Lack of Finite Characterizations for the Distance-based Revision*

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Abstract

Lehmann, Magidor, and Schlechta developed an approach to belief revision based on distances between any two valuations. Suppose we are given such a distance $\mathcal{D}$. This defines an operator $|_\mathcal{D}$, called a *distance operator*, which transforms any two sets of valuations V and W into the set $V|_\mathcal{D}W$ of all those elements of W that are closest to V. This operator $|_\mathcal{D}$ defines naturally the revision of K by α as the set of all formulas satisfied in $M_K|_\mathcal{D}M_\alpha$ (i.e. the set of all those models of α that are closest to the models of K). This constitutes a *distance-based revision operator*. Lehmann *et al.* characterized families of them using a "loop" condition of arbitrarily big size. An interesting question is whether this loop condition can be replaced by a finite one. Extending the results of Schlechta, we will provide elements of negative answer. In fact, we will show that for families of distance operators, there is no "normal" characterization. Approximatively, a characterization is normal iff it contains only finite and universally quantified conditions. Though they are negative, these results have an interest of their own for they help to understand more clearly the limits of what is possible in this area. In addition, we are quite confident that they can be used to show that for families of distance-based revision operators, there is no normal characterization either. For instance, the families of Lehmann *et al.* might well be concerned with this, which suggests that their big loop condition cannot be replaced by a finite and universally quantified condition.

Introduction

Belief revision is the study of how an intelligent agent may replace its current epistemic state by another one which is non-trivial and incorporates new information. In (Alchourrón, Gärdenfors, & Makinson 1985), the well-known AGM approach was proposed. An epistemic state is modelled there by a deductively closed set of formulas K and new information by a formula α. A revision operator is then a function that transforms K and α into a new set of formulas (intuitively, the revised epistemic state).

One of the contributions of the AGM approach is that it provides well-known postulates that any reasonable revision operator should satisfy. These postulates have been defended by their authors. But, doubts have been expressed as to their "soundness", e.g. (Katsuno & Mendelzon 1992), and especially "completeness", e.g. (Freund & Lehmann 1994), (Darwiche & Pearl 1994), (Lehmann 1995), and (Darwiche & Pearl 1997). In particular, to be accepted, an operator never needs to put some coherence between the revisions of two different sets K and K'. As a consequence, some operators are accepted though they are not well-behaved when iterated. In addition, modelling an epistemic state by just a deductively closed set of formulas has been rejected by many researchers, e.g. (Boutilier & Goldszmidt 1993), (Boutilier 1993), (Darwiche & Pearl 1997), (Williams 1994), and (Nayak *et al.* 1996). In (Lehmann 1995) and (Friedman & Halpern 1996), it is argued that this modelling is not sufficient in many AI applications.

This provides motivations for another approach, based on distances between any two valuations, introduced in (Schlechta, Lehmann, & Magidor 1996) and investigated further in (Lehmann, Magidor, & Schlechta 2001). Their approach is in line with the AGM modelling of an epistemic state, but it defines well-behaved iterated revisions. More precisely, suppose we have at our disposal a distance $\mathcal{D}$ between any two valuations. This defines an operator $|_\mathcal{D}$, called a *distance operator*, which transforms any ordered pair (V, W) of sets of valuations into the set $V|_\mathcal{D}W$ of all those elements of W that are closest to V according to $\mathcal{D}$.

This operator $|_\mathcal{D}$ defines naturally the revision of K by α as the set of all formulas satisfied in $M_K|_\mathcal{D}M_\alpha$ (i.e. the set of all those models of α that are closest to the models of K). This constitutes a *distance-based revision operator*, which is interesting for its natural aspect and for it is well-behaved when iterated. This is due to the fact that the revisions of the different K's are all defined by the same distance, which ensures a strong coherence between them. Note that this is not the case with other definitions. For instance, with sphere systems (Grove 1988) and epistemic entrenchment relations (Gärdenfors & Makinson 1988), the revision of each K is defined by a different structure without any "glue" relating them.

In (Lehmann, Magidor, & Schlechta 2001), several families of distance-based revision operators were characterized by the AGM postulates together with new ones that deal with iterated revisions. However, the latter postulates include a "loop" condition of arbitrarily big size. An interesting question is whether it can be replaced by a finite

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condition. Elements of negative answer were provided in (Schlechta 2004). Approximatively, Schlechta call normal a characterization containing only conditions which are finite, universally quantified (like e.g. the AGM postulates), and simple (i.e. using only elementary operations like e.g. $\cup$, $\cap$, $\setminus$). Then, he showed that for families of distance operators, there is no normal characterization.

Now, there is a strong connexion between the distance operators (which apply to valuations) and the distance-based revision operators (which apply to formulas). It is quite reasonable to think that the work of Schlechta can be continued to show that for families of distance-based revision operators, there is no normal characterization either. The families investigated in (Lehmann, Magidor, & Schlechta 2001) might well be concerned with this, which suggests that the arbitrarily big loop condition cannot be replaced by a finite, universally quantified, and simple condition.

The contribution of the present paper is to extend the work of Schlechta in two directions. First, we will use the word "normal" in a larger sense. Indeed, we will call normal a characterization containing only conditions which are finite and universally quantified, but not necessarily simple (i.e. the conditions can involve complex structures or functions, etc., we are not limited to elementary operations). Then, we will show that the families which Schlechta investigated still do not admit a normal characterization, in our larger sense. This is therefore a generalization of his negative results. Second, we will extend the negative results (always in our sense) to new families of distance operators, in particular to some that respect the Hamming distance.

We are quite confident that the present work can be continued, like the work of Schlechta, to show that for families of distance-based revision operators, there is no normal characterization either. But, we will cover more families and with a more general definition of a normal characterization. This is the main motivation. In addition, the impossibility results of the present paper already help to understand more clearly the limits of what is possible in this area. They have therefore an interest of their own.

First, we will present the distance-based revision and the characterizations of Lehmann *et al.* Second, we will define formally the normal characterizations. Third, we will show the impossibility results. And finally, we will conclude.

Background

Pseudo-distances

In many circumstances, it is reasonable to assume that an agent can evaluate for any two valuations v and w, how far is the situation described by w from the situation described by v, or how difficult or unexpected the transition from v to w is, etc. In (Lehmann, Magidor, & Schlechta 2001), this is modelled by pseudo-distances:

Definition 1 Let $\mathcal{V}$ be a set.
$\mathcal{D}$ is a *pseudo-distance* on $\mathcal{V}$ iff $\mathcal{D} = \langle C, \prec, d \rangle$, where C is a non-empty set, $\prec$ is a strict total order on C, and d is a function from $\mathcal{V} \times \mathcal{V}$ to C.

Intuitively, $\mathcal{V}$ is a set of valuations. Each element of C represents a "cost". $c \prec c'$ means the cost c is strictly smaller than the cost c'. And, $d(v, w)$ is the cost of the move from v to w. Natural properties that come to mind are those of usual distances. Before introducing them, we need standard notations:

Notation 2 $\mathcal{P}$ denotes the power set operator.
For every set S, $|S|$ denotes the cardinality of S.
$\mathbb{N}$, $\mathbb{N}^+$, $\mathbb{R}$, and $\mathbb{R}^+$ denote respectively the natural, positive natural, real, and positive real numbers.
Let $r \in \mathbb{R}$. Then, $abs(r)$ denotes the absolute value of r.
Let $n, m \in \mathbb{N}$. Then, $[n, m]$ denotes the set of every k in $\mathbb{N}$ (not in $\mathbb{R}$) such that $n \leq k \leq m$.

Definition 3 Suppose $\mathcal{D} = \langle C, \prec, d \rangle$ is a pseudo-distance on a set $\mathcal{V}$.
$\mathcal{D}$ is *symmetric* iff $\forall\, v, w \in \mathcal{V}$, $d(v, w) = d(w, v)$.
$\mathcal{D}$ is *identity respecting* (IR) iff
(1) $C = \mathbb{R}$;
(2) $\prec$ is the usual strict total order on $\mathbb{R}$;
(3) $\forall\, v, w \in \mathcal{V}$, $d(v, w) = 0$ iff $v = w$.
$\mathcal{D}$ is *positive* iff (1), (2), and
(4) $\forall\, v, w \in \mathcal{V}$, $0 \preceq d(v, w)$.
$\mathcal{D}$ is *triangle-inequality respecting* (TIR) iff (1), (2), and
(5) $\forall\, v, w, x \in \mathcal{V}$, $d(v, x) \preceq d(v, w) + d(w, x)$.

These properties have not been imposed from start because natural circumstances could then no longer be modelled. For instance, non-symmetric pseudo-distances are useful when moving from v to w may be "cheaper" than moving from w to v. There are also circumstances where staying the same requires effort and then non-IR pseudo-distances will be helpful. We can also imagine scenarios where some costs can be seen as "benefits", we will then turn to non-positive pseudo-distances, etc.

In addition, the costs are not required to be necessarily the real numbers. Indeed, for instance, we could need $|\mathbb{N}|$ to model an "infinite cost" useful when a move is impossible or extremely difficult. Provided one accepts the infinite cost $|\mathbb{N}|$, we can define naturally "liberal" versions of identity respect, positivity, and triangle-inequality respect:

Definition 4 Suppose $\mathcal{D} = \langle C, \prec, d \rangle$ is a pseudo-distance on a set $\mathcal{V}$.
$\mathcal{D}$ is *liberally* IR iff
(1) $C = \mathbb{R} \cup \{|\mathbb{N}|\}$;
(2) $\forall\, c, c' \in C$, $c \prec c'$ iff $(c, c' \in \mathbb{R}$ and $c < c')$ or $(c \in \mathbb{R}$ and $c' = |\mathbb{N}|)$;
(3) $\forall\, v, w \in \mathcal{V}$, $d(v, w) = 0$ iff $v = w$.
$\mathcal{D}$ is *liberally positive* iff (1), (2), and
(4) $\forall\, v, w \in \mathcal{V}$, $0 \preceq d(v, w)$.
$\mathcal{D}$ is *liberally* TIR iff (1), (2), and
(5) $\forall\, v, w, x \in \mathcal{V}$: if $d(v, x), d(v, w), d(w, x) \in \mathbb{R}$, then $d(v, x) \preceq d(v, w) + d(w, x)$;
if $d(v, x) = |\mathbb{N}|$, then $d(v, w) = |\mathbb{N}|$ or $d(w, x) = |\mathbb{N}|$.

The Hamming distance between propositional valuations has been considered in (Dalal 1988) and investigated further by many authors. Respecting this distance is an important property. We need before to present the matrices for a propositional language (Urquhart 2001):

Definition 5 Let $\mathcal{L} = \langle \mathcal{A}, \mathcal{C} \rangle$ be a propositional language ($\mathcal{A}$ denotes the atoms and $\mathcal{C}$ the connectives), let $\mathcal{F}$ be the

set of all well-formed formulas (wffs) of $\mathcal{L}$, and $\forall \diamond \in \mathcal{C}$, let $n(\diamond)$ be the arity of $\diamond$.
$\mathcal{M}$ is a *matrix* on $\mathcal{L}$ iff $\mathcal{M} = \langle T, E, f\rangle$, where T is a set, E is a non-empty proper subset of T, and f is a function (whose domain is $\mathcal{C}$) such that $\forall \diamond \in \mathcal{C}$, $f_\diamond$ (i.e. $f(\diamond)$) is a function from $T^{n(\diamond)}$ to T.
v is a $\mathcal{M}$*-valuation* iff v is a function from $\mathcal{F}$ to T such that $\forall \diamond \in \mathcal{C}$, $\forall \alpha_1, \ldots, \alpha_{n(\diamond)} \in \mathcal{F}$, $v(\diamond(\alpha_1, \ldots, \alpha_{n(\diamond)})) = f_\diamond(v(\alpha_1), \ldots, v(\alpha_{n(\diamond)}))$.

Intuitively, T is a set of truth values and E contains all the designated truth values.

Definition 6 Let $\mathcal{L} = \langle \mathcal{A}, \mathcal{C}\rangle$ be a propositional language, $\mathcal{M}$ a matrix on $\mathcal{L}$, $\mathcal{V}$ the set of all $\mathcal{M}$-valuations, and $\mathcal{D} = \langle C, \prec, d\rangle$ a pseudo-distance on $\mathcal{V}$.
We use the following notation: $\forall\, v, w \in \mathcal{V}$,
$h(v, w) := \{p \in \mathcal{A} : v(p) \neq w(p)\}$.
$\mathcal{D}$ is *Hamming-inequality respecting* (HIR) iff $\forall v, w, x \in \mathcal{V}$, if $|h(v, w)| < |h(v, x)|$, then $d(v, w) \prec d(v, x)$.

Recall that $h(v, w)$ may be infinite and thus $<$ should be understood as the usual order on the cardinal numbers.

We turn to crucial operators introduced in (Lehmann, Magidor, & Schlechta 2001). They are central in the definition of the distance-based revision. They transform any two sets of valuations V and W into the set of every element w of W such that a global move from V to w is of minimal cost. Note that concerning this point, (Lehmann, Magidor, & Schlechta 2001) has its roots in (Katsuno & Mendelzon 1992) and especially in (Lewis 1973).

Definition 7 Suppose $\mathcal{D} = \langle C, \prec, d\rangle$ is a pseudo-distance on a set $\mathcal{V}$.
We denote by $|_\mathcal{D}$ the binary operator on $\mathcal{P}(\mathcal{V})$ such that $\forall\, V, W \subseteq \mathcal{V}$, we have $V|_\mathcal{D}W =$

$$\{w \in W : \exists v \in V, \forall v' \in V, \forall w' \in W, d(v, w) \preceq d(v', w')\}.$$

Distance-based revision operators

The ontological commitments endorsed in (Lehmann, Magidor, & Schlechta 2001) are close to the AGM ones: a classical propositional language is considered and both epistemic states and new information are modelled by consistent sets of formulas (not necessarily deductively closed).

Notation 8 We denote by $\mathcal{L}_c$ some classical propositional language and by $\vdash_c$, $\mathcal{V}_c$, $\models_c$, and $\mathcal{F}_c$ respectively the classical consequence relation, valuations, satisfaction relation, and wffs of $\mathcal{L}_c$. Let $\Gamma, \Delta \subseteq \mathcal{F}_c$ and $V \subseteq \mathcal{V}_c$, then:
$\Gamma \vee \Delta := \{\alpha \vee \beta : \alpha \in \Gamma, \beta \in \Delta\}$;
$\vdash_c(\Gamma) := \{\alpha \in \mathcal{F}_c : \Gamma \vdash_c \alpha\}$;
$M_\Gamma := \{v \in \mathcal{V}_c : \forall\, \alpha \in \Gamma, v \models_c \alpha\}$;
$T(V) := \{\alpha \in \mathcal{F}_c : V \subseteq M_\alpha\}$;
$\mathbf{C} := \{\Gamma \subseteq \mathcal{F}_c : \vdash_c(\Gamma) \neq \mathcal{F}_c\}$;
$\mathbf{D} := \{V \subseteq \mathcal{V}_c : \exists\, \Gamma \subseteq \mathcal{F}_c, V = M_\Gamma\}$.

In this classical framework, two new properties for pseudo-distances can be defined. They convey natural meanings. Their importance has been put in evidence in (Lehmann, Magidor, & Schlechta 2001).

Definition 9 Let $\mathcal{D} = \langle C, \prec, d\rangle$ be a pseudo-distance on $\mathcal{V}_c$.
$\mathcal{D}$ is *definability preserving* (DP) iff
$\forall\, V, W \in \mathbf{D}$, $V|_\mathcal{D}W \in \mathbf{D}$.
$\mathcal{D}$ is *consistency preserving* (CP) iff
$\forall\, V, W \in \mathcal{P}(\mathcal{V}_c) \setminus \{\emptyset\}, V|_\mathcal{D}W \neq \emptyset$.

Now, suppose we are given a pseudo-distance $\mathcal{D}$ on $\mathcal{V}_c$. Then, the revision of a consistent set of formulas Γ by a second one Δ can be defined naturally as the set of all formulas satisfied in $M_\Gamma|_\mathcal{D}M_\Delta$:

Definition 10 Let $\star$ be an operator from $\mathbf{C} \times \mathbf{C}$ to $\mathcal{P}(\mathcal{F}_c)$. We say that $\star$ is a *distance-based revision operator* iff there exists a pseudo-distance $\mathcal{D}$ on $\mathcal{V}_c$ such that $\forall\, \Gamma, \Delta \in \mathbf{C}$,

$$\Gamma \star \Delta = T(M_\Gamma|_\mathcal{D}M_\Delta).$$

In addition, if $\mathcal{D}$ is symmetric, IR, DP etc., then so is $\star$.

The authors of (Lehmann, Magidor, & Schlechta 2001) rewrote the AGM postulates in their framework as follows. Suppose $\star$ is an operator from $\mathbf{C} \times \mathbf{C}$ to $\mathcal{P}(\mathcal{F}_c)$ Then, define the following properties: $\forall\, \Gamma, \Gamma', \Delta, \Delta' \in \mathbf{C}$,

($\star$0) if $\vdash_c(\Gamma) = \vdash_c(\Gamma')$ and $\vdash_c(\Delta) = \vdash_c(\Delta')$, then $\Gamma \star \Delta = \Gamma' \star \Delta'$;

($\star$1) $\Gamma \star \Delta \in \mathbf{C}$ and $\Gamma \star \Delta = \vdash_c(\Gamma \star \Delta)$;

($\star$2) $\Delta \subseteq \Gamma \star \Delta$;

($\star$3) if $\Gamma \cup \Delta \in \mathbf{C}$, then $\Gamma \star \Delta = \vdash_c(\Gamma \cup \Delta)$;

($\star$4) if $(\Gamma \star \Delta) \cup \Delta' \in \mathbf{C}$, then $\Gamma \star (\Delta \cup \Delta') = \vdash_c((\Gamma \star \Delta) \cup \Delta')$.

Then, it can be checked that every positive, IR, CP and DP distance-based revision operator $\star$ satisfies ($\star$0)-($\star$4), i.e. the AGM postulates. More importantly, $\star$ satisfies also certain properties that deal with iterated revisions. This is not surprising as the revisions of the different Γ's are all defined by a unique pseudo-distance, which ensures a strong coherence between them. For example, $\star$ satisfies two following properties: $\forall\, \Gamma, \Delta, \{\alpha\}, \{\beta\} \in \mathbf{C}$,

- if $\gamma \in (\Gamma \star \{\alpha\}) \star \Delta$ and $\gamma \in (\Gamma \star \{\beta\}) \star \Delta$, then $\gamma \in (\Gamma \star \{\alpha \vee \beta\}) \star \Delta$;
- if $\gamma \in (\Gamma \star \{\alpha \vee \beta\}) \star \Delta$, then $\gamma \in (\Gamma \star \{\alpha\}) \star \Delta$ or $\gamma \in (\Gamma \star \{\beta\}) \star \Delta$.

These properties are not entailed by the AGM postulates, a counter-example can be found in (Lehmann, Magidor, & Schlechta 2001). But, they seem intuitively justified. Indeed, take three sequences of revisions that differ only at some step in which the new information is α in the first sequence, β in the second, and $\alpha \vee \beta$ in the third. Now, suppose γ is concluded after both the first and the second sequences. Then, it should intuitively be the case that γ is concluded after the third sequence too. Similar arguments can be given for the second property. Now, to characterize the full distance-based revision more is needed. This is discussed in the next section.

Characterizations

The authors of (Lehmann, Magidor, & Schlechta 2001) provided characterizations for families of distance-based revision operators. They proceed in two steps. First, they defined the distance operators, in a very general framework:

Definition 11 Let $\mathcal{V}$ be a set, $\mathbf{V}, \mathbf{W}, \mathbf{X} \subseteq \mathcal{P}(\mathcal{V})$, and $|$ an operator from $\mathbf{V} \times \mathbf{W}$ to $\mathbf{X}$.
$|$ is a *distance operator* iff there exists a pseudo-distance $\mathcal{D}$ on $\mathcal{V}$ such that $\forall\, V \in \mathbf{V}$, $\forall\, W \in \mathbf{W}$, $V|W = V|_{\mathcal{D}}W$.
In addition, if $\mathcal{D}$ is symmetric, HIR, DP, etc., then so is $|$.

Then, they characterized families of such distance operators (with the least possible assumptions about $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{X}$). This is the essence of their work. Here is an example:

Proposition 12 (Lehmann, Magidor, & Schlechta 2001) Suppose $\mathcal{V}$ is a non-empty set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$ (such that $\emptyset \notin \mathbf{V}$ and $\forall\, V, W \in \mathbf{V}$, we have $V \cup W \in \mathbf{V}$ and if $V \cap W \neq \emptyset$, then $V \cap W \in \mathbf{V}$ too), and $|$ an operator from $\mathbf{V} \times \mathbf{V}$ to $\mathbf{V}$. Then, $|$ is a symmetric distance operator iff $\forall\, k \in \mathbb{N}^+$ and $\forall\, V_0, V_1, \ldots, V_k \in \mathbf{V}$, we have $V_0|V_1 \subseteq V_1$ and

$$(|loop)\ \text{ if } \begin{cases} (V_1|(V_0 \cup V_2)) \cap V_0 \neq \emptyset, \\ (V_2|(V_1 \cup V_3)) \cap V_1 \neq \emptyset, \\ \ldots, \\ (V_k|(V_{k-1} \cup V_0)) \cap V_{k-1} \neq \emptyset, \end{cases}$$
then $(V_0|(V_k \cup V_1)) \cap V_1 \neq \emptyset$.

In a second step only, they applied these results to characterize families of distance-based revision operators. For instance, they applied Proposition 12 to get Proposition 13 below. We should say immediately that they chose a classical framework to define the distance-based revision. But, if we choose now another framework, there are quite good chances that Proposition 12 can be still applied, thanks to its algebraic nature.

Proposition 13 (Lehmann, Magidor, & Schlechta 2001) Let $\star$ be an operator from $\mathbf{C} \times \mathbf{C}$ to $\mathcal{P}(\mathcal{F}_c)$.
Then, $\star$ is a symmetric CP DP distance-based revision operator iff $\star$ satisfies $(\star 0)$, $(\star 1)$, $(\star 2)$, and $\forall\, k \in \mathbb{N}^+$, $\forall\, \Gamma_0, \Gamma_1, \ldots, \Gamma_k \in \mathbf{C}$,

$$(\star loop)\ \text{ if } \begin{cases} \Gamma_0 \cup (\Gamma_1 \star (\Gamma_0 \vee \Gamma_2)) \in \mathbf{C}, \\ \Gamma_1 \cup (\Gamma_2 \star (\Gamma_1 \vee \Gamma_3)) \in \mathbf{C}, \\ \ldots, \\ \Gamma_{k-1} \cup (\Gamma_k \star (\Gamma_{k-1} \vee \Gamma_0)) \in \mathbf{C}, \end{cases}$$
then $\Gamma_1 \cup (\Gamma_0 \star (\Gamma_k \vee \Gamma_1)) \in \mathbf{C}$.

Normal characterizations

Let $\mathcal{V}$ be a set, $\mathcal{O}$ a set of binary operators on $\mathcal{P}(\mathcal{V})$, and $|$ a binary operator on $\mathcal{P}(\mathcal{V})$. Approximatively, in (Schlechta 2004), a characterization of $\mathcal{O}$ is called normal iff it contains only conditions which are universally quantified, apply $|$ only a finite number of times, and use only elementary operations (like e.g. $\cup$, $\cap$, $\setminus$), see Section 1.6.2.1 of (Schlechta 2004) for details. Here is an example of such a condition:

$(C1)\ \forall\, V, W \in \mathbf{U} \subseteq \mathcal{P}(\mathcal{V}),\, V|((V \cup W)|W) = \emptyset.$

Now, we introduce a new, more general, definition with an aim of providing more general impossibility results. Approximatively, in the present paper, a characterization of $\mathcal{O}$ will be called normal iff it contains only conditions which are universally quantified and apply $|$ only a finite number of times. Then, the conditions can involve complex structures or functions, etc., we are not limited to elementary operations. More formally:

Definition 14 Suppose $\mathcal{V}$ is a set and $\mathcal{O}$ a set of binary operators on $\mathcal{P}(\mathcal{V})$.
$\mathcal{C}$ is a *normal characterization* of $\mathcal{O}$ iff $\mathcal{C} = \langle n, \Phi \rangle$ where $n \in \mathbb{N}^+$ and Φ is a relation on $\mathcal{P}(\mathcal{V})^{3n}$ such that for every binary operator $|$ on $\mathcal{P}(\mathcal{V})$, we have $| \in \mathcal{O}$ iff
$\forall\, V_1, \ldots, V_n, W_1, \ldots, W_n \subseteq \mathcal{V}$,
$(V_1, \ldots, V_n, W_1, \ldots, W_n, V_1|W_1, \ldots, V_n|W_n) \in \Phi$.

Note that Φ is a relation in the purely set-theoretic sense. Now, suppose there is no normal characterization of $\mathcal{O}$. Here are examples (i.e. $(C1)$, $(C2)$, and $(C3)$ below) that will give the reader a good idea which conditions cannot characterize $\mathcal{O}$. This will therefore make clearer the range of our impossibility results (Propositions 15 and 16 below).

To begin, $(C1)$ cannot characterize $\mathcal{O}$. Indeed, suppose it does, i.e. $| \in \mathcal{O}$ iff $\forall\, V, W \in \mathbf{U}$, $V|((V \cup W)|W) = \emptyset$.
Then, take $n = 4$ and Φ such that
$(V_1, \ldots, V_4, W_1, \ldots, W_4, X_1, \ldots, X_4) \in \Phi$ iff

$$\begin{cases} V_1, V_2 \in \mathbf{U}, \\ V_3 = V_1 \cup V_2, \\ W_3 = V_2, \\ V_4 = V_1, \\ W_4 = X_3 \end{cases} \quad \text{entail } X_4 = \emptyset.$$

Then, $\langle 4, \Phi \rangle$ is a normal characterization of $\mathcal{O}$. We give the easy proof of this, so that the reader can check that a convenient relation Φ can be found immediately for all simple conditions like $(C1)$.

Proof Direction: "$\rightarrow$".
Suppose $| \in \mathcal{O}$.
Then, $\forall\, V, W \in \mathbf{U}$, $V|((V \cup W)|W) = \emptyset$.
Let $V_1, \ldots, V_4, W_1, \ldots, W_4 \subseteq \mathcal{V}$. We show:
$(V_1, \ldots, V_4, W_1, \ldots, W_4, V_1|W_1, \ldots, V_4|W_4) \in \Phi$.
Suppose $V_1, V_2 \in \mathbf{U}$, $V_3 = V_1 \cup V_2$, $W_3 = V_2$, $V_4 = V_1$, and $W_4 = V_3|W_3$.
Then, as $V_1, V_2 \in \mathbf{U}$, we get $V_1|((V_1 \cup V_2)|V_2) = \emptyset$.
But, $V_1|((V_1 \cup V_2)|V_2) = V_1|(V_3|W_3) = V_4|W_4$.

Direction: "$\leftarrow$".
Suppose $\forall\, V_1, \ldots, V_4, W_1, \ldots, W_4 \subseteq \mathcal{V}$,
$(V_1, \ldots, V_4, W_1, \ldots, W_4, V_1|W_1, \ldots, V_4|W_4) \in \Phi$.
We show $| \in \mathcal{O}$. Let $V, W \in \mathbf{U}$.
Take $V_1 = V$, $V_2 = W$, $V_3 = V_1 \cup V_2$, $W_3 = V_2$, $V_4 = V_1$, $W_4 = V_3|W_3$. Take any values for W_1 and W_2.
Then, $V_1 \in \mathbf{U}$, $V_2 \in \mathbf{U}$, $V_3 = V_1 \cup V_2$, $W_3 = V_2$, $V_4 = V_1$, and $W_4 = V_3|W_3$.
But, $(V_1, \ldots, V_4, W_1, \ldots, W_4, V_1|W_1, \ldots, V_4|W_4) \in \Phi$
Therefore, by definition of Φ, $V_4|W_4 = \emptyset$.
But, $V_4|W_4 = V_1|((V_1 \cup V_2)|V_2) = V|((V \cup W)|W)$. ∎

At this point, we excluded all those conditions which are excluded by (the nonexistence of a normal characterization of $\mathcal{O}$ in the sense of) Schlecha, i.e. all conditions like $(C1)$. But actually, more complex conditions are also excluded. For instance, let f be any function from $\mathcal{P}(\mathcal{V})$ to $\mathcal{P}(\mathcal{V})$. Then, the following condition:

$(C2)\ \forall\, V, W \in \mathbf{U},\, f(V)|((V \cup W)|W) = \emptyset.$

cannot characterize $\mathcal{O}$. Indeed, suppose it characterizes $\mathcal{O}$. Then, take $n = 4$ and Φ such that
$(V_1, \ldots, V_4, W_1, \ldots, W_4, X_1, \ldots, X_4) \in \Phi$ iff

$$\left\{\begin{array}{l} V_1, V_2 \in \mathbf{U}, \\ V_3 = V_1 \cup V_2, \\ W_3 = V_2, \\ V_4 = f(V_1), \\ W_4 = X_3 \end{array}\right. \quad \text{entail } X_4 = \emptyset.$$

Then, $\langle 4, \Phi \rangle$ is a normal characterization of $\mathcal{O}$. We leave the easy proof of this to the reader. On the other hand, $(C2)$ is not excluded by Schlechta, if f cannot be constructed from elementary operations. But, even if there exists such a construction, showing that it is indeed the case might well be a difficult problem.

We can even go further combining universal (not existential) quantifiers and functions like f. For instance, suppose $\mathcal{G}$ is a set of functions from $\mathcal{P}(\mathcal{V})$ to $\mathcal{P}(\mathcal{V})$ and consider the following condition:

$(C3)$ $\forall f \in \mathcal{G}, \forall V, W \in \mathbf{U}, f(V)|((V \cup W)|W) = \emptyset.$

Then, $(C3)$ cannot characterize $\mathcal{O}$. Indeed, suppose $(C3)$ characterizes $\mathcal{O}$. Then, take $n = 4$ and Φ such that $(V_1, \ldots, V_4, W_1, \ldots, W_4, X_1, \ldots, X_4) \in \Phi$ iff

$$\forall f \in \mathcal{G}, \text{ if } \left\{\begin{array}{l} V_1, V_2 \in \mathbf{U}, \\ V_3 = V_1 \cup V_2, \\ W_3 = V_2, \\ V_4 = f(V_1), \\ W_4 = X_3, \end{array}\right. \quad \text{then } X_4 = \emptyset.$$

Then, $\langle 4, \Phi \rangle$ is a normal characterization of $\mathcal{O}$. The easy proof is left to the reader. On the other hand, $(C3)$ is not excluded by Schlechta.

Finally, a good example of a condition which is not excluded (neither by us nor by Schlechta) is of course the arbitrary big loop condition $(|loop)$.

Impossibility results

We provide our first impossibility result. It generalizes Proposition 4.2.11 of (Schlechta 2004). Our proof will be based on a slight adaptation of a particular pseudo-distance invented by Schlechta (called "Hamster Wheel").

Proposition 15 Let $\mathcal{V}$ be an infinite set, $\mathcal{N}$ the set of all symmetric IR positive TIR distance operators from $\mathcal{P}(\mathcal{V})^2$ to $\mathcal{P}(\mathcal{V})$, and $\mathcal{O}$ a set of distance operators from $\mathcal{P}(\mathcal{V})^2$ to $\mathcal{P}(\mathcal{V})$ such that $\mathcal{N} \subseteq \mathcal{O}$.
Then, there does not exist a normal characterization of $\mathcal{O}$.

Proof Suppose the contrary, i.e. suppose there is $n \in \mathbb{N}^+$ and a relation Φ on $\mathcal{P}(\mathcal{V})^{3n}$ such that

(0) for every binary operator $|$ on $\mathcal{P}(\mathcal{V})$, we have $| \in \mathcal{O}$ iff
$\forall V_1, \ldots, V_n, W_1, \ldots, W_n \subseteq \mathcal{V}$,
$(V_1, \ldots, V_n, W_1, \ldots, W_n, V_1|W_1, \ldots, V_n|W_n) \in \Phi$.

As $\mathcal{V}$ is infinite, there are distinct $v_1, \ldots, v_m$, $w_1, \ldots, w_m$ in $\mathcal{V}$, with $m = n + 3$.
Let $X = \{v_1, \ldots, v_m, w_1, \ldots, w_m\}$.
Let $\mathcal{D}$ be the pseudo-distance on $\mathcal{V}$ such that $\mathcal{D} = \langle \mathbb{R}, <, d \rangle$, where $<$ is the usual order on $\mathbb{R}$ and d is the function defined as follows. Let $v, w \in \mathcal{V}$. Consider the cases that follow:
Case 1: $v = w$.
Case 2: $v \neq w$.
Case 2.1: $\{v, w\} \not\subseteq X$.
Case 2.2: $\{v, w\} \subseteq X$.
Case 2.2.1: $\{v, w\} \subseteq \{v_1, \ldots, v_m\}$.
Case 2.2.2: $\{v, w\} \subseteq \{w_1, \ldots, w_m\}$.
Case 2.2.3: $\exists\, i, j \in [1, m],\ \{v, w\} = \{v_i, w_j\}$.
Case 2.2.3.1: $i = j$.
Case 2.2.3.2: $abs(i - j) \in \{1, m - 1\}$.
Case 2.2.3.3: $1 < abs(i - j) < m - 1$.
Then,

$$d(v, w) = \left\{\begin{array}{ll} 0 & \text{if Case 1 holds;} \\ 1 & \text{if Case 2.1 holds;} \\ 1.1 & \text{if Case 2.2.1 holds;} \\ 1.1 & \text{if Case 2.2.2 holds;} \\ 1.4 & \text{if Case 2.2.3.1 holds;} \\ 2 & \text{if Case 2.2.3.2 holds;} \\ 1.2 & \text{if Case 2.2.3.3 holds.} \end{array}\right.$$

Note that $\mathcal{D}$ is essentially, but not exactly, the Hamster Wheel of (Schlechta 2004). The main difference is Case 2.1, which was not treated by Schlechta. The reader can find a picture of $\mathcal{D}$ in Figure 1.

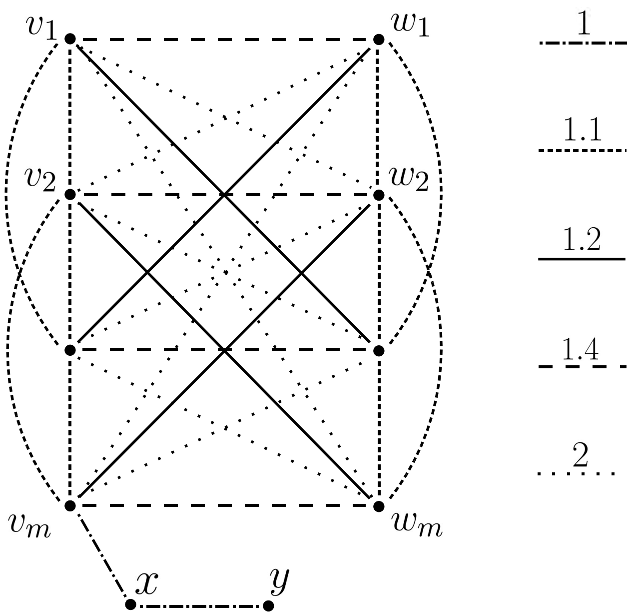

Figure 1: A slight adaptation of Hamster Wheel.

Let $|$ be the binary operator on $\mathcal{P}(\mathcal{V})$ such that $\forall V, W \subseteq \mathcal{V}$,

$$V|W = \left\{\begin{array}{ll} \{w_m\} & \text{if } V = \{v_m, v_1\}, W = \{w_m, w_1\}; \\ \{v_m\} & \text{if } V = \{w_m, w_1\}, W = \{v_m, v_1\}; \\ V|_{\mathcal{D}}W & \text{otherwise.} \end{array}\right.$$

The difference between $|$ and $|_{\mathcal{D}}$ is strong enough so that:

(1) $|$ is not a distance operator.

The proof will be given later. Thus, $| \notin \mathcal{O}$. Thus, by (0):

(2) $\exists V_1, \ldots, V_n, W_1, \ldots, W_n \subseteq \mathcal{V}$,
$(V_1, \ldots, V_n, W_1, \ldots, W_n, V_1|W_1, \ldots, V_n|W_n) \notin \Phi$.

In addition, we took m sufficiently big so that:

(3) $\exists\, r \in [1, m-1]$ such that
$\forall i \in [1, n], \{V_i, W_i\} \neq \{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}$.

We will give the proof later.
Let $|'$ be the binary operator on $\mathcal{P}(\mathcal{V})$ such that $\forall\, V, W \subseteq \mathcal{V}$,

$$V|'W = \begin{cases} \{w_{r+1}\} & \text{if } V = \{v_r, v_{r+1}\}, W = \{w_r, w_{r+1}\}; \\ \{v_{r+1}\} & \text{if } V = \{w_r, w_{r+1}\}, W = \{v_r, v_{r+1}\}; \\ V|W & \text{otherwise.} \end{cases}$$

The difference between $|'$ and $|$ is "invisible" for Φ.
More formally, $\forall\, i \in [1, n]$, $V_i|'W_i = V_i|W_i$.
The proof of this is obvious by (3).
Therefore, by (2), we get:
$(V_1, \ldots, V_n, W_1, \ldots, W_n, V_1|'W_1, \ldots, V_n|'W_n) \notin \Phi$.
Thus, by (0), we obtain:

(4) $|' \notin \mathcal{O}$.

But, at the same time, there is a convenient pseudo-distance that represents $|'$. Indeed, let $\mathcal{D}'$ be the pseudo-distance on $\mathcal{V}$ such that $\mathcal{D}' = \langle \mathbb{R}, <, d' \rangle$, where d' is the function such that $\forall\, v, w \in \mathcal{V}$,

$$d'(v, w) = \begin{cases} 1.3 & \text{if } \exists i \in [r+1, m], \{v, w\} = \{v_i, w_i\}; \\ d(v, w) & \text{otherwise.} \end{cases}$$

Then, we will show:

(5) $|' = |_{\mathcal{D}'}$.

But, $\mathcal{D}'$ is obviously symmetric, IR, and positive.
In addition, $\mathcal{D}'$ is TIR, because $\mathcal{D}'$ is IR and
$\forall\, v, w \in \mathcal{V}$, $d'(v, w) = 0$ or $1 \leq d'(v, w) \leq 2$.
Thus, $|'$ is a symmetric IR positive TIR distance operator.
Consequently, $|' \in \mathcal{N}$ and thus

(6) $|' \in \mathcal{O}$.

So, we get a final contradiction by (4) and (6).

Proof of (1). Suppose the contrary, i.e. suppose there is a pseudo-distance $\mathcal{S} = \langle C, \prec, g \rangle$ on $\mathcal{V}$ such that $| = |_{\mathcal{S}}$.
Then, we will show:
(1.1) $\quad \forall\, i \in [1, m-1]$, $g(v_i, w_i) = g(v_{i+1}, w_{i+1})$.
On the other hand, we will show:
(1.2) $\quad g(v_m, w_m) \prec g(v_1, w_1)$.
But, by (1.1) and (1.2), we get an obvious contradiction.

Proof of (1.1). Suppose $i \in [1, m-1]$. Then:
$\{v_i, v_{i+1}\}|_{\mathcal{S}}\{w_i, w_{i+1}\} = \{v_i, v_{i+1}\}|_{\mathcal{D}}\{w_i, w_{i+1}\} = \{w_i, w_{i+1}\}$.
Case 1: $g(v_i, w_i) \prec g(v_{i+1}, w_{i+1})$.
We have $\{v_i\}|_{\mathcal{S}}\{w_i, w_{i+1}\} = \{v_i\}|_{\mathcal{D}}\{w_i, w_{i+1}\} = \{w_i\}$.
Thus, $w_{i+1} \notin \{v_i\}|_{\mathcal{S}}\{w_i, w_{i+1}\}$.
Therefore, $g(v_i, w_i) \prec g(v_i, w_{i+1})$.
Thus, $w_{i+1} \notin \{v_i, v_{i+1}\}|_{\mathcal{S}}\{w_i, w_{i+1}\}$, which is impossible.
Case 2: $g(v_{i+1}, w_{i+1}) \prec g(v_i, w_i)$.
We have $\{v_{i+1}\}|_{\mathcal{S}}\{w_i, w_{i+1}\} = \{v_{i+1}\}|_{\mathcal{D}}\{w_i, w_{i+1}\} = \{w_{i+1}\}$.
Therefore, $w_i \notin \{v_{i+1}\}|_{\mathcal{S}}\{w_i, w_{i+1}\}$.
Consequently, $g(v_{i+1}, w_{i+1}) \prec g(v_{i+1}, w_i)$.
Thus, $w_i \notin \{v_i, v_{i+1}\}|_{\mathcal{S}}\{w_i, w_{i+1}\}$, which is impossible.
Case 3: $g(v_i, w_i) \not\prec g(v_{i+1}, w_{i+1})$ and $g(v_{i+1}, w_{i+1}) \not\prec g(v_i, w_i)$.
Then, as $\prec$ is total, $g(v_i, w_i) = g(v_{i+1}, w_{i+1})$.

Proof of (1.2). We have $\{v_m, v_1\}|_{\mathcal{S}}\{w_m, w_1\} = \{v_m, v_1\}|\{w_m, w_1\} = \{w_m\}$.
Therefore, $w_1 \notin \{v_m, v_1\}|_{\mathcal{S}}\{w_m, w_1\}$. Thus:
$\exists\, v \in \{v_m, v_1\}$, $\exists\, w \in \{w_m, w_1\}$, $g(v, w) \prec g(v_1, w_1)$.
Case 1: $g(v_m, w_m) \prec g(v_1, w_1)$. We are done.
Case 2: $g(v_m, w_1) \prec g(v_1, w_1)$.
We have $\{v_m\}|_{\mathcal{S}}\{w_m, w_1\} = \{v_m\}|_{\mathcal{D}}\{w_m, w_1\} = \{w_m\}$.
Therefore, $w_1 \notin \{v_m\}|_{\mathcal{S}}\{w_m, w_1\}$.
Thus, $g(v_m, w_m) \prec g(v_m, w_1)$.
Thus, by transitivity of $\prec$, $g(v_m, w_m) \prec g(v_1, w_1)$.
Case 3: $g(v_1, w_m) \prec g(v_1, w_1)$.
Then, $\{v_1\}|_{\mathcal{S}}\{w_m, w_1\} = \{w_m\}$.
However, $\{v_1\}|_{\mathcal{S}}\{w_m, w_1\} = \{v_1\}|_{\mathcal{D}}\{w_m, w_1\} = \{w_1\}$, which is impossible.
Case 4: $g(v_1, w_1) \prec g(v_1, w_1)$.
Impossible by irreflexivity of $\prec$.

Proof of (3). For all $s \in [1, m-1]$, define:
$I_s := \{i \in [1, n] : \{V_i, W_i\} = \{\{v_s, v_{s+1}\}, \{w_s, w_{s+1}\}\}\}$.
Suppose the opposite of what we want to show, i.e. suppose $\forall\, s \in [1, m-1]$, $I_s \neq \emptyset$.
As $v_1, \ldots, v_m, w_1, \ldots, w_m$ are distinct, $\forall\, s, t \in [1, m-1]$, if $s \neq t$, then $I_s \cap I_t = \emptyset$.
Therefore, $m - 1 \leq |I_1 \cup \ldots \cup I_{m-1}|$.
On the other hand, $\forall\, s \in [1, m-1]$, $I_s \subseteq [1, n]$.
Thus, $|I_1 \cup \ldots \cup I_{m-1}| \leq n$.
Thus, $m - 1 \leq n$, which is impossible as $m = n + 3$.

Proof of (5). Let $V, W \subseteq \mathcal{V}$.
Case 1: $V = \{v_r, v_{r+1}\}$ and $W = \{w_r, w_{r+1}\}$.
Then, $V|'W = \{w_{r+1}\} = V|_{\mathcal{D}'}W$.
Case 2: $V = \{w_r, w_{r+1}\}$ and $W = \{v_r, v_{r+1}\}$.
Then, $V|'W = \{v_{r+1}\} = V|_{\mathcal{D}'}W$.
Case 3: $V = \{v_m, v_1\}$ and $W = \{w_m, w_1\}$.
Then, $V|'W = V|W = \{w_m\} = V|_{\mathcal{D}'}W$.
Case 4: $V = \{w_m, w_1\}$ and $W = \{v_m, v_1\}$.
Then, $V|'W = V|W = \{v_m\} = V|_{\mathcal{D}'}W$.
Case 5: $\{V, W\} \notin$
$\{\{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}, \{\{v_m, v_1\}, \{w_m, w_1\}\}\}$.
Then, $V|'W = V|W = V|_{\mathcal{D}}W$.
Case 5.1: $V = \emptyset$ or $W = \emptyset$.
Then, $V|_{\mathcal{D}}W = \emptyset = V|_{\mathcal{D}'}W$.
Case 5.2: $V \cap W \neq \emptyset$.
Then, $V|_{\mathcal{D}}W = V \cap W = V|_{\mathcal{D}'}W$.
Case 5.3: $V \neq \emptyset$, $W \neq \emptyset$, and $V \cap W = \emptyset$.
Case 5.3.1: $V \not\subseteq X$.
Then, $V|_{\mathcal{D}}W = W = V|_{\mathcal{D}'}W$.
Case 5.3.2: $V \subseteq X$.
Case 5.3.2.1: $W \not\subseteq X$.
Then, $V|_{\mathcal{D}}W = W \setminus X = V|_{\mathcal{D}'}W$.
Case 5.3.2.2: $W \subseteq X$.
Case 5.3.2.2.1: $V \not\subseteq \{v_1, \ldots, v_m\}$ and $V \not\subseteq \{w_1, \ldots, w_m\}$.
Then, $V|_{\mathcal{D}}W = W = V|_{\mathcal{D}'}W$.
Case 5.3.2.2.2: $V \subseteq \{v_1, \ldots v_m\}$ and $W \not\subseteq \{w_1, \ldots w_m\}$.
Then, $V|_{\mathcal{D}}W = W \cap \{v_1, \ldots, v_m\} = V|_{\mathcal{D}'}W$.
Case 5.3.2.2.3: $V \subseteq \{v_1, \ldots v_m\}$ and $W \subseteq \{w_1, \ldots w_m\}$.
Case 5.3.2.2.3.1: $\exists\, v_i \in V$, $\exists\, w_j \in W$,
$1 < abs(i - j) < m - 1$.
Then, $V|_{\mathcal{D}}W =$

$\{w_j \in W : \exists\, v_i \in V, 1 < abs(i-j) < m-1\} = V|_{\mathcal{D}'}W$.
Case 5.3.2.2.3.2: $\forall\, v_i \in V$, $\forall\, w_j \in W$,
$abs(i-j) \in \{0, 1, m-1\}$.
Case 5.3.2.2.3.2.1: $|V \cup W| \geq 5$.
As $m \geq 4$, $\exists\, v_i \in V$, $\exists\, w_j \in W$, $1 < abs(i-j) < m-1$, which is impossible.
Case 5.3.2.2.3.2.2: $|V \cup W| \in \{2, 3, 4\}$.
Case 5.3.2.2.3.2.2.1: $\{k \in [1, m] : v_k \in V, w_k \in W\} = \emptyset$.
Then, $V|_{\mathcal{D}}W = W = V|_{\mathcal{D}'}W$.
Case 5.3.2.2.3.2.2.2: $\exists\, i \in [1, m]$ such that
$\{k \in [1, m] : v_k \in V, w_k \in W\} = \{i\}$.
Then, $V|_{\mathcal{D}}W = \{w_i\} = V|_{\mathcal{D}'}W$.
Case 5.3.2.2.3.2.2.3: $\exists\, i, j \in [1, m]$ such that $i < j$ and
$\{k \in [1, m] : v_k \in V$ and $w_k \in W\} = \{i, j\}$.
Then, $V = \{v_i, v_j\}$ and $W = \{w_i, w_j\}$.
Case 5.3.2.2.3.2.2.3.1: $r < i$ or $j \leq r$.
Then, $V|_{\mathcal{D}}W = \{w_i, w_j\} = V|_{\mathcal{D}'}W$.
Case 5.3.2.2.3.2.2.3.2: $i \leq r < j$.
We have $abs(i-j) \in \{1, m-1\}$. Thus, $\langle V,\ W\rangle \in \{\langle\{v_r,\ v_{r+1}\},\ \{w_r,\ w_{r+1}\}\rangle, \langle\{v_1,\ v_m\},\ \{w_1,\ w_m\}\rangle\}$, which is impossible.
Case 5.3.2.2.3.2.2.4: $|\{k \in [1, m] : v_k \in V, w_k \in W\}| \geq 3$.
Then, $|V \cup W| \geq 6$, which is impossible.
Case 5.3.2.2.4: $V \subseteq \{w_1, \ldots w_m\}$ and $W \not\subseteq \{v_1, \ldots v_m\}$.
Then, $V|_{\mathcal{D}}W = W \cap \{w_1, \ldots, w_m\} = V|_{\mathcal{D}'}W$.
Case 5.3.2.2.5: $V \subseteq \{w_1, \ldots w_m\}$ and $W \subseteq \{v_1, \ldots v_m\}$.
Similar to Case 5.3.2.2.3. ∎

We extend the negative results to the "liberal" and Hamming properties. The proof will be based on an adaptation of the Hamster Wheel. Note that the Hamming distance is a realistic distance which has been investigated by many researchers. This strengthen the importance of Proposition 16 below in the sense that not only abstract but also concrete cases do not admit a normal characterization.

Proposition 16 Let $\mathcal{L} = \langle \mathcal{A}, \mathcal{C}\rangle$ be a propositional language with $\mathcal{A}$ infinite and countable, $\mathcal{M}$ a matrix on $\mathcal{L}$, $\mathcal{V}$ the set of all $\mathcal{M}$-valuations, $\mathcal{N}$ the set of all symmetric, HIR, liberally IR, liberally positive, and liberally TIR distance operators from $\mathcal{P}(\mathcal{V})^2$ to $\mathcal{P}(\mathcal{V})$, and $\mathcal{O}$ a set of distance operators from $\mathcal{P}(\mathcal{V})^2$ to $\mathcal{P}(\mathcal{V})$ such that $\mathcal{N} \subseteq \mathcal{O}$.
Then, there does not exist a normal characterization of $\mathcal{O}$.

Proof Suppose the contrary, i.e. suppose there are $n \in \mathbb{N}^+$ and a relation Φ on $\mathcal{P}(\mathcal{V})^{3n}$ such that

(0) for every binary operator $|$ on $\mathcal{P}(\mathcal{V})$, we have $| \in \mathcal{O}$ iff
$\forall\, V_1, \ldots, V_n, W_1, \ldots, W_n \subseteq \mathcal{V}$,
$(V_1, \ldots, V_n, W_1, \ldots, W_n, V_1|W_1, \ldots, V_n|W_n) \in \Phi$.

As $\mathcal{A}$ is infinite, there are distinct $p_1, \ldots, p_m, q_1, \ldots, q_m$ in $\mathcal{A}$, with $m = n + 3$.
Let's pose $\mathcal{M} = \langle T, D, f\rangle$.
As $D \neq \emptyset$ and $T \setminus D \neq \emptyset$, there are distinct $0, 1 \in T$.
Now, $\forall\, i \in [1, m]$, let v_i be the $\mathcal{M}$-valuation that assigns 1 to p_i and 0 to each other atom of $\mathcal{A}$.
Similarly, $\forall\, i \in [1, m]$, let w_i be the $\mathcal{M}$-valuation that assigns 1 to q_i and 0 to each other atom of $\mathcal{A}$.
Let $X = \{v_1, \ldots, v_m, w_1 \ldots, w_m\}$.
Note that $\forall\, v, w \in X$, with $v \neq w$, we have $|h(v, w)| = 2$.
Finally, let $\mathcal{D}$ be the pseudo-distance on $\mathcal{V}$ such that $\mathcal{D} = \langle \mathbb{R} \cup \{|\mathbb{N}|\}, \prec, d\rangle$, where $\prec$ and d are defined as follows.
Let $c, c' \in \mathbb{R} \cup \{|\mathbb{N}|\}$. Then, $c \prec c'$ iff ($c, c' \in \mathbb{R}$ and $c < c'$) or ($c \in \mathbb{R}$ and $c' = |\mathbb{N}|$).
Let $v, w \in \mathcal{V}$ and consider the cases which follow:
Case 1: $v = w$.
Case 2: $v \neq w$.
Case 2.1: $\{v, w\} \not\subseteq X$.
Case 2.1.1: $|h(v, w)| = 1$.
Case 2.1.2: $|h(v, w)| \geq 2$.
Case 2.2: $\{v, w\} \subseteq X$.
Case 2.2.1: $\{v, w\} \subseteq \{v_1, \ldots, v_m\}$.
Case 2.2.2: $\{v, w\} \subseteq \{w_1, \ldots, w_m\}$.
Case 2.2.3: $\exists\, i, j \in [1, m],\ \{v, w\} = \{v_i, w_j\}$.
Case 2.2.3.1: $i = j$.
Case 2.2.3.2: $abs(i-j) \in \{1, m-1\}$.
Case 2.2.3.3: $1 < abs(i-j) < m-1$.
Then,

$$d(v, w) = \begin{cases} 0 & \text{if Case 1 holds;} \\ 1.4 & \text{if Case 2.1.1 holds;} \\ |h(v, w)| & \text{if Case 2.1.2 holds;} \\ 2.1 & \text{if Case 2.2.1 holds;} \\ 2.1 & \text{if Case 2.2.2 holds;} \\ 2.4 & \text{if Case 2.2.3.1 holds;} \\ 2.5 & \text{if Case 2.2.3.2 holds;} \\ 2.2 & \text{if Case 2.2.3.3 holds.} \end{cases}$$

Note that $\mathcal{D}$ is an adaptation of the Hamster Wheel of (Schlechta 2004). The reader can find a picture of $\mathcal{D}$ in Figure 2.

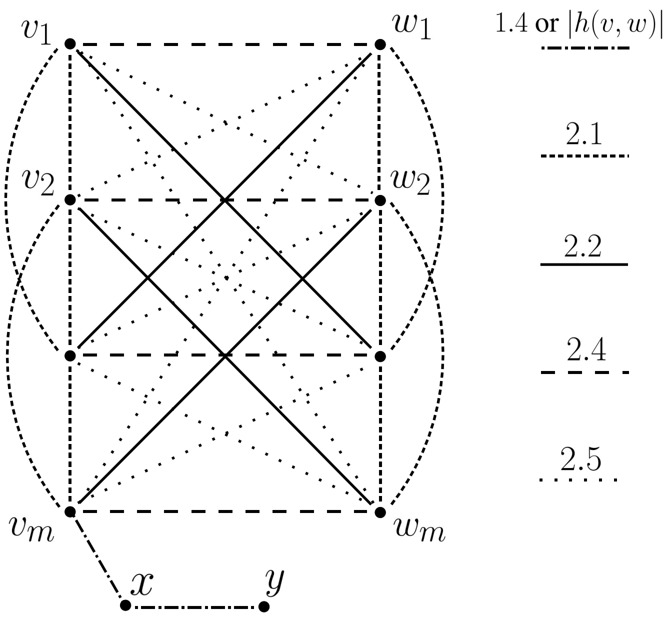

Figure 2: An adaptation of Hamster Wheel.

Let $|$ be the binary operator on $\mathcal{P}(\mathcal{V})$ defined as follows.
Let $V, W \subseteq \mathcal{V}$ and consider the cases that follow:
Case 1: $\forall v \in V, \forall w \in W, \{v, w\} \subseteq X$ or $3 \leq |h(v, w)|$.
Case 1.1: $V \cap X = \{v_m, v_1\}$ and $W \cap X = \{w_m, w_1\}$.

Case 1.2: $V \cap X = \{w_m, w_1\}$ and $W \cap X = \{v_m, v_1\}$.
Case 1.3: $\{V \cap X, W \cap X\} \neq \{\{v_m, v_1\}, \{w_m, w_1\}\}$.
Case 2: $\exists\, v \in V, \exists\, w \in W, \{v, w\} \not\subseteq X$ and $|h(v, w)| < 3$.
Then,

$$V|W = \begin{cases} \{w_m\} & \text{if Case 1.1 holds;} \\ \{v_m\} & \text{if Case 1.2 holds;} \\ V|_{\mathcal{D}}W & \text{if Case 1.3 or Case 2 holds.} \end{cases}$$

The difference between $|$ and $|_{\mathcal{D}}$ is sufficiently strong so that $|$ is not a distance operator. The proof is verbatim the same as for (1) in the proof of Proposition 15.
Consequently, $| \notin \mathcal{O}$, thus, by (0), we get that

(1) $\exists V_1, \ldots, V_n, W_1, \ldots, W_n \subseteq \mathcal{V}$,
$(V_1, \ldots, V_n, W_1, \ldots, W_n, V_1|W_1, \ldots, V_n|W_n) \notin \Phi$.

Moreover, we chose m sufficiently big so that:

(2) $\exists\, r \in [1, m-1], \forall\, i \in [1, n]$,
$\{V_i \cap X, W_i \cap X\} \neq \{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}$.

The proof is verbatim the same as for (3) in the proof of Proposition 15, except that V_i and W_i are replaced by $V_i \cap X$ and $W_i \cap X$.
Let $|'$ be the binary operator on $\mathcal{P}(\mathcal{V})$ defined as follows.
Let $V, W \subseteq \mathcal{V}$ and consider the cases that follow:
Case 1: $\forall v \in V, \forall w \in W, \{v, w\} \subseteq X$ or $3 \leq |h(v, w)|$.
Case 1.1: $V \cap X = \{v_r, v_{r+1}\}$ and $W \cap X = \{w_r, w_{r+1}\}$.
Case 1.2: $V \cap X = \{w_r, w_{r+1}\}$ and $W \cap X = \{v_r, v_{r+1}\}$.
Case 1.3: $\{V \cap X, W \cap X\} \neq \{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}$.
Case 2: $\exists\, v \in V, \exists\, w \in W, \{v, w\} \not\subseteq X$ and $|h(v, w)| < 3$.
Then,

$$V|'W = \begin{cases} \{w_{r+1}\} & \text{if Case 1.1 holds;} \\ \{v_{r+1}\} & \text{if Case 1.2 holds;} \\ V|W & \text{if Case 1.3 or Case 2 holds.} \end{cases}$$

The difference between $|'$ and $|$ is "invisible" for Φ.
More formally, $\forall\, i \in [1, n]$, $V_i|'W_i = V_i|W_i$.
The proof is obvious by (2). Thus, by (1), we get:
$(V_1, \ldots, V_n, W_1, \ldots, W_n, V_1|'W_1, \ldots, V_n|'W_n) \notin \Phi$.
Therefore, by (0), we get:

(3) $|' \notin \mathcal{O}$.

But, in parallel, there is a convenient pseudo-distance that represents $|'$. Indeed, let $\mathcal{D}'$ be the pseudo-distance on $\mathcal{V}$ such that $\mathcal{D}' = \langle \mathbb{R} \cup \{|\mathbb{N}|\}, \prec, d' \rangle$, where d' is the function such that $\forall\, v, w \in \mathcal{V}$,

$$d'(v, w) = \begin{cases} 2.3 & \text{if } \exists i \in [r+1, m], \{v, w\} = \{v_i, w_i\}; \\ d(v, w) & \text{otherwise.} \end{cases}$$

Note that $\forall\, v, w \in \mathcal{V}$, we have:
$|h(v, w)| \in \mathbb{N}$ iff $d(v, w) \in \mathbb{R}$ iff $d'(v, w) \in \mathbb{R}$.
Thus, $|h(v, w)| = |\mathbb{N}|$ iff $d(v, w) = |\mathbb{N}|$ iff $d'(v, w) = |\mathbb{N}|$.
Note again that $\forall\, v, w \in \mathcal{V}$, with $|h(v, w)| \in \mathbb{N}$, we have:
$|h(v, w)| \leq d'(v, w) \leq d(v, w) \leq |h(v, w)| + 0.5$.
We will show:

(4) $|' = |_{\mathcal{D}'}$.

But, $\mathcal{D}'$ is symmetric, liberally IR, and liberally positive.
In addition, we will show:

(5) $\mathcal{D}'$ is HIR;

(6) $\mathcal{D}'$ is liberally TIR.

So, $|'$ is a symmetric, liberally IR, liberally positive, liberally TIR, and HIR distance operator.
Therefore, $|' \in \mathcal{N}$ and thus:

(7) $|' \in \mathcal{O}$.

Finally, (3) and (7) entail a contradiction.

Proof of (4). Let $V, W \subseteq \mathcal{V}$.
Case 1: $\forall v \in V, \forall w \in W, \{v, w\} \subseteq X$ or $3 \leq |h(v, w)|$.
Case 1.1: $V \cap X = \{v_r, v_{r+1}\}$ and $W \cap X = \{w_r, w_{r+1}\}$.
Then, $V|'W = \{w_{r+1}\} = V|_{\mathcal{D}'}W$.
Case 1.2: $V \cap X = \{w_r, w_{r+1}\}$ and $W \cap X = \{v_r, v_{r+1}\}$.
Then, $V|'W = \{v_{r+1}\} = V|_{\mathcal{D}'}W$.
Case 1.3: $V \cap X = \{v_m, v_1\}$ and $W \cap X = \{w_m, w_1\}$.
Then, $V|'W = \{w_m\} = V|_{\mathcal{D}'}W$.
Case 1.4: $V \cap X = \{w_m, w_1\}$ and $W \cap X = \{v_m, v_1\}$.
Then, $V|'W = \{v_m\} = V|_{\mathcal{D}'}W$.
Case 1.5: $\{V \cap X, W \cap X\} \notin$
$\{\{\{v_m, v_1\}, \{w_m, w_1\}\}, \{\{v_r, v_{r+1}\}, \{w_r, w_{r+1}\}\}\}$.
Then, $V|'W = V|W = V|_{\mathcal{D}}W$.
Case 1.5.1: $V \cap W \neq \emptyset$.
Then, $V|_{\mathcal{D}}W = V \cap W = V|_{\mathcal{D}'}W$.
Case 1.5.2: $V \cap W = \emptyset$.
Case 1.5.2.1: $V \cap X = \emptyset$ or $W \cap X = \emptyset$.
Then, $\forall\, v \in V, \forall\, w \in W, d'(v, w) = d(v, w)$.
Therefore, $V|_{\mathcal{D}}W = V|_{\mathcal{D}'}W$.
Case 1.5.2.2: $V \cap X \neq \emptyset$ and $W \cap X \neq \emptyset$.
Then, we will show:

(4.1) $V|_{\mathcal{D}}W = V \cap X|_{\mathcal{D}}W \cap X$;

(4.2) $V|_{\mathcal{D}'}W = V \cap X|_{\mathcal{D}'}W \cap X$.

But, we have $V \cap X|_{\mathcal{D}}W \cap X = V \cap X|_{\mathcal{D}'}W \cap X$.
The proof of this is verbatim the same as for Case 5.3.2.2, in the proof of (5), in the proof of Proposition 15, except that V and W are replaced by $V \cap X$ and $W \cap X$.
Case 2: $\exists\, v \in V, \exists\, w \in W, \{v, w\} \not\subseteq X$ and $|h(v, w)| \leq 2$.
Then, $V|'W = V|W = V|_{\mathcal{D}}W$.
Case 2.1. $V \cap W \neq \emptyset$.
Then, $V|_{\mathcal{D}}W = V \cap W = V|_{\mathcal{D}'}W$.
Case 2.2. $V \cap W = \emptyset$.
Case 2.2.1. $\exists\, v' \in V, \exists\, w' \in W, |h(v, w)| = 1$.
Then, $V|_{\mathcal{D}}W = \{w \in W : \exists\, v \in V, |h(v, w)| = 1\} = V|_{\mathcal{D}'}W$.
Case 2.2.2. $\forall\, v' \in V, \forall\, w' \in W, |h(v, w)| \geq 2$.
Then, $V|_{\mathcal{D}}W =$
$\{w \in W : \exists\, v \in V, \{v, w\} \not\subseteq X$ and $|h(v, w)| = 2\} = V|_{\mathcal{D}'}W$.

Proof of (4.1). Direction: "$\subseteq$".
Let $w \in V|_{\mathcal{D}}W$.
Then, $\exists\, v \in V, \forall\, v' \in V, \forall\, w' \in W, d(v, w) \preceq d(v', w')$.
Case 1: $\{v, w\} \subseteq X$.
Then, $w \in V \cap X|_{\mathcal{D}}W \cap X$.
Case 2: $\{v, w\} \not\subseteq X$.
We have $\exists\, v' \in V \cap X$ and $\exists\, w' \in W \cap X$.
In addition, $d(v', w') \in \mathbb{R}$ and $d(v', w') \leq 2.5$.
Case 2.1: $|h(v, w)| = |\mathbb{N}|$.
Then, $d(v, w) = |\mathbb{N}|$.

Therefore, $d(v', w') \prec d(v, w)$, which is impossible.
Case 2.2: $|h(v, w)| \in \mathbb{N}$.
Then, $d(v, w) \in \mathbb{R}$ and $3 \leq |h(v, w)| \leq d(v, w)$.
Therefore, $d(v', w') < d(v, w)$.
Thus, $d(v', w') \prec d(v, w)$, which is impossible.

Direction: "$\supseteq$".
Let $w \in V \cap X|_{\mathcal{D}} W \cap X$.
Then, $\exists\, v \in V \cap X$ such that
$\forall\, v' \in V \cap X, \forall\, w' \in W \cap X$, $d(v, w) \preceq d(v', w')$.
Let $v' \in V$, $w' \in W$.
Case 1: $\{v', w'\} \subseteq X$.
Then, $d(v, w) \preceq d(v', w')$.
Case 2: $\{v', w'\} \not\subseteq X$.
As $v, w \in X$, we have $d(v, w) \in \mathbb{R}$ and $d(v, w) \leq 2.5$.
Case 2.1: $|h(v', w')| = |\mathbb{N}|$.
Then, $d(v', w') = |\mathbb{N}|$. Thus, $d(v, w) \prec d(v', w')$.
Case 2.2: $|h(v', w')| \in \mathbb{N}$.
Then, $d(v', w') \in \mathbb{R}$ and $3 \leq |h(v', w')| \leq d(v', w')$.
Therefore, $d(v, w) < d(v', w')$.
Thus, $d(v, w) \prec d(v', w')$.
Consequently, in all cases, $d(v, w) \preceq d(v', w')$.
Thus, $w \in V|_{\mathcal{D}} W$.

Proof of (4.2). Verbatim the proof of (4.1), except that $|_{\mathcal{D}}$ and d are replaced by $|_{\mathcal{D}'}$ and d'.

Proof of (5). Let $v, w, x \in \mathcal{V}$ with $|h(v, w)| < |h(v, x)|$.
Case 1: $|h(v, x)| = |\mathbb{N}|$.
Then, $|h(v, w)| \in \mathbb{N}$.
Thus, $d'(v, w) \in \mathbb{R}$ and $d'(v, x) = |\mathbb{N}|$.
Therefore, $d'(v, w) \prec d'(v, x)$.
Case 2: $|h(v, x)| \in \mathbb{N}$.
Then, $|h(v, w)| \in \mathbb{N}$.
Therefore, $d'(v, x) \in \mathbb{R}$, $d'(v, w) \in \mathbb{R}$, and $d'(v, w) \leq |h(v, w)| + 0.5 < |h(v, w)| + 1 \leq |h(v, x)| \leq d'(v, x)$.
Thus, $d'(v, w) \prec d'(v, x)$.

Proof of (6). Let $v, w, x \in \mathcal{V}$.
Note that $h(v, x) \subseteq h(v, w) \cup h(w, x)$.
Therefore, $|h(v, x)| \leq |h(v, w) \cup h(w, x)|$.
Case 1: $d'(v, x) = |\mathbb{N}|$.
Then, $|h(v, x)| = |\mathbb{N}|$.
Now, suppose $d'(v, w) \in \mathbb{R}$ and $d'(w, x) \in \mathbb{R}$.
Then, $|h(v, w)|, |h(w, x)| \in \mathbb{N}$.
Thus, $|h(v, w) \cup h(w, x)| \in \mathbb{N}$.
Therefore, $|h(v, x)| \in \mathbb{N}$, which is impossible.
Thus, $d'(v, w) = |\mathbb{N}|$ or $d'(w, x) = |\mathbb{N}|$.
Case 2: $d'(v, x), d'(v, w), d'(w, x) \in \mathbb{R}$.
Case 2.1: $|h(v, w)| = 0$ or $|h(w, x)| = 0$. Trivial.
Case 2.2: $|h(v, w)| \geq 1$ and $|h(w, x)| \geq 1$.
Case 2.2.1: $|h(v, w)| \geq 2$ or $|h(w, x)| \geq 2$.
Case 2.2.1.1: $|h(v, x)| \in \{0, 1, 2\}$.
Then, $d'(v, x) \leq |h(v, x)| + 0.5 \leq 2.5 < 3 \leq |h(v, w)| + |h(w, x)| \leq d'(v, w) + d'(w, x)$.
Case 2.2.1.2: $|h(v, x)| \geq 3$.
Then, $d'(v, x) = |h(v, x)| \leq |h(v, w)| + |h(w, x)| \leq d'(v, w) + d'(w, x)$.
Case 2.2.2: $|h(v, w)| = 1$ and $|h(w, x)| = 1$.
Case 2.2.2.1: $|h(v, x)| \in \{0, 1, 2\}$.
Then, $d'(v, x) \leq |h(v, x)| + 0.5 \leq 2.5 < 1.4 + 1.4 = d'(v, w) + d'(w, x)$.
Case 2.2.2.2: $|h(v, x)| \geq 3$.
Then, $|h(v, x)| > |h(v, w)| + |h(w, x)|$, impossible. ∎

Conclusion

We laid the focus on the question to know whether $(\star loop)$ can be replaced by a finite condition in Proposition 13. Obviously, the presence of $(\star loop)$ is due to the presence of $(|loop)$. So, to solve the problem one might attack its source, i.e. try to replace $(|loop)$ by a finite condition in Proposition 12. But, we showed in the present paper that for families of distance operators, there is no normal characterization. The symmetric family is concerned with this and therefore $(|loop)$ cannot be replaced by a finite and universally quantified condition.

Now, we can go further. Indeed, there is a strong connexion between the distance operators and the distance-based revision operators. Lehmann *et al.* used this connexion to get their results on the latter from their results on the former. It is reasonable to think that the same thing can be done with our negative results, i.e this paper can certainly be continued in future work to show that for families of distance-based revision operators, there is no normal characterization either. For instance, the family which is symmetric, CP, and DP might well be concerned with this, which suggests that $(\star loop)$ cannot be replaced by a finite and universally quantified condition.

In addition, this direction for future work can still be followed if we define the distance-based revision in a non-classical framework. Indeed, as Lehmann *et al.* did, we worked in a general framework. For instance, if we define the revision in the $\mathcal{FOUR}$ framework —$\mathcal{FOUR}$ is a well-known paraconsistent logic from (Belnap 1977b) and (Belnap 1977a) — then we can probably use the results of (Lehmann, Magidor, & Schlechta 2001) and our results respectively to show characterizations of revision operators and show that they cannot be really improved.

Moreover, most of the approaches to belief revision treat in a trivial way inconsistent sets of beliefs (if they are treated at all). However, people may be rational despite inconsistent beliefs (there may be overwhelming evidence for both something and its contrary). There are also inconsistencies in principle impossible to eliminate like the "Paradox of the Preface" (Makinson 1965). The latter says that a conscientious author has reasons to believe that everything written in his book is true. But, because of human imperfection, he is sure that his book contains errors, and thus that something must be false. Consequently, he has (in the absolute sense) both reasons to believe that everything is true and that something is false. So, principles of rational belief revision must work on inconsistent sets of beliefs. Standard approaches to belief revision (e.g. AGM) all fail to do this as they are based on classical logic. Paraconsistent logics (like e.g. $\mathcal{FOUR}$) could be the bases of more adequate approaches.

Another advantage of such approaches is that they will not be forced to eliminate a contradiction even when there is no good way to do it. Contradictions could be tolerated until new information eventually comes to justify one or another way of elimination.

Finally, such approaches will benefit from an extended field of application which includes multi-agent systems where the agents can have individually inconsistent beliefs. Furthermore, it is easy to see that these perspectives for belief revision can be transposed to belief merging.

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