}^*(\lambda) \equiv \tilde{C}_1 - \lambda \tilde{C}_2.
\]

To compute \([ \tilde{I} \tilde{J}] \), our idea is to use the generalized eigenvalue invariance stated in §3.3. Without loss of generality, we assign to the supporting plane some Euclidean representation such that the canonical forms (2) hold, and choose \( \mu_j \) in (2) such that \( \det C_j^* = 1 \), \( j \in \{1, 2\} \). It can be verified that \( |\mu_j| = (a_j b_j)^{-2/3} \), involving the semi-minor axis:

\[
b_j = |a_j^2 - c_j^2|^{1/2},
\]

with \( \mu_j < 0 \) (resp. \( > 0 \)) if \( C_j \) is an ellipse (resp. hyperbola).

Furthermore, we have \( \mu_1 / \mu_2 = \phi^{2/3} \), introducing the ratio:

\[
\phi \equiv \frac{a_2 b_2}{a_1 b_1},
\]

which will be referred to as the ratio of semi-axis product.

Eventually, assume that \( \det C_j^* = 1 \) so all conditions (5) hold. Therefore, the problem of determining the parameter of \([ \tilde{I} \tilde{J}] \) can be solved indifferently w.r.t. \( C^*(\lambda) \) or \( C^*(\lambda) \).

On the one hand, it can be shown that:

\[
\lambda = \phi^{2/3} \left( \begin{array}{ccc}
a_1^2 / a_2^2 & b_1^2 / b_2^2 & 1 \\
1 & 1 & 1
\end{array} \right)^\top,
\]

\[
Z = I_{3 \times 3},
\]

\(^3\)It is easy to show that the bilinear forms \((\tilde{f}^\top \tilde{C}_0 \tilde{f})\) and \((\tilde{g}^\top \tilde{C}_0 \tilde{g})\) in the radicand are always of same sign.

\(^4\)e.g., such to have their third homogeneous coordinates equal to 1.
are the generalized eigenvalue vector and eigenvector matrix of \( \mathbf{C}_1, \mathbf{C}_2 \), with both algebraic and geometric multiplicities one. On the other hand, \( \mathbf{\tilde{Z}} = \mathbf{HZ} \text{ diag}(\xi_1, \xi_2, \xi_3) \), where \( \xi_k \in \mathbb{R} \setminus \{0\} \), are the corresponding vector and matrix of \( \mathbf{C}_1, \mathbf{C}_2 \).

We now establish the important proposition:

**Proposition 3** Given the images of two central conics \( \mathbf{\tilde{I}}, \mathbf{\tilde{J}} \) \( \rho \) of (8), the image of the circular point-envelope is:

\[
[\mathbf{\tilde{I}J}] \sim \mathbf{\tilde{C}_1}^\ast - \varphi^{2/3} \mathbf{\tilde{C}_2}^\ast.
\]

**Proof.** Only the third generalized eigenvector satisfies \( \text{Null}[\mathbf{I}J] | z_3 = 0 \) i.e., \( z_3 \sim 1 \). It follows that \( \lambda_3 = \varphi^{2/3} \) is the parameter of \( [\mathbf{IJ}] \). By invariance of the generalized eigenvalues, Eq. (11) holds. \( \blacksquare \)

So far, we can compute \([\mathbf{\tilde{I}J}]\), providing the ratio of semi-axis product \( \varphi \) in (8) is known.

### 4.3. Two Unknown Confocal Conics

Now, let the ratio of semi-axis product (8) be unknown.

**Proposition 4** Given the images \( \mathbf{\tilde{C}_1}, \mathbf{\tilde{C}_2} \) of two unknown central conics, the image of the circular point-envelope is:

\[
[\mathbf{\tilde{I}J}] \sim \mathbf{\tilde{C}_1} - \min(\lambda)\mathbf{\tilde{C}_2},
\]

where \( \lambda \) is the vector of generalized eigenvalues of \( \mathbf{C}_1, \mathbf{C}_2 \).

**Proof.** We know that \( \lambda = \mathbf{\tilde{C}_1} \), as given in (12), up a permutation of elements. In any case, \( \varphi^{2/3} > 0 \) since \( \mathbf{C}_1, \mathbf{C}_2 \) are of same “conic type”. Since, by hypothesis, \( a_1 > a_2 \) and, so, \( b_1 > b_2 \), we have \( \lambda_1 > \lambda_3 \) and \( \lambda_2 > \lambda_3 \). \( \blacksquare \)

In other words, (12) is a closed-form solution for \([\mathbf{\tilde{I}J}]\), only depending on \( \mathbf{\tilde{C}_1} \) and \( \mathbf{\tilde{C}_2} \).

**Algorithm for Computing the Euclidean Structure.**

We now describe a four-step (MATLAB-like) algorithm for computing the image \([\mathbf{\tilde{I}J}]\) of the circular point-envelope, from one view containing the images \( \mathbf{\tilde{C}_1}, \mathbf{\tilde{C}_2} \) of two confocal conics.

1. Detect and estimate the matrices \( \mathbf{\tilde{C}_1}, \mathbf{\tilde{C}_2} \) e.g., using [4]
2. Normalize \( \mathbf{\tilde{C}_1}, \mathbf{\tilde{C}_2} \) such that \( \text{det}(\mathbf{C}_1) = \text{det}(\mathbf{C}_2) = 1 \)
3. Compute \( \lambda_m = \min(\text{eig}(\mathbf{C}_1, \mathbf{C}_2)) \)
4. Compute \([\mathbf{\tilde{I}J}] = \mathbf{\tilde{C}_1} - \lambda_m \mathbf{\tilde{C}_2} \)

If the ratio of semi-axis product \( \varphi \), defined in (8), is known, then we can replace step 3 by \( \lambda_m = \varphi^{2/3} \).

The images \( \mathbf{\tilde{I}}, \mathbf{\tilde{J}} \) of the circular points are given by \( T(1, \pm i, 0)^\top \), where \( T \in \mathbb{R}^{3 \times 3} \) is the homography obtained from the decomposition \( \pm[\mathbf{\tilde{I}J}] = T \text{ diag}(1, 1, 0)T^\top \); the inverse \( T^{-1} \) being a metric-rectifying homography [10]. Note that in the presence of noise, the images of concentric circles will be rectified using previous works either as confocal conics [9] or as non-concentric circles [18]. Here, based on proposition 1, the images of confocal conics are exactly rectified as confocal conics. With regard to what must geometrically ensure a “metric rectification”, this had to be mentioned.

### 5. Detecting Images of Confocal Conics

The first step of the proposed algorithm, as described in §4.3, requires to detect and estimate conics in the view. Dedicated algorithms may detect conics with no physical existence i.e., artifacts or “real” conics which are not the ones we are looking for. Actually, we claim that detecting the images of confocal conics is much simpler than detecting conics in general because confocal conics have important projectively invariant properties [13].

Designing some efficient algorithm for detecting the images of confocal conics goes outside the scope of this work. Nevertheless, we state an important geometric property which can be of great interest for this purpose.

**Proposition 5** The centres of the images of confocal conics are aligned.

**Proof.** Referring to §4, the image of a variable envelope of a conic range can be written as \( \mathbf{\tilde{C}_m} = [\mathbf{IJ}] - \lambda_m[\mathbf{fg}] \), for some \( \lambda_m \). Hence, since \([\mathbf{IJ}]\) and \([\mathbf{fg}]\) are fixed for all conics of the range, by right-multiplying \( \mathbf{C}_m \) by some line vector \( \mathbf{l} \), we obtain the pole \( \mathbf{C}_m^\ast \) which lies on some fixed line, through the poles \([\mathbf{IJ}]\) and \([\mathbf{fg}]\). Now, let \( \mathbf{l} = (0, 0, 1)^\top \) be the image plane’s line at infinity so that \( \mathbf{c}_m \sim \mathbf{\tilde{C}_m}(0, 0, 1)^\top \) denotes the centre of \( \mathbf{\tilde{C}_m} \); this centre lies on a fixed line. \( \blacksquare \)

If we want to design a conic-based pattern which is easily detectable, let us imagine a range of \( M \) confocal ellipses, say ellipses to simplify matters, as in Fig. 5(a). All the centres of the imaged ellipses in one view are aligned; in the example of Fig. 5(c), these centres are grouped together in a small part of the view. Albeit, we will not prove it here, we claim that these centres are ordered on the line through the centres in the same order as the ellipses are ordered in the 3D plane, by considering the relation of “inclusion” (we will say that the inner ellipse is included in the outer ellipse). In some ways, this constraint provides a discriminant signature of a set of confocal ellipses.

### 6. Experimental Results

The theoretical results stated in previous sections and their application to camera calibration have been implemented and experimented on synthetic and real images.

**Synthetic Images.** We have compared our conic-based method (“ELL”) to the “well-known” point-based method (“PTS”) [15][19] by assessing their performances with regard to calibration errors on both intrinsic and extrinsic parameters. The difference between methods only lies in the way the 2D Euclidean structure is computed in one view. Method “PTS” linearly estimates the world-to-image
homography $H$ that maps the 100 points of a 10-by-10 checkerboard to their projections, as described in [15][19]. Method “ELL” relies on the algorithm described at the end of §4.3, using two confocal ellipses as pattern. It consists in computing the image of the circular-point envelope. Once it has been recovered in each view, calibration and pose estimation algorithms are carried out, exactly as in [15][16][19], to compute the intrinsic and extrinsic parameters (see also [9][18]). No non-linear refinements of parameters are run. We point out that all the used patterns are designed to have equal areas.

Figure 3. (a) Confocal ellipse-based calibration pattern vs. checkerboard-like pattern. (b) A synthetic view of both.

About the simulated conditions, the camera roughly fixes the centre of the pattern, from approximately a constant distance, from randomly generated camera orientations, in terms of azimuth, elevation and swing (i.e., rotation about the optical axis) angles, varying by ±30°. The simulated zero-skewed camera has a 512 × 512 pixel resolution and constant internal parameters ($f = 1200$, $u0 = 280$, $v0 = 230$, $\tau = 0.95$). Gaussian noise of zero mean and standard deviation $\sigma_{\text{pixel}}$ is added to the pixel coordinates of both ellipse and checkerboard points. The digitizations of ellipses are achieved at pixel resolution to have the number of sampled pixels approximately equal to its perimeter.

Due to space limitation, we are not to present all the results on synthetic data. To highlight the algorithm behaviours in presence of minimal data, only the case of three views and $\sigma_{\text{pixel}} = 1$ is treated. When adding more views or increasing $\sigma_{\text{pixel}}$, the performances do change smoothly.

Our aim is about determining which is the “best” pair of confocal ellipses. We imagine a scenario in which the pattern width is fixed to $120\,\text{mm}$ and is about $600\,\text{mm}$ away from the camera. We fix the semi-minor axis of the outer ellipse $E_1$ to $b_1 = 60$. We will investigate 3 possible patterns, corresponding to 3 outer ellipses $E_1^{(n)}$, $n \in \{1, 2, 3\}$, with ratios of semi axes $a_1^{(n)}/b_1^{(n)} \in \{1.05, 1.2, 1.5\}$ so that $a_1^{(n)} \in \{63, 72, 90\}$ and $c^{(n)} \in \{19.21, 39.80, 67.08\}$, cf. (7) about notations. For the pattern number $n$, we investigate how vary the errors w.r.t. a variable inner confocal ellipse $E_2^{(n,m)}$, $m \in \{1..M\}$, specified as follows. The distance $\Delta^{(n)} = a_1^{(n)} - c^{(n)}$ (i.e., the semi-major axis of $E_1^{(n)}$ minus half the distance between foci) is divided in $(M - 1)$ equal parts of length $\delta^{(n)} = \Delta^{(n)}/(M - 1)$ such that the semi-major axis of $E_2^{(n,m)}$ is $a_2^{(n,m)} = c^{(n)} + m \delta^{(n)}$. The pattern number 4 i.e., with $a_1/b_1 = 1.5$, is shown for $M = 12$ in Fig.3(a), along with a 10-by-10 checkerboard having equal area. In Fig.3(b), you can see a typical simulated image. The tests for the ellipse-based pattern number $n$ are repeated 500 times.

The calibration results are plotted in Fig.4. The results obtained by “PTS” correspond to horizontal lines. Calibration using “ELL” seems to work better than using “PTS”, even with 100 points. In Fig.4(e), failure rates are given, which indicate when no correct intrinsic parameters can be estimated e.g., when the estimated squared focal length is negative. Again, better results are for “ELL”. To distinguish the results obtained using the ellipse-based pattern number $n$, we draw plots such that the darker its colour is, the larger $n$ (and so $a_1^{(n)}$ is). Remind that, regarding the variable inner ellipse, all $m$, $a_2^{(n,m)}$ and $a_2^{(n,m)}/b_2^{(n,m)}$ vary in increasing order, while degeneracies predictably occur when $m$ tends to $M$. Table 1 summarizes the parameters for the 3 possible ellipse-based patterns ($m = 4$) that yield the best calibration results. From these experiments, the best ellipse-based pattern seems to be associated with the pair $(n = 3, m = 4)$ i.e., with the third row in table 1. To get an idea, refer to Fig.3(a) for the best two ellipses, drawn in bold.

Table 1. Parameters of the best ellipse-based patterns.

<table>
<thead>
<tr>
<th>$a_1/b_1$</th>
<th>$a_1$</th>
<th>$b_1$</th>
<th>$a_2$</th>
<th>$b_2$</th>
<th>$\varphi^{\pi/5}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.05</td>
<td>63</td>
<td>60</td>
<td>31.12</td>
<td>24.28</td>
<td>0.3419</td>
</tr>
<tr>
<td>1.2</td>
<td>72</td>
<td>60</td>
<td>48.67</td>
<td>28.02</td>
<td>0.4636</td>
</tr>
<tr>
<td>1.5</td>
<td>90</td>
<td>60</td>
<td>73.57</td>
<td>30.22</td>
<td>0.5534</td>
</tr>
</tbody>
</table>

We point out that we ran the proposed algorithms using known and unknown ratio of semi-axis product, cf. Eq. (8). We have obtained very similar accuracies, with so little differences that they can hardly be seen on the plots. To illustrate this point, we have computed the relative errors on the generalized eigenvalues of the matrix-pair, as shown in Fig.4(e). Notice that these quantities, which are projectively invariant, can be computed with very high accuracies.

Real Images. To illustrate the performance of the proposed algorithm and to show off the interest of using confocal conics as simple artificial markers, we processed a sequence of 209 real images, with size $720 \times 576$, for augmented reality purposes (see Fig.5(a)). The images were captured using a CANON digital video Camcorder XMc2. After edge detection using Canny operator [2], ellipses are detected (see Fig.5(b)) using the function available from the Open Source Computer Vision Library (OpenCV) and then

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www.intel.com/technology/computing/opencv/
fitted using Fitzgibbon’s algorithm [4]. The ellipse outliers and artifacts can be eliminated using a method based on the constraint, given as proposition 5 in §5. Fig. 5(c) shows how the centres of images of confocal ellipses are aligned. A video covering this experiment can be seen at the URL [1].

7Take a string of some length $l_1$, drive two stakes into the ground and use them as foci to draw a first ellipse using a stick, by moving it along the string in such the way the string is kept stiff. Repeat it with a string with length $l_2 < l_1$ and you will obtain a second ellipse confocal with the first.

7Conclusion

In this paper, we investigate the geometric properties of confocal conics for calibration purposes. We show that two confocal conics naturally encode all the metric information of the supporting plane i.e., its 2D Euclidean structure, since the circular-point envelope, sometimes referred to as “2D absolute conic” [13, p. 121], is a degenerate member of any confocal conic range. We show that the parameter of the circular-point envelope with regard to a confocal conic-pair can be determined thanks to the projective invariance of generalized eigenvalues of the corresponding matrix-pair.

We investigate three practical scenarios: one conic with known foci, two confocal conics with known ratio of semi-axis product, and two unknown confocal conics. We show that in all these cases the Euclidean structure can be estimated as a closed-form solution. A compact algorithm is detailed for the most general case of unknown confocal conics. Once the induced 2D Euclidean structures are recovered in all views, the camera can be calibrated. A crucial point is that, with concentric circles, the pose of the supporting plane can only be estimated up to 2D similarity. With confocal conics, the rotation and translation can be fixed, even if actually two solution exist, like with any symmetric object. Calibration performances are analysed and we show that the proposed algorithms obtain better accuracies for both intrinsic and extrinsic camera parameters than point-based algorithms. Lastly, we show that there is a geometric constraint on the centres of images of confocal conics that can be used for the automatic detection of confocal conics; we experiment this idea in some augmented reality application [1], as shown in Fig. 5.

We only dealt with central conics but the case of two confocal parabolas does not introduce major difficulties to be treated in the proposed framework, besides dealing with some special considerations. However, owing to lack of space, this could hardly be included here.

Eventually, the reader may wonder where to find confocal conics in the real world. We make the point that two confocal ellipses can be easily constructed on any planar surface, using the “gardener’s technique“7. Thus, plane-based calibration can be carried out on the ground, with a simple piece of string and two stakes.

References

Figure 4. Calibration errors ($\sigma_{\text{pixel}} = 1$), from three views, either using a 10-by-10 checkerboard (PTS) or two confocal ellipses (ELL), with equal surface areas. The semi-major axis of the outer ellipse is $a_1 \in \{1.05, 1.2, 1.5\}$ and the darker the color is, the larger $a_1$ is. The abscissa is the subscript $m$ of a variable inner ellipse, with increasing semi-major axis $a_2 \in [c, ..., a_1 - c]$ (see text).

Figure 5. (a) Original image. (b) Ellipses detected in the edge image, including artifacts. (c) Selection of ellipses whose centers are aligned (property of images of confocal conics). (d) Augmented reality from calibrated camera [1].