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Evaluating Trustworthiness from Past Performances: Interval-based Approaches

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Abstract In many multi-agent systems, especially in the field of e-commerce, the users have to decide whether they sufficiently trust an agent to achieve a certain goal. To help users to make such decisions, an increasing number of trust systems have been developed. By trust system, we mean a system that gathers information about an agent and evaluates its trustworthiness on the basis of this information. The aim of the present paper is to develop, and analyze from an axiomatic point of view, new trust systems based on intervals. More precisely, we assume that a set of grades describing the past performances of an agent is given. Then, the goal is to construct an interval that summarizes these grades. In our opinion, such an interval gives a good account of the trustworthiness of the agent. In addition, this kind of representation format overcomes certain limitations (at a certain cost) of the approaches that represent trustworthiness by a single number. We establish seven axioms that should be satisfied by a summarizing method. Next, we develop two new methods. The first one is based on the idea that certain concentrations of grades are strong enough to pull the bounds of the summarizing interval towards themselves. The second one represents data in the setting of possibility theory, and then computes lower and upper expected values. Finally, we check that

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our methods satisfy the axioms introduced before, which provide theoretical justifications for them.

Keywords Trust · Distrust · Intervals · Possibility Theory

1 Introduction

In many multi-agent systems, especially in the field of e-commerce, the users have decisions to make about the agents. In particular, a user has to decide whether he (or she) believes that an agent is more trustworthy than another, or whether an agent is sufficiently trustworthy in the absolute sense. This is the case for example in the famous auction system Ebay. Indeed, a buyer has to decide whether he sufficiently trusts a seller to be honest and competent. In order to help users to make such decisions, an increasing number of *trust systems* have been developed, see e.g. [21] for a review on trust and reputation systems, see e.g. [20] for an experiment that evaluates the influence of the reputation system integrated into Ebay.

By trust system, we mean a system that gathers information about a particular agent (called the trustee), and then constructs on the basis of this information, an object giving a good account of its trustworthiness. For example, Ebay is equipped with a system that collects the ratings a seller has received from buyers, and then attributes to that seller a certain score on the basis of these ratings. This score gives an indication of how reliable the seller is.

Developing a trust system is a challenging problem that raises questions such as: how trustworthiness may be represented, and from which information it may be estimated? The aim of this paper is to develop, and analyze from an axiomatic point of view, new trust systems based on intervals. More precisely, we assume that a set of *grades* describing the past performances of the trustee is given. By convention, the value of a grade belongs to a discrete scale starting with 0 and ending with 1. Each grade describes how well the trustee has achieved a certain past goal. Naturally, 0 means that the goal has not at all been achieved, whilst 1 means on the contrary that it has been achieved perfectly.

Given a set of grades, the goal is to construct a real interval included in $[0, 1]$ that summarizes the grades. Such a *summarizing interval* unavoidably describes the past behavior of the trustee with less accuracy than the grades, but nevertheless it gives a good account of the essential parts of this behavior. In our opinion, a summarizing interval constitutes a good overview of the trustworthiness of the trustee, and thus provides trustors with a handy tool for judging and comparing trustees.

To our knowledge, in many approaches, the trustworthiness of an agent is represented by a single number, usually a real number. An advantage of this representation format is that it is obvious for a trustor how to compare two agents. On the other hand, one might consider that a single number is a too concise description of an agent, that is, it does not allow trustors to distinguish between certain agents that are quite different. For example, if the trustworthiness of an agent is defined as the mean of the grades this agent has received, then we cannot distinguish between a first agent with very good and very bad grades, and a second one with only average grades.

Our approach overcomes this limitation. On the other hand, an interval is a slightly more complex object than a single number, thus it requires slightly more efforts from trustors to compare two agents. To summarize, our summarizing intervals constitute a good compromise, that is, they are sufficiently simple for trustors to easily make decisions about trustees, and in parallel, they are sufficiently complex for trustors to distinguish between two very different trustees.

Concerning the contributions of this paper, we first establish seven properties that should be satisfied by a *summarizing method*, that is, a method that transforms any set of grades into an interval summarizing them. Next, we develop two new methods. The first one is based on the idea that certain concentrations of grades are strong enough to pull the bounds of the summarizing interval towards themselves. This is called the *pulling principle*. The second one is based on possibility theory. More precisely, we view the grades as a basis for building a possibility distribution, and then we compute lower and upper expected values from it. Finally, we check that our methods satisfy all or part of the axioms introduced before, which provides theoretical justifications for them. This constitutes the core of the paper.

Next, we study three extensions of this work. In the first one, we propose a method for constructing a *trust interval* from a summarizing one. By trust interval, we mean an interval such that it is rational to believe (on the basis of the past grades) that the future grades will essentially fall on it. In our opinion, such an interval can be obtained by taking a summarizing one and adding an adequate margin of error.

In the second extension, the idea is to compute, on the basis of the aforementioned possibility distribution, a *level of trust* as the certainty (on the basis of the past grades) that a future grade will be good, and a *level of distrust* as the fear that a future grade may be bad. These levels provide additional indications on the trustee.

Finally, the third extension consists in studying several ways to compare two sets of grades. More precisely, we first present a particular pre-order called stochastic dominance. Next, we introduce several bipolar variants of it that distinguish between good and bad grades. And finally, we introduce pre-orders based on summarizing intervals.

This paper is a revised and extended version of the conference paper [3]. Concerning the difference with the conference paper, the journal paper has been completely restructured in such a way that there is now a main line (containing two summarizing methods) and three extensions. Several points have been reworked or developed, in particular in the summarizing method based on possibility theory, which is now described in much more detail and contains a formal proposition and several examples. The two last extensions (levels of trust and distrust, and comparison of grade structures) have also been reworked and extended. Finally, we have added a new axiom called shifting.

The rest of the paper is structured as follows. In Section 2, we discuss different research trends on the problem of evaluating trustworthiness. In Section 3 and 4, we define formally what particular kind of trust system we investigate, and we introduce seven axioms for the summarizing methods. In Section 5, we develop a particular method based on the pulling principle. In Section 6, we develop a second method that transforms the grades into a possibility distribution, and then computes expected values on the basis of it. We show that our methods satisfy all or part of the axioms. Finally, we investigate three different extensions of this work. In Section 7, we explain how summarizing intervals may be used to construct trust intervals. In Section 8, we show how to construct levels of trust and distrust on the basis of the aforementioned possibility distribution. Finally, in Section 9, we investigate different ways to compare two sets of grades, including a method based on summarizing intervals.

2 Evaluating trustworthiness: different research trends

The problem of evaluating the trustworthiness of an agent has several facets. A first question is how to represent trustworthiness. Different representation formats have been investigated. For example, trustworthiness may be represented by a number, an interval, or even a fuzzy interval.

In many approaches, trustworthiness is represented by a single number. One of the first such approaches is [14]. Another example is PageRank [17], which is used by the Google search engine. PageRank evaluates the trustworthiness of each web page. This system is based on a certain step-by-step procedure in which a web surfer is jumping from pages to pages. Initially, the surfer is at some page chosen at random (using a uniform distribution). In each step, the surfer jumps from the page p it is at to a page q. We will not detail how q is chosen, but the important point is that a page has more chances to be chosen if p links to it. Finally, the trustworthiness of a page p is, roughly speaking, the probability that the surfer will be at p after a large number of steps. An important version of PageRank adapted to peer-to-peer systems can be found in [13].

Next, the same representation format may have different understandings. For example, in some approaches (e.g. [12] and [26]), an agent is either trustworthy or not, and the authors manipulate a number that indicates the probability or the belief that it is trustworthy. In other approaches (e.g. [4]), an agent is trustworthy to a certain degree, and the authors work with a number that indicates to which degree. But, the probability of being (fully) trustworthy is not the same thing as a degree of trustworthiness. For example, suppose the trustworthiness of an agent is 0.5. In the first case, this means that one goal out of two has been well achieved by the agent, while in the second case, this means that all goals have been half-well achieved.

Other approaches motivate interval-based representation of trustworthiness by the poorness of the information available, e.g. [18]. This view is compatible with the understanding that trustworthiness is binary and the probability of being trustworthy is imprecise, see e.g. [19]. Alternatively, it is also compatible with the understanding that trustworthiness is graded and the degree of trustworthiness is ill-known.

Another facet of the problem is that it may refer to quite different types of input data. For example, these data may take the form of opinions of agents about others, e.g. [1], [25], [16], and [11]. This is also the case in [23] and [24], where trust evaluation is based on direct opinions, that is, opinions obtained from past interactions. Then, indirect opinions are computed by chaining and combining direct ones, either by means of inference rules [24] or by path semirings [23]. Both approaches associate an uncertainty estimate to their trust values. Another option is to view trust assessment as a matter of argumentation, that is, the idea is to balance arguments in favor of deciding to trust an agent with arguments against this choice, see e.g. [18] and [22].

On the other hand, one may consider that the input data consist of measures or observations which are supposed to be rather objective. This is the kind of information considered in this paper. More precisely, we evaluate trustworthiness from grades reflecting past performances. These grades are supposed to be objective and harmonized. Although purely statistical methods may be considered if enough data are available, we investigate other roads here since data are not necessarily numerous in practice.

3 A framework based on grades and intervals

First of all, we define formally what we mean by a grade structure and a summarizing method.

Definition 1 We fix a natural number $\mathcal{N} > 0$ and we denote by $\mathcal{S}$ the following discrete scale: $\{\frac{k}{\mathcal{N}} : 0 \leq k \leq \mathcal{N}\}$. A *grade structure* $\mathcal{G}$ is an ordered pair

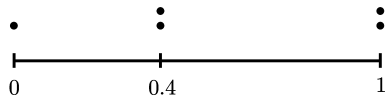
$\langle G, v\rangle$, where G is a non-empty and finite set of grades and v is a function from a superset of G to $\mathcal{S}$. We call $v(g)$ the value of the grade g.

A summarizing method I is a function that transforms any grade structure into a real interval included in $[0, 1]$.

The reader may wonder why the domain of v is a superset of G, while G would be sufficient. The reason is simply that it helps to increase the readability of certain definitions and proofs.

A grade structure can be visualized as a set of points above the real interval $[0, 1]$, each point being located above its value. For example, the structure $\mathcal{G}_0$ consisting of 0, 0.4, 0.4, 1, and 1 can be visualized as below. This structure $\mathcal{G}_0$ will be used as a running example through the paper.

Example 1



Now, the goal is to summarize a grade structure by an interval. What properties should be satisfied by a summarizing method? In the next section, we give some elements to answer this question.

4 Axioms

We introduce desirable properties for summarizing methods. For every $x \in \mathcal{S}$, we denote by $wei_{\mathcal{G}}(x)$ the *weight* of x in the grade structure $\mathcal{G}$. More precisely, $wei_{\mathcal{G}}(x) = |\{g \in G : v(g) = x\}|$. When the context is clear we may drop the subscript $_{\mathcal{G}}$. The same goes for all notations.

Here is a first obvious axiom: if two structures lead to the same values and to the same weights, then they should naturally lead to the same interval.

Definition 2 (Equivalence) Let $\mathcal{G} = \langle G, v\rangle$ and $\mathcal{G}' = \langle G', v'\rangle$ be two grade structures. We say that $\mathcal{G}$ and $\mathcal{G}'$ are *equivalent* (in symbols $\mathcal{G} \equiv \mathcal{G}'$) iff $v(G) = v'(G')$ and $\forall\, x \in v(G)$, $wei_{\mathcal{G}}(x) = wei_{\mathcal{G}'}(x)$.

A summarizing method I respects *equivalence* iff for all grade structures $\mathcal{G}$ and $\mathcal{G}'$, if $\mathcal{G} \equiv \mathcal{G}'$, then $I(\mathcal{G}) = I(\mathcal{G}')$.

Another obvious property is that the summarizing interval should not exceed the limits of the zone in which the grades are located.

Definition 3 (Confinement) A summarizing method I respects *confinement* iff for any grade structure $\mathcal{G}$, $I(\mathcal{G}) \subseteq [min(v(G)), max(v(G))]$.

Next, assume that the grades are regularly scattered over some distance. Then, the summarizing interval should cover exactly that distance.

Definition 4 (Regularity) A grade structure $\mathcal{G}$ is *regular* iff the following holds:
$-\ \exists\, r, s \in \mathbb{R}$, $\exists\, n \in \mathbb{N}$, $v(G) = \{r, r+s, r+2s, \ldots, r+ns\}$;
$-\ \forall\, x, y \in v(G)$, $wei(x) = wei(y)$.

A summarizing method I respects *regularity* iff for any regular grade structure $\mathcal{G}$, $I(\mathcal{G}) = [min(v(G)), max(v(G))]$.

The second condition ($\forall\, x, y \in v(G)$, $wei(x) = wei(y)$) might seem too strong at first glance. But it is necessary. Take for instance the following set of grades: $\{0, 1, 1\}$. Without this condition, regularity would say that the summarizing interval should be $[0, 1]$, which would be weird since the weight of 1 is greater than that of 0, and the interval should take this difference into account, i.e. the summarizing interval obtained from $\{0, 1, 1\}$ should be of the form $[\alpha, 1]$ with $0 < \alpha$. Now, if the aforementioned condition is satisfied, then regularity makes sense. For example, if the set of grades is $\{0, 0, 1, 1\}$, then it is obvious that the summarizing interval should be $[0, 1]$.

The next property says the following: if the grades are symmetric with respect to some axis, then so should be the summarizing interval. Let $x, a \in \mathbb{R}$. We denote by $mir_a(x)$ is the *mirror image* of x with respect to a, that is, $mir_a(x) = 2a - x$.

Definition 5 (Symmetry) A grade structure $\mathcal{G}$ is *symmetric* with respect to $a \in \mathbb{R}$ iff for any $x \in v(G)$, $mir_a(x) \in v(G)$ and $wei(x) = wei(mir_a(x))$.

A summarizing method I respects *symmetry* iff for any grade structure $\mathcal{G}$ and $\forall\, a \in \mathbb{R}$, if $\mathcal{G}$ is symmetric with respect to a, then $mir_a(minI(\mathcal{G})) = maxI(\mathcal{G})$.

Note that we sometimes write $minX$, vX, etc. instead of $min(X)$, $v(X)$, etc. We do it in order to increase readability. Next, assume that a grade structure $\mathcal{G}'$ can be obtained from a grade structure $\mathcal{G}$ by shifting all grades to the right by a quantity r. Then, the bounds of $I(\mathcal{G}')$ should be obtained from those of $I(\mathcal{G})$ by shifting them to the right by r. The same goes for left. This axiom is due to Yonatan Aumann (Bar Ilan University) following personal communication.

Definition 6 (Shifting) Let $\mathcal{G} = \langle G, v\rangle$ and $\mathcal{G}' = \langle G', v'\rangle$ be two grade structures, and let r be a positive or negative real number. We say that $\mathcal{G}'$ is a r-*shift* of $\mathcal{G}$ iff $v'(G') = \{x+r : x \in v(G)\}$ and $\forall x \in v(G)$, $wei_{\mathcal{G}}(x) = wei_{\mathcal{G}'}(x+r)$.

Note that, since $\mathcal{G}'$ is a grade structure, we have $0 \leq x + r$. A summarizing method I respects *shifting* iff for all grade structures $\mathcal{G}$ and $\mathcal{G}'$, and for any positive or negative real number r, the following holds: if $\mathcal{G}'$ is a r-shift of $\mathcal{G}$, then $minI(\mathcal{G}') = minI(\mathcal{G}) + r$ and $maxI(\mathcal{G}') = maxI(\mathcal{G}) + r$.

Next, we denote by $mean_{\mathcal{G}}$ the mean of the grades of $\mathcal{G}$ and by $cen_{\mathcal{G}}$ the center of the zone in which they are located, that is, $cen_{\mathcal{G}}$ is the middle of $[min(v(G)), max(v(G))]$. Now, assume that *mean* is to the right of *cen*. Then, intuitively, $\mathcal{G}$ is leaning to the right. The summarizing interval should reflect this asymmetry, that is, it should "forget" at least a bit the grades on the extreme left. The same goes when $\mathcal{G}$ is leaning to the left.

Definition 7 (Leaning) A summarizing method I respects *leaning* iff for any grade structure $\mathcal{G}$, the following holds:
− if $cen < mean$, then $min(v(G)) < minI(\mathcal{G})$;
− if $mean < cen$, then $maxI(\mathcal{G}) < max(v(G))$.

Finally, take a grade structure $\mathcal{G}$, add new grades to the right, and call $\mathcal{G}'$ the structure so obtained. Then, the bounds of $I(\mathcal{G}')$ should be at least as to the right as the bounds of $I(\mathcal{G})$. In addition, if some new grades are strictly more to the right than the old ones, then the right bound of $I(\mathcal{G}')$ should be strictly more to the right than the right bound of $I(\mathcal{G})$. The same goes for left.

Definition 8 Let $\mathcal{G} = \langle G, v\rangle$ and $\mathcal{G}' = \langle G', v'\rangle$ be two grade structures.
We denote by $\preceq_r$ the relation such that:

$$\mathcal{G} \preceq_r \mathcal{G}' \quad \text{iff} \quad G' = G \cup H, \;\; v|_G = v'|_G, \;\; \text{and} \;\; \forall\, g \in G, \forall\, h \in H, \; v'(g) \leq v'(h)$$

We denote by $\prec_r$ the relation such that:

$$\mathcal{G} \prec_r \mathcal{G}' \quad \text{iff} \quad \mathcal{G} \preceq_r \mathcal{G}' \;\; \text{and} \;\; \exists\, g' \in G', \forall\, g \in G, \; v(g) < v'(g')$$

The definitions of $\preceq_l$ and $\prec_l$ are obtained by replacing $\leq$ by $\geq$, and $<$ by $>$.

Intuitively, $\mathcal{G} \preceq_r \mathcal{G}'$ means that $\mathcal{G}'$ can be obtained from $\mathcal{G}$ by adding new grades to the right, and $\mathcal{G} \prec_r \mathcal{G}'$ means that some new grades are strictly more to the right than the old ones. The meanings of $\preceq_l$ and $\prec_l$ are similar, just replace "right" by "left". Note that $v|_G$ denotes the restriction of v to G.

Definition 9 (Coherence) A summarizing method I is *coherent* iff for all grade structures $\mathcal{G}$ and $\mathcal{G}'$, the following holds:
− if $\mathcal{G} \preceq_r \mathcal{G}'$, then $minI(\mathcal{G}) \leq minI(\mathcal{G}')$ and $maxI(\mathcal{G}) \leq maxI(\mathcal{G}')$;
− if $\mathcal{G} \prec_r \mathcal{G}'$, then $maxI(\mathcal{G}) < maxI(\mathcal{G}')$;
− if $\mathcal{G} \preceq_l \mathcal{G}'$, then $minI(\mathcal{G}') \leq minI(\mathcal{G})$ and $maxI(\mathcal{G}') \leq maxI(\mathcal{G})$;
− if $\mathcal{G} \prec_l \mathcal{G}'$, then $minI(\mathcal{G}') < minI(\mathcal{G})$.

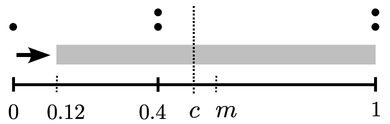
5 A summarizing method based on the pulling principle

Assume that a grade structure $\mathcal{G} = \langle G, v\rangle$ is given. We construct an interval $I_p(\mathcal{G})$ that summarizes $\mathcal{G}$. Initially, $I_p(\mathcal{G})$ is $[min(v(G)), max(v(G))]$, the smallest interval that contains (the values of) all grades, which seems natural. Next, the idea is to identify the "strong" groups of grades, that is, the groups able to pull the bounds of $I_p(\mathcal{G})$ towards themselves, despite the resistance of certain grades. In the end, we move each bound of $I_p(\mathcal{G})$ to the farthest value x such that there exists a group able to pull it to x.

Here is a first simple way to identify a group able to pull the left bound of $I_p(\mathcal{G})$ towards the right (the case of the right bound is similar). Suppose that the arithmetic mean $m = mean_{\mathcal{G}}$ of the grades is to the right of the center $c = cen_{\mathcal{G}}$ of $I_p(\mathcal{G})$. In our opinion, this means that the points above the right half of $I_p(\mathcal{G})$ constitute a group S able to pull the left bound towards the right. The more m is far from c, the more S is able to give a hard pull. More precisely, S can pull the left bound "until c reaches m", that is, it can pull it to the value x such that if the left bound was equal to x, then c would be equal to m.

For example, let $\mathcal{G}_0$ be the grade structure given in Example 1. In $\mathcal{G}_0$, the two 1's are able to pull the left bound of $I_p(\mathcal{G}_0)$ to 0.12. Indeed, the middle of $[0.12, 1]$ is equal to 0.56, the mean of $\{0, 0.4, 0.4, 1, 1\}$. Put differently, $0.12 = 2 \times 0.56 - 1$.

Example 2

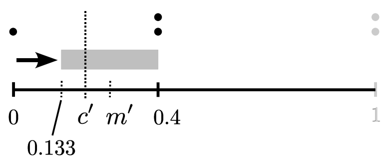


We turn to a more general way to identify a group able to pull the left bound of $I_p(\mathcal{G})$. Take some limit l, ignore the points of $\mathcal{G}$ to the right of l, and let $\mathcal{G}'$ be the grade structure so obtained. Next, let $I_p(\mathcal{G}')$ be the smallest interval that contains all points of $\mathcal{G}'$. Suppose that the mean $m' = mean_{\mathcal{G}'}$ of the grades of $\mathcal{G}'$ is to the right of the center $c' = cen_{\mathcal{G}'}$ of $I_p(\mathcal{G}')$. Then, the points above the right half of $I_p(\mathcal{G}')$ constitute a group S' able to pull the left bound of $I_p(\mathcal{G}')$ until c' reaches m'. But, if the group S' can pull the left bound of $I_p(\mathcal{G}')$ to a certain value x, then it can pull the left bound of $I_p(\mathcal{G})$ to x as well.

Indeed, if a group of points can pull the left bound towards the right in a certain context, then it can do it in any context obtained by adding new points to the right of this group. In other words, if a group of points is able to pull the left bound of $I_p(\mathcal{G}')$ to a certain value, then it is able to pull the left bound of $I_p(\mathcal{G})$ to the same value as well. This principle is based on the intuition that he who can do more can do less.

For example, let $\mathcal{G}'_0$ be the structure obtained from $\mathcal{G}_0$ by removing the two 1's. Then, the two 0.4's can pull the left bound of $I_p(\mathcal{G}'_0)$ to 0.133. Indeed, the middle of $[0.133, 0.4]$ is equal to 0.266, the mean of $\{0, 0.4, 0.4\}$. Put differently, $0.133 = 2 \times 0.266 - 0.4$. Consequently, the two 0.4's can pull the left bound of $I_p(\mathcal{G}_0)$ to 0.133 as well.

Example 3

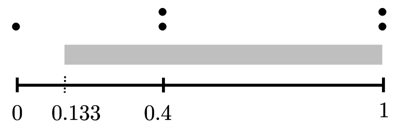


In $\mathcal{G}_0$, there are only two groups able to pull the left bound of $I_p(\mathcal{G}_0)$, namely the two 1's (to 0.12) and the two 0.4's (to 0.133). So, we finally move the left bound to 0.133. Note that any group (determined by some limit l) is potentially able to influence the final position of the left bound. It is not the case, for example, that the farthest group is necessarily the one that can move the left bound to the farthest position. Indeed, in $\mathcal{G}_0$ for example, the two 0.4's are able to pull the left bound to a farther value than the two 1's. At first glance, this may seem surprising since the second group is farther than the first one, but actually this is normal. Indeed, the two 1's are farther but they face a stronger resistance, that is, when pulling the left bound, they are opposed to 0, 0.4, and 0.4, while the two 0.4's are only opposed to 0. So, we have to check every group determined by some limit l.

To summarize, we begin with the smallest interval $I_p(\mathcal{G})$ that contains all points of $\mathcal{G}$, then we identify the groups of points able to pull the left bound of $I_p(\mathcal{G})$, and finally we move it to the farthest value x such that there exists a group able to pull it to x. The same goes for the right bound.

Concerning the example of $\mathcal{G}_0$, no group is able to pull the right bound of $I_p(\mathcal{G}_0)$. So, here is the final interval $I_p(\mathcal{G}_0)$ obtained from $\mathcal{G}_0$ by the pulling method:

Example 4



Definition 10 Let $\mathcal{G} = \langle G, v \rangle$ be a grade structure. We denote by $I_p^s(\mathcal{G})$ the interval that superficially summarizes $\mathcal{G}$ according to the pulling method, that is, the interval obtained by this method when no limit l is considered. More formally:

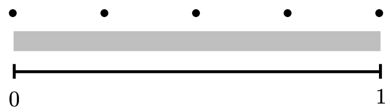
$$I_p^s(\mathcal{G}) = \begin{cases} [min(vG), \quad max(vG)] & \text{if } cen = mean \\ [mir_{mean}(max(vG)), \quad max(vG)] & \text{if } cen < mean \\ [min(vG), \quad mir_{mean}(min(vG))] & \text{if } mean < cen \end{cases}$$

Definition 11 Let $\mathcal{G} = \langle G, v \rangle$ be a grade structure. We denote by $L(\mathcal{G})$ the set of every $L \subseteq G$ such that $\exists\, l \in \mathbb{R}$, $L = \{g \in G : v(g) \leq l\}$. Similarly, we denote by $R(\mathcal{G})$ the set of every $R \subseteq G$ such that $\exists\, l \in \mathbb{R}$, $R = \{g \in G : l \leq v(g)\}$. Finally, we denote by $I_p(\mathcal{G})$ the interval that summarizes $\mathcal{G}$ according to the pulling method, that is,

$$I_p(\mathcal{G}) = [max\{minI_p^s\langle L, v\rangle : L \in L(\mathcal{G})\}, \quad min\{maxI_p^s\langle R, v\rangle : R \in R(\mathcal{G})\}]$$

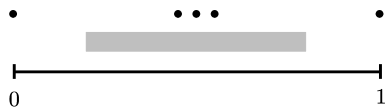
Here are some additional examples to give the reader a better idea how I_p behaves. We scattered five grades regularly over some distance. Then, as desired, the interval obtained by the pulling method covers exactly the five grades.

Example 5



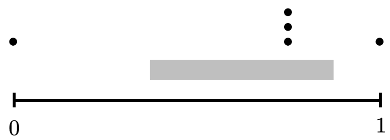
Next, suppose that we bring closer together the three grades which are closest to the middle, while keeping a symmetric situation. Then, the newly formed concentration of grades makes us forget a bit the two extreme grades.

Example 6



If we do the same thing, but without keeping a symmetric situation, then the concentration of grades makes us forget an extreme grade more than the other.

Example 7



More formally, the summarizing method based on the pulling principle satisfies all the axioms introduced in Section 4.

Proposition 1 *The summarizing method I_p based on the pulling principle satisfies equivalence, confinement, regularity, symmetry, shifting, leaning, and coherence.*

Proof Equivalence, confinement, regularity, shifting, and leaning are easy.

Proof for symmetry. Assume $\mathcal{G} = \langle G, v \rangle$ is symmetric with respect to $a \in [0, 1]$.
We will show later that:
(0) $\forall\, L \in L(\mathcal{G}),\ \exists\, R \in R(\mathcal{G}),\ mir(minI_p^s\langle L, v\rangle) = maxI_p^s\langle R, v\rangle$
(1) $\forall\, R \in R(\mathcal{G}),\ \exists\, L \in L(\mathcal{G}),\ mir(maxI_p^s\langle R, v\rangle) = minI_p^s\langle L, v\rangle$
By definition, $\exists\, L \in L(\mathcal{G}),\ minI_p(\mathcal{G}) = minI_p^s\langle L, v\rangle$.
By (0), $\exists\, R \in R(\mathcal{G}),\ mir(minI_p^s\langle L, v\rangle) = maxI_p^s\langle R, v\rangle$.
Therefore, $maxI_p(\mathcal{G}) \leq maxI_p^s\langle R, v\rangle = mir(minI_p(\mathcal{G}))$.
We show $mir(minI_p(\mathcal{G})) \leq maxI_p(\mathcal{G})$.

Suppose $maxI_p(\mathcal{G}) < mir(minI_p(\mathcal{G}))$.
By definition, $\exists\, R \in R(\mathcal{G})$, $maxI_p(\mathcal{G}) = maxI^s_p\langle R, v\rangle$.
Thus, by (1), $\exists\, L \in L(\mathcal{G})$, $mir(maxI^s_p\langle R, v\rangle) = minI^s_p\langle L, v\rangle$.
Thus, $minI_p(\mathcal{G}) = mir(mir(minI_p(\mathcal{G}))) < mir(maxI_p(\mathcal{G})) = mir(maxI^s_p\langle R, v\rangle) = minI^s_p\langle L, v\rangle$, which is impossible.

Proof of (0) *((1) is similar).* Let $L \in L(\mathcal{G})$. Then, $\exists\, l \in \mathbb{R}$, $L = \{g \in G : v(g) \leq l\}$.
Let $R = \{g \in G : mir(l) \leq v(g)\}$. We show $mir(v(L)) = v(R)$.
"$\subseteq$". Let $y \in mir(v(L))$. Then, $\exists\, x \in v(L)$, $y = mir(x)$. But, $x \leq l$.
Thus, $mir(l) \leq y$. In addition, by symmetry, $\exists\, g \in G$, $v(g) = y$.
But, $g \in R$. Therefore, $y \in v(R)$.
"$\supseteq$". Let $x \in v(R)$. Then, $mir(l) \leq x$. Thus, $mir(x) \leq mir(mir(l)) = l$.
By symmetry, $\exists\, g \in G$, $v(g) = mir(x)$. But, $g \in L$. Thus, $mir(x) \in v(L)$.
Therefore, $x = mir(mir(x)) \in mir(v(L))$.
In addition, $L \neq \emptyset$. Thus, $R \neq \emptyset$. Therefore, $R \in R(\mathcal{G})$.
Let c, m, min, max be shorthands for $cen_{\langle L,v\rangle}$, $mean_{\langle L,v\rangle}$, $min(v(L))$, $max(v(L))$.
We show $mir(c) = cen_{\langle R,v\rangle}$.
By definition, $\overrightarrow{c\,min} + \overrightarrow{c\,max} = \overrightarrow{0}$. But $\forall\, x, y \in \mathbb{R}$, $\overrightarrow{xy} = \overrightarrow{mir(y)mir(x)}$.
Therefore, $\overrightarrow{mir(min)mir(c)} + \overrightarrow{mir(max)\,mir(c)} = \overrightarrow{0}$.
But, $mir(v(L)) = v(R)$. So, $mir(min) = max(v(R))$ and $mir(max) = min(v(R))$.
Thus, $\overrightarrow{max(v(R))mir(c)} + \overrightarrow{min(v(R))mir(c)} = \overrightarrow{0}$. So, $mir(c) = cen_{\langle R,v\rangle}$.
We show $mir(m) = mean_{\langle R,v\rangle}$.
By definition, $\sum_{x\in v(L)} wei(x)\overrightarrow{m\,x} = \overrightarrow{0}$. Therefore,
$\sum_{x\in v(L)} wei(x)\overrightarrow{mir(x)mir(m)} = \sum_{x\in v(L)} wei(mir(x))\overrightarrow{mir(x)mir(m)} = \overrightarrow{0}$.
Consequently, $\sum_{y\in v(R)} wei(y)\overrightarrow{y\,mir(m)} = \overrightarrow{0}$. Thus, $mir(m) = mean_{\langle R,v\rangle}$.
Finally, we show $mir_a(minI^s_p\langle L, v\rangle) = maxI^s_p\langle R, v\rangle$.
Case 1: $c \leq m$. Then, $minI^s_p\langle L, v\rangle = mir_m(max)$ and $mir_a(m) \leq mir_a(c)$.
Therefore, $maxI^s_p\langle R, v\rangle = mir_{mir_a(m)}(min(vR)) = mir_{mir_a(m)}(mir_a(max))$
$= mir_a(mir_m(max))$.
Case 2: $m < c$. Then, $minI^s_p\langle L, v\rangle = min$ and $mir_a(c) < mir_a(m)$.
Thus, $maxI^s_p\langle R, v\rangle = max(v(R)) = mir_a(min)$.

Proof for coherence. Let $\mathcal{G} = \langle G, v\rangle$ and $\mathcal{G}' = \langle G', v'\rangle$ be two grade structure.
Suppose $\mathcal{G} \preceq_r \mathcal{G}'$. We show $minI_p(\mathcal{G}) \leq minI_p(\mathcal{G}')$.
There exists $L \in L(\mathcal{G})$ such that $minI_p(\mathcal{G}) = minI^s_p\langle L, v\rangle$.
Let $L' = \{g \in G' : v'(g) \leq max(vL)\}$. Then, $L' \in L(\mathcal{G}')$.
Let m, m', c, c' be shorthands for $mean_{\langle L,v\rangle}$, $mean_{\langle L',v'\rangle}$, $cen_{\langle L,v\rangle}$, $cen_{\langle L',v'\rangle}$.
Then, $min(vL) = min(v'L')$, $max(vL) = max(v'L')$, $c = c'$ and $m \leq m'$.
Case 1: $c \leq m$ and $c' \leq m'$.
Then, $minI^s_p\langle L, v\rangle = mir_m(max(vL)) \leq mir_{m'}(max(v'L')) = minI^s_p\langle L', v'\rangle$.
Case 2: $c \leq m$ and $m' < c'$. Then, $m' < m$, which is impossible.

Case 3: $m < c$ and $c' \leq m'$.
Then, $minI_p^s\langle L, v\rangle = min(vL) = min(v'L') \leq mir_{m'}(max(v'L')) = minI_p^s\langle L', v'\rangle$.
Case 4: $m < c$ and $m' < c'$.
Then, $minI_p^s\langle L, v\rangle = min(vL) = min(v'L') = minI_p^s\langle L', v'\rangle$.
We show $maxI_p(\mathcal{G}) \leq maxI_p(\mathcal{G}')$.
There exists $R' \in R(\mathcal{G}')$ such that $maxI_p(\mathcal{G}') = maxI_p^s\langle R', v'\rangle$.
Let $R = \{g \in G : min(v'R') \leq v(g)\}$.
Case 1: $R = \emptyset$. Then, $maxI_p(\mathcal{G}) \leq max(vG) < min(v'R') \leq maxI_p(\mathcal{G}')$.
Case 2: $R \neq \emptyset$. Then, $R \in R(\mathcal{G})$. Thus, $maxI_p(\mathcal{G}) \leq maxI_p^s\langle R, v\rangle$.
Let m, m', c, c' be shorthands for $mean_{\langle R,v\rangle}$, $mean_{\langle R',v'\rangle}$, $cen_{\langle R,v\rangle}$, $cen_{\langle R',v'\rangle}$.
Then, $min(vR) = min(v'R')$, $max(vR) \leq max(v'R')$, and $m \leq m'$.
Case 2.1: $c \leq m$ and $c' \leq m'$.
Then, $maxI_p^s\langle R, v\rangle = max(vR) \leq max(v'R') = maxI_p^s\langle R', v'\rangle$.
Case 2.2: $c \leq m$ and $m' < c'$. Then, $maxI_p^s\langle R, v\rangle = max(vR) \leq mir_m(min(vR))$ $\leq mir_{m'}(min(v'R')) = maxI_p^s\langle R', v'\rangle$.
Case 2.3: $m < c$ and $c' \leq m'$.
Then, $maxI_p^s\langle R, v\rangle = mir_m(min(vR)) < max(vR) \leq max(v'R') = maxI_p^s\langle R', v'\rangle$.
Case 2.4: $m < c$ and $m' < c'$.
Then, $maxI_p^s\langle R, v\rangle = mir_m(min(vR)) \leq mir_{m'}(min(v'R')) = maxI_p^s\langle R', v'\rangle$.
Suppose $\mathcal{G} \prec_r \mathcal{G}'$. We show $maxI_p(\mathcal{G}) < maxI_p(\mathcal{G}')$.
The proof is similar to that of $maxI_p(\mathcal{G}) \leq maxI_p(\mathcal{G}')$.
The difference is that this time we have $max(vR) < max(v'R')$ and $m < m'$.
This difference allows us to derive in all cases $maxI_p^s\langle R, v\rangle < maxI_p^s\langle R', v'\rangle$.
The proofs for $\preceq_l$ and $\prec_l$ and similar to those for $\preceq_r$ and $\prec_r$. □

6 A summarizing method based on expected values

We now investigate another possible summarizing method based on an intuition quite different from the pulling principle. A grade structure is now perceived, through a probability-possibility transformation, as a fuzzy interval describing the range of the grade values obtained by an agent, where the degree of possibility of a value somewhat reflects the plausibility of getting this value for a grade. Then a summary can be obtained under the form of a classical interval which is the expected value of the fuzzy interval.

Recall that the information available about the past behavior of an agent is supposed to take the form of a grade structure $\mathcal{G} = \langle G, v\rangle$ where the values of the $|G|$ grades belong to a discrete scale $\mathcal{S} = \{\frac{k}{\mathcal{N}} : 0 \leq k \leq \mathcal{N}\}$. Let $w_k = wei(\frac{k}{\mathcal{N}}) = |\{g \in G : v(g) = \frac{k}{\mathcal{N}}\}|$, for $0 \leq k \leq \mathcal{N}$, be the number of times the grade $\frac{k}{\mathcal{N}}$ was obtained. Note that if the grades are directly taking their values in the real interval $[0, 1]$, it is then always possible to partition the unit interval into $\mathcal{N}+1$ subintervals $[0, \frac{1}{2\mathcal{N}}], (\frac{1}{2\mathcal{N}}, \frac{3}{2\mathcal{N}}], \ldots, (\frac{(2k-1)}{2\mathcal{N}}, \frac{(2k+1)}{2\mathcal{N}}], \ldots,$ $(\frac{(2\mathcal{N}-3)}{2\mathcal{N}}, \frac{(2\mathcal{N}-1)}{2\mathcal{N}}], (\frac{(2\mathcal{N}-1)}{2\mathcal{N}}, 1]$, and to count the grades that are in each subinterval. These subintervals, except the ones at the two extremities whose length is half, have the same length ($\frac{1}{\mathcal{N}}$), and can be associated with the value of their middle point $x_k = \frac{k}{\mathcal{N}}$.

For performing the first step, we begin by normalizing the weighting structure in a probabilistic-like manner, and then apply a probability-possibility transformation, preserving as much information as possible. Namely, let

$$p_k = \frac{w_k}{|G|}, \text{ for } k = 0, \ldots, \mathcal{N} \quad (1)$$

Then, the probability-possibility transformation [7], [10], [5] is defined by

$$\pi_k = \sum \{p_j : p_j \leq p_k\}, \text{ for } k = 0, \ldots, \mathcal{N} \quad (2)$$

Clearly, when $\mathcal{N}$ is too small, viewing π_k as a probability estimate is debatable. The above expression yields the smallest possibility distribution π such that the associated possibility measure $\Pi(A) = \max_{x \in A} \pi(x)$ of event A dominates the corresponding probability $P(A) = \sum_{x \in A} p(x)$, i.e. $\forall A\ \Pi(A) \geq P(A)$ holds. It can be checked that $p_i \leq p_j$ entails $\pi_i \leq \pi_j$, which expresses that the transformation is faithful with respect to the shape of the distribution. In particular a uniform probability distribution is transformed into a uniform possibility distribution (which takes the value 1 when it is non zero). The possibility degrees $\pi_k = \frac{1}{|G|} \sum \{w_j : w_j \leq w_k\}$ thus obtained directly reflect that a value is all the less possible as it is observed with a low frequency and that no other value is observed with such a frequency (or a smaller one).

Example 8 Consider Example 1 again. Namely $\mathcal{N} = 10$, where $\mathcal{G}$ is defined by $|G| = 5$, with $w_0 = 1$, $w_4 = 2$, $w_{10} = 2$. Then we get $\pi_0 = 1/5 = 0.2$, $\pi_4 = \pi_{10} = 1$. Taking $\mathcal{G}'$ defined by $|G'| = 8$, with $w_0 = 1$, $w_4 = 2$, $w_5 = 1$, $w_6 = 1$, $w_7 = 1$, $w_{10} = 2$, yields $\pi_0 = \pi_5 = \pi_6 = \pi_7 = 4/8 = 0.5$, $\pi_4 = \pi_{10} = 1$, while $\mathcal{G}''$ defined by $|G''| = 8$, with $w_0 = 1$, $w_4 = 2$, $w_5 = 3$, $w_{10} = 2$, gives $\pi_0 = 1/8 = 0.125$, $\pi_4 = \pi_{10} = 5/8 = 0.625$, $\pi_5 = 1$. This illustrates the fact that in this view π_k depends not only on the frequency with which x_k is observed, but also on the existence of other observations with similar or smaller frequencies.

Using the possibility distribution π, one may look for the smallest interval $[\underline{x}, \overline{x}]$ such that we are certain that any value restricted by π is greater than $\underline{x}$ and cannot be greater than $\overline{x}$. It amounts to compute the largest value $\underline{x}$ such that $N(\underline{x} \leq) = 1$ (or more generally $N(\underline{x} \leq) \geq \theta$), and the smallest $\overline{x}$ such that $\Pi(\overline{x} \leq) = 0$ (or more generally $\Pi(\overline{x} \leq) \leq \rho$), where θ and ρ are thresholds, Π and N are possibility and necessity measures with $\Pi(x \leq) = \max_{u: x \leq u} \pi(u)$ and $N(x \leq) = 1 - \Pi(x >) = 1 - \max_{u: x > u} \pi(u)$. This provides a summarizing interval $[\underline{x}, \overline{x}]$. However, it amounts to take the convex hull of the support of π, i.e. the interval $[\min\{x : \pi(x) > 0\}, \max\{x : \pi(x) > 0\}]$ for $\theta = 1$ and $\rho = 0$, which may seem too large and does not really take into account the relative values of the possibility degrees. Choosing other values for the two thresholds would introduce arbitrariness.

However, let us introduce the two possibility distributions

$$\pi_*(x) = \max_{t \leq x} \pi(t) \text{ and } \pi^*(x) = \max_{t \geq x} \pi(t) \tag{3}$$

associated with π, which respectively represent the fuzzy sets of values that are greater and the fuzzy sets of values that are less than the imprecisely known value fuzzily restricted by π. Note that, while here π is defined on $\mathcal{S} = \{\frac{k}{\mathcal{N}} : 0 \leq k \leq \mathcal{N}\}$, π_* and π^* are defined on the interval $[0, 1]$. Then, $\widehat{\pi} = min(\pi_*, \pi^*)$ can be seen as a kind of fuzzy version of a summarizing interval in the sense of Section 5. This fuzzy interval may be transformed into a classical interval representing its mean or expected value [8]. The bounds of the interval are computed as the lower and upper expected values $E_*(\pi)$ and $E^*(\pi)$ (using Choquet integrals) by the following expressions:

$$E_*(\pi) = \sum_{k=1}^{\mathcal{N}} x_k(\pi_*(x_k) - \pi_*(x_{k-1})); \quad E^*(\pi) = \sum_{k=0}^{\mathcal{N}-1} x_k(\pi^*(x_k) - \pi^*(x_{k+1})) \tag{4}$$

These expected values are used to define a summarizing interval. More formally:

Definition 12 We denote by $I_e(\mathcal{G})$ the interval that summarizes $\mathcal{G}$ according to the method based on possibility distribution transformation, that is,

$$I_e(\mathcal{G}) = [E_*(\pi), E^*(\pi)]$$

where $E_*(\pi)$ and $E^*(\pi)$ are defined via (4) from π_* and π^*, which are defined via (3) from π, which is itself defined via (1) and (2) from $\mathcal{G}$.

Example 9 Consider Example 1 again with $\mathcal{N} = 10$, $\mathcal{G}$ is defined by $|G| = 5$, with $w_0 = 1$, $w_4 = 2$, $w_{10} = 2$, and $\pi_0 = 1/5 = 0.2$, $\pi_4 = \pi_{10} = 1$. We get $E_*(\pi) = 0.4 \times (1 - 0.2) = 0.32$, and $E^*(\pi) = 1$.

It can be shown that the summarizing method I_e satisfies most of the axioms presented in Section 4, namely we have:

Proposition 2 *The summarizing method I_e based on possibility distribution transformation satisfies equivalence, confinement, regularity, symmetry, shifting and coherence. But, leaning does not hold.*

Proof Equivalence, symmetry, and shifting are straightforward. Confinement and regularity follow easily from the properties of $E_*(\pi)$ and $E_*(\pi)$.

More precisely, $[E_*(\pi), E^*(\pi)] \subseteq \{x : \widehat{\pi}(x) > 0\}$, which ensures confinement.

Besides, $[E_*(\pi_{[a,b]}), E^*(\pi_{[a,b]})] = [a, b]$, with $\pi_{[a,b]}(x) = 1$ if $x \in [a, b]$ and $\pi_{[a,b]}(x) = 0$ otherwise. It holds as well as for $\pi_{[a,b]}(x) = 1$ if $x \in [a, b] \cap \mathcal{S}$ and $\pi_{[a,b]}(x) = 0$ otherwise, provided that $\exists r, \exists s, a = \frac{r}{\mathcal{N}}, b = \frac{s}{\mathcal{N}}$. This ensures regularity.

Coherence also holds. To see it, it is enough to look on the effect on π_* of the addition of h new observations having the smallest already observed value, where π_* is associated with a grade structure $\mathcal{G}$ and $|G| = g$ in the way explained above. Namely, let x_s be the smallest grade with $\pi_*(x_s) > 0$ and x_t be the smallest grade such that $\pi_*(x_t) = 1$. We have $\pi_*(x_s) = \pi(x_s) = \frac{1}{g}\sum\{w_j : w_j = w_s\}$. For $x_s < x < x_t$, we have $0 < \pi_*(x_s) \leq \pi_*(x) \leq \pi_*(x_t) = 1$. Let $\mathcal{G}'$ be a new grade structure obtained from $\mathcal{G}$ by adding h new observations having value x_s. Then the new associated possibility distribution is such that $\pi'_*(x_s) = \pi'(x_s) = \frac{1}{(g+h)}\sum\{w_j : w_j \leq w_s + h\}$. Again $0 < \pi'_*(x_s) \leq \pi'_*(x) \leq \pi'_*(x_t) = 1$ for $x_s < x < x_t$; let $\pi_*(x) = w/g$, then $\pi'_*(x) = (w+h)/(g+h)$. Since $w \leq g$, $(w+h)/(g+h) \geq w/g$ holds. As a consequence, for $x_s \leq x < x_t$, $\pi'_*(x) > \pi_*(x)$. From which, it follows that $E_*(\pi') < E_*(\pi)$. A similar situation take place if the new observations have values $x < x_s$. A symmetrical analysis can be done for $E^*(\pi)$ when new values greater or equal to the greatest previously observed value are reported.

This is illustrated by the following example. Let $\mathcal{N} = 10$, with $\mathcal{G}^1$ defined by $|G^1| = 9$, with $w_1 = 1$, $w_2 = 1$, $w_4 = 2$, $w_5 = 4$, $w_7 = 1$, we get $\pi_1 = \pi_2 = \pi_7 = 3/9$, $\pi_4 = 5/9$, $\pi_5 = 1$, and $E_*(\pi^1) = 0.1 \times (3/9 - 0) + 0.2 \times (3/9 - 3/9) + 0.4 \times (5/9 - 3/9) + 0.5 \times (1 - 5/9) = (0.3 + 0.8 + 2)/9 = 3.1/9 = 0.344$. Consider now $|G^2| = 10$, with $w_1 = 2$, $w_2 = 1$, $w_4 = 2$, $w_5 = 4$, $w_7 = 1$, we get $\pi_2 = \pi_7 = 2/10 = 0.2$, $\pi_1 = \pi_4 = 6/10 = 0.6$, $\pi_5 = 1$, and $E_*(\pi^2) = 0.1 \times (0.6 - 0) + 0.2 \times (0.6 - 0.6) + 0.4 \times (0.6 - 0.6) + 0.5 \times (1 - 0.6) = 0.26$.

Finally, leaning does not hold. A counter-example can be easily built by considering two grade structures, one where all values have a weight of 1, $\langle G, v\rangle$, and an unsymmetrical one $\langle G', v'\rangle$ such that $v(G) = \{\frac{r}{\mathcal{N}}, 0, \ldots, 0, \frac{(r+2k)}{\mathcal{N}}\}$ and $v'(G') = \{\frac{r}{\mathcal{N}}, \frac{(r+1)}{\mathcal{N}}, \ldots, \frac{(r+k)}{\mathcal{N}}, 0, \ldots, 0, \frac{(r+2k)}{\mathcal{N}}\}$. Indeed these two grade structures have the same expected interval summary $[\frac{r}{\mathcal{N}}, \frac{(r+2k)}{\mathcal{N}}]$, while it would not be the case with the summarizing method based on the pulling principle. The reason is that the approach ignores what is between the two extreme values, which plays a central role in the leaning axiom. □

Due to the failure of the leaning axiom for the possibility distribution-based method, there is a noticeable difference between the summarizing interval computed in Section 5 with the one computed above. Although their behavior is basically the same for unimodal distributions $\{p_k : 0 \leq k \leq \mathcal{N}\}$ (or equivalently possibility distributions π), i.e. for distributions such that $\nexists(r, s, t), r < s < t$ and $p_s < \min(p_r, p_t)$, the two types of summary may behave differently for multi modal distributions. Indeed $[E_*(\pi_{[a,b]}), E^*(\pi_{[a,b]})] = [E_*(\pi_{\{a,b\}}), E^*(\pi_{\{a,b\}})]$, where $\pi_{\{a,b\}}(x) = 1$ if $x = a$, or $x = b$ and $\pi_{\{a,b\}}(x) = 0$ otherwise. More generally, $[E_*(\pi), E^*(\pi)] = [E_*(\mathring{\pi}), E^*(\mathring{\pi})]$, where $\mathring{\pi}(x) = \pi(x)$ if $x \notin (a, b)$ where $[a, b] = [\min\{x_k : \pi(x_k) = 1\}, \max\{x_k : \pi(x_k) = 1\}]$, and $\mathring{\pi}(x) = 0$ otherwise. This means that the existence of "hole(s)" in the distribution makes no difference for the expected interval summary. Expected interval summaries only account for worst and best grades and their relative plausibility, while "pulling-based" interval summaries also account in general for the existence, or not, of grades in between. Still this is not always the case, due to the following exception: the two grade structures $\langle G, v\rangle$ and $\langle G', v'\rangle$ such that $v(G) = \{\frac{r}{\mathcal{N}}, \frac{s}{\mathcal{N}}\}$, $v'(G') = \{\frac{r}{\mathcal{N}}, \frac{(r+1)}{\mathcal{N}}, \ldots, \frac{(s-1)}{\mathcal{N}}, \frac{s}{\mathcal{N}}\}$, $r + 1 < s$ and all values have a weight of 1, have the same summary $[\frac{r}{\mathcal{N}}, \frac{s}{\mathcal{N}}]$, with the the above approach and the approach proposed in Section 5, due to the uniformity of the distribution of the values (see Example 5).

Generally speaking, the possibility distribution-based approach favors the "emerging" values, while the multiplicity of equally observed values make them more possible. This is illustrated by the following examples:

Example 10 As already computed $E_*(\pi) = 0.32$ and $E^*(\pi) = 1$ for $\mathcal{G}$ defined by $|G| = 5$ and $w_0 = 1$, $w_4 = 2$, $w_{10} = 2$.

Consider now $\mathcal{G}'$ defined by $|G'| = 8$, with $w_0 = 1$, $w_4 = 2$, $w_5 = 1$, $w_6 = 1$, $w_7 = 1$, $w_{10} = 2$, which is obtained from $\mathcal{G}$ adding 3 observations having distinct values that are between the most observed values x_4 and x_{10}. Then, we get a larger interval, namely $E_*(\pi') = 0.4 \times (1 - 0.5) = 0.20$ and $E^*(\pi') = 1$. This may thought surprising, except if we consider that x_4 and x_{10} are now less emerging with respect to the wider set of equal possibilities x_0, x_5, x_6, w_7.

In contrast, if we take $\mathcal{G}$" defined by $|G"| = 8$, with $w_0 = 1$, $w_4 = 2$, $w_5 = 3$, $w_{10} = 2$, where the 3 extra observations with respect to $\mathcal{G}$ are now concentrated on one value x_5, between the values x_4 and x_{10} that are only observed 2 times. Then, we have $\pi_0 = 1/8 = 0.125$, $\pi_4 = \pi_{10} = 5/8 = 0.625$, $\pi_5 = 1$, and we get $E_*(\pi") = 0.1 \times 1/8 + 0.4 \times (5/8 - 1/8) + 0.5 \times (1 - 5/8) = 0.40$, and $E^*(\pi") = 0.5 \times (1 - 5/8) + 1 \times (5/8 - 0) = 0.8125$, which is now more narrow than the interval obtained with $\mathcal{G}$. As an expected interval, it may quite narrow with respect to the maximal range of values, which is here the whole unit interval.

7 From summaries to trust intervals

In certain circumstances, given a grade structure $\mathcal{G} = \langle G, v\rangle$ describing how well a certain trustee has achieved certain past goals, the trustors need a trust interval rather than a summarizing one to make decisions about the trustee. A trust interval is an interval such that it is rational to believe (on the basis of the past grades) that the future grades will essentially fall on it. We think that we can get such an interval $T(\mathcal{G})$ by taking a summarizing interval $I(\mathcal{G})$ and then adding an adequate margin of error.

The question is of course: what is an adequate margin of error? The more the number of past grades is big, the more the margin should be small. A solution is for example to define the left bound of $T(\mathcal{G})$ as the weighted mean of two components: 0 on the one hand, and the left bound of $I(\mathcal{G})$ on the other hand. The weight of the first component is 1, while that of the second component is the number of past grades, that is, $|G|$. Similarly, the right bound of $T(\mathcal{G})$ is defined as the weighted mean of 1 (with weight 1) and the right bound of $I(\mathcal{G})$ (with weight $|G|$).

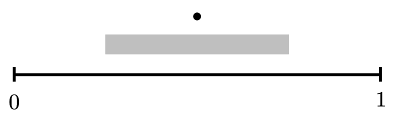
Definition 13 Let $\mathcal{G} = \langle G, v\rangle$ be a grade structure and I a summarizing method. We denote by $T_I(\mathcal{G})$ the trust interval obtained from $\mathcal{G}$ and I, that is,

$$T_I(\mathcal{G}) = [\frac{|G| minI(\mathcal{G})}{1+|G|}, \ \frac{1+|G| maxI(\mathcal{G})}{1+|G|}]$$

T_I and I behave almost in the same way. The essential difference is that two structures may lead to the same summarizing interval, but to different trust intervals. This is possible because there exist structures that contain different numbers of grades, and yet lead to the same summarizing interval.

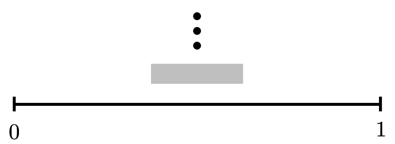
Here is an example. Suppose that $\mathcal{G}$ consists of only one grade of value 0.5. Then, $T_{I_p}(\mathcal{G})$ is centered and strictly smaller than $[0, 1]$ (see Example 11 below), which reflects the idea that we are expecting future grades close to 0.5, but we remain cautious. Concerning $I_p(\mathcal{G})$, it is equal to $[0.5, 0.5]$.

Example 11



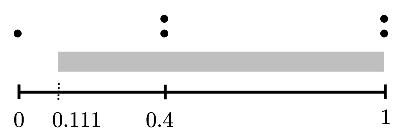
Next, add two other 0.5's. Then, $T_{I_p}(\mathcal{G})$ is still centered but smaller than before (see Example 12 below), which reflects the idea that we are now less cautious. Concerning $I_p(\mathcal{G})$, it is still equal to $[0.5, 0.5]$.

Example 12



As a last example, here is $T_{I_p}(\mathcal{G}_0)$, where $\mathcal{G}_0$ is the grade structure of Example 1:

Example 13



Provided that our view of a trust interval is convincing ("trust = summary + margin of error"), T_{I_p} and T_{I_e} provide trustors with an indication of what they can expect from trustees.

To summarize, the approaches discussed in this paper for providing summarizing intervals do not really take into account the fact that a grade structure $\mathcal{G}$ may be based on a more or less large set of observations. Indeed two grade structures $\mathcal{G}$ and $\mathcal{G}'$ may have the same summarizing interval, while $|G|$ is significantly higher than $|G'|$, leading to be more confident in the predictive value of the summary for the agent associated with $\mathcal{G}$. This has led to propose the notion of trust interval in this section. Another way to handle this question might be the use of methods for prediction based on poor statistics [15]. Note that these methods assume a unimodal distribution of the grades.

8 Levels of trust and distrust as expectations of good or bad grades

In the previous section, the idea of trust was pertaining to the building of an interval for which one may have a reasonable confidence that it will include future grades of the considered agent, provided that the agent continues to behave in the same way. We now discuss another type of evaluations, where the idea of trust is related to levels of expectation, and which thus takes its values in an abstract scale, and no longer lies in the set where grades range. Namely, we propose to define:

– a *level of trust* as the certainty (on the basis of the past grades) that a future grade will be good, and

– a *level of distrust* as the fear that a future grade may be bad.

The idea is that the level of trust should be high if *always* grades with good values are reported, while the level of distrust should be high as soon as *some* grades with bad values are reported. For computing such levels of trust and distrust, we follow a two-step approach:

i) transform the grade structure $\mathcal{G}$ into a possibility distribution, as in Section 6. This distribution is then viewed as restricting the possible value of the next outcome;

ii) on this basis, compute the level of trust as the *necessity* measure that a grade with a good value will be obtained, and the level of distrust as the *possibility* that a bad grade value will be obtained, where "good" and "bad" refer to the fuzzy sets *Good* and *Bad*. Here "good" means "good enough", while its antonym "bad" corresponds to the idea of "bad enough", a notion that in general may be more restrictive than the idea of "not good enough" (which covers bad and not sufficiently good cases). These fuzzy sets are respectively defined by increasing and decreasing membership functions on $\mathcal{S} = \{\frac{k}{\mathcal{N}} : 0 \leq k \leq \mathcal{N}\}$, and are such that $Good(0) = 0$, $Good(1) = 1$, while $Bad(0) = 1$, and $Bad(1) = 0$. We may, for the example, simply take *Bad* as the complement of *Good*, i.e. $Bad(x) = 1 - Good(x)$.

The second step amounts to the computation of the necessity and the possibility of fuzzy events *Good* and *Bad*. These levels are computed from past performances, represented by the distribution π, which are found good or bad, in agreement with possibility theory. That is,

$$trust(\mathcal{G}) = \min_x \max(Good(x), 1 - \pi(x)) \quad distrust(\mathcal{G}) = \max_x \min(Bad(x), \pi(x))$$

Example 14 Consider Example 1 again with $\mathcal{N} = 10$, $\mathcal{G}$ is defined by $|G| = 5$, with $w_0 = 1$, $w_4 = 2$, $w_{10} = 2$, and $\pi_0 = 1/5 = 0.2$, $\pi_4 = \pi_{10} = 1$. We have

$trust(\mathcal{G}) = \min(\max(Good(x_0), 1 - \pi(x_0)), \max(Good(x_4), 1 - \pi(x_4)), \max(Good(x_{10}), 1 - \pi(x_{10}))) = \min(\max(0, 0.8), \max(0.4, 0), \max(1, 0)) = 0.4.$

$distrust(\mathcal{G}) = \max(\min(Bad(x_0), \pi(x_0)), \min(Bad(x_4), \pi(x_4)), \min(Bad(x_{10}), \pi(x_{10}))) = \max(\min(1, 0.2), \min(0, 6, 1), \min(0, 1)) = 0.6.$

This reflects the fact that the considered grade structure includes a value x_4 which is quite far to be really good and which is considered as fully possible.

The level of distrust is high as soon as there exists a really bad grade value that is highly plausible. The level of trust is high as soon as any bad grade value (including the less bad ones) is impossible or almost impossible. Such definitions acknowledge the fact that one should be afraid by bad performances in trust evaluation, which have a greater impact on the opinion of the trustor than the good ones.

In this view, a high level of trust will decrease (and the level of distrust will increase) as soon as really bad grades are reported repeatedly, thus making higher and higher the possibility of a (new) bad grade (as computed in Section 6). A small level of trust will increase if a sufficient number of really good grades are reported thus decreasing progressively the possibility of a (new) bad grade (and the level of distrust).

It can be checked that the sum of these two levels is always less or equal to 1 (as in most models of trust and distrust, e.g. [4]). This due to the fact that we have not assumed here that the fuzzy sets *Good* and *Bad* are necessarily complement of each other. Namely it is not assumed that $Bad(x) = 1 - Good(x)$, otherwise, we would have $distrust(\mathcal{G}) = 1 - trust(\mathcal{G})$ in agreement with the duality between the possible and the necessary in possibility theory for opposite events. Rather *Good* and *Bad* may be viewed as antonyms whose representation only obey to $Good(1 - x) = Bad(x)$, with a fully empty intersection, in such a way that $\forall x, Good(x) + Bad(x) \leq 1$.

Thus, from these two levels an interval pertaining to trust (resp. to distrust) can be built as $[trust(\mathcal{G}), 1 - distrust(\mathcal{G})]$ (resp. $[distrust(\mathcal{G}), 1 - trust(\mathcal{G})]$). This interval has nothing to do with the trust interval computed in Section 7 which was pertaining to the range of grades. Clearly, these levels (and associated intervals) not only involve the grade structure information $\mathcal{G}$, but also a graded view of goodness and badness, and belong to another scale.

Remark: Note that the above notion of distrust may be found quite weak in the sense that distrust appears as soon as a really bad grade is regarded as highly possible (which is in fact just a pessimistic attitude). This situation might be viewed as a mere lack of trust (as long as actually good grades are also highly possible), and might be rather termed *mistrust* (this may be of interest in contexts where trust representations need to tolerate situations where people are allowed to have malicious intentions). Then, a probably too strong a notion of distrust would be captured here by the *certainty* of getting bad grades, which is estimated by $\min_x \max(Bad(x), 1 - \pi(x))$.

9 Comparing grade structures

Two grade structures may be directly compared without necessarily summarizing them in a first step. Let us consider two grade structures $\mathcal{G} = \langle G, v\rangle$ and $\mathcal{G}' = \langle G', v'\rangle$ respectively characterized by the data sets $(p_k = \frac{w_k}{|G|} : k = 0, \ldots, \mathcal{N})$ and $(p'_k = \frac{w'_k}{|G'|} : k = 0, \ldots, \mathcal{N})$. Intuitively speaking, $\mathcal{G}$ may be preferred to $\mathcal{G}'$ if the proportion of good (resp. bad) grades in the former is higher (resp. smaller) than in the latter.

A natural way to express such a preference is to use stochastic dominance. Namely,

$$\mathcal{G} \gg_{stdo} \mathcal{G}' \text{ iff } \forall i = 0, \ldots, \mathcal{N},\ \sum_{k=0}^{i} p_k \leq \sum_{k=0}^{i} p'_k$$

Indeed $\sum_{k=0}^{i} p_k$ represents the proportion of grades that are less or equal to $\frac{i}{\mathcal{N}}$ and that are thus rather bad (all the worst as i is small). Similarly $\sum_{k=i}^{\mathcal{N}} p_k$ is the proportion of grades that are at least equal to $\frac{i}{\mathcal{N}}$ and that are thus rather good (all the best as i is large). It can be checked that we have equivalently (since $\sum_{k=0}^{\mathcal{N}} p_k = 1$)

$$\mathcal{G} \gg_{stdo} \mathcal{G}' \text{ iff } \forall i = 0, \ldots, \mathcal{N},\ \sum_{k=i}^{\mathcal{N}} p_k \geq \sum_{k=i}^{\mathcal{N}} p'_k$$

Clearly, $\gg_{stdo}$ is a partial order. Other partial orders may be of interest here. In particular, one may consider the "good" and the "bad" grades separately. This means that the discrete scale $\mathcal{S} = \{\frac{k}{\mathcal{N}} : 0 \leq k \leq \mathcal{N}\}$ is now viewed as a bipolar univariate scale where the two kinds of grades are distinguished and possibly separated by one (or several) neutral value(s). This amounts to identify a subset of bad grades in $\mathcal{S}$, namely $Bad = \{\frac{j}{\mathcal{N}} : 0 \leq j \leq m\}$ and a subset of good grades $Good = \{\frac{k}{\mathcal{N}} : n \leq k \leq \mathcal{N}\}$ with $m < n$. Then $\mathcal{G} \gg_{bipo} \mathcal{G}'$ iff

$$\sum_{i=0}^{\mathcal{N}} Good(\frac{i}{\mathcal{N}}) \cdot p_i \geq \sum_{i=0}^{\mathcal{N}} Good(\frac{i}{\mathcal{N}}) \cdot p'_i \text{ and } \sum_{j=0}^{\mathcal{N}} Bad(\frac{j}{\mathcal{N}}) \cdot p_j \leq \sum_{j=0}^{\mathcal{N}} Bad(\frac{j}{\mathcal{N}}) \cdot p'_j \quad (5)$$

where $Good(x)$ (resp. $Bad(x)$) equals 1 or 0 according as x belongs or not to $Good$ (resp. Bad). It can be checked that $\mathcal{G} \gg_{stdo} \mathcal{G}'$ entails $\mathcal{G} \gg_{bipo} \mathcal{G}'$.

Clearly, $\mathcal{G} \gg_{bipo} \mathcal{G}'$ expresses that the grade structure $\mathcal{G}$ is at least as trustworthy as $\mathcal{G}'$ since $\mathcal{G}$ has a higher proportion of grades with good values and a smaller proportion of grades with bad values than $\mathcal{G}'$. A more refined, actually complete, order can then be obtained by taking the difference of the global amounts of grades with good values and bad values. Then $\mathcal{G} \gg_{bipo\text{-}diff} \mathcal{G}'$ iff

$$\sum_{i=0}^{\mathcal{N}} Good(\frac{i}{\mathcal{N}}) \cdot p_i - \sum_{j=0}^{\mathcal{N}} Bad(\frac{j}{\mathcal{N}}) \cdot p_j \geq \sum_{i=0}^{\mathcal{N}} Good(\frac{i}{\mathcal{N}}) \cdot p'_i - \sum_{j=0}^{\mathcal{N}} Bad(\frac{j}{\mathcal{N}}) \cdot p'_j \quad (6)$$

However, Equation (6) makes no difference between a structure with a few good grade values and no bad ones, and a structure with some bad ones and a few more good ones. Note also that the two above expressions (5) and (6) still make sense when *Good* and *Bad* are fuzzy sets as in Section 8. We then recognize in (6) the expression of fuzzy (relative) cardinalities (see, e.g. [6]).

Besides, summarizing intervals provide another basis for comparing grade structures. More precisely, given a summarizing method I and a pre-order $\preceq$ over intervals (e.g., $[a, b] \preceq [c, d]$ iff $a \leq c$ and $b \leq d$), we can define a pre-order over grade structures:

$$\mathcal{G} \gg_{inter\text{-}sum\ I} \mathcal{G}' \text{ iff } I(\mathcal{G}') \preceq I(\mathcal{G})$$

Such a summary-based pre-order remains partial. Note that it is not necessarily compatible with the mean-based *scalar* summary:

$$\mathcal{G} \gg_{mean} \mathcal{G}' \text{ iff } \frac{1}{|G|} \sum_{i=0}^{\mathcal{N}} w_i \cdot \frac{i}{\mathcal{N}} \geq \frac{1}{|G'|} \sum_{i=0}^{\mathcal{N}} w'_i \cdot \frac{i}{\mathcal{N}}$$

which is weaker than $\gg_{stdo}$.

10 Conclusion

The main contribution of the present paper is that we have developed two intuitive methods for summarizing a set of grades by an interval, which provides a handy tool for judging and comparing agents. In addition, we have showed that our methods satisfy several desirable properties, which provides theoretical justifications for them. We have also showed how to use summarizing intervals to construct trust intervals, estimate levels of trust and distrust, and compare two sets of grades.

Concerning the perspectives, new postulates could be investigated, for example, a postulate expressing some agreement between stochastic dominance and the pre-orders based on summarizing intervals. Indeed, stochastic dominance seems to be very cautious and therefore rather uncontroversial. Extensions of stochastic dominance [2], or approaches for determining typical intervals that summarize (when they make sense) a set of data are also worth investigating directions for further research [9]. Other lines for further research include a deeper comparison of the two approaches, the validation of the models from a cognitive psychology point of view, and modifications in order to take into account the freshness of the information.

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