Topological spatio-temporal reasoning and representation

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Abstract

We present here a theory of motion from a topological point of view, in a symbolic perspective. Taking space-time histories of objects as primitive entities, we introduce temporal and topological relations on the thus defined space-time to characterize classes of spatial changes. The theory thus accounts for qualitative spatial information, dealing with underspecified, symbolic information when accurate data is not available or unnecessary. We show that these structures give a basis for commonsense spatio-temporal reasoning by presenting a number of significant deductions in the theory. This can serve as a formal basis for languages describing motion events in a qualitative way.

Topic

Knowledge representation, spatio-temporal reasoning, motion description.

1 Introduction

1.1 Space, Motion and Common Sense

The work we present in this paper is a study of motion from the point of view of qualitative reasoning and the representation of human spatial knowledge, in a computational perspective. Spatial and temporal knowledge are central in several domains of Artificial Intelligence, whether in natural language processing, man-machine interaction, automated reasoning, geographical systems or high-level vision. A significant effort has been done in the past few years to develop formal models for the handling of spatial data in contexts where exact information is not available, or is not easily processed. The amount of quantitative data is indeed often a problem in some contexts, and being numerical in essence, they are sometimes an obstacle to human-computer communication, as they are somewhat remote to human cognition and experience. Qualitative Spatial Reasoning (henceforth QSR) (Cohn, 1996; Vieu, 1997) is a rather recent field in knowledge representation, which has focused on such problems for automated reasoning, and within which we place our study.

Little work has been devoted to the problem of representing motion in the more cognitive kind of approaches that characterizes the processing of spatial information in QSR, besides the work of (Galton, 1993; Galton, 1997), followed by (Muller, 1998b) and more recently (Davis, 2001; Cohn and Hazarika, 2001). The purpose of this paper is to address the representation of motion in that perspective and overcome some of the formal limitations of the previously mentioned studies. Dealing with motion is essential to spatial information systems as most of them handle changing data. Taking time into account is thus a central issue in geographical information systems (Claramunt *et al.*, 1997; Frank and Kuhn, 1995; Hirtle and Frank, 1997), and a lot of effort in this area is devoted to providing useful, well-grounded qualitative models to be used as high level description of spatial and spatio-temporal information[1]. The cultural gap between the artificial intelligence community and people working specifically on spatial databases is thus being reduced, as papers start to investigate the use of formal models for the description of objects in databases (Erwig *et al.*, 1999; Erwig and Schneider, to appear), taking over ideas that have been around in the qualitative spatial reasoning community. The issue of representing motion qualitatively also arises in

[1]The 7th international symposium on Advances in spatial and temporal databases presents no less than six papers devoted to the question of moving objects, see for instance (Tøssebro and Güting, 2001).

vision, where the interfacing with human operators depends on cognitively adequate representations (Pinhanez and Bobick, 1996). In a somewhat related problem, (Kalita and Lee, 1997) have proposed a semantics for the representation of motion verbs that was used for help in image synthesis, pointing out to the need for symbolic representations for motion events. Lastly, a need has also emerged for symbolic, models in video databases, e.g. for indexing sequences, as reflected in (Li *et al.*, 1997), who propose a set of qualitative relations for motion in a video.

We thus aim at modeling some properties of space and time in a formalism which allows for the representation of relations between moving entities and we want to be able to reason symbolically about these situations in underspecified contexts. The model should be powerful enough to express categories of motion that can be useful in a qualitative context, and be kept as simple as possible so that characterising its properties is still possible in a precise way. Only then can we have a principled representation for motion that could unifiy the various needs for the aforementioned situations.

From a methodological point of view, we have tried to take into account cognitive aspects related to the concepts we model; we want thus to ensure the relevance of the symbolic representation for commonsense reasoning. To do this, we have considered the expression and representation of motion in natural language.

1.2 Classical Representations of Motion

We are going to examine in this section the various approaches towards the representation of motion in past AI work and in related domains.

The Newtonian conception of space and time has exerted a strong influence on views about motion even outside physics. However a lot of the approaches close to our concerns depart somewhat from the conception of motion as a continuous function from time (seen as the real line) to space, isomorphic to $\mathbf{R}^3$. [2]

The different approaches can be distinguished with respect to a few key choices about the ontology of space, time, and thus motion:

1. The choice of an absolute space (persisting through time and existing independently of the objects in it) vs. a relative space, where only physical objects have an existence and are located with respect to one another.

[2]By focusing on representation problems, we leave out a lot work in philosophy or psychology, e.g., on the perception of motion.

2. The choice of extended regions as primitive objects vs. the choice of dimensionless points, either for time, space or both (hybrid solutions are not uncommon in the literature).

3. The choice of expressing motion as relative to other entities or as absolute in a coordinate system.

4. The choice of a discrete or dense or continuous time and/or space; a fully, explicit, discrete model of motion is rare, but see a proposal in (Forrest, 1995).

5. The choice of a primitive space-time vs. two separate domains for space and time.

Most of those choices can be done independently, as is demonstrated by the various attempts found in qualitative physics, linguistics or philosophy. Obviously, absolute, Euclidean space and a separate continuous time form the basis of pre-relativistic physics (the primitive objects being points in space and instants in time). This conception is also at work in robotics and in studies grouped under the "qualitative physics" label (Forbus *et al.*, 1987; Forbus, 1995; Faltings, 1990).

The relative nature of space is on the contrary advocated for in most cognitively oriented approaches, in linguistics for instance (Talmy, 1975; Herskovits, 1982; Asher and Sablayrolles, 1995), although it does not necessarily entail that motion in such spaces should be regarded purely in terms of relations (cf. (Asher and Sablayrolles, 1995)).

Among the proponents of extended objects as primitives (either regions for space or intervals for time) can be found supporters of an absolute pre-existing space (Galton, 1997; Borgo *et al.*, 1996) or of a relative space (Asher and Vieu, 1995; Clarke, 1981). Some of these are in fact hybrid as they admit also points as objects (Galton, 1997; Eschenbach and Heydrich, 1993; Claramunt *et al.*, 1997; Clarke, 1981).

As for the last point, most approaches prefer to consider space and time as independent and express motion as a relation between the two, the exception being (Hayes, 1985a) which deals with spatio-temporal histories of objects; such entities are also regarded by (Clarke, 1981; Vieu, 1991) as possible models for their theory although they are not fully characterized as such, and are proposed as a philosophically grounded ontological basis by (Heller, 1990).

2 Representation of motion on a qualitative basis

In the perspective of the representation of motion, we should therefore ask ourselves what structures we think relevant in the previous list. Reflecting a quite common methodology in QSR, we have made the following choices:

- we represent knowledge in an axiomatic approach; concepts under study are modeled in a first-order theory; we then examine its expressive power and inferential properties.

- for the sake of ontological economy, we wish to avoid the proliferation of different entity types. Furthermore, we want to limit the number of entities as much as possible, assuming the least possible a priori (reflecting contexts in which knowledge of the world is partial or imperfect). A region-based ontology for space, for instance, assumes the existence of a few given entities and allows for the construction of other ones (with operators, such as complementation, or the sum of two entities), and doesn't assume the implicit existence of arbitrary entities.

- there is a kind of hierarchy on the complexity of spatial relations that places more elementary concepts at the base of most models. For space, it seems that the notion of inclusion is considered the most basic, then comes topology, and above are considered distance, orientation, shape, ... (Cohn, 1996).

According to these general principles, it seems rather natural to consider space-time as an homogeneous domain, peopled by "histories" in the sense of (Hayes, 1985b), i.e, regions of space-time, rather than try to combine separate theories of space and time, as in (Galton, 1993; Li *et al.*, 1997). This is the same path followed by Allen (Allen, 1984; Allen and Hayes, 1985) for his theory of time, and Randell et al. (Randell *et al.*, 1992) for their theory of space (RCC). We then need to model the structural properties at the most basic levels, which we think is the mereological and the topological, following what was proposed first in (Clarke, 1985). We do so in a more formal way than what Hayes did, who admitted to be more interested in the breadth of coverage of concepts than in the precision of the axiomatizing itself. We, on the contrary, think that breadth of coverage is only possible when the foundations are sound enough to build upon.
The main interest of taking space-time as primitive is to allow for a certain level of underspecification, which in turn allows for an easier expression of global constraints. Some problems related to the identity of objects are indeed intrinsically

settled by such an ontology, as is discussed in (Heller, 1990; Muller, 1998c). Moreover it has allowed us to formally characterize properties linked to the continuity of motion in space-time on a qualitative basis, something only hinted at by previous work (Galton, 1993; Galton, 1997; Cui *et al.*, 1992). The first step in that direction in (Muller, 1998b) was somewhat over-interpreted and was completed in (Muller, 1998a) (written in french however, explaining why the flaws of the first proposal were later indicated in (Davis, 2001) and (Cohn and Hazarika, 2001)); we sum up those results section (8)[3].

We can then sum up our representation choices:

- Primitives objects of our theory are extended in both space and time.
- Knowledge about these entities is only expressed in terms of relations.

We will now present the theory that accounts for the properties of spatio-temporal objects in a commonsense manner. We build upon the mereo-topology of extended objects in the tradition of Clarke (Clarke, 1981), add temporal relations in the spirit of event logics (Kamp, 1979) and formally characterize the links between those primitives.

3 The topology of space-time

We will now sum up the topological theory we use as a basis for our theory of spatio-temporal entities. It is taken from Asher and Vieu's (Asher and Vieu, 1995). Objects of this theory are regarded as spatio-temporal referents of physical objects or events. We have only kept the part dealing with the notions of mereology and classical topology, leaving aside the definition given by the authors for a notion of natural contact between two objects. The good side of this theory is that it has been shown to be consistent and the class of models it axiomatizes has been characterized in (Asher and Vieu, 1995). We will see how the way we enrich it bears on the models. Its relations are generally interpreted as bearing on regions of the plane or volumes of three dimensional space (see figure 1), although the domain can be of arbitrary dimension. Clarke (Clarke, 1981) and Vieu (Vieu,

[3]It is interesting to note that people from the database community proposed to follow the same kind of path for representing motion events in databases in what seemed an independent manner in (Erwig *et al.*, 1999) and by making explicit references to earlier work in QSR, in (Erwig and Schneider, to appear).

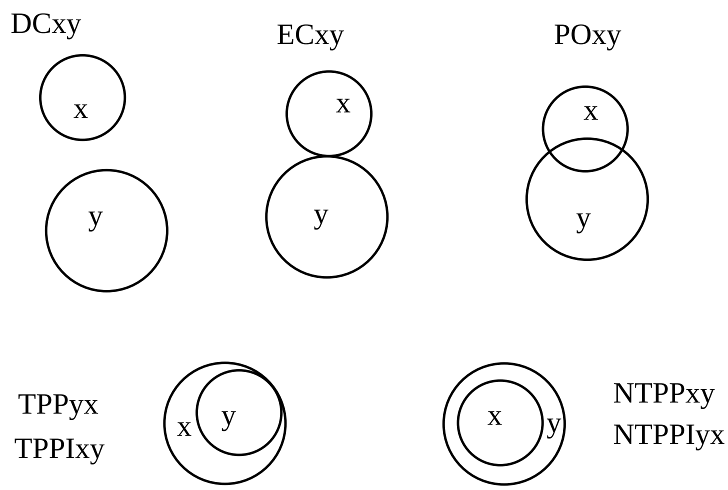

Figure 1: The eight mereo-topological relations RCC.

1991) proposed to consider the entities of their theories as regions of space-time, by adding temporal relations, albeit in an incomplete manner. It is this spatio-temporal interpretation that is intended here and we will see how to go beyond the limitations of these studies. Figure 2 shows for instance the intuitive spatio-temporal interpretation of a relation of topological overlap. The horizontal axis corresponds to the spatial dimension and the temporal evolution is shown along the vertical axis. Space here is unidimensional for convenience, but could be of dimension 2, 3, ... n.

The mereo-topological theory of (Asher and Vieu, 1995) is built on a unique primitive: a relation of connection, noted C, interpreted here as connection between spatio-temporal regions (that we will sometimes call “histories”). In the following, we adopt a few conventions for readability sake: any universal quantifier whose scope is the whole formula will be omitted; predication parentheses will be omitted when it is unambiguous; variables are denoted by lower case letters; the only constant is noted a. Symbols $\neg$, $\wedge$, $\vee$, $\rightarrow$ classically denotes logical negation, “and”, “or”, and material implication, while $\triangleq$ introduces a definition.

The following definitions and axioms, from (A-1) to (A-9) are taken from (Asher and Vieu, 1995). The C relation is reflexive, symmetric and extensional:

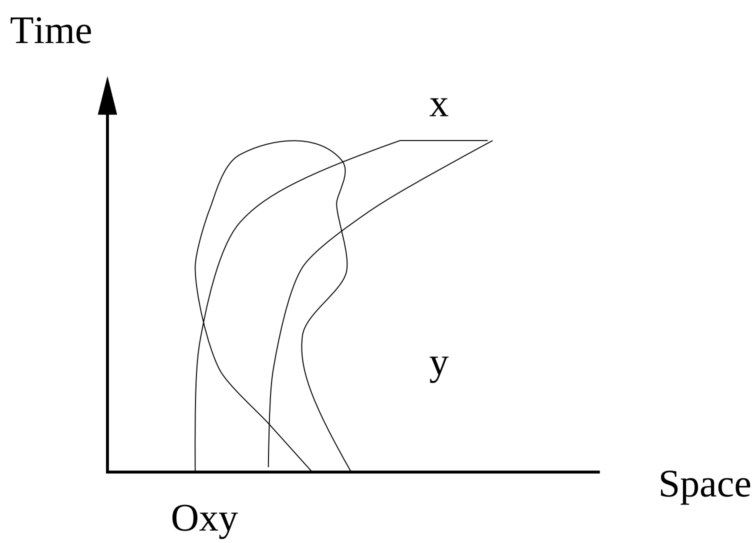

Figure 2: A spatio-temporal interpretation of O(verlap)

A 1 $\mathrm{C}xx$

A 2 $\mathrm{C}xy \rightarrow \mathrm{C}yx$

A 3 $(\forall z\ (\mathrm{C}zx \leftrightarrow \mathrm{C}zy) \rightarrow x = y)$

The following relations have standard definition:

D 1 $\mathrm{P}xy \triangleq \forall z\ \ (\mathrm{C}zx \rightarrow \mathrm{C}zy)$ (part of)

D 2 $\mathrm{DC}xy \triangleq \neg \mathrm{C}xy$ (disconnection)

D 3 $\mathrm{PP}xy \triangleq \mathrm{P}xy\ \wedge\ \neg \mathrm{P}xy$ (proper part of)

D 4 $\mathrm{O}xy \triangleq \exists z\ (\mathrm{P}zx \wedge \mathrm{P}zy)$ (overlap)

D 5 $\mathrm{PO}xy \triangleq \mathrm{O}xy\ \wedge\ \neg \mathrm{P}xy\ \wedge\ \neg \mathrm{P}yx$ (partial overlap)

D 6 $\mathrm{EC}xy \triangleq \mathrm{C}xy\ \wedge\ \neg \mathrm{O}xy$ (contact)

D 7 $\mathrm{TP}xy \triangleq \mathrm{P}xy\ \wedge\ \exists z\ (\mathrm{EC}zx \wedge \mathrm{EC}zy)$ (tangential part)

D 8 NTP$xy \triangleq$ P$xy \ \wedge\ \neg\exists z$ (EC$zx \wedge$ ECzy) (non tangential part)

(N)TPP then is (non) tangential proper part. Some existence axioms allow for the existence of classical operators[4]:

A 4 $\forall x \forall y \exists z\ \forall u\ \ (\mathrm{C}uz \leftrightarrow (\mathrm{C}ux \vee \mathrm{C}uy))$

(z is the sum, noted x+y)

A 5 $\forall x(\exists y\ \neg\mathrm{C}yx) \rightarrow \exists z\ \forall u(\mathrm{C}uz \leftrightarrow \exists v\ (\neg\mathrm{C}vx \wedge \mathrm{C}vu))$

(z: complement, noted $-x$)

A 6 $\exists x\ \forall u\ \ \mathrm{C}ux$

(existence of a universe, noted a)

A 7 $\mathrm{O}xy \rightarrow \exists z\ \forall u(\mathrm{C}uz \leftrightarrow \exists v\ (\mathrm{P}xy \wedge \mathrm{P}vy \wedge \mathrm{C}vu))$

(z: intersection, noted $x \cdot y$)

A 8 $\forall x \exists y\ \forall u\ \ (\mathrm{C}uy \leftrightarrow \exists v\ (\mathrm{NTP}vx \wedge \mathrm{C}vu))$

(y: interior, noted $i(x)$)

Topological closure is then defined by:

D 9 $cx \triangleq\ -i(-x)$

A 9 $ca = a$

To which must be added the classical property of opens (the intersection of opens is open):

D 10 OP$x \triangleq\ (ix = x)$

A 10 (OP$x \wedge$ OP$y \wedge \mathrm{O}xy) \rightarrow$OP$(x \cdot y)$

And "separateness" and "self-connectedness" are straightforwardly defined as:

D 11 SP$xy \triangleq \neg\mathrm{C}cxcy$

D 12 CON$x \triangleq \neg(\exists x_1 \exists x_2(x = x_1 + x_2 \wedge \mathrm{SP}x_1x_2))$

[4] These operators are partial.

4 Temporal order

The previously introduced mereo-topology is general and does not force the dimension of the topological space considered. It needs further structural specification to be regarded as a proper spatio-temporal theory. As our primitive objects are extended both in space and time, the appropriate logics for temporal relations will be close to the "event logics" of Kamp (Kamp, 1979), where contemporaneous entities need not be equal. Besides a classical precedence relation we will need to express the notions of temporal inclusion and overlap; the first proposal in that direction of research was made in (Vieu, 1991) but was not pursued further. Inclusion and overlap are not sufficient to express spatial transitions in time, however, as shown by Galton (Galton, 1993). It is indeed necessary to be able to distinguish an overlap from a simple temporal "contact" (which can be seen as an instantaneous transition, such as the coming in contact of two bodies). Allen's relation "meets" can make the distinction, but is restricted to temporally convex time intervals, a restriction too strong when dealing with spatio-temporal entities. To overcome those problems, we introduce a primitive of temporal connection, noted $\asymp$, which has more or less the same behaviour as C, only on a temporal level. We thus avoid the solution of having a non-homogenous domain including instants, as in (Galton, 1993).

We propose the following axiomatization of $\asymp$ and $<$:

A 11 $x \asymp y \rightarrow y \asymp x$ (symmetry)

A 12 $x \asymp x$ (reflexivity)

A 13 $x \asymp y \rightarrow \neg x < y$ (non compatibility of $\asymp$ and $<$)

A 14 $x < y \rightarrow \neg y < x$ (antisymetry of $<$)

A 15 $(x < y \ \wedge \ y \asymp z \ \wedge \ z < t) \rightarrow x < t$ (transfer between $\asymp$ and $<$)

By axiom 12 and axiom 15, $<$ is transitive. We can then define classical relations (see figure 3 for an illustration):

D 13 $x \subseteq_t y \triangleq \forall z (z \asymp x \rightarrow z \asymp y)$ (temporal inclusion)

D 14 $x \sigma y \triangleq \exists z (z \subseteq_t y \ \wedge \ z \subseteq_t x)$ (temporal overlap)

D 15 $(x \equiv_t y) \triangleq x \subseteq_t y \wedge y \subseteq_t x$ (temporal equivalence)

And add :

A 16 $x < y \to (\forall z\ (z \subseteq_t x \to z < y)\ \wedge\ (z \subseteq_t y \to x < z))$
(monotony of $\subseteq_t$ w.r.t. $<$)

Those properties subsumes axioms of other event logics (van Benthem, 1995), except for the linearity of time, to which we come back in section 5.3[5]:

Th 1 $x \subseteq_t x$ (def. of $\subseteq_t$)

Th 2 $x \sigma y \to y \sigma x$ (def. of σ)

Th 3 $x < y \to \neg x \sigma y$ (Ax. 13)

Th 4 $(x < y\ \wedge\ y \sigma z\ \wedge\ z < t) \to x < t$ (Ax. 15)

Th 5 $(x < y\ \wedge\ y \subseteq_t z\ \wedge\ z < t) \to x < t$ (Ax. 15)

Th 6 $(x \subseteq_t y\ \wedge\ y \subseteq_t z) \to x \subseteq_t z$ (def. of $\subseteq_t$)

Th 7 $x \subseteq_t y \to \forall z\ (z \sigma x \to z \sigma y)$ (Th.6)

Th 8 $x \subseteq_t y \to \forall z\ ((z < y \to z < x)\ \wedge\ (y < z \to x < z))$ (Ax. 16)

To be able to recover a notion of temporal linearity, we will introduce the following entities:

A 17 $\forall x((\exists y(x < y)) \to (\exists z \forall u(x < u \leftrightarrow Puz)))$ (existence of a future of x)

A 18 $\forall x((\exists y(y < x)) \to (\exists z \forall u(u < x \leftrightarrow Puz)))$ (existence of a past of x)

We will note the future z of x ($f(x)$) and its past ($p(x)$); their uniqueness can be shown straightforwardly:
e.g. for the future, let's assume $\exists z_1 \forall u(x < u \to Puz_1)))$ and $\exists z_2 \forall u(x < u \to Puz_2)))$. With P$z_1 z_1$ we get $x < z_1$ and therefore P$z_1 z_2$ and symetrically we get P$z_2 z_1$ so $z_1 = z_2$.
The following properties are easily derived:

Th 9 $x < f(x)$

Th 10 $p(x) < x$

[5] We indicate for each theorem which axioms or theorems are involved in its proof; full proofs for the theorems are to be found in the appendix.

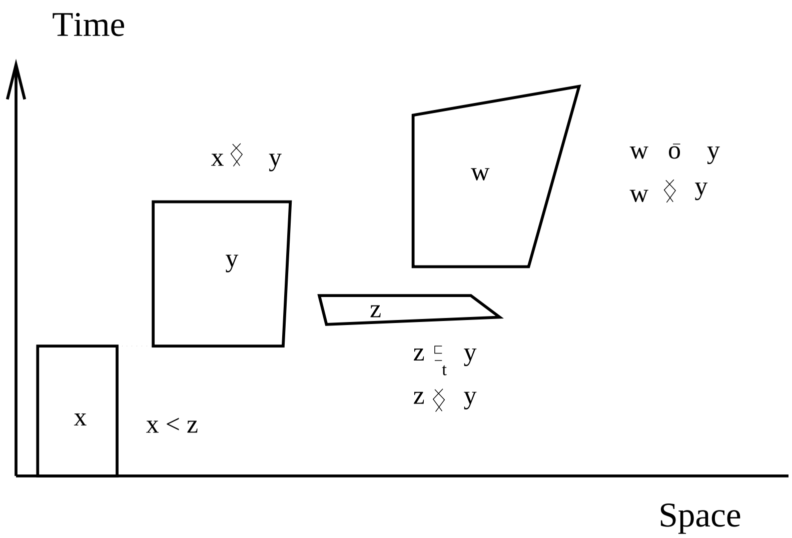

Figure 3: Temporal relations

5 Constraints on space-time

5.1 Links between time and space-time

In this section, we describe the links between the temporal and the spatio-temporal predicates.

First, spatio-temporal connection must imply temporal connection:

A 19 $\mathrm{C}xy \rightarrow x \gtrless y$

The following axiom accounts for mereological correspondance between time and space-time (inclusion implies temporal inclusion):

A 20 $\mathrm{P}xy \rightarrow x \subseteq_t y$

In order to achieve a truly multi-dimensional structure, the connection relations must be distinct, hence the following axioms:

A 21 $\exists x \exists y \; x \gtrless y \; \wedge \; \neg \mathrm{C}xy$

A 22 $\exists x \exists y \; x < y$

The temporal non-convexity of entities impose the following constraints on the sum operator and the temporal relations:

A 23 $(x < y \wedge z < y) \leftrightarrow (x + z) < y$

A 24 $(x+y) \mathrel{\asymp} z \leftrightarrow\ x \mathrel{\asymp} z\ \vee\ y \mathrel{\asymp} z$

Last, we characterise the link between temporal connection and the interior function as follows: a temporal contact with an open region must imply an overlap, so:

A 25 $ix \mathrel{\asymp} y \rightarrow x\sigma y$

Then, we have the following properties:

Th 11 $\mathrm{O}xy \rightarrow x\sigma y$ (Ax. 20)

Th 12 $(x < y \wedge \mathrm{P}zx \wedge \mathrm{P}ty) \rightarrow z < t$ (Th. 8, Ax. 20)

Th 13 $(x\sigma y \wedge \mathrm{P}xz \wedge \mathrm{P}yt) \rightarrow z\sigma t$ (Th. 7, Ax. 20)

Th 14 $(x+y)\sigma z \leftrightarrow\ x\sigma z\ \vee\ y\sigma z$ (Ax. 24)

We can also introduce a notion of "strong connection" between two entities (illustrated figure 4). It intuitively corresponds to a connection by more than a " point" in a classical interpretation; we give the following definition:

D 16 $\mathrm{SC}xy \triangleq \mathrm{C}xy\ \wedge\ \exists z\ (\mathrm{NTP}cz(x+y) \wedge \mathrm{O}zx \wedge \mathrm{O}zy \wedge \mathrm{CON}z)$

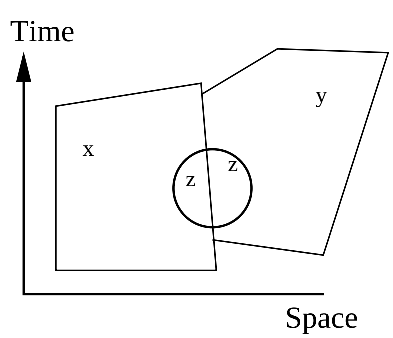

Figure 4: Example of strong connection.

This says that two regions are strongly connected when they are connected and there is a simply-connected region that overlap them both, and whose closure is a non-tangential part of their sum. We can then define a notion of "strong connectedness" from SC, in the same way connectedness is defined from C (corresponding to the notion of simple region in (S. Borgo and Masolo, 1996), where it is taken as a primitive for a mereo-topology):

D 17 $\mathrm{SR}x \triangleq \forall x_1 \forall x_2\ (x = x_1 + x_2 \rightarrow \mathrm{SC}cx_1cx_2)$

We will use these notions in section 8.

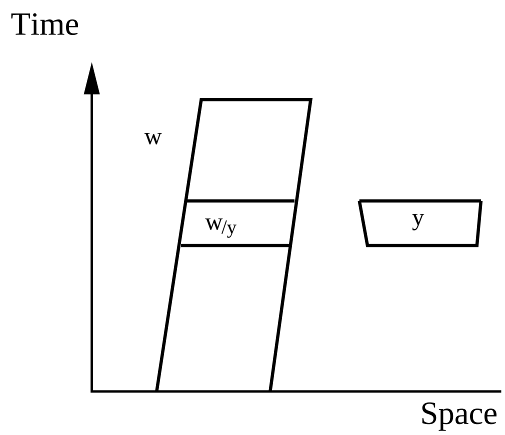

Figure 5: Temporal slice : x is a slice of w.

5.2 Temporal Parts

In order to define relations changing through time and to recover some concepts of spatial (relative) localisation, we are going to define a concept of temporal part, called a temporal slice. A temporal slice of a spatio-temporal entity is the maximum part of a spatio-temporal region corresponding to the lifespan of another one (an episode of a larger history). We thus assume any entity can have a temporal slice. Temporal parts sometimes appear in mereological theories, and are always associated with durations; a slice would be then a zero-duration temporal part (Simons, 1987; Carnap, 1958). In order to stay completely in a mereo-topological framework our definition will dispense with such a notion. We will adopt the following definition: a temporal slice x is a part of an entity y such that any part of y that is temporally included in x is a part of x[6].

D 18 $\mathrm{TS}xy \triangleq \mathrm{P}xy \wedge \forall z \; ((\mathrm{P}zy \wedge z \subseteq_t x) \rightarrow \mathrm{P}zx)$

This definition implies the following properties:

Th 15 $\mathrm{TS}xx$ (reflexivity)

Th 16 $(\mathrm{TS}xy \wedge \mathrm{TS}yx) \rightarrow x = y$ (antisymetry)

Th 17 $(\mathrm{TS}xy \wedge \mathrm{TS}yz) \rightarrow \mathrm{TS}xz$ (transitivity)

[6]TS denote a Temporal Slice.

Moreover, a slice x of an object y temporally included in another slice z of y is also a slice of z:

Th 18 $(\mathrm{TS}xy \wedge \mathrm{TS}zy \ \wedge \ x \subseteq_t z) \rightarrow \mathrm{TS}xz$

We come now to the question of what temporal parts of objects should be allowed in our theory. In order to avoid an unnecessary proliferation of arbitrary entities, we want to introduce a minimal number of parts. It seems that temporal parts should be at least determined with respect to entities one wants to compare with one another on the temporal level. Thus any object should have a temporal part corresponding to the temporal extent of a contemporaneous entity. Hence the following axiom:

A 26 $y \subseteq_t x \rightarrow \exists u \ (\mathrm{TS}ux \ \wedge \ u \equiv_t y)$
(every entity x has a temporal slice u temporally equivalent to any entity y temporally included in x)

Consequently:
Th 19 $\forall x, y(x\sigma y \rightarrow \exists u(\mathrm{TS}ux \ \wedge \ u \subseteq_t y))$ (if two histories temporally overlap then there is a slice of one included in the other).

Th 20 $\mathrm{P}xy \rightarrow \exists z(\mathrm{TS}zy \ \wedge \ z \equiv_t x)$
(for any entity y, there is a slice of y temporally equivalent to any part of y).

Th 21 $(\mathrm{TS}xy \wedge \mathrm{TS}zy \ \wedge \ x \equiv_t z) \rightarrow x = z$
(this slice is unique).

We will note $x_{/y}$ this corresponding slice whenever it exists: $x_{/y}$ is the part of x corresponding to the lifespan of y when $y \subseteq_t x$. For instance, if x stands for America's history, and y for Clinton's presidency, $x_{/y}$ is the history of America during Clinton's presidency.

5.3 Temporal slices and the temporal structure

The temporal properties of the spatio-temporal theory can now be characterised more precisely. First we can show that:
Th 22 $\mathrm{TS}(f(x), a) \wedge \mathrm{TS}(p(x), a)$
(the past and future of any entity are slices of the universe a).

A slice of the universe can be regarded as an episode of the world we consider. This theorem has for immediate consequence the equality of the past and future of two temporally equivalent entity:

Th 23 $(x \equiv_t y \ \wedge \ \exists z \ (x < z)) \rightarrow f(x) = f(y)$

Th 24 $(x \equiv_t y \ \wedge \ \exists z \ (z < x)) \rightarrow p(x) = p(y)$

Some useful temporal notions can be defined, such as:

D 19 $\mathrm{CON}_t x \triangleq \neg(\exists x_1 \exists x_2 (x = x_1 + x_2 \ \wedge \ \neg(cx_1 \between cx_2)))$

(temporal self- connectedness)

A 27 $(\mathrm{TS}ua \ \wedge \ cu \between cv) \rightarrow \mathrm{C}cucv$

This property (which we call "normality of the universe") ensures that there is no spatio-temporal leaps in the universe (no entity can be temporally connected to a slice of the universe without being connected spatio-temporally). This property has to do with the continuity of space-time[7]. We will come back to it section (8).

In order to further constraint our model, we want to consider a world with a linear underlying temporal order. This means for instance that there should be an ordering between temporally self-connected entities resembling that of Kamp's logics where overlap is replaced with our temporal connection:

$$(\mathrm{CON}_t x \wedge \mathrm{CON}_t y) \rightarrow (x < y \ \vee \ x \between y \ \vee \ y < x)$$

However it is not enough to characterize relations between arbitrary entities, as they can be non self-connected. What we need is a stronger notion. We will use the following definitions:

D 20 $\mathrm{ORD}(x, y) \triangleq x < y \ \vee \ x > y$

This relation (let's call it general ordering) is obviously symetric.

D 21 $\mathrm{PMCT}(x, y) \triangleq \mathrm{P}xy \ \wedge \ \neg\exists z \ (\mathrm{P}zy \wedge \mathrm{CON}_t(x + z))$

This relation expresses that x is a part of y maximally connected temporally (a temporally connected component). To impose linearity, we will state that such components of any two given entities must be ordered, using a relation of "betweenness":

[7]The expression $cu \between cv$ is necessary to exclude the case of an open region jumping away from another one and for which we could not have $u \between v$.

D 22 BETW$(y, x) \triangleq \neg$ORD(x, y) $\wedge$
$\forall y'$(PMCT$y'y \rightarrow \exists x_1 \exists x_2$ $(x = x_1 + x_2 \wedge$ ORD$x_1 y' \wedge$ ORD$x_2 y')$

An example of the configuration covered by that definition is given figure 6. This relation expresses that we can always find parts of x and y that are ordered if x and y are not connected temporally. Note that we can have at the same time BETWxy and BETWyx. Our linearity then corresponds to :

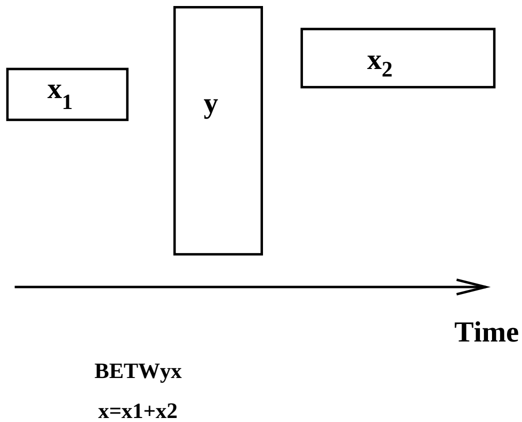

Figure 6: Betweenness of two entities

A 28 $\forall x, y$(BETW$(x, y) \vee$ BETW$(y, x) \vee$ ORD(x, y) $\vee$ $x \gtrless y)$

And we will add that any entity has at least a PMCT:

A 29 $\forall x \exists x'$ PMCT$x'x$

Giving the following consequences:

Th 25 BETW$yx \rightarrow \exists y'$(PMCT$y'y \rightarrow \exists x_1 \exists x_2$ $(x = x_1 + x_2$ $\wedge$ $x_1 < y' < x_2)$

This means that connected components of an entity can be ordered with respect to the parts of another one which stands in the between relation with the first. This gives us the linearity condition we were looking for:

Th 26 (CON$_t x \wedge$ CON$_t y) \rightarrow (x < y$ $\vee$ $x \gtrless y$ $\vee$ $y < x)$

Th 27 $\forall x[(\exists y(y < x)) \rightarrow (p(x) < -p(x))]$

(the past of an entity is before its complement.)

Th 28 $\forall x[(\exists y(x < y)) \to (-f(x) < f(x))]$ (the future of an entity is after its complement.)

The proof of these last two theorems uses the linearity axiom 28 *and* the regularity axiom 27.
These theorems have the following corollaries:
Th 29 $(\exists y(x < y)) \to -f(x) = p(f(x))$ (the complement of a future is its own past)
Th 30 $(\exists y(y < x)) \to -p(x) = f(p(x))$ (the complement of a past is its own future)

Put another way :

Th 31 $a = p(x) + f(p(x))$
Th 32 $a = f(x) + p(f(x))$

6 Models of the theory

We are going to present here what define the class of models of our theory ST, corresponding to axioms 1-29.

6.1 Definition of the class of intended models

We consider a classical tpological space $\langle \mathcal{E}, T \rangle$, where $\mathcal{E}$ is a set of points and T is a set of open sets of $\mathcal{E}$.
Now we consider a structure $\langle \mathcal{E}, T, \mathcal{G}, \prec, [\![\cdot]\!] \rangle$ such that $\mathcal{G} \subseteq \mathcal{P}(\mathcal{E})$. Topological operators will be noted "int" and "closed". Classical union and intersection are noted $\cap$ and $\cup$. The set $\mathcal{G}$ must verify the following conditions:

(a) $\mathcal{E} \in \mathcal{G}$.

(b) Regularity: the interior of an element of $\mathcal{G}$ is not empty, is "full" ($int(cl(X)) = int(X)$) and is in $\mathcal{G}$; its closure is regular ($cl(int(X)) = cl(X)$) and belongs to $\mathcal{G}$.
The operator $\cup^*$ and $\cap^*$ then denotes union and intersection operators preserving the "regularity" of interiors and closures:
$x \cup^* y = x \cup y \cup int(cl(x \cup y))$
$x \cap^* y = x \cap y \cap cl(int(x \cap y))$

(c) if $X \in \mathcal{G}$ and $X \neq \mathcal{E}$ then $(\mathcal{C}(X)/_{\mathcal{E}}) \in \mathcal{G}$ (if the complement of X is not empty, it is in $\mathcal{G}$).

(d) if $X \in \mathcal{G}$ and $Y \in \mathcal{G}$ and $int(X \cap Y) \neq \emptyset$, then $X \cap^* Y \in \mathcal{G}$. (The intersection $\cap^*$ of two elements of $\mathcal{G}$ is in $\mathcal{G}$ if it has a non empty interior).

(e) if $X \in \mathcal{G}$ and $Y \in \mathcal{G}$, $X \cup^* Y \in \mathcal{G}$. (The union of two elements of $\mathcal{G}$ is in $\mathcal{G}$).

These properties characterize models of the axiomatization of C that is taken from (Asher and Vieu, 1995), leaving aside weak contact.
The structure we consider is more constrained as a partial order on points is added (noted $\prec$) to reflect temporal order. The relation $\approx_t$ will denote the following equivalence relation on points of $\mathcal{E}$[8] :

for all $\alpha \in \mathcal{E}$ and $\beta \in \mathcal{E}$, $\alpha \approx_t \beta$ if and only if : for all γ $(\gamma \prec \alpha \Leftrightarrow \gamma \prec \beta)$ and $(\alpha \prec \gamma \Leftrightarrow \beta \prec \gamma)$

This structure must have the additional properties listed below, where greek letters always denote elements of $\mathcal{E}$:

(f) for all $\alpha \in \mathcal{E} \quad \alpha \not\prec \alpha$

(g) for all $\alpha \in \mathcal{E}, \beta \in \mathcal{E}, \gamma \in \mathcal{E}$, $\alpha \prec \beta$ and $\beta \prec \gamma$ imply $\alpha \prec \gamma$

(h) for all $\alpha \in \mathcal{E}, \beta \in \mathcal{E}, \alpha \approx_t \beta \ \vee \ \beta \prec \alpha \ \vee \ \alpha \prec \beta$

To ease notation, we now define the following functions from $\mathcal{G}$ on $\mathcal{G}$:

TPS(X)=$\{ \alpha \mid \exists \beta \in X \ \beta \approx_t \alpha \}$
$f^*(X) \ =\{ \alpha \mid \forall \beta \in X \ \beta \prec \alpha \}$
$p^*(X) \ =\{ \alpha \mid \forall \beta \in X \ \alpha \prec \beta \}$

And the following relation:

$X < Y$ if and only if $\forall \alpha \in X \ \forall \beta \in Y$, it is true that $\alpha \prec \beta$

And we also have the following constraints:

[8]It is straightforward to check it is indeed an equivalence relation.

(i) for all $X \in \mathcal{G}$ and $Y \in \mathcal{G}$ such that TPS(X) $\subseteq$TPS(Y) it is also true that $Y \cap$TPS$(X) \in \mathcal{G}$
This is compatible with the other constraints since $Y \cap$TPS(X) has a non-empty interior and since Y and X (and therefore TPS(X)) have non-empty interiors.

(j) for all $X \in \mathcal{G}$ and $Y \in \mathcal{G}$ such that $X < Y$ we also have $f^*(X) \in \mathcal{G}$ and $p^*(Y) \in \mathcal{G}$. Again, objects thus introduced necessarily have a non-empty interior and a regular closure. This condition corresponds to existence axioms for past and future of spatio-temporal entities.

(k) there is an $X \in \mathcal{G}$ and $Y \in \mathcal{G}$, such that $X < Y$

(l) there is an $X \in \mathcal{G}$ and $Y \in \mathcal{G}$ such that TPS$(X) \cap$TPS$(Y) \neq \emptyset$ and $X \cap Y = \emptyset$ (this corresponds to axiom 21).

(m) "normality" of the universe : for all $X \in \mathcal{G}$, such that there is Y with $X =$TPS(Y) and for all $Z \in \mathcal{G}$ such that $cl(\text{TPS}(X)) \cap cl(\text{TPS}(Z)) \neq \emptyset$, we have $X \cap Z \neq \emptyset$.
This corresponds to the axiom of regularity 27.

6.2 Semantics

Let $[\![\cdot]\!]$ be an interpretation function over the domain $\mathcal{G}$. It assigns a denotation to the terms of the language of ST in $\mathcal{G}$, and a truth value to its propositions. If we note g a function assigning values $\mathcal{G}$ to variables, we give the following interpretation to ST primitives:

$[\![x < y]\!]_g$ = true if and only if $[\![x]\!]_g < [\![y]\!]_g$
$[\![x \asymp y]\!]_g$ = true if and only if TPS$([\![x]\!]_g) \cap$TPS$([\![y]\!]_g) \neq \emptyset$
$[\![Cxy]\!]_g$ = true if and only if $[\![x]\!]_g \cap [\![y]\!]_g \neq \emptyset$

Let's call S the class of structures with constraints (a)–(m) and with the semantics defined above.
It can be shown that we have then the intended interpretations:

$[\![x + y]\!]_g = [\![x]\!]_g \cup^* [\![y]\!]_g$
$[\![x \cdot y]\!]_g = [\![x]\!]_g \cap^* [\![y]\!]_g$
$[\![-x]\!]_g = \mathcal{C}_{/\mathcal{E}}([\![x]\!]_g)$
$[\![ix]\!]_g = int([\![x]\!]_g)$

$\llbracket cx \rrbracket_g \quad = cl(\llbracket x \rrbracket_g)$
$\llbracket p(x) \rrbracket_g = p^*(\llbracket x \rrbracket_g)$
$\llbracket f(x) \rrbracket_g = f^*(\llbracket x \rrbracket_g)$
$\llbracket x/y \rrbracket_g = \llbracket x \rrbracket_g \cap^* \text{TPS}(\llbracket y \rrbracket_g)$
$\llbracket x \subseteq_t y \rrbracket_g$= true iff $\text{TPS}(\llbracket x \rrbracket_g) \subset \text{TPS}(\llbracket y \rrbracket_g)$

The proof of consistency and completeness of the theory with respect to these structures has been made in (Muller, 1998a).

7 Spatial Relations

In order to compare a spatio-temporal approach to approaches where time and space are separated, we need to express "spatial relations" in our framework in a way that captures what is usually intended in spatial representations. Even though we make use of relations similar to the RCC8 relations of (Randell *et al.*, 1992), their interpretation is not purely spatial. Indeed there is no "space" per se in a spatio-temporal theory as was presented, since histories are only defined relatively to one another.

In a qualitative setting that distinguishes only a specific set of relations between objects, the intended interpretation of a spatial relation (when time enters the picture) must be that the relation holds for a certain time and doesn't change to some other (disjoint) relation during that time. In our framework that can be translated as:

(P) a spatial relation is a relation holding between all temporal slices of two entities during the relevant period.

In the perspective of a comparison with an approach where RCC8 are the basic spatial relations, we only need to express equivalent for those relations and show that they have the same kind of properties. We will only go as far as showing they are disjoint, but of course they cannot be exhaustive in a spatio-temporal framework. These definitions will then be used in section 8 to define several notions of qualitative continuity.

Property (P) is already true in the case of DC when it holds between contemporaneous entities (since it is a theorem that any two parts of disconnected entities are also disconnected).

Th 33 $(\text{DC}xy \ \wedge \ x \equiv_t y) \rightarrow \forall u(\text{TS}ux \rightarrow \text{DC}x_{/u}y_{/u})$

So we can define a "spatial disconnection" as:

D 23 $\text{DC}_{sp}xy \triangleq \text{DC}xy \ \wedge \ x \equiv_t y$

We will adopt the following definitions for the other classical RCC8 relations (R_{sp} denotes the purely spatial equivalent of R):

D 24 $\mathrm{EC}_{sp}xy \triangleq x \equiv_t y \ \wedge\ \forall u(\mathrm{TS}ux \rightarrow \mathrm{EC}x_{/u}y_{/u})$

For overlap it is enough to state:

D 25 $\mathrm{O}_{sp}xy \triangleq x \equiv_t y \wedge \mathrm{O}xy \ \wedge\ x \equiv_t (x \cdot y)$

As we now have:

Th 34 $(\mathrm{O}_{sp}xy \rightarrow \forall u(\mathrm{TS}ux \rightarrow \mathrm{O}x_{/u}y_{/u})$

Thus, in the same way, partial overlap is defined as (we add a condition on the difference of x and y to force its existence during the whole of x and y, thus ensuring the partiality of the overlap):

D 26 $\mathrm{PO}_{sp}xy \triangleq x \equiv_t y \wedge \mathrm{PO}xy \ \wedge\ x \equiv_t (x \cdot y) \ \wedge\ x \equiv_t (x - y)$

As:

Th 35 $\mathrm{PO}_{sp}xy \rightarrow \forall u(\mathrm{TS}ux \rightarrow \mathrm{PO}x_{/u}y_{/u})$

We can also state

D 27 $\mathrm{P}_{sp}xy \triangleq \mathrm{P}xy \ \wedge\ x \equiv_t y$

since we have:

Th 36 $(\mathrm{P}xy \ \wedge\ x \equiv_t y) \rightarrow \forall u(\mathrm{TS}ux \rightarrow \mathrm{P}x_{/u}y_{/u})$

The definition of $\mathrm{TPP}_{sp}xy$ is of a different style, since taking temporally closed slices of an entity x is going to give us tangential parts of x every time, no matter what the configurations are between the "spatial" borders[9]. To define it in a non-trivial manner, we will thus define $\mathrm{TPP}_{sp}xy$ as TPP holding on every "temporally" interior definable slices on x and y:

D 28 $\mathrm{TPP}_{sp}xy \triangleq x \equiv_t y \ \wedge\ \forall u(\mathrm{TS}ux \rightarrow \mathrm{TPP}x_{/(iu)}y_{/(iu)})$

The definition of $\mathrm{NTPP}_{sp}xy$ is also slightly different, because it should be possible to hold on a closed time period. However, with $\mathrm{TS}zx \rightarrow \mathrm{NTPP}x_{/z}y_{/z}$, and with $z = x$ we have $\mathrm{NTPP}xy$, and this means it is not possible for both $x \equiv_t y$ and $cy = y$ to hold. We have thus chosen the following definition, stating that NTPP holds between every pair of contemporaneous slices of the interior of x and y:

D 29 $\mathrm{NTPP}_{sp}xy \triangleq x \equiv_t y \ \wedge\ \forall z(\mathrm{TS}z(ix) \rightarrow \mathrm{NTPP}(x_{/z})(y_{/z})$

The spatial counterparts of TPP^{-1} and NTPP^{-1} can be defined in a straight-forward way.

[9]We wish to thank one of the reviewer for pointing this out to us.

8 Qualitative continuity of motion

We have already mentioned the importance of continuity in our intuitive understanding of motion (motion is a perceptually continuous spatial change), and how this notion pervades a lot of work on spatial relations, the most obvious being the transition graphs of RCC8 (Randell *et al.*, 1992); indeed, changes of state in RCC has been analyzed through transition graphs in which the relations form so-called "conceptual neighborhoods", via potential motion. Only certain changes are allowed, assuming continuous change between relations (this means that spatial changes for closed regions are restricted to the edges of such graphs, see figure 8). For instance two regions cannot be disconnected at a time and overlap at some later time without being in between in external connection. Or, a region cannot be part of another and then be externally connected. Continuity is the central notion here, but remains implicitly assumed without a formal definition; only the work of Galton (Galton, 1993; Galton, 1997) has begun to address what continuity implies for a common-sense theory of motion. Still, this kind of work characterizes continuity as a set of logical constraints on the transitions in a temporal framework and does not add much insight to the already existing transition graph (it is more descriptive than explanatory), and falls short of an explicit, generic characterization of spatio-temporal continuity. It led the author to propose a definition of qualitative continuity in (Muller, 1998b) that would be a characterisation of a similar notion from within the mereo-topological theory itself. The continuity we have in mind is different from the mathematical sense where arbitrarily fine distinctions can be made about space and time. We want something closer to intuition and at the same time covering the properties shown to be necessary for a theory of motion. The continuity we propose can be defined within the theory without stating separately the possible transitions, contrarily to what Galton does in (Galton, 1993)[10], and this provides a more general account of qualitative continuous change; it is thus moreover much easier to check the consistency of such a theory.

We will see that the definition of (Muller, 1998b) was not constrained enough to characterise all the properties that would have made it a definition characterising the conceptual neighborhood of RCC8. This was already corrected in (Muller, 1998a), only available in french, and the problems of that definition were also stated in (Davis, 2001), who proposed to characterise the same kind of notion in a geometrical framework, something we want to avoid in a qualitative context for

[10] Moreover we express the continuity of transition for any entity whereas Galton focused on rigid objects, a concept which cannot be expressed in a topological theory, and which eliminates some transitions such as NTPP to =.

the reasons presented in the introduction. Two papers (Cohn and Hazarika, 2001; Hazarika and Cohn, 2001) took over a simplified version of our space-time theory to propose another definition of continuity. Their stengthened version of continuity follows the same kind of intuition that was in (Muller, 1998a) (see the definition of T-continuity below), but it is difficult to assess what they achieved since they add a spatial connection to express spatial relations within the spatio-temporal theory and their interpretation of spatial connection between two objects is that their spatial projections are connected. Since this means two objects can never have been connected during a given interval and still be considered spatially connected, it is hard to compare their results with ours.

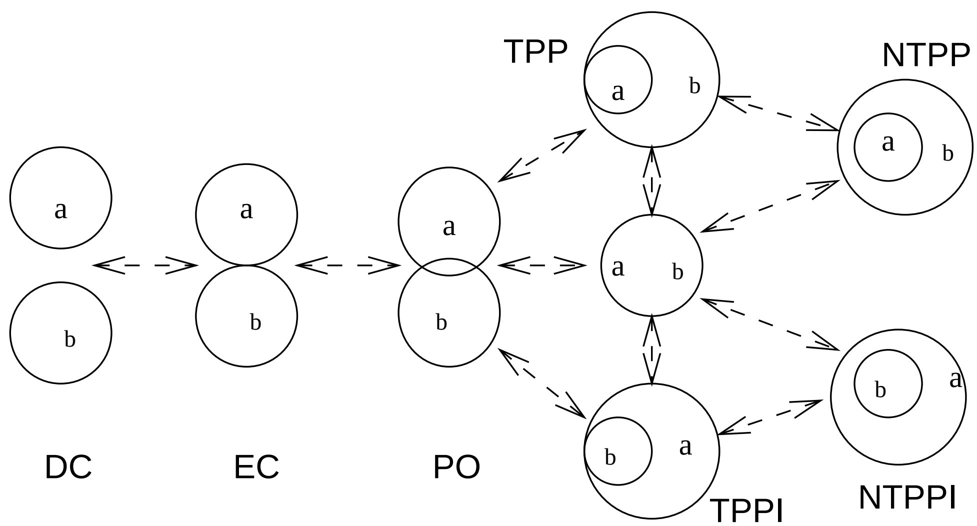

Figure 7: RCC8 Conceptual Neighborhoods

We can specify a first kind of constraint on spatial change by imposing that it is temporally convex and that there is no succesive slices which are disconnected. Let's call it A-continuity and define it as follows:

D 30 A_CONT$w \triangleq$CON$_t w \ \wedge\ \forall x \forall u((\mathrm{TS}xw\ \wedge\ cx \lessgtr cu \wedge \mathrm{TS}uw) \rightarrow \mathrm{C}cxcu)$

This definition says that a region is A-continuous when it is temporally convex and when every two temporally connected closed slices are spatio-temporally connected. Among the cases which are temporally convex but not A-continuous are the change of location of an institution, such as the moving of the capital of Germany from Bonn to Berlin. The previous definition is obviously very coarse and allows for spatio-temporal evolutions that are not seen as continuous in many contexts. For instance, the loss of a province by a country (as the loss of Alsace by France in 1871) is A-continuous, while it corresponds to the loss of a part. We are going to define a continuity with respect to that kind of mereological change by changing slightly the definition of A-continuity to the following P-continuity (for Part-continuity):

D 31 $\mathrm{P\_CONT}w \triangleq \mathrm{CON}_t w \ \wedge\ \forall x \forall u((\mathrm{TS}xw \ \wedge\ cx \between cu \wedge \mathrm{P}uw) \rightarrow \mathrm{C}cxcu)$

This says that a ST-region is P-continuous when every closed part temporally connected to a slice is also connected. A counterexample, which might illustrate the return of Alsace within France in 1918, is shown figure 8. The definition of P-continuity rules out the case where there is a sudden loss or gain of an entity and the cases where there is a gain or loss, even gradual, of a self-connected component, such as the apparition of an island in an archipelago.

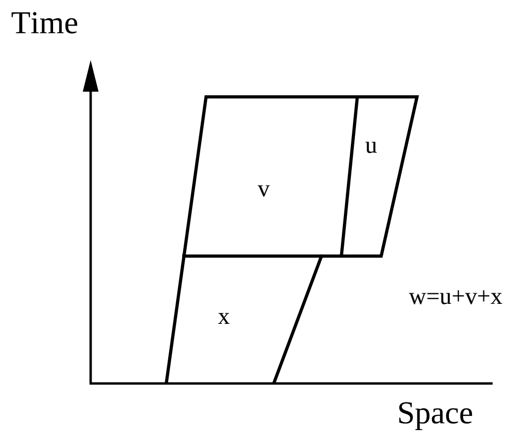

Figure 8: Example of a non P-continuous change

Finally, the last kind of continuity we want to introduce is close to the notion of continuous motion as usually understood, but in a topological context. P-continuity already provides an approximation of the conceptual neighborhood graph: if one considers relations between contemporaneous slices of two histories as spatial relations whenever they are the same for each sub-slices (cf section 7), it can be easily be shown that the definition excludes cases involving disconnected parts appearing, as in a change from $\neg$C to PP.

It is not enough to recover all constraints of the topological neighborhood graph, however. Indeed the definition accepts the history of x shown figure 9 which can be seen as the gradual shrinking of an entity to a spatial "point" and back to an extended region again (like a puddle drying up and reforming right afterwards), allowing for leaps in the topological configuration. In this case a slice of x is a part of y for a time, then the next slice of x is in external connection with y, and no intermediary slice of x is ever in relation of overlap with a slice of y in the process. In order to rule out those cases of "born-again" regions, we could impose the strong connectedness on regions; it does seem a little heavy handed however, as we only want to rule out the kind of "temporal" points that cause the trouble, and it is perfectly acceptable to have spatio-temporally disconnected entities (like an archipelago). We only want to exclude jumps of non strongly connected parts, and thus we need a notion of temporal strong connectedness. We

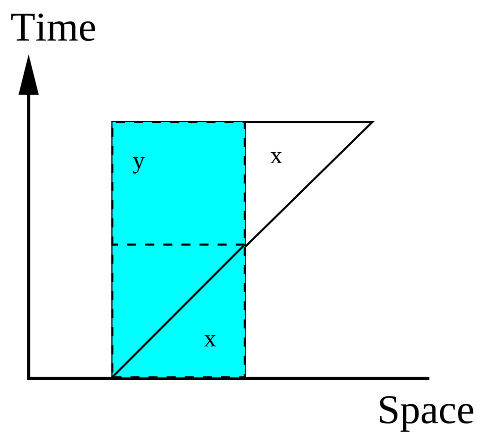

Figure 9: From "part-of" to external connection without overlap

first define a relation of strong temporal connection between two entities as the strong connection between corresponding slices of the universe a:

D 32 $\mathrm{STC}xy \triangleq \mathrm{SC}(a_{/x})y \wedge \mathrm{SC}x(a_{/y})$

The notion of temporal strong connectedness follows:

D 33 $\mathrm{STCR}x \triangleq \forall x_1 \forall x_2 (x = x_1 + x_2 \rightarrow \mathrm{STC}cx_1cx_2)$

And the last kind of strengthened continuity can be defined:

D 34 $\mathrm{T\_CONT}w \triangleq \mathrm{STCR}w \ \wedge \ \forall x \forall u((\mathrm{TS}xw \ \wedge \ cx \between cu \wedge \mathrm{P}uw) \rightarrow \mathrm{C}xu)$

We can easily see that T-continuity implies P-continuity. Thus we have a hierarchy of spatial changes, as P-continuity implies A-continuity, which implies temporal connectedness.

P-continuity rules out transition from DC to any Part-relation (TPP, NTPP and their converses), and T-continuity rules out transition from EC to TPP or NTPP. We have thus recovered most of the transition graph of RCC8. The transition from DC to PO is not ruled out by such a definition as it depends on the kind of atoms that the mereotopological theory allows[11].

9 Categories of common sense motion

Many linguistic studies have tried to classify motion into classes, stemming from the study of some motion verbs (cf. e.g. (Talmy, 1983) for geometric characteristics of motion descriptions through verbs). For instance (Sablayrolles, 1995) considers motion as essentially a complete characterisation of the position of an object with respect to another one during three successive phases (at the beginning, during, or at the end of the event). We will on the contrary take over the study of motion verbs presented in (Muller and Sarda, 1997) in order to isolate the essential aspects of motion as expressed in natural language. We will try to use this as a guide to what can be expressed in the formal theory stated above in order to represent the most "basic" common sense motion events. Indeed, on the basis of the spatio-temporal mereo-topology ST, one can define very different trajectories in space-time, and this expressivity should be somewhat tamed in order to reason efficiently. A reasonable objective in the perspective of qualitative reasoning would then be to define a set of spatio-temporal relations that would be rich enough to represent various situations, and constrained enough to be of practical use. From this point of view, transitive motion verbs describe relations between two entities (the subject and object of the verb) in motion w.r.t one another and provide a natural set of relations which can be used to represent motion events, even though on a non-exhautive basis.

It seems that the relevant information contained in motion described by such verbs is focused on one of the phases of the motion described rather than completely describing the successive locations of the object(s) involved. In (Muller and Sarda, 1999) it is shown that three main features can be isolated to characterize motion

[11] Counter-examples are exhibited in (Muller, 1998a), and could only be ruled out by dealing with the shape of atoms. This goes beyond the scope of our study here, which is merely mereological and topological.

described by the verbs considered:

- a *polarity*, that is, the phase of the motion on which the verb semantics focuses. It can be initial (as in *to leave*), final (*to reach*) or median (*to cross*).
- the topological relation involved between the two entites related by the motion event during the phase defined by the polarity of motion. It can be *inclusion* (*to leave the city*), *contact* (*to hit the wall*), or a type of non connexion that can be further refined.
- the change of this topological relation during the motion (compare *leave the city* with *wander in the city*).

It appears that six non-empty classes of verbs can be isolated when these features are combined: internal/initial, internal/final, contact/final, internal/median, medians with change, and non-topological medians.

We are now going to define the classes of topological change that correspond to these verbs, taking into account topological aspects only. Our lexical study must thus be seen as only a guide in the definition of cognitively significant motion classes, as we leave out a lot of the semantics of motion verbs. Nonetheless this could easily be taken as a basis for motion verbs semantics in the manner of (Kalita and Lee, 1997), but in a more principled manner at the topological level (although there would still be a need for ways of translating between region-based and point-based representations).

10 Representing natural motions

10.1 Some spatio-temporal concepts

We need first to introduce a few useful definitions.
We will make use of a specific partial overlap corresponding to a spatio-temporal overlap (a slice of x has to be completely disconnected with y for a moment):

D 35 $\mathrm{STPO}xy \triangleq \mathrm{PO}xy \ \wedge \ \exists u(\mathrm{TS}ux \ \wedge \ u \subseteq_t y \ \wedge \ \neg \mathrm{C}uy)$

This definition indeed implies the following property:

Th 37 $\mathrm{STPO}xy \rightarrow \exists u(\mathrm{TS}ux \ \wedge \ \neg \mathrm{C}uy)$

We now introduce a relation of inclusion of slices:

D 36 $\text{LOC}xy \triangleq \exists z(\text{TS}zx \wedge \text{P}zy)$ (a slice of x is part of y)

And a relation of "temporary inclusion":

D 37 $\text{TEMP\_IN}xy \triangleq \text{LOC}xy \wedge \text{STPO}xy$ (x is temporarily included in y)

We can also define counterparts of Allen's temporal relations, bearing on spatio-temporal individuals[12]:

D 38 $\text{MEETS}xy \triangleq ix < iy \wedge x \between y$

D 39 $\text{START}xy \triangleq x \subseteq_t y \wedge \forall z(z < x \rightarrow z < y) \wedge \neg y \subseteq_t x$

D 40 $\text{FINISH}xy \triangleq x \subseteq_t y \ \wedge \ \forall z(x < z \rightarrow y < z) \ \wedge \ \neg y \subseteq_t x$

To avoid confusion, Allen's concept of overlap will be noted O_t; it is different from σ as it is not a symetrical relation.

D 41 $\text{O}_t xy \triangleq x \sigma y \ \wedge \ \exists u(\text{START}uy \wedge \text{FINISH}ux)$

D 42 $\text{DURING}xy \triangleq x \subseteq_t y \ \wedge \ \neg\text{FINISH}xy \ \wedge \ \neg\text{START}xy$

Inverse relations can be defined in an obvious way and will be noted MEETS_i, START_i, FINISH_i, O_i, DURING_i.
These relations are slightly different from Allen's, as they bear on entities which are not necessarily convex temporally. Moreover, temporal equivalence is different from logical equality.

10.2 Natural classes

At this stage, we can define the classes of topological change taking into account considerations seen in 10.1. The following classes relate three spatio-temoral entities: the first one (z) correspond to the entity that establish the time of the motion event: the relation holds *during* the lifespan of z. The other two entities x and y are entities in relative motion. Thus, such relations actually bear on $x_{/z}$ and $y_{/z}$. We define six classes of motion according to the lexical study, noted LEAVE, REACH, HIT, CROSS, INTERNAL and EXTERNAL; they intuitively correspond to the simplified topological behaviours of the eponymous verbs, internal and external corresponding to verbs such as *wander around in* and *go around something* respectively.

[12] Note that temporal = and "before" already correspond to $\equiv_t$ and $<$.

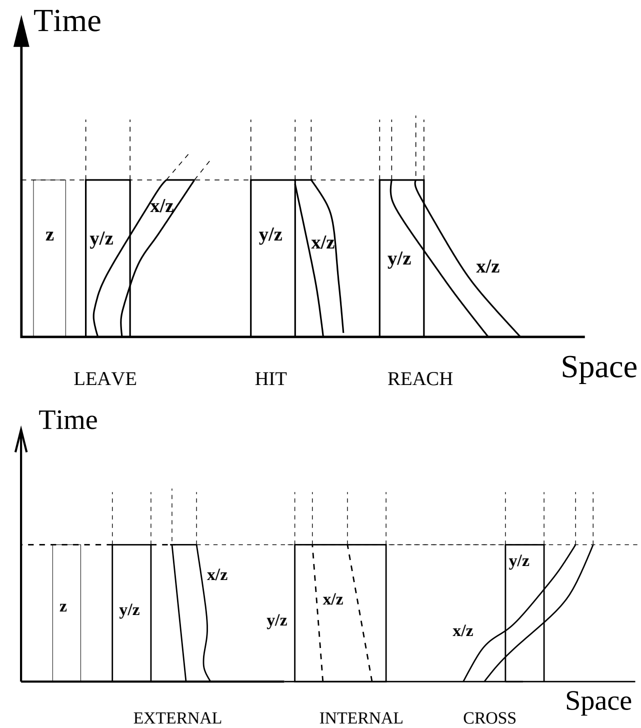

Figure 10: The six common sense motion relations.

D 43 REACH$zxy \triangleq$TEMP_IN$x_{/z}y_{/z} \wedge$ FINISH$(x_{/z} \cdot y_{/z})z$

D 44 LEAVE$zxy \triangleq$TEMP_IN$x_{/z}y_{/z} \wedge$ START$(x_{/z} \cdot y_{/z})z$

D 45 INTERNAL$zxy \triangleq$PP$x_{/z}y_{/z}$

D 46 HIT$zxy \triangleq$EC$x_{/z}y_{/z} \wedge$
$\forall x_1, y_1[($P$x_1x_{/z}\wedge$P$y_1y_{/z}\wedge$EC$x_1y_1) \rightarrow ($FINISH$x_1z\wedge$FINISH$y_1z)]$

D 47 EXTERNAL$zxy \triangleq \neg$C$x_{/z}y_{/z}$

D 48 CROSS$zxy \triangleq \exists z_1, z_2(z = z_1+z_2 \wedge$ MEETS$z_1z_2 \wedge$ REACH$z_1xy \wedge$ LEAVE$z_2xy)$

If we want to have some kind of temporal symetry we can also define the initial analogous of HIT, which we call SPLIT:
D 49 SPLIT$zxy \triangleq$EC$x_{/z}y_{/z} \wedge$
$\forall x_1, y_1[($P$x_1x_{/z}\wedge$P$y_1y_{/z}\wedge$EC$x_1y_1) \rightarrow ($START$x_1z\wedge$START$y_1z)]$

Obviously these classes are not exhaustive, but give a reasonable set for qualitatively describing motion events, and (Muller, 1998b) has shown some simple examples of reasoning that can be made with constraints using this set.

11 Conclusion

We have presented here a commonsense theory of motion from a topological perspective. The use of first-order logic allows for a explicit characterisation of the properties that seem desirable for a model of space and time in that context: the object domain comprises only regions of space-time, denoting the histories of physical objects and spatio-temporal events. A set of relations can be expressed between such regions, that correspond to relative motions of spatial entities, and we have shown how reasoning can be done on these relations, along with temporal and purely spatial information. We thus believe we have provided a sound basis for the representation of intuitive concepts related to space and time in a symbolic perspective where exact geometric information is not available or not necessary. In so doing, we have restricted our model to topological information, leaving out what could be a future enrichment, i.e. more detailed spatial information such as orientation relations or relative distances. Besides, the inferential properties of our theory have still to be completely characterised to be able to deal with all types of deduction on spatio-temporal information. Our definition of spatio-temporal continuity has corrected the problems of (Muller, 1998b), remaining a lot simpler to define that the proposed change in (Cohn and Hazarika, 2001) and avoiding the use of a point-based system as in (Davis, 2001). The language we have presented and formally investigated can be seen as a validation of the proposals of the kind of (Erwig *et al.*, 1999) for spatio-temporal databases, as it makes explicit the properties one can expect of databases representations; they nonetheless keep an intrinsically point-based semantics in (Erwig and Schneider, to appear) by extending Egenhofer's work for spatial relations (Egenhofer and Franzosa, 1991), thus limiting their formal models to very specific kinds of histories, namely convex histories without holes. Our proposal's domain is thus more general. Moreover the work of (Tøssebro and Güting, 2001) has shown how such a language could be used from temporally discrete point-based data for spatial regions by proposing interpolation methods that would give wholesome spatio-temporal objects in a higher dimension, thus bypassing the claim made by other authors that region-based representations are not practically usable. There is still clearly a lot to investigate at this level for spatial representation theories.

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List of Figures

Proofs of theorems of sections 4 and 5

We begin by showing a few useful lemmas :

Lemma 1 $x \subseteq_t y \rightarrow x \between y$

Proof

1 $x \subseteq_t y \to \forall u\ (u \between x \to u \between y)$ (def. of $\subseteq_t$)

2 $x \between x$ (Ax. 12)

3 $x \between y$ (with (1) and (2))

$\square$

Lemma 2 $x\sigma y \to x \between y$

Proof

1 $x\sigma y \to \exists z(z \subseteq_t y\ \wedge\ z \subseteq_t x)$

2 $\forall u\ (u \between z \to u \between y)$ (1 and def. of $\subseteq_t$)

3 $\forall u\ (u \between z \to u \between x)$ (1 and def. of $\subseteq_t$)

4 $z \between z$ (Ax 11)

5 $z \between x$ (4 and 2)

6 $x \between y$ (Ax 4.12, 5 and 3)

$\square$

Th 3 $x < y \to \neg x\sigma y$

We show the contrapositive:

Proof

1 $x\sigma y \to x \between y$ (Lemma 2)

2 $\neg x < y$ (Ax 4.13 and 1)

$\square$

Th 4 $(x < y\ \wedge\ y\sigma z\ \wedge\ z < t) \to x < t$

Proof

1 $y\sigma z \to y \between z$ (Lemma 2)

2 $(x < y\ \wedge\ y \between z\ \wedge\ z < t) \to x < t$ (Ax. 15)

3 $(x < y\ \wedge\ y\sigma z\ \wedge\ z < t) \to x < t$ (from 1 and 2)

$\square$

Th 5 $(x < y\ \wedge\ y \subseteq_t z\ \wedge\ z < t) \to x < t$

Proof

1 $y \subseteq_t z \to y \asymp z$ (Lemma 1)

2 $(x < y \ \wedge \ y \asymp z \ \wedge \ z < t) \to x < t$ (Ax. 15)

3 $(x < y \ \wedge \ y \subseteq_t z \ \wedge \ z < t) \to x < t$ (from 1 and 2)

□

Th 6 $(x \subseteq_t y \ \wedge \ y \subseteq_t z) \to x \subseteq_t z$

Proof

1 $x \subseteq_t y \to \forall u \ (u \asymp x \to u \asymp y)$ (def. of $\subseteq_t$)

2 $y \subseteq_t z \to \forall u \ (u \asymp y \to u \asymp z)$ (def. of $\subseteq_t$)

3 $\forall u \ (u \asymp x \to u \asymp z)$ which is the definition of $x \subseteq_t z$ (from 1 and 2)

□

Th 7 $x \subseteq_t y \to \forall z \ (z\sigma x \to z\sigma y)$

Proof

1 $z\sigma x \to \exists u \ (u \subseteq_t x \ \wedge \ u \subseteq_t z)$ (def. of σ)

2 $(u \subseteq_t x \ \wedge \ x \subseteq_t y) \to u \subseteq_t y$ (Th.)

3 $\exists u \ (u \subseteq_t y \ \wedge \ u \subseteq_t z) \to z\sigma y$ (1, 2 and def. of σ)

□

Th 11 $\mathrm{O}xy \to x\sigma y$

Proof

1 $\exists z \ (\mathrm{P}zy \wedge \mathrm{P}zx)$ ($\mathrm{O}xy$)

2 $\mathrm{P}zy \to z \subseteq_t y$ (Ax 20 and 1)

3 $\mathrm{P}zx \to z \subseteq_t x$ (1)

4 $x\sigma y$ (def. of σ and 2 and 3)

□

Th 12 $(x < y \wedge \mathrm{P}zx \wedge \mathrm{P}ty) \to z < t$

Proof

1 $\mathrm{P}zx \to z \subseteq_t x$ (Ax. 20)
2 $(x < y \ \wedge\ z \subseteq_t x) \to z < y$ (Ax. 16)
3 $\mathrm{P}ty \to t \subseteq_t y$ (Ax. 20)
4 $(z < y \ \wedge\ t \subseteq_t y) \to z < t$ (Ax. 16 and 1, 2, 3)

□

Th 13 $(x\sigma y \wedge \mathrm{P}xz \wedge \mathrm{P}yt) \to z\sigma t$

Proof

1 $\mathrm{P}xz \to x \subseteq_t z$ (Ax. 20)
2 $\mathrm{P}yt \to y \subseteq_t t$ (Ax. 20)
3 $x\sigma y \to \exists u\ (u \subseteq_t x \ \wedge\ u \subseteq_t y)$ (def. of σ)
4 $(u \subseteq_t x \ \wedge\ x \subseteq_t z) \to u \subseteq_t z$ (Ax. 6)
5 $(u \subseteq_t y \ \wedge\ y \subseteq_t t) \to u \subseteq_t t$ (Ax. 6)
6 $(x\sigma y \wedge \mathrm{P}xz \wedge \mathrm{P}yt) \to z\sigma t$ (def. of σ and 1, 2, 3, 4, 5)

□

Th 14 $(x+y)\sigma z \leftrightarrow\ x\sigma z \ \vee\ y\sigma z$

First, we show the following lemma::

Lemma 3 $u \subseteq_t (x+y) \to (u\sigma x \ \vee\ u\sigma y)$

Proof

1 $\mathrm{C}(iu)u$ (def. of interior)
2 $iu \lessgtr u$ (1 and ax. 19)
3 $iu \lessgtr (x+y)$ (hypothesis, def. of $\subseteq_t$ and 2)
4 $iu \lessgtr x \ \vee\ iu \lessgtr y$ (3 and ax.24)
5 $iu \lessgtr x \to u\sigma x$ (ax. 25)
6 $u \lessgtr x \ \vee\ u \lessgtr y$ (4 and 5) □

Then we can show the direct sense of the theorem:

Proof

1 $\exists u\ (u \subseteq_t (x+y) \ \wedge\ u \subseteq_t z)$ (hypothesis and def. of σ)
2 $u\sigma x \ \vee\ u\sigma y$ (1 and lemma 3)
3 $\exists v\ (v \subseteq_t u \ \wedge\ v \subseteq_t x) \ \vee\ \exists v\ (v \subseteq_t u \ \wedge\ v \subseteq_t y)$ (def. of σ and 2)

4 $(v \subseteq_t u \ \wedge\ u \subseteq_t z) \rightarrow v \subseteq_t z$ (th. 6)
5 $z\sigma x \ \vee\ z\sigma y$ (3 and 4) □

The converse is shown by:

Proof
1 $\mathrm{P}x(x+y) \rightarrow x \subseteq_t (x+y)$ (ax. 20)
2 $x\sigma z \rightarrow \exists u\ (u \subseteq_t x \ \wedge\ u \subseteq_t z)$ (def. of σ)
3 $u \subseteq_t (x+y)$ (from 1, 2 and th. 6)
4 $z\sigma(x+y)$ (from 1 and 2)
5 it can be shown likewise that $y\sigma z \rightarrow z\sigma(x+y)$.

□

Th 15 $\mathrm{TS}xx$

We have $\mathrm{P}xx$ and $\forall z(\mathrm{P}zx \rightarrow \mathrm{P}zx)$ so by definition from TS, $\forall x$ $\mathrm{TS}xx$.

Th 16 $(\mathrm{TS}xy \wedge \mathrm{TS}yx) \rightarrow x = y$

Straightforward by definition of TS, as $\mathrm{P}xy \wedge \mathrm{P}yx \rightarrow x = y$.

Th 17 $(\mathrm{TS}xy \wedge \mathrm{TS}yz) \rightarrow \mathrm{TS}xz$

Proof
1 $\mathrm{TS}xy \rightarrow \mathrm{P}xy$ (def. of TS)
2 $\mathrm{TS}yz \rightarrow \mathrm{P}yz$ (def. of TS)
3 $(\mathrm{P}xy \wedge \mathrm{P}yz) \rightarrow \mathrm{P}xz$ (transitivity of P)
4 $\mathrm{TS}xy \rightarrow \forall u\ ((\mathrm{P}uy \ \wedge\ u \subseteq_t x) \rightarrow \mathrm{P}ux)$ (def. of TS)
5 $\mathrm{TS}yz \rightarrow \forall u\ ((\mathrm{P}uz \ \wedge\ u \subseteq_t y) \rightarrow \mathrm{P}uy)$ (def. of TS)
6 $\mathrm{P}xy \rightarrow x \subseteq_t y$ (Ax 20)
7 $(u \subset_t x \ \wedge\ x \subseteq_t y) \rightarrow u \subseteq_t y$ (6)
8 $(\mathrm{P}uz \ \wedge\ u \subseteq_t y) \rightarrow \mathrm{P}uy$ (5)
9 $(\mathrm{P}uy \ \wedge\ u \subseteq_t x) \rightarrow \mathrm{P}ux$ (4)
10 $(\mathrm{P}uz \ \wedge\ u \subseteq_t x) \rightarrow \mathrm{P}ux$ (6, 7, 8, and 9)
11 3 and 10 yield $\mathrm{TS}xz$ by definition of TS.

□

Th 18 $(\mathrm{TS}xy \wedge \mathrm{TS}zy \ \wedge \ x \subseteq_t z) \rightarrow \mathrm{TS}xz$

Proof

1 $\mathrm{TS}xy \rightarrow \mathrm{P}xy$
2 $(\mathrm{P}xy \ \wedge \ x \subseteq_t z \wedge \mathrm{TS}zy) \rightarrow \mathrm{P}xz$ (def. of TS.)
3 $\mathrm{TS}yz \rightarrow \mathrm{P}zy$ (def. of TS.)
4 $(\mathrm{P}uz \wedge \mathrm{P}zy) \rightarrow \mathrm{P}uy$ (transitivity of P)
5 $(\mathrm{P}uy \ \wedge \ u \subseteq_t x) \rightarrow \mathrm{P}ux$ (TSxy)
6 $(\mathrm{P}uz \ \wedge \ u \subseteq_t x) \rightarrow \mathrm{P}ux$ (4 and 5)
7 from 2 and 6 we get by definition that TSxz

□

Th 19 $\forall x, y(x\sigma y \rightarrow \exists u(\mathrm{TS}ux \ \wedge \ u \subseteq_t y))$

Proof

1 $x\sigma y \rightarrow \exists z \ (z \subseteq_t x \ \wedge \ z \subseteq_t y)$
2 $z \subseteq_t x \rightarrow \exists u \ (\mathrm{TS}ux \ \wedge \ z \equiv_t u)$ (Ax. 26)
3 $(z \equiv_t u \ \wedge \ z \subseteq_t y) \rightarrow u \subseteq_t y$ □

Th 20 $\mathrm{P}xy \rightarrow \exists z(\mathrm{TS}zy \ \wedge \ z \equiv_t x)$

Proof

1 $\mathrm{P}xy \rightarrow x \subseteq_t y$ (Ax. 20)
2 $x\sigma y \rightarrow \exists z \ (\mathrm{TS}zy \ \wedge \ z \equiv_t x)$ (Ax. 26)

□

Th 21 $(\mathrm{TS}xy \wedge \mathrm{TS}zy \ \wedge \ x \equiv_t z) \rightarrow x = z$

Proof

1 $\forall u(\mathrm{P}uy \ \wedge \ u \subseteq_t x) \to \mathrm{P}ux$ (TSxy)
2 TS$zy \to \mathrm{P}zy$
3 $x \equiv_t z \to z \subseteq_t x$ (def of $\equiv_t$)
4 Pzx (from 1, 2 and 3)
5 by symmetry of the hypotheses with respect to x and z, we have likewise Pxz, hence $x = z$.

□

Th 22 TS$(f(x), a)$

Proof

1 P$f(x)a$ (def. of a)
2 $x < f(x)$ (th. 9)
3 $\forall u \ (u \subseteq_t f(x) \to x < u)$ (2 and th. 8)
4 $\forall u \ (x < u \to \mathrm{P}uf(x))$ (def. of $f(x)$)
5 $\forall u \ (\mathrm{P}ua \ \wedge \ u \subseteq_t f(x) \to \mathrm{P}uf(x))$ (4)
6 from 1 and 5 we get TS$(f(x), a)$ by definition of TS.

□

It can be shown likewise that TS$(p(x), a)$.

Th 23 $(x \equiv_t y \ \wedge \ \exists z \ (x < z)) \to f(x) = f(y)$

Proof

1 $x < f(x)$ (def. of $f(x)$)
2 $(x < f(x) \ \wedge \ y \subseteq_t x) \to y < f(x)$ (th. 8)
3 $y < f(x) \to \mathrm{P}f(x)f(y)$ (def. of $f(y)$)
4 by symmetry of the hypotheses with respect to x and y we also have P$f(x)f(y)$ therefore $f(x) = f(y)$.

□

Likewise $(x \equiv_t y \ \wedge \ \exists z \ (z < x)) \to p(x) = p(y)$

Lemma 4 AT$x \to \neg$BETWyx

We show the contrapositive:

Proof

1 $\exists y', x_1, x_2\ (x = x_1 + x_2\ \wedge\ x_1 < y' < x_2)$ (def. of BETW and th. 25)
2 $x_1 < y' < x_2 \rightarrow \neg \mathrm{C} x_1 x_2$ (ax. 13 and ax. 19)
3 $\neg(x_1 = x_2)$ (2)
4 $(x = x_1 + x_2\ \wedge\ \neg(x_1 = x_2)) \rightarrow \neg \mathrm{AT} x$ (def. of AT)

□

Th 25 $(\mathrm{AT}x \wedge \mathrm{AT}y) \rightarrow (\mathrm{ORD}(x, y)\ \vee\ x \equiv_t y)$

We are going to show that $(\mathrm{AT}x \wedge \mathrm{AT}y\ \wedge\ \neg\mathrm{ORD}(x, y)) \rightarrow x \equiv_t y$.

Proof

1 $(\mathrm{AT}x \wedge \mathrm{AT}y\ \wedge\ \neg\mathrm{ORD}(x, y)) \rightarrow x \between y)$ (previous lemma and ax. 28)
2 $\mathrm{AT}x \rightarrow ix = x$ (def. of atom and $\mathrm{P}(ix)x$)
3 $ix \between y$ (1 and 2)
4 $x \sigma y$ (3 and ax. 25)
5 $\exists u\ (u \subseteq_t x\ \wedge\ u \subseteq_t y)$ (4 and def. of σ)
6 $\exists v_1, v_2\ (v_1 = x/u\ \wedge\ v_2 = y/u\ \wedge\ v_1 \equiv_t v_2)$ (ax. 26 and 5)
7 $(\mathrm{AT}x \wedge \mathrm{AT}y) \rightarrow (v_1 = x\ \wedge\ v_2 = y)$ (def. of AT and $\mathrm{P}(x/u)x$ and $\mathrm{P}(y/u)y$)
8 $x \equiv_t y$ (6 and 7)

□

Th 26 $(\mathrm{CON}_t x \wedge \mathrm{CON}_t y) \rightarrow (x < y\ \vee\ x \between y\ \vee\ y < x)$

This is a corollary of the following lemma (by axiom 28), whose contrapositive is shown below:

Lemma 5 $\mathrm{CON}_t x \rightarrow \neg \mathrm{BETW} y x$

Proof

1
2 $\exists y', x_1, x_2\ (x = x_1 + x_2\ \wedge\ x_1 < y' < x_2)$ (def de BETW and th. 25)
3 $x_1 < y' < x_2 \rightarrow \neg \mathrm{C} c x_1 c x_2$ (ax. 13 and ax. 19)
4 $\neg \mathrm{CON}_t x$ (def. of CON_t and 3)

□

Th 27 $\forall x[(\exists y(y < x)) \to (p(x) < -p(x))]$

Th 28 $\forall x[(\exists y(x < y)) \to (-f(x) < f(x))]$

There are only the following possibilities: either

1 $BETW(-p(x), p(x)) : \exists u_1, u_2, v(Pu_1p(x) \ \wedge \ Pu_2p(x) \ \wedge \ Pv(-(p(x))) \ \wedge \ u_1 < v < u_2)$
from $Pu_2p(x)$ we draw $u_2 < x$ and then $v < u_2 < x$
and $Pv(p(x))$ and $Op(x)(\text{-}p(x))$: contradiction.

2 or $BETW(p(x), -p(x)) : \exists u_1, u_2, v(Pu_1{-}p(x) \wedge Pu_2{-}p(x) \wedge Pv(p(x)) \wedge u_1 < v < u_2)$
$Pv(p(x))$ implies $v < x$ and then $u_1 < x$ d'où $Pu_1p(x)$
and then $Op(x)(-p(x))$: contradiction.

3 or $-p(x) < p(x)$: so $-p(x) < x$ and then $P(-p(x))p(x)$.

4 or $p(x) \asymp -(p(x))$: with the "normality" axiom and because TS$p(x)a$, we get $Cp(x)(-p(x))$ which gives a contradiction.

This only remaining possibility then is $p(x) < -p(x)$. The similar theorem for f is shown along the same line.

Th 29 $(\exists y(x < y)) \to -f(x) = p(f(x))$
Th 30 $(\exists y(y < x)) \to -p(x) = f(p(x))$

Proof

1	$-f(x) < f(x)$	(th. 28)
2	$\mathrm{P}(-f(x), p(f(x)))$	(1 and def. of $p(f(x))$)
3	$p(f(x)) < f(x)$	(def. of $p()$)
4	$p(f(x)) < f(x) \to \neg f(x) \asymp p(f(x))$	(ax. 13)
5	$\neg f(x) \asymp p(f(x)) \to \neg \mathrm{C}(f(x), p(f(x)))$	(ax. 19)
6	$\mathrm{P}(p(f(x)), -f(x))$	(5 and def. of complement)
7	$-f(x) = p(f(x))$	(2 and 6)

□

Likewise $-p(x) = f(p(x))$.
The following are just rewriting of these two theorems since for any x,
$a = x + (-x)$.

Th 31 $a = p(x) + f(p(x))$

Th 32 $a = f(x) + p(f(x))$

Th 33 $(\text{DC}xy \ \wedge \ x \equiv_t y) \rightarrow \forall u(\text{TS}ux \rightarrow \text{DC}x_{/u}y_{/u})$

By the definition of P and DC, DC$xy \wedge$ P$ux \rightarrow$DCuy and since slices of an entity are parts by definition, the theorem follows.

Th 34 $(\text{O}_{sp}xy \rightarrow \forall u(\text{TS}ux \rightarrow \text{O}x_{/u}y_{/u})$

Proof

H	TSux	
1	TS$ux \rightarrow u \subseteq_t x$	(H and def of TS)
2	$x \equiv_t x \cdot y$	(def of O$_{sp}$)
3	$u \subseteq_t x \cdot y$	(1,2 and def of $\equiv_t$)
4	$\exists x'(\text{TS}x'(x \cdot y) \ \wedge \ x' \equiv_t u)$ (3 and ax.(26)	
5	P$x'x$	(def of TS and 4)
6	$x' \subseteq_t u$	(4 and def of $\equiv_t$)
7	P$x'u$	(H, def of TS and 5 and 6)
8	P$x'y$	(def of TS and 4)
9	Ouy	(7,8, def of O)
10	u is a slice of x so by definition $x_{/u} = u$	
11	$u \subseteq_t y \rightarrow u \equiv_t y_{/u}$	(ax 26)
12	O$x_{/u}y_{/u}$	(9, 10, 11)

□

Th 35 $(\text{PO}_{sp}xy \rightarrow \forall u(\text{TS}ux \rightarrow \text{PO}x_{/u}y_{/u})$

This is shown in a similar way to the previous theorem, first by stating O$_{sp}xy$, then by considering $(x - y)$ instead of $(x \cdot y)$ in the proof. Thus we prove there is always a $(x - y)_{/u}$ for all slices u of x and that the corresponding overlap is partial.

Th 36 $(\text{P}xy \ \wedge \ x \equiv_t y) \rightarrow \forall u(\text{TS}ux \rightarrow \text{P}x_{/u}y_{/u})$

Proof

1 TSux →Pux

H Pxy

2 Puy (transitivity of P)

3 P$u(y_{/u})$ (2 and def of $y_{/u}$)

4 u is a slice of x so by definition $x_{/u} = u$

5 P$(x_{/u})(y_{/u})$ (3 and 4)

□