Pivotal and Pivotal-discriminative Consequence Relations *

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Abstract

In the present paper, we investigate consequence relations that are both paraconsistent and plausible (but still monotonic). More precisely, we put the focus on pivotal consequence relations, i.e. those relations that can be defined by a pivot (in the style of e.g. D. Makinson). A pivot is a fixed subset of valuations which are considered to be the important ones in the absolute sense. We worked with a general notion of valuation that covers e.g. the classical valuations as well as certain kinds of many-valued valuations. In the many-valued cases, pivotal consequence relations are paraconsistant (in addition to be plausible), i.e. they are capable of drawing reasonable conclusions which contain contradictions. We will provide in our general framework syntactic characterizations of several families of pivotal relations. In addition, we will provide, again in our general framework, characterizations of several families of pivotal-discriminative consequence relations. The latter are defined exactly as the plain version, but contradictory conclusions are rejected. We will also answer negatively a representation problem that was left open by Makinson. Finally, we will put in evidence a connexion with X-logics from Forget, Risch, and Siegel. The motivations and the framework of the present paper are very close to those of a previous paper of the author which is about preferential consequence relations.

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1 Introduction

One of the main motivations of this paper is to combine some tools used in Paraconsistent Reasoning on the one hand and Plausible Reasoning on the other hand to deal with both incomplete and inconsistent information. Actually, the motivations of the present paper are very close to those of a previous paper of the author which is about preferential consequence relations [BN05].

Many-valued consequence relations have been developed with the aim of dealing with inconsistent information. These relations are defined in frameworks where valuations can assign more than two different truth values to formulas. In fact, they tolerate contradictions within the conclusions, but reject the principle of explosion according to which a single contradiction entails the deduction of every formula.

Independently, plausible (generally non-monotonic) consequence relations have been developed with the aim of dealing with incomplete information. *Choice functions* are central tools to define plausible relations. Indeed, suppose we have at our disposal a function μ, called a choice function, which chooses in any set of valuations V, those elements which are preferred, not necessarily in the absolute sense, but when the valuations in V are the only ones under consideration. Then, we can define a plausible consequence relation in the following natural way: a formula α follows from a set of formulas Γ iff every model for Γ chosen by μ is a model for α.

In the present paper, we put the focus on a particular family of choice functions. Let us present it. Suppose some valuations are considered to be the important ones in the absolute sense and collect them in a set $\mathcal{I}$, called a pivot. This defines naturally a choice function. Indeed, simply choose in any set of valuations, those elements that belong to $\mathcal{I}$. Those choice functions which can be defined in this manner constitute the aforementioned family. The consequence relations defined by this family are called *pivotal consequence relations*. Their importance has been put in evidence by D. Makinson in [Mak03, Mak05] where it is shown that they constitute an easy conceptual passage between classical and plausible non-monotonic relations. Indeed, they are perfectly monotonic but already display some of the distinctive features (i.e. the choice functions) of plausible non-monotonic relations.

For a long time, research efforts on paraconsistent relations and plausible relations were separated. However, in many applications, the information is both incomplete and inconsistent. For instance, the semantic web or big databases inevitably contain inconsistencies. This can be due to human or material imperfections as well as contradictory sources of information. On the other hand, neither the web nor big databases can contain "all" information. Indeed, there are rules of which the exceptions cannot be enumerated. Also, some information might be left voluntarily vague or in concise form. Consequently, consequence relations that are both paraconsistent and plausible are useful to reason in such applications.

Such relations first appear in e.g. [Pri91, Bat98, KL92, AA00, KM02]. The idea begins by taking a many-valued framework to get paraconsistency. Then, only those models that are most preferred according to some particular binary preference relation on valuations (in the style of [Sho88, Sho87]) are relevant for making inference, which provides plausibility (and in fact also non-monotonicity). In [AL01b, AL01a], A. Avron and I. Lev generalized the study to families of binary preference relations which compare two valuations using, for each of them, this part of a certain set of formulas it satisfies. The present paper follows this line of research by combining many-valued frameworks and choice functions.

More explicitly, we will investigate pivotal consequence relations in a general framework. According to the different assumptions which will be made about the latter, it will cover various kinds of frameworks, including e.g. the classical propositional one as well as some many-valued ones. Moreover, in the many-valued frameworks, pivotal relations lead to non-trivial conclusions is spite of the presence of contradictions and are thus useful to deal with both incomplete and inconsistent information. However, they will not satisfy the Disjunctive Syllogism (from α and $\neg\alpha \vee \beta$ we can conclude β), whilst they satisfy it in classical frameworks.

In addition, we will investigate *pivotal-discriminative consequence relations*. They are defined exactly as the plain version, but any conclusion such that its negation is also a conclusion is rejected. In the classical framework, they do not bring something really new. Indeed, instead of concluding everything in the face of inconsistent information, we will simply conclude nothing. On the other hand, in the many-valued frameworks, where the conclusions are rational even from inconsistent information, the discriminative version will reject the contradictions among them, rendering them all the more rational.

As a first contribution, we will characterize, in our general framework, several families of pivotal(-discriminative) consequence relations. To do so, we will use techniques very similar to those of a previous paper of the author [BN05]. The latter is about another family of choice functions. Let us present it. Suppose we are given a binary preference relation $\prec$ on states labelled by valuations (in the style of e.g. [KLM90, LM92, Sch04]). This defines naturally a choice function. Indeed, choose in any set of valuations V, each element which labels a state which is $\prec$-preferred among those states which are labelled by the elements of V. Those choice functions which can be defined in this manner constitute the aforementioned family. The consequence relations defined by this family of choice functions are called *preferential(-discriminative) consequence relations*. In fact, the present paper provides an example of how the techniques developed in [BN05] (especially, in the discriminative case) can be adapted to new families of choice functions. Note that, in the non-discriminative case, the techniques of [BN05] are themselves strongly inspired by the work of K. Schlechta [Sch04].

In many cases, our characterizations will be purely syntactic. This has a lot of advantages, let us quote some important ones. Take a set of syntactic conditions that characterizes a family of pivotal consequence relations. This gives a syntactic point of view on this family defined semantically, which enables us to compare it to conditions known on the "market", and thus to other consequence relations. This can also give rise to questions like: if we modified the conditions in such and such a natural-looking way, what would happen on the semantic side? More generally, this can open the door to questions that would not easily come to mind otherwise or to techniques of proof that could not have been employed in the semantic approach.

Some characterizations of pivotal consequence relations, valid in classical frameworks, can be found in the literature, e.g. [Rot01, Mak03, Mak05]. But, to the author knowledge, the present paper contains the first systematic work of characterization for them in non-classical frameworks. Similarly, it seems that the author is the first to investigate pivotal-discriminative consequence relations.

As a second contribution, we will answer negatively a representation problem that was left open by Makinson, namely, in an infinite classical framework, there does not exist a "normal" characterization for the family of all pivotal consequence relations. Approximatively, a characterization is called normal iff it contains only conditions universally quantified and of limited size. This constitutes the more innovative part of the paper. A last contribution is that a certain family of pivotal consequence relations will be shown to be precisely a certain family of X-logics, which were introduced by Forget, Risch, and Siegel [FRS01].

The rest of the paper is organized as follows. In Section 2.1, we introduce our general framework and the different assumptions which will sometimes be made about it. We will see that it covers in particular the many-valued frameworks of the well-known paraconsistent logics $\mathcal{FOUR}$ and J_3. In Section 2.2, we present choice functions and some of their well-known properties. We will see which properties characterize those choice functions that can be defined by a pivot. In Section 2.3, we define pivotal(-discriminative) consequence relations and give examples of them in both the classical and the many-valued frameworks. In Section 3, we provide our characterizations. In Section 4, we answer negatively the problem that was left open by Makinson. In Section 5, we put in evidence a connexion with X-logics. Finally, we conclude in Section 6.

2 Background

2.1 Semantic structures

2.1.1 Definitions and properties

The framework is exactly the one presented in [BN05]. We will work with general formulas, valuations, and satisfaction. A similar approach has been taken in two well-known papers [Mak05, Leh01].

Definition 1 We say that $\mathcal{S}$ is a *semantic structure* iff $\mathcal{S} = \langle \mathcal{F}, \mathcal{V}, \models \rangle$ where $\mathcal{F}$ is a set, $\mathcal{V}$ is a set, and $\models$ is a relation on $\mathcal{V} \times \mathcal{F}$.

Intuitively, $\mathcal{F}$ is a set of formulas, $\mathcal{V}$ a set of valuations for these formulas, and $\models$ a satisfaction relation for these objects (i.e. $v \models \alpha$ means the formula α is satisfied in the valuation v, i.e. v is a model for α).

Notation 2 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure, $\Gamma \subseteq \mathcal{F}$, and $V \subseteq \mathcal{V}$. Then,
$M_\Gamma := \{v \in \mathcal{V} : \forall\, \alpha \in \Gamma, v \models \alpha\}$,
$T(V) := \{\alpha \in \mathcal{F} : V \subseteq M_\alpha\}$,
$\mathbf{D} := \{V \subseteq \mathcal{V} : \exists\, \Gamma \subseteq \mathcal{F}, M_\Gamma = V\}$.
Suppose $\mathcal{L}$ is a language, $\neg$ a unary connective of $\mathcal{L}$, and $\mathcal{F}$ the set of all wffs of $\mathcal{L}$. Then,
$T_d(V) := \{\alpha \in \mathcal{F} : V \subseteq M_\alpha$ and $V \not\subseteq M_{\neg\alpha}\}$,
$\mathbf{C} := \{V \subseteq \mathcal{V} : \forall\, \alpha \in \mathcal{F}, V \not\subseteq M_\alpha$ or $V \not\subseteq M_{\neg\alpha}\}$.

Intuitively, M_Γ is the set of all models for Γ and $T(V)$ the set of all formulas satisfied in V. Roughly speaking, $T_d(V)$ is this part of $T(V)$ that is not contradictory. $\mathbf{D}$ is the set of all those sets of valuations that are definable by a set of formulas and $\mathbf{C}$ the set of all those sets of valuations that do not satisfy both a formula and its negation. As usual, $M_{\Gamma,\alpha}$, $T(V, v)$ stand for respectively $M_{\Gamma\cup\{\alpha\}}$, $T(V \cup \{v\})$, etc.

Remark 3 The notations M_Γ, $T(V)$, etc. should contain the semantic structure on which they are based. To increase readability, we will omit it. There will never be any ambiguity. We will omit similar things with other notations in the sequel, for the same reason.

A semantic structure defines a basic consequence relation:

Notation 4 We denote by $\mathcal{P}$ the power set operator.
Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure.
We denote by $\vdash$ the relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\forall\, \alpha \in \mathcal{F}$,

$$\Gamma \vdash \alpha \text{ iff } M_\Gamma \subseteq M_\alpha.$$

Let $\mid\!\sim$ be a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then,
$\mid\!\sim(\Gamma) := \{\alpha \in \mathcal{F} : \Gamma \mid\!\sim \alpha\}$.
Suppose $\mathcal{L}$ is a language, $\neg$ a unary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, and $\Gamma \subseteq \mathcal{F}$.
Then, we say that Γ is *consistent* iff $\forall\, \alpha \in \mathcal{F}$, $\Gamma \nvdash \alpha$ or $\Gamma \nvdash \neg\alpha$.

The following trivial facts hold, we will use them implicitly in the sequel:

Remark 5 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and $\Gamma, \Delta \subseteq \mathcal{F}$. Then:
$M_{\Gamma,\Delta} = M_\Gamma \cap M_\Delta$;
$\vdash(\Gamma) = T(M_\Gamma)$;
$M_\Gamma = M_{\vdash(\Gamma)}$;
$\Gamma \subseteq \vdash(\Delta)$ iff $\vdash(\Gamma) \subseteq \vdash(\Delta)$ iff $M_\Delta \subseteq M_\Gamma$.

Sometimes, we will need to make some of the following assumptions about a semantic structure:

Definition 6 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure.
The define the following assumptions:

$(A0)$ $M_{\mathcal{F}} = \emptyset$;

$(A1)$ $\mathcal{V}$ is finite.

Suppose $\mathcal{L}$ is a language, $\neg$ a unary connective of $\mathcal{L}$, and $\mathcal{F}$ the set of all wffs of $\mathcal{L}$. Then, define:

$(A2)$ $\forall\, \Gamma \subseteq \mathcal{F}, \forall\, \alpha \in \mathcal{F}$, if $\alpha \notin T(M_\Gamma)$ and $\neg\alpha \notin T(M_\Gamma)$, then $M_\Gamma \cap M_\alpha \not\subseteq M_{\neg\alpha}$.

Suppose $\vee$ and $\wedge$ are binary connectives of $\mathcal{L}$. Then, define:

$(A3)$ $\forall\, \alpha, \beta \in \mathcal{F}$, we have:
$M_{\alpha\vee\beta} = M_\alpha \cup M_\beta$;
$M_{\alpha\wedge\beta} = M_\alpha \cap M_\beta$;
$M_{\neg\neg\alpha} = M_\alpha$;
$M_{\neg(\alpha\vee\beta)} = M_{\neg\alpha\wedge\neg\beta}$;
$M_{\neg(\alpha\wedge\beta)} = M_{\neg\alpha\vee\neg\beta}$.

Clearly, those assumptions are satisfied by classical semantic structures, i.e. structures where $\mathcal{F}$, $\mathcal{V}$, and $\models$ are classical. In addition, we will see, in Sections 2.1.2 and 2.1.3, that they are also satisfied by certain many-valued semantic structures.

2.1.2 The semantic structure defined by $\mathcal{FOUR}$

The logic $\mathcal{FOUR}$ was introduced by N. Belnap in [Bel77a, Bel77b]. This logic is useful to deal with inconsistent information. Several presentations are possible, depending on the language under consideration. For the needs of the present paper, a classical propositional language will be sufficient. The logic has been investigated intensively in e.g. [AA94, AA96, AA98], where richer languages, containing an implication connective $\supset$ (first introduced by A. Avron [Avr91]), were considered.

Notation 7 We denote by $\mathcal{A}$ a set of propositional symbols (or atoms).
We denote by $\mathcal{L}_c$ the classical propositional language containing $\mathcal{A}$, the usual constants $false$ and $true$, and the usual connectives $\neg$, $\vee$, and $\wedge$.
We denote by $\mathcal{F}_c$ the set of all wffs of $\mathcal{L}_c$.

We briefly recall a meaning for the logic $\mathcal{FOUR}$ (more details can be found in [CLM99, Bel77a, Bel77b]). Consider a system in which there are, on the one hand, sources of information and, on the other hand, a processor that listens to them. The sources provide information about the atoms only, not about the compound formulas. For each atom p, there are exactly four possibilities: either the processor is informed (by the sources, taken as a whole) that p is true; or he is informed that p is false; or he is informed of both; or he has no information about p.

Notation 8 Denote by 0 and 1 the classical truth values and define:
$\mathbf{f} := \{0\}$; $\quad \mathbf{t} := \{1\}$; $\quad \top := \{0, 1\}$; $\quad \bot := \emptyset$.

The global information given by the sources to the processor can be modelled by a function s from $\mathcal{A}$ to $\{\mathbf{f}, \mathbf{t}, \top, \bot\}$. Intuitively, $1 \in s(p)$ means the processor is informed that p is true, whilst $0 \in s(p)$ means he is informed that p is false.

Then, the processor naturally builds information about the compound formulas from s. Before he starts to do so, the situation can be be modelled by a function v from $\mathcal{F}_c$ to $\{\mathbf{f}, \mathbf{t}, \top, \bot\}$ which agrees with s about the atoms and which assigns $\bot$ to all compound formulas. Now, take p and q in $\mathcal{A}$ and suppose $1 \in v(p)$ or $1 \in v(q)$. Then, the processor naturally adds 1 to $v(p \vee q)$. Similarly, if $0 \in v(p)$ and $0 \in v(q)$, then he adds 0 in $v(p \vee q)$. Of course, such rules hold for $\neg$ and $\wedge$ too.

Suppose all those rules are applied recursively to all compound formulas. Then, v represents the "full" (or developed) information given by the sources to the processor. Now, the valuations of the logic $\mathcal{FOUR}$ can be defined as exactly those functions that can be built in this manner (i.e. like v) from some information sources. More formally,

Definition 9 We say that v is a *four-valued valuation* iff v is a function from $\mathcal{F}_c$ to $\{\mathbf{f}, \mathbf{t}, \top, \bot\}$ such that $v(true) = \mathbf{t}$, $v(false) = \mathbf{f}$ and $\forall\, \alpha, \beta \in \mathcal{F}_c$,
$1 \in v(\neg\alpha)$ iff $0 \in v(\alpha)$;
$0 \in v(\neg\alpha)$ iff $1 \in v(\alpha)$;
$1 \in v(\alpha \vee \beta)$ iff $1 \in v(\alpha)$ or $1 \in v(\beta)$;
$0 \in v(\alpha \vee \beta)$ iff $0 \in v(\alpha)$ and $0 \in v(\beta)$;
$1 \in v(\alpha \wedge \beta)$ iff $1 \in v(\alpha)$ and $1 \in v(\beta)$;
$0 \in v(\alpha \wedge \beta)$ iff $0 \in v(\alpha)$ or $0 \in v(\beta)$.
We denote by $\mathcal{V}_4$ the set of all four-valued valuations.

The definition may become more accessible if we see the four-valued valuations as those functions that satisfy Tables 1, 2, and 3 below:

$v(\alpha)$	$v(\neg\alpha)$
$\mathbf{f}$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$
$\bot$	$\bot$

Table 1.

$v(\alpha \vee \beta)$

$v(\alpha)$ \ $v(\beta)$	$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\mathbf{t}$	$\top$	$\mathbf{t}$
$\bot$	$\bot$	$\mathbf{t}$	$\mathbf{t}$	$\bot$

Table 2.

$v(\alpha \wedge \beta)$

$v(\alpha)$ \ $v(\beta)$	$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\top$	$\bot$
$\top$	$\mathbf{f}$	$\top$	$\top$	$\mathbf{f}$
$\bot$	$\mathbf{f}$	$\bot$	$\mathbf{f}$	$\bot$

Table 3.

In the logic $\mathcal{FOUR}$, a formula α is considered to be satisfied iff the processor is informed that it is true (it does not matter whether he is also informed that α is false).

Notation 10 We denote by $\models_4$ the relation on $\mathcal{V}_4 \times \mathcal{F}_c$ such that $\forall\, v \in \mathcal{V}_4$, $\forall\, \alpha \in \mathcal{F}_c$, we have $v \models_4 \alpha$ iff $1 \in v(\alpha)$.

Proof systems for the consequence relation $\vdash$ based on the semantic structure $\langle \mathcal{F}_c, \mathcal{V}_4, \models_4 \rangle$ (i.e. the semantic structure defined by $\mathcal{FOUR}$) can be found in e.g. [AA94, AA96, AA98].

Note that the $\mathcal{FOUR}$ semantic structure satisfies $(A0)$ and $(A3)$. In addition, if $\mathcal{A}$ is finite, then $(A1)$ is also satisfied. However, $(A2)$ is not satisfied by this structure. In Section 2.1.3, we turn to a many-valued semantic structure which satisfies $(A2)$.

2.1.3 The semantic structure defined by J_3

The logic J_3 was introduced in [DdC70] to answer a question posed in 1948 by S. Jaśkowski, who was interested in systematizing theories capable of containing contradictions, especially if they occur in dialectical reasoning. The step from informal reasoning under contradictions and formal reasoning with databases and information was done in [CMdA00] (also specialized for real database models in [dACM02]), where another formulation of J_3 called **LFI1** was introduced, and its first-order version, semantics and proof theory were studied in detail. Investigations of J_3 have also been made in e.g. [Avr91], where richer languages than our $\mathcal{L}_c$ were considered.

The valuations of the logic J_3 can be given the same meaning as those of the logic $\mathcal{FOUR}$, except that the consideration is restricted to those sources which always give some information about an atom. More formally,

Definition 11 We say that v is a *three-valued valuation* iff v is a function from $\mathcal{F}_c$ to $\{\mathbf{f}, \mathbf{t}, \top\}$ such that $v(true) = \mathbf{t}$, $v(false) = \mathbf{f}$ and $\forall\, \alpha, \beta \in \mathcal{F}_c$,
$1 \in v(\neg\alpha)$ iff $0 \in v(\alpha)$;
$0 \in v(\neg\alpha)$ iff $1 \in v(\alpha)$;
$1 \in v(\alpha \vee \beta)$ iff $1 \in v(\alpha)$ or $1 \in v(\beta)$;
$0 \in v(\alpha \vee \beta)$ iff $0 \in v(\alpha)$ and $0 \in v(\beta)$;
$1 \in v(\alpha \wedge \beta)$ iff $1 \in v(\alpha)$ and $1 \in v(\beta)$;
$0 \in v(\alpha \wedge \beta)$ iff $0 \in v(\alpha)$ or $0 \in v(\beta)$.
We denote by $\mathcal{V}_3$ the set of all three-valued valuations.

As previously, the definition may become more accessible if we see the three-valued valuations as those functions that satisfy Tables 4, 5, and 6 below:

$v(\alpha)$	$v(\neg\alpha)$
$\mathbf{f}$	$\mathbf{t}$
$\mathbf{t}$	$\mathbf{f}$
$\top$	$\top$

Table 4.

$v(\alpha \vee \beta)$: $v(\alpha)$ \ $v(\beta)$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$	$\mathbf{t}$
$\top$	$\top$	$\mathbf{t}$	$\top$

Table 5.

$v(\alpha \wedge \beta)$: $v(\alpha)$ \ $v(\beta)$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$	$\mathbf{f}$
$\mathbf{t}$	$\mathbf{f}$	$\mathbf{t}$	$\top$
$\top$	$\mathbf{f}$	$\top$	$\top$

Table 6.

We turn to the satisfaction relation.

Notation 12 We denote by $\models_3$ the relation on $\mathcal{V}_3 \times \mathcal{F}_c$ such that $\forall\, v \in \mathcal{V}_3$, $\forall\, \alpha \in \mathcal{F}_c$, we have $v \models_3 \alpha$ iff $1 \in v(\alpha)$.

Proof systems for the consequence relation $\vdash$ based on the semantic structure $\langle \mathcal{F}_c, \mathcal{V}_3, \models_3 \rangle$ (i.e. the semantic structure defined by J_3) have been provided in e.g. [Avr91, DdC70] and chapter IX of [Eps90]. The J_3 structure satisfies $(A0)$, $(A3)$ and $(A2)$. In addition, if $\mathcal{A}$ is finite, then it satisfies $(A1)$ too.

2.2 Choice functions

2.2.1 Definitions and properties

In many situations, an agent has some way to choose in any set of valuations V, those elements that are preferred (the bests, the more normal, etc.), not necessarily in the absolute sense, but when the valuations in V are the only ones under consideration. In Social Choice, this is modelled by choice functions [Che54, Arr59, Sen70, AM81, Leh02, Leh01].

Definition 13 Let $\mathcal{V}$ be a set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$, and μ a function from $\mathbf{V}$ to $\mathbf{W}$.
We say that μ is a *choice function* iff $\forall\, V \in \mathbf{V}$, $\mu(V) \subseteq V$.

Several properties for choice functions have been put in evidence by researchers in Social Choice. For the sake of completeness, we present two important ones though we will not investigate them in the present paper (a better presentation can be found in [Leh01]).

Suppose W is a set of valuations, V is a subset of W, and $v \in V$ is a preferred valuation of W. Then, a natural requirement is that v is a preferred valuation of V. Indeed, in many situations, the larger a set is, the harder it is to be a preferred element of it, and he who can do the most can do the least. This property appears in [Che54] and has been given the name Coherence in [Mou85].

We turn to the second property. Suppose W is a set of valuations, V is a subset of W, and suppose all the preferred valuations of W belong to V. Then, they are expected to include all the preferred valuations of V. The importance of this property has been put in evidence by [Aiz85, AM81] and has been given the name Local Monotonicity in e.g. [Leh01].

In [Sch00], Schlechta showed that Coherence and Local Monotonicity characterize those choice functions that can be defined by a binary preference relation on states labelled by valuations (in the style of e.g. [KLM90]).

Now, we turn to properties relevant for the paper, i.e. properties which characterize those choice functions that can be defined by a pivot (in the style of e.g. D. Makinson [Mak03, Mak05]). A pivot is a fixed subset of valuations which are considered to be the important ones in the absolute sense. Details will be given in Section 2.2.2.

Definition 14 Let $\mathcal{V}$ be a set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$, and μ a choice function from $\mathbf{V}$ to $\mathbf{W}$.
We say that μ is *strongly coherent* (SC) iff $\forall\, V, W \in \mathbf{V}$,

$$\mu(W) \cap V \subseteq \mu(V).$$

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ is a semantic structure.
We say that μ is *definability preserving* (DP) iff

$$\forall\, V \in \mathbf{V} \cap \mathbf{D},\ \mu(V) \in \mathbf{D}.$$

In addition, suppose $\mathcal{V} \in \mathbf{V}$.
We say that μ is *universe-codefinable* (UC) iff

$$\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D}.$$

Definability Preservation has been put in evidence first in [Sch92]. One of its advantages is that when the choice functions under consideration satisfy it, we will provide characterizations with purely syntactic conditions. To the author knowledge, Strong Coherence and Universe-codefinability are first introduced in the present paper. An advantage of Universe-codefinability is that it provides a link with X-logics [FRS01]. We will see it in Section 5.

Now, we turn to a last property:

Definition 15 Suppose $\mathcal{L}$ is a language, $\neg$ a unary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$, and μ be a choice function from $\mathbf{V}$ to $\mathbf{W}$.
We say that μ is *coherency preserving* (CP) iff

$$\forall\, V \in \mathbf{V} \cap \mathbf{C},\ \mu(V) \in \mathbf{C}.$$

To the author knowledge, Coherency Preservation has been first introduced in [BN05]. An advantage of it is that when the choice functions under consideration satisfy it, we will not need to assume $(A2)$ to show our characterizations (in the discriminative case).

2.2.2 Pivots

Suppose some valuations are considered to be the important ones in the absolute sense and collect them in a set $\mathcal{I}$, called a pivot. Then, $\mathcal{I}$ defines naturally a choice function $\mu_{\mathcal{I}}$ which chooses in any set of valuations, simply those elements which belong to $\mathcal{I}$. More formally,

Definition 16 Let $\mathcal{V}$ be a set.
We say that $\mathcal{I}$ is a pivot on $\mathcal{V}$ iff $\mathcal{I} \subseteq \mathcal{V}$.
Let $\mathcal{I}$ be a pivot on $\mathcal{V}$.
We denote by $\mu_{\mathcal{I}}$ the function from $\mathcal{P}(\mathcal{V})$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, V \subseteq \mathcal{V}$,

$$\mu_{\mathcal{I}}(V) = V \cap \mathcal{I}.$$

Pivots have been investigated extensively by D. Makinson in [Mak03, Mak05]. In the present section, we show that the properties of Strong Coherence, Definability Preservation, and Universe-codefinability characterize those choice functions that can be defined by a pivot. More precisely:

Proposition 17 Let $\mathcal{V}$ be a set, $\mathbf{V}, \mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$, and μ a choice function from $\mathbf{V}$ to $\mathbf{W}$. Then:

(0) μ is SC iff there exists a pivot $\mathcal{I}$ on $\mathcal{V}$ such that $\forall\, V \in \mathbf{V}$, $\mu(V) = \mu_{\mathcal{I}}(V)$.

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ is a semantic structure and $\mathcal{V} \in \mathbf{V}$. Then:

(1) μ is SC and DP iff there exists a pivot $\mathcal{I}$ on $\mathcal{V}$ such that $\mathcal{I} \in \mathbf{D}$ and $\forall\, V \in \mathbf{V}$, $\mu(V) = \mu_{\mathcal{I}}(V)$;

(2) μ is SC and UC iff there exists a pivot $\mathcal{I}$ on $\mathcal{V}$ such that $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$ and $\forall V \in \mathbf{V}$, $\mu(V) = \mu_{\mathcal{I}}(V)$.

Proof *Proof of* (0). Direction: "$\rightarrow$".
Let $\mathcal{I} = \{v \in \mathcal{V} : \exists\, V \in \mathbf{V}, v \in \mu(V)\}$ and suppose $V \in \mathbf{V}$.
If $v \in \mu(V)$, then $v \in V$ and, by definition of $\mathcal{I}$, $v \in \mathcal{I}$. Consequently, $\mu(V) \subseteq V \cap \mathcal{I}$.
If $v \in V \cap \mathcal{I}$, then $\exists\, W \in \mathbf{V}$, $v \in \mu(W)$, thus, by SC, $v \in \mu(W) \cap V \subseteq \mu(V)$.
Consequently, $V \cap \mathcal{I} \subseteq \mu(V)$.
Direction: "$\leftarrow$".
There exists $\mathcal{I} \subseteq \mathcal{V}$ such that $\forall\, V \in \mathbf{V}$, $\mu(V) = V \cap \mathcal{I}$.
We show that μ satisfies SC.
Let $V, W \in \mathbf{V}$. Then, $\mu(W) \cap V = W \cap \mathcal{I} \cap V \subseteq \mathcal{I} \cap V = \mu(V)$.

Proof of (1). Direction: "$\rightarrow$".
Take the same $\mathcal{I}$ as for (0). Then, by verbatim the same proof, $\forall\, V \in \mathbf{V}$, $\mu(V) = V \cap \mathcal{I}$.
It remains to show that $\mathcal{I} \in \mathbf{D}$.
As $M_{\emptyset} = \mathcal{V}$, $\mathcal{V} \in \mathbf{D}$. Thus, as μ is DP, $\mu(\mathcal{V}) \in \mathbf{D}$. But, $\mu(\mathcal{V}) = \mathcal{V} \cap \mathcal{I} = \mathcal{I}$.
Direction: "$\leftarrow$".
Verbatim the proof of (0), except that in addition $\mathcal{I} \in \mathbf{D}$.
We show that μ is DP. Let $V \in \mathbf{V} \cap \mathbf{D}$.
Then, $\exists\, \Gamma \subseteq \mathcal{F}$, $M_{\Gamma} = V$. Similarly, as $\mathcal{I} \in \mathbf{D}$, $\exists\, \Delta \subseteq \mathcal{F}$, $M_{\Delta} = \mathcal{I}$.
Therefore, $\mu(V) = V \cap \mathcal{I} = M_{\Gamma} \cap M_{\Delta} = M_{\Gamma \cup \Delta} \in \mathbf{D}$.

Proof of (2). Direction: "$\to$".
Take the same $\mathcal{I}$ as for (0). Then, by verbatim the same proof, $\forall\, V \in \mathbf{V}$, $\mu(V) = V \cap \mathcal{I}$.
It remains to show $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$. As μ is UC, $\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D}$. But, $\mathcal{V} \setminus \mu(\mathcal{V}) = \mathcal{V} \setminus (\mathcal{V} \cap \mathcal{I}) = \mathcal{V} \setminus \mathcal{I}$.
Direction: "$\leftarrow$".
Verbatim the proof of (0), except that in addition $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$.
We show that μ is UC: $\mathcal{V} \setminus \mu(\mathcal{V}) = \mathcal{V} \setminus (\mathcal{V} \cap \mathcal{I}) = \mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$. ∎

2.3 Pivotal(-discriminative) consequence relations

2.3.1 Definitions

Suppose we are given a semantic structure and a choice function μ on the valuations. Then, it is natural to conclude a formula α from a set of formulas Γ iff every model for Γ chosen by μ is a model for α. More formally:

Definition 18 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
We say that $\mid\!\sim$ is a *pivotal consequence relation* iff there exists a SC choice function μ from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\forall\, \alpha \in \mathcal{F}$,

$$\Gamma \mid\!\sim \alpha \text{ iff } \mu(M_\Gamma) \subseteq M_\alpha$$

In addition, if μ is DP, CP, etc., then so is $\mid\!\sim$.

We called these relations "pivotal" because, in the light of Proposition 17, they can be defined equivalently with pivots, instead of SC choice functions. Their importance has been put in evidence by D. Makinson in e.g. [Mak03, Mak05], where he showed that they constitute easy conceptual passage from basic to plausible non-monotonic consequence relations. Indeed, they are perfectly monotonic but already display some of the distinctive features (i.e. the choice functions) of plausible non-monotonic relations. Note that pivotal (resp. DP pivotal) consequence relations correspond to Makinson's pivotal-valuation (resp. pivotal-assumption) relations. We will give an example of how they can be used to draw plausible conclusions from incomplete information in Section 2.3.2.

Moreover, if a many-valued semantic structure is considered, they lead to rational and non-trivial conclusions is spite of the presence of contradictions and are thus useful to treat both incomplete and inconsistent information. However, they will not satisfy the Disjunctive Syllogism. We will give an example with the $\mathcal{FOUR}$ semantic structure in Section 2.3.3.

Characterizations of pivotal consequence relations, valid in classical frameworks, can be found in the literature. For instance, the following result appears to be part of folklore for decades: DP pivotal consequence relations correspond precisely to those supraclassical closure operations that are compact and satisfy Disjunction in the premisses. For more details see e.g. [Rot01, Mak03, Mak05].

Now, we turn to a qualified version of pivotal consequence. It captures the idea that the contradictions in the conclusions should be rejected.

Definition 19 Let $\mathcal{L}$ be a language, $\neg$ a unary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
We say that $\mid\!\sim$ is a *pivotal-discriminative consequence relation* iff there exists a SC choice function μ from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\forall\, \alpha \in \mathcal{F}$,

$$\Gamma \mid\!\sim \alpha \text{ iff } \mu(M_\Gamma) \subseteq M_\alpha \text{ and } \mu(M_\Gamma) \not\subseteq M_{\neg\alpha}$$

In addition, if μ is DP, CP, etc., then so is $\mid\!\sim$.

If a classical semantic structure is considered, the discriminative version does not bring something really new. Indeed, the only difference will be to conclude nothing instead of everything in the face of inconsistent information. On the other hand, with a many-valued structure, the conclusions are rational even from inconsistent information. The discriminative version will then reject the contradictions in the conclusions, rendering the latter all the more rational.

2.3.2 Example in the classical framework

Let $\mathcal{L}$ be a classical propositional language of which the atoms are r, q, and p. Intuitively, r means Nixon is a republican, q means Nixon is a quaker, and p means Nixon is a pacifist. Let $\mathcal{F}$ be the set of all wffs of $\mathcal{L}$, $\mathcal{V}$ the set of all classical two-valued valuations of $\mathcal{L}$, and $\models$ the classical satisfaction relation for these objects. Then, $\mathcal{V}$ is the set of the 8 following valuations: v_0, v_1, v_2, v_3, v_4, v_5, v_6, and v_7, which are defined in the obvious way by the following table:

	r	q	p
v_0	0	0	0
v_1	0	0	1
v_2	0	1	0
v_3	0	1	1
v_4	1	0	0
v_5	1	0	1
v_6	1	1	0
v_7	1	1	1

Now, consider the class of all republicans and the class of all quakers. Consider that a republican is normal iff he is not a pacifist and that a quaker is normal iff he is a pacifist. And, consider that a valuation is negligible iff (in it) Nixon is a non-normal individual of some class. Then, collect the non-negligible valuations in a pivot $\mathcal{I}$. More formally:

$$\mathcal{I} = \{v \in \mathcal{V} : \text{if } v \models r, \text{ then } v \models \neg p; \text{ and if } v \models q, \text{ then } v \models p\}.$$

Finally, let $\mid\!\sim$ be the pivotal consequence relation defined by the SC choice function $\mu_{\mathcal{I}}$.

Then, $\mid\!\sim$ leads to "jump" to plausible conclusions from incomplete information. For instance, $r \mid\!\sim \neg p$ and $q \mid\!\sim p$. But, we fall into triviality if we face new information that contradict previous "hasty" conclusions. For instance, $\{r, p\} \mid\!\sim \alpha$, $\forall\, \alpha \in \mathcal{L}$, and $\{q, \neg p\} \mid\!\sim \alpha$, $\forall\, \alpha \in \mathcal{L}$. This is the price to pay for being monotonic, whereas conclusions that are only plausible are accepted.

In addition, $\mid\!\sim$ is not paraconsistent and some sets of formulas are rendered useless because there is no model in the pivot for them, though there are models for them. For instance, $\{q, r\} \mid\!\sim \alpha$, $\forall\, \alpha \in \mathcal{L}$.

2.3.3 Example in the $\mathcal{FOUR}$ framework

Consider the $\mathcal{FOUR}$ semantic structure $\langle \mathcal{F}_c, \mathcal{V}_4, \models_4 \rangle$ and suppose $\mathcal{A} = \{r, q, p\}$ (these objects have been defined in Section 2.1.2). In addition, make the same considerations about Nixon, the classes, normality, etc., as in Section 2.3.2, except that this time a valuation is considered to be negligible iff (in it) the processor is informed that Nixon is an individual of some class, but he is not informed that Nixon is a normal individual of that class. See Section 2.1.2 for recalls about the sources-processor systems. Again, collect the non-negligible valuations in a pivot $\mathcal{I}$. More formally:

$$\mathcal{I} = \{v \in \mathcal{V}_4 : \text{if } v \models r, \text{ then } v \models \neg p; \text{ and if } v \models q, \text{ then } v \models p\}.$$

Let $\mid\!\sim$ be the pivotal consequence relation defined by the SC choice function $\mu_{\mathcal{I}}$.

Then, again $\mid\!\sim$ leads to "jump" to plausible conclusions from incomplete information. For instance, $r \mid\!\sim \neg p$ and $q \mid\!\sim p$. Moreover, though "hasty" conclusions are never withdrawn, we do not fall into triviality when we face new information that contradict them. For instance, $\{r, p\} \mid\!\sim p$ and $\{r, p\} \mid\!\sim \neg p$ and $\{r, p\} \mid\!\sim r$ and $\{r, p\} \not\mid\!\sim \neg r$.

In addition, $\mid\!\sim$ is paraconsistent. For instance, $\{p, \neg p, q\} \mid\!\sim p$ and $\{p, \neg p, q\} \mid\!\sim \neg p$ and $\{p, \neg p, q\} \mid\!\sim q$ and $\{p, \neg p, q\} \not\mid\!\sim \neg q$. And, less sets of formulas are rendered useless because there is no model in the pivot for them, though there are models for them. For instance, this time, $\{q, r\} \mid\!\sim p$ and $\{q, r\} \mid\!\sim \neg p$ and $\{q, r\} \mid\!\sim q$ and $\{q, r\} \not\mid\!\sim \neg q$ and $\{q, r\} \mid\!\sim r$ and $\{q, r\} \not\mid\!\sim \neg r$.

However, $\mid\!\sim$ does not satisfy the Disjunctive Syllogism. Indeed, for instance, $\{\neg r, r \vee q\} \not\mid\!\sim q$.

3 Characterizations

The first contributions of the paper are characterizations (in many cases, with purely syntactic conditions) of several families of pivotal and pivotal-discriminative consequence relations. Sometimes, we will need to make some assumptions about the semantic structure under consideration. However, no assumption will be needed for the two following families:

- the pivotal consequence relations (Section 3.2);
- the DP pivotal consequence relations (Section 3.1).

We will assume $(A0)$ for:

- the UC pivotal consequence relations (Section 3.2).

We will need $(A3)$ and $(A1)$ for:

- the CP pivotal-discriminative consequence relations (Section 3.4);
- the CP DP pivotal-discriminative consequence relations (Section 3.3).

We will assume $(A3)$, $(A1)$, and $(A2)$ for:

- the pivotal-discriminative consequence relations (Section 3.4);
- the DP pivotal-discriminative consequence relations (Section 3.3).

We will assume $(A0)$, $(A3)$, and $(A1)$ for:

- the CP UC pivotal-discriminative consequence relations (Section 3.4).

We will need $(A0)$, $(A3)$, $(A1)$, and $(A2)$ for:

- the UC pivotal-discriminative consequence relations (Section 3.4).

3.1 The non-discriminative and definability preserving case

In the present section, we provide, in a general framework, a characterization for the family of all DP pivotal consequence relations. We will use techniques very similar to those of [BN05] (see the DP and non-discriminative case). The latter are themselves strongly inspired by the work, in a classical propositional framework, of K. Schlechta (see Proposition 3.1 of [Sch00]). The idea is to get to the remarkable equality: $\mu(M_\Gamma) = M_{\mid\sim(\Gamma)}$. Thanks to it, properties like Strong Coherence can be easily translated in syntactic terms (i.e. using only the language, $\vdash$, $\mid\!\sim$, etc.).

Definition 20 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and $\mid\!\sim$ be a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then, consider the following conditions: $\forall\, \Gamma, \Delta \subseteq \mathcal{F}$,

($\mid\!\sim 0$) if $\vdash(\Gamma) = \vdash(\Delta)$, then $\mid\!\sim(\Gamma) = \mid\!\sim(\Delta)$;

($\mid\!\sim 1$) $\mid\!\sim(\Gamma) = \vdash(\mid\!\sim(\Gamma))$;

($\mid\!\sim 2$) $\Gamma \subseteq \mid\!\sim(\Gamma)$;

($\mid\!\sim 3$) $\mid\!\sim(\Gamma) \subseteq \vdash(\mid\!\sim(\Delta), \Gamma)$.

Note that those conditions are purely syntactic when there is a proof system available for $\vdash$ (which is the case with e.g. the classical, $\mathcal{FOUR}$, and J_3 semantic structures).

Proposition 21 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and $\mid\!\sim$ be a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then, $\mid\!\sim$ is an DP pivotal consequence relation iff $\mid\!\sim$ satisfies ($\mid\!\sim 0$), ($\mid\!\sim 1$), ($\mid\!\sim 2$), and ($\mid\!\sim 3$).

Proof Direction: "$\rightarrow$".
There exists an DP SC choice function μ from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma))$.
We will show:
(0) $\forall\, \Gamma \subseteq \mathcal{F}$, $\mu(M_\Gamma) = M_{\mid\sim(\Gamma)}$;
(1) $\mid\!\sim$ satisfies ($\mid\!\sim 0$);
(2) $\mid\!\sim$ satisfies ($\mid\!\sim 1$);
(3) $\mid\!\sim$ satisfies ($\mid\!\sim 2$);
(4) $\mid\!\sim$ satisfies ($\mid\!\sim 3$).
Direction: "$\leftarrow$".
Suppose $\mid\!\sim$ satisfies ($\mid\!\sim 0$), ($\mid\!\sim 1$), ($\mid\!\sim 2$), and ($\mid\!\sim 3$).
Let μ be the function from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\mu(M_\Gamma) = M_{\mid\sim(\Gamma)}$.
We will show:
(5) μ is well-defined;
(6) μ is a DP choice function;
(7) μ is SC;
(8) $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma))$.

Proof of (0). Let $\Gamma \subseteq \mathcal{F}$. As μ is DP, $\mu(M_\Gamma) \in \mathbf{D}$. Thus, $\exists\, \Delta \subseteq \mathcal{F}$, $\mu(M_\Gamma) = M_\Delta$.
Therefore, $\mu(M_\Gamma) = M_\Delta = M_{T(M_\Delta)} = M_{T(\mu(M_\Gamma))} = M_{\mid\sim(\Gamma)}$.

Proof of (1). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $\vdash(\Gamma) = \vdash(\Delta)$.
Then, $M_\Gamma = M_\Delta$. Thus, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma)) = T(\mu(M_\Delta)) = \mid\!\sim(\Delta)$.

Proof of (2). Let $\Gamma \subseteq \mathcal{F}$. Then, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(M_{\mid\sim(\Gamma)}) = \vdash(\mid\!\sim(\Gamma))$.

Proof of (3). Let $\Gamma \subseteq \mathcal{F}$. Then, $\Gamma \subseteq T(M_\Gamma) \subseteq T(\mu(M_\Gamma)) = \mid\!\sim(\Gamma)$.

Proof of (4). Let $\Gamma, \Delta \subseteq \mathcal{F}$. Then, by (0) and SC,
$M_{\mid\sim(\Delta),\Gamma} = M_{\mid\sim(\Delta)} \cap M_\Gamma = \mu(M_\Delta) \cap M_\Gamma \subseteq \mu(M_\Gamma) = M_{\mid\sim(\Gamma)}$.
Therefore, by ($\mid\!\sim 1$), we get $\mid\!\sim(\Gamma) = \vdash(\mid\!\sim(\Gamma)) = T(M_{\mid\sim(\Gamma)}) \subseteq T(M_{\mid\sim(\Delta),\Gamma}) = \vdash(\mid\!\sim(\Delta), \Gamma)$.

Proof of (5). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $M_\Gamma = M_\Delta$.

Then, $\vdash(\Gamma) = \vdash(\Delta)$. Thus, by $(\mid\!\sim 0)$, $M_{\mid\sim(\Gamma)} = M_{\mid\sim(\Delta)}$.

Proof of (6). Let $\Gamma \subseteq \mathcal{F}$. Then, by $(\mid\!\sim 2)$, $\mu(M_\Gamma) = M_{\mid\sim(\Gamma)} \subseteq M_\Gamma$.
Consequently, μ is a choice function. In addition, μ is obviously DP.

Proof of (7). Let $\Gamma, \Delta \subseteq \mathcal{F}$.
Then, by $(\mid\!\sim 3)$, we get $\mu(M_\Delta) \cap M_\Gamma = M_{\mid\sim(\Delta)} \cap M_\Gamma = M_{\mid\sim(\Delta),\Gamma} \subseteq M_{\mid\sim(\Gamma)} = \mu(M_\Gamma)$.

Proof of (8). Let $\Gamma \subseteq \mathcal{F}$. Then, by $(\mid\!\sim 1)$, $\mid\!\sim(\Gamma) = \vdash(\mid\!\sim(\Gamma)) = T(M_{\mid\sim(\Gamma)}) = T(\mu(M_\Gamma))$. ∎

3.2 The non-discriminative and not necessarily definability preserving case

In the present section, we will investigate in particular the family of all pivotal consequence relations. Unlike in Section 3.1, the choice functions considered here are not necessarily definability preserving. As a consequence, we will no longer have at our disposal the remarkable equality: $\mu(M_\Gamma) = M_{\mid\sim(\Gamma)}$. Therefore, we cannot translate properties like Strong Coherence in syntactic terms. Moreover, we will put in evidence, in Section 4, some limits of what can be done in this area. Approximatively, we will show, in an infinite classical framework, that there does not exist a characterization (of the aforementioned family) made of conditions which are universally quantified and of limited size.

We provide a solution with semi-syntactic conditions. To do so, we will use techniques very similar to those of [BN05] (see the non-DP and non-discriminative case). The latter are themselves strongly inspired by the work of K. Schlechta (see Proposition 5.2.5 of [Sch04]). Technically, the idea begins by building from any function f, a SC choice function μ_f such that whenever f "covers" some SC choice function, it necessarily covers μ_f.

Definition 22 Let $\mathcal{V}$ be a set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$ and f a function from $\mathbf{V}$ to $\mathbf{W}$.
We denote by μ_f the function from $\mathbf{V}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, V \in \mathbf{V}$,

$$\mu_f(V) = \{v \in V : \forall\, W \in \mathbf{V},\ \text{if } v \in W,\ \text{then } v \in f(W)\}.$$

Lemma 23 Let $\mathcal{V}$ be a set, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, $\mathbf{W} \subseteq \mathcal{P}(\mathcal{V})$ and f a function from $\mathbf{V}$ to $\mathbf{W}$.
Then, μ_f is a SC choice function.

Proof μ_f is obviously a choice function. We show that it satisfies Strong Coherence.
Suppose the contrary, i.e. suppose $\exists\, V, W \in \mathbf{V}$ and $\exists\, v \in \mu_f(W) \cap V$ such that $v \notin \mu_f(V)$.
Then, as $v \in V$ and $v \notin \mu_f(V)$, we have $\exists\, Z \in \mathbf{V}$, $v \in Z$, and $v \notin f(Z)$.
Therefore, simply by definition of μ_f, $v \notin \mu_f(W)$, which is impossible. ∎

Lemma 24 Let $\mathcal{V}$ be a set, $\mathbf{V}$, $\mathbf{W}$, and $\mathbf{X}$ subsets of $\mathcal{P}(\mathcal{V})$, f a function from $\mathbf{V}$ to $\mathbf{W}$, and μ a SC choice function from $\mathbf{V}$ to $\mathbf{X}$ such that $\forall\, V \in \mathbf{V}$, $f(V) = M_{T(\mu(V))}$. Then:

(0) $\forall\, V \in \mathbf{V}$, $f(V) = M_{T(\mu_f(V))}$.

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ is a semantic structure satisfying $(A0)$, $\mathbf{D} \subseteq \mathbf{V}$, and μ is UC. Then:

(1) $\mu_f(\mathcal{V}) = \mu(\mathcal{V})$.

Proof *Proof of* (0). Suppose $V \in \mathbf{V}$. We show $f(V) = M_{T(\mu_f(V))}$.
Case 1: $\exists\, v \in \mu(V)$, $v \notin \mu_f(V)$.
As $\mu(V) \subseteq V$, we have $v \in V$.
Thus, by definition of μ_f, $\exists\, W \in \mathbf{V}$, $v \in W$, and $v \notin f(W) = M_{T(\mu(W))} \supseteq \mu(W)$.
On the other hand, as μ is SC, $\mu(V) \cap W \subseteq \mu(W)$. Thus, $v \in \mu(W)$, which is impossible.
Case 2: $\mu(V) \subseteq \mu_f(V)$.
Case 2.1: $\exists\, v \in \mu_f(V)$, $v \notin f(V)$.
Then, $\exists\, W \in \mathbf{V}$, $v \in W$, and $v \notin f(W)$. Indeed, just take V itself for the choice of W.
Therefore, by definition of μ_f, $v \notin \mu_f(V)$, which is impossible.
Case 2.2: $\mu_f(V) \subseteq f(V)$.
Then, $f(V) = M_{T(\mu(V))} \subseteq M_{T(\mu_f(V))} \subseteq M_{T(f(V))} = M_{T(M_{T(\mu(V))})} = M_{T(\mu(V))} = f(V)$.

Proof of (1). Direction: "$\subseteq$".
Suppose the contrary, i.e. suppose $\exists\, v \in \mu_f(\mathcal{V})$, $v \notin \mu(\mathcal{V})$.
Then, $v \in \mathcal{V} \setminus \mu(\mathcal{V})$. But, as μ is UC, $\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D} \subseteq \mathbf{V}$.
On the other hand, as $v \in \mu_f(\mathcal{V})$, we get $\forall\, W \in \mathbf{V},$ if $v \in W,$ then $v \in f(W)$.
Therefore, $v \in f(\mathcal{V} \setminus \mu(\mathcal{V})) = M_{T(\mu(\mathcal{V} \setminus \mu(\mathcal{V})))}$.
But, we will show:
$(1.0)\quad \mu(\mathcal{V} \setminus \mu(\mathcal{V})) = \emptyset$.
Therefore, $M_{T(\mu(\mathcal{V} \setminus \mu(\mathcal{V})))} = M_{T(\emptyset)} = M_{\mathcal{F}}$.
But, by $(A0)$, $M_{\mathcal{F}} = \emptyset$. Therefore, $v \in \emptyset$, which is impossible.
Direction: "$\supseteq$".
Suppose the contrary, i.e. suppose $\exists\, v \in \mu(\mathcal{V})$, $v \notin \mu_f(\mathcal{V})$.
As $v \in \mathcal{V}$ and $v \notin \mu_f(\mathcal{V})$, we get $\exists\, W \in \mathbf{V}$, $v \in W$ and $v \notin f(W) = M_{T(\mu(W))} \supseteq \mu(W)$.
But, as μ is SC, $\mu(\mathcal{V}) \cap W \subseteq \mu(W)$. Therefore, $v \in \mu(W)$, which is impossible.

Proof of (1.0). Suppose the contrary, i.e. suppose $\exists\, v \in \mu(\mathcal{V} \setminus \mu(\mathcal{V}))$.
As μ is SC, $\mu(\mathcal{V} \setminus \mu(\mathcal{V})) \cap \mathcal{V} \subseteq \mu(\mathcal{V})$. Thus, $v \in \mu(\mathcal{V})$. Therefore, $v \notin \mathcal{V} \setminus \mu(\mathcal{V})$.
But, $\mu(\mathcal{V} \setminus \mu(\mathcal{V})) \subseteq \mathcal{V} \setminus \mu(\mathcal{V})$. Thus, $v \notin \mu(\mathcal{V} \setminus \mu(\mathcal{V}))$, which is impossible. ∎

Definition 25 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and $\mid\!\sim$ be a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
Then, consider the following conditions: $\forall\, \Gamma \subseteq \mathcal{F}$,

$(\mid\!\sim 4)$ $\mid\!\sim(\Gamma) = T(\{v \in M_\Gamma : \forall\, \Delta \subseteq \mathcal{F},\ \text{if } v \in M_\Delta,\ \text{then } v \in M_{\mid\sim(\Delta)}\})$;

$(\mid\!\sim 5)$ $\mathcal{V} \setminus \{v \in \mathcal{V} : \forall\, \Delta \subseteq \mathcal{F},\ \text{if } v \in M_\Delta,\ \text{then } v \in M_{\mid\sim(\Delta)}\} \in \mathbf{D}$.

Proposition 26 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then:

(0) $\mid\!\sim$ is a pivotal consequence relation iff $\mid\!\sim$ satisfies $(\mid\!\sim 4)$.

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies $(A0)$. Then:

(1) $\mid\!\sim$ is a UC pivotal consequence relation iff $\mid\!\sim$ satisfies $(\mid\!\sim 4)$ and $(\mid\!\sim 5)$.

Proof *Proof of* (0). Direction: "$\rightarrow$"
There exists a SC choice function μ from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma))$.
Let f be the function from $\mathbf{D}$ to $\mathbf{D}$ such that $\forall\, V \in \mathbf{D}$, we have $f(V) = M_{T(\mu(V))}$.
By Lemma 24, $\forall\, V \in \mathbf{D}$, we have $f(V) = M_{T(\mu_f(V))}$.
Note that $\forall\, \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{T(\mu(M_\Gamma))} = M_{\mid\sim(\Gamma)}$.

We show that $(\mid\!\sim 4)$ holds. Let $\Gamma \subseteq \mathcal{F}$.
Then, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(f(M_\Gamma)) = T(M_{T(\mu_f(M_\Gamma))}) = T(\mu_f(M_\Gamma)) =$
$T(\{v \in M_\Gamma : \forall\, W \in \mathbf{D}$, if $v \in W$, then $v \in f(W)\}) =$
$T(\{v \in M_\Gamma : \forall\, \Delta \subseteq \mathcal{F}$, if $v \in M_\Delta$, then $v \in f(M_\Delta)\}) =$
$T(\{v \in M_\Gamma : \forall\, \Delta \subseteq \mathcal{F}$, if $v \in M_\Delta$, then $v \in M_{\mid\sim(\Delta)}\})$.

Direction: "$\leftarrow$".
Suppose $\mid\!\sim$ satisfies $(\mid\!\sim 4)$.
Let f be the function from $\mathbf{D}$ to $\mathbf{D}$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, we have $f(M_\Gamma) = M_{\mid\sim(\Gamma)}$.
Note that f is well-defined. Indeed, if $\Gamma, \Delta \subseteq \mathcal{F}$ and $M_\Gamma = M_\Delta$, then, by $(\mid\!\sim 4)$, $\mid\!\sim(\Gamma) = \mid\!\sim(\Delta)$.
In addition, by $(\mid\!\sim 4)$, we clearly have $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T(\mu_f(M_\Gamma))$.
And finally, by Lemma 23, μ_f is a SC choice function.

Proof of (1). Direction: "$\rightarrow$".
Verbatim the proof of (0), except that in addition $(A0)$ holds and μ is UC.
We show that $\mid\!\sim$ satisfies $(\mid\!\sim 5)$. As μ is UC, $\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D}$. But, by Lemma 24, $\mu(\mathcal{V}) = \mu_f(\mathcal{V}) =$
$\{v \in \mathcal{V} : \forall\, W \in \mathbf{D}$, if $v \in W$, then $v \in f(W)\} =$
$\{v \in \mathcal{V} : \forall\, \Delta \subseteq \mathcal{F}$, if $v \in M_\Delta$, then $v \in f(M_\Delta)\} =$
$\{v \in \mathcal{V} : \forall\, \Delta \subseteq \mathcal{F}$, if $v \in M_\Delta$, then $v \in M_{\mid\sim(\Delta)}\}$.

Direction: "$\leftarrow$".
Verbatim the proof of (0), except that in addition $(A0)$ holds and $\mid\!\sim$ satisfies $(\mid\!\sim 5)$.
But, because of $(\mid\!\sim 5)$, $\mathcal{V} \setminus \mu_f(\mathcal{V}) \in \mathbf{D}$. Therefore μ_f is UC.
Note that in this direction $(A0)$ is not used. ∎

3.3 The discriminative and definability preserving case

In the present section, we will characterize certain families of DP pivotal-discriminative consequence relations. We need an inductive construction introduced in [BN05]:

Notation 27 $\mathbb{N}$ denotes the natural numbers: $\{0, 1, 2, \ldots\}$ and
$\mathbb{N}^+$ the positive natural numbers: $\{1, 2, \ldots\}$.

Definition 28 Let $\mathcal{L}$ be a language, $\neg$ a unary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$, and $\Gamma \subseteq \mathcal{F}$. Then,

$$H_1(\Gamma) := \{\neg\beta \in \mathcal{F} : \beta \in \vdash(\Gamma, \mid\!\sim(\Gamma)) \setminus \mid\!\sim(\Gamma) \text{ and } \neg\beta \notin \vdash(\Gamma, \mid\!\sim(\Gamma))\}.$$

Let $i \in \mathbb{N}$ with $i \geq 2$. Then,

$$H_i(\Gamma) := \{\neg\beta \in \mathcal{F} : \left\{ \begin{array}{l} \beta \in \vdash(\Gamma, \mid\!\sim(\Gamma), H_1(\Gamma), \ldots, H_{i-1}(\Gamma)) \setminus \mid\!\sim(\Gamma) \text{ and} \\ \neg\beta \notin \vdash(\Gamma, \mid\!\sim(\Gamma), H_1(\Gamma), \ldots, H_{i-1}(\Gamma)) \end{array} \right. \}.$$

$$H(\Gamma) := \bigcup_{i \in \mathbb{N}^+} H_i(\Gamma).$$

We turn to the representation result:

Definition 29 Suppose $\mathcal{L}$ is a language, $\neg$ a unary connective of $\mathcal{L}$, $\vee$ a binary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
Then, consider the following conditions: $\forall\, \Gamma, \Delta \subseteq \mathcal{F}$, $\forall\, \alpha, \beta \in \mathcal{F}$,

($\mid\!\sim 6$) if $\beta \in \vdash(\Gamma, \mid\!\sim(\Gamma)) \setminus \mid\!\sim(\Gamma)$ and $\neg\alpha \in \vdash(\Gamma, \mid\!\sim(\Gamma), \neg\beta)$, then $\alpha \notin \mid\!\sim(\Gamma)$;

($\mid\!\sim 7$) if $\alpha \in \vdash(\Gamma, \mid\!\sim(\Gamma)) \setminus \mid\!\sim(\Gamma)$ and $\beta \in \vdash(\Gamma, \mid\!\sim(\Gamma), \neg\alpha) \setminus \mid\!\sim(\Gamma)$, then $\alpha \vee \beta \notin \mid\!\sim(\Gamma)$;

($\mid\!\sim 8$) if $\alpha \in \mid\!\sim(\Gamma)$, then $\neg\alpha \notin \vdash(\Gamma, \mid\!\sim(\Gamma))$;

($\mid\!\sim 9$) $\mid\!\sim(\Gamma) \cup H(\Gamma) \subseteq \vdash(\Delta, \mid\!\sim(\Delta), H(\Delta), \Gamma)$;

($\mid\!\sim 10$) if Γ is consistent, then $\mid\!\sim(\Gamma)$ is consistent, $\Gamma \subseteq \mid\!\sim(\Gamma)$, and $\vdash(\mid\!\sim(\Gamma)) = \mid\!\sim(\Gamma)$.

Note that those conditions are purely syntactic when there is a proof system available for $\vdash$

Proposition 30 Suppose $\mathcal{L}$ is a language, $\neg$ a unary connective of $\mathcal{L}$, $\vee$ and $\wedge$ binary connectives of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying $(A3)$ and $(A1)$, and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then:

(0) $\mid\!\sim$ is a CP DP pivotal-discriminative consequence relation iff $\mid\!\sim$ satisfies ($\mid\!\sim 0$), ($\mid\!\sim 6$), ($\mid\!\sim 7$), ($\mid\!\sim 8$), ($\mid\!\sim 9$), and ($\mid\!\sim 10$).

Suppose $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies $(A2)$. Then:

(1) $\mid\!\sim$ is a DP pivotal-discriminative consequence relation iff $\mid\!\sim$ satisfies ($\mid\!\sim 0$), ($\mid\!\sim 6$), ($\mid\!\sim 7$), ($\mid\!\sim 8$) and ($\mid\!\sim 9$).

Before we show Proposition 30, we need to introduce Lemmas 31 and 32 below, taken from [BN05]:

Lemma 31 From [BN05].
Suppose $\mathcal{L}$ is a language, $\neg$ a unary connective of $\mathcal{L}$, $\vee$ and $\wedge$ binary connectives of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying $(A3)$ and $(A1)$, and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$ satisfying ($\mid\!\sim 6$), ($\mid\!\sim 7$), and ($\mid\!\sim 8$).
Then, $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T_d(M_{\Gamma, \mid\sim(\Gamma), H(\Gamma)})$.

Lemma 32 From [BN05].
Suppose $\mathcal{L}$ is a language, $\neg$ a unary connective of $\mathcal{L}$, $\vee$ and $\wedge$ binary connectives of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying $(A3)$ and $(A1)$, $\mathbf{V} \subseteq \mathcal{P}(\mathcal{V})$, μ a DP choice function from $\mathbf{D}$ to $\mathbf{V}$, $\mid\!\sim$ the relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T_d(\mu(M_\Gamma))$, and $\Gamma \subseteq \mathcal{F}$. Then:

(0) $\mid\!\sim$ satisfies ($\mid\!\sim 6$), ($\mid\!\sim 7$), and ($\mid\!\sim 8$);

(1) if $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies $(A2)$ too, then $\mu(M_\Gamma) = M_{\Gamma, \mid\sim(\Gamma), H(\Gamma)}$;

(2) if μ is coherency preserving, then $\mu(M_\Gamma) = M_{\Gamma, \mid\sim(\Gamma), H(\Gamma)}$.

We come to the proof of **Proposition 30**.

Proof *Proof of* (0). Direction: "$\rightarrow$".
There exists a CP DP SC choice function μ from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that
$\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T_d(\mu(M_\Gamma))$.
We will show:
(0.0) $\mid\!\sim$ satisfies ($\mid\!\sim 0$).
By Lemma 32 (0), $\mid\!\sim$ satisfies ($\mid\!\sim 6$), ($\mid\!\sim 7$), and ($\mid\!\sim 8$).

By Lemma 32 (2) and Strong Coherence of μ, $\mid\!\sim$ satisfies $(\mid\!\sim 9)$.
We will show:
(0.1) $\mid\!\sim$ satisfies $(\mid\!\sim 10)$.
Direction: "$\leftarrow$".
Suppose $\mid\!\sim$ satisfies $(\mid\!\sim 0)$, $(\mid\!\sim 6)$, $(\mid\!\sim 7)$, $(\mid\!\sim 8)$, $(\mid\!\sim 9)$, and $(\mid\!\sim 10)$.
Then, let μ be the function from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\mu(M_\Gamma) = M_{\Gamma, \mid\sim(\Gamma), H(\Gamma)}$.
We will show:
(0.2) μ is well-defined.
Clearly, μ is a DP choice function.
In addition, as $\mid\!\sim$ satisfies $(\mid\!\sim 9)$, μ is strongly coherent.
We will show:
(0.3) μ is CP.
And finally, by Lemma 31, $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T_d(\mu(M_\Gamma))$.

Proof of (0.0). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $\vdash(\Gamma) = \vdash(\Delta)$. Then, $M_\Gamma = M_\Delta$.
Therefore, $\mid\!\sim(\Gamma) = T_d(\mu(M_\Gamma)) = T_d(\mu(M_\Delta)) = \mid\!\sim(\Delta)$.

Proof of (0.1). Let $\Gamma \subseteq \mathcal{F}$ and suppose Γ is consistent.
Then, $M_\Gamma \in \mathbf{D} \cap \mathbf{C}$. Thus, as μ is CP, $\mu(M_\Gamma) \in \mathbf{C}$. Therefore, $T_d(\mu(M_\Gamma)) = T(\mu(M_\Gamma))$.
Consequently, $\Gamma \subseteq T(M_\Gamma) \subseteq T(\mu(M_\Gamma)) = T_d(\mu(M_\Gamma)) = \mid\!\sim(\Gamma)$.
In addition, $M_{\mid\sim(\Gamma)} = M_{T_d(\mu(M_\Gamma))} = M_{T(\mu(M_\Gamma))}$. But, $\mu(M_\Gamma) \in \mathbf{C}$. Thus, $M_{T(\mu(M_\Gamma))} \in \mathbf{C}$.
Consequently, $\mid\!\sim(\Gamma)$ is consistent.
And finally, $\mid\!\sim(\Gamma) = T_d(\mu(M_\Gamma)) = T(\mu(M_\Gamma)) = T(M_{T(\mu(M_\Gamma))}) = T(M_{\mid\sim(\Gamma)}) = \vdash(\mid\!\sim(\Gamma))$.

Proof of (0.2). Let $\Gamma, \Delta \subseteq \mathcal{F}$ and suppose $M_\Gamma = M_\Delta$.
Then, $\vdash(\Gamma) = \vdash(\Delta)$. Thus, by $(\mid\!\sim 0)$, $\mid\!\sim(\Gamma) = \mid\!\sim(\Delta)$.
Consequently, $H(\Gamma) = H(\Delta)$. Therefore, $M_{\Gamma, \mid\sim(\Gamma), H(\Gamma)} = M_{\Delta, \mid\sim(\Delta), H(\Delta)}$.

Proof of (0.3). Suppose $V \in \mathbf{D} \cap \mathbf{C}$. Then, $\exists\, \Gamma \subseteq \mathcal{F}$, $V = M_\Gamma$.
Case 1: $H_1(\Gamma) \neq \emptyset$.
Thus, $\exists\, \beta \in \mathcal{F}$, $\beta \notin \mid\!\sim(\Gamma)$ and $M_{\Gamma, \mid\sim(\Gamma)} \subseteq M_\beta$.
By $(\mid\!\sim 10)$, $\Gamma \subseteq \mid\!\sim(\Gamma)$ and $\vdash(\mid\!\sim(\Gamma)) = \mid\!\sim(\Gamma)$. Thus, $M_{\Gamma, \mid\sim(\Gamma)} = M_{\mid\sim(\Gamma)}$. Thus, $M_{\mid\sim(\Gamma)} \subseteq M_\beta$.
Therefore, $\beta \in T(M_{\mid\sim(\Gamma)}) = \vdash(\mid\!\sim(\Gamma)) = \mid\!\sim(\Gamma)$, which is impossible.
Case 2: $H_1(\Gamma) = \emptyset$.
Then, $H(\Gamma) = \emptyset$. Thus, $\mu(V) = \mu(M_\Gamma) = M_{\Gamma, \mid\sim(\Gamma), H(\Gamma)} = M_{\mid\sim(\Gamma)}$.
But, by $(\mid\!\sim 10)$, $\mid\!\sim(\Gamma)$ is consistent. Therefore, $M_{\mid\sim(\Gamma)} \in \mathbf{C}$.

Proof of (1). Direction: "$\rightarrow$".
Verbatim the proof of (0), except that μ is no longer CP, whilst $(A2)$ now holds.
Note that, in (0), CP was used only to show $(\mid\!\sim 9)$ and $(\mid\!\sim 10)$.
But, $(\mid\!\sim 10)$ is no longer required to hold and we are going to get $(\mid\!\sim 9)$ by another mean.
Indeed, by Lemma 32 (1) and Strong Coherence of μ, $(\mid\!\sim 9)$ holds.
Direction: "$\leftarrow$".
Verbatim the proof of (0), except that $(\mid\!\sim 10)$ does no longer hold, whilst $(A2)$ now holds.
However, in (0), $(\mid\!\sim 10)$ was used only to show that μ is CP, which is no longer required.
Note that we do not need to use $(A2)$ in this direction. ∎

3.4 The discriminative and not necessarily definability preserving case

Unlike in Section 3.3, the conditions of the present section will not be purely syntactic. The translation of properties like Strong Coherence in syntactic terms is blocked because we do no longer have the following useful equality: $\mu(M_\Gamma) = M_{\Gamma,\mid\!\sim(\Gamma),H(\Gamma)}$, which hold when the choice functions under consideration are definability preserving (but this is not the case here). Thanks to Lemmas 23 and 24 (stated in Section 3.2), we will provide a solution with semi-syntactic conditions.

Definition 33 Let $\mathcal{L}$ be a language, $\neg$ a unary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure, and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
Then, consider the following conditions: $\forall\, \Gamma \subseteq \mathcal{F}$,

$(\mid\!\sim 11)$ $\vdash(\Gamma, \mid\!\sim(\Gamma), H(\Gamma)) = T(\{v \in M_\Gamma : \forall\, \Delta \subseteq \mathcal{F},\ \text{if } v \in M_\Delta,\ \text{then } v \in M_{\mid\sim(\Delta),H(\Delta)}\})$;

$(\mid\!\sim 12)$ $\mathcal{V} \setminus \{v \in \mathcal{V} : \forall\, \Delta \subseteq \mathcal{F},\ \text{if } v \in M_\Delta,\ \text{then } v \in M_{\mid\sim(\Delta),H(\Delta)}\} \in \mathbf{D}$.

Proposition 34 Suppose $\mathcal{L}$ is a language, $\neg$ a unary connective of $\mathcal{L}$, $\vee$ and $\wedge$ binary connectives of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying $(A3)$ and $(A1)$, and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$. Then:

(0) $\mid\!\sim$ is a CP pivotal-discriminative consequence relation iff $\mid\!\sim$ satisfies $(\mid\!\sim 0)$, $(\mid\!\sim 6)$, $(\mid\!\sim 7)$, $(\mid\!\sim 8)$, $(\mid\!\sim 10)$, and $(\mid\!\sim 11)$.

If $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies $(A0)$ too, then:

(1) $\mid\!\sim$ is a CP UC pivotal-discriminative consequence relation iff $\mid\!\sim$ satisfies $(\mid\!\sim 0)$, $(\mid\!\sim 6)$, $(\mid\!\sim 7)$, $(\mid\!\sim 8)$, $(\mid\!\sim 10)$, $(\mid\!\sim 11)$, and $(\mid\!\sim 12)$.

If $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies $(A2)$ too, then:

(2) $\mid\!\sim$ is a pivotal-discriminative consequence relation iff $\mid\!\sim$ satisfies $(\mid\!\sim 0)$, $(\mid\!\sim 6)$, $(\mid\!\sim 7)$, $(\mid\!\sim 8)$, and $(\mid\!\sim 11)$.

If $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ satisfies $(A0)$ and $(A2)$ too, then:

(3) $\mid\!\sim$ is a UC pivotal-discriminative consequence relation iff $\mid\!\sim$ satisfies $(\mid\!\sim 0)$, $(\mid\!\sim 6)$, $(\mid\!\sim 7)$, $(\mid\!\sim 8)$, $(\mid\!\sim 11)$, and $(\mid\!\sim 12)$.

Proof *Proof of* (2). Direction: "$\rightarrow$".
There exists a SC choice function μ from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T_d(\mu(M_\Gamma))$.
Then, $\mid\!\sim$ satisfies obviously $(\mid\!\sim 0)$.
Let f be the function from $\mathbf{D}$ to $\mathbf{D}$ such that $\forall\, V \in \mathbf{D}$, $f(V) = M_{T(\mu(V))}$.
Then, by Lemma 24, $\forall\, V \in \mathbf{D}$, $f(V) = M_{T(\mu_f(V))}$.
Moreover, $\forall\, \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{T(\mu(M_\Gamma))} \subseteq M_{T(M_\Gamma)} = M_\Gamma$.
Therefore, f is a choice function.
Obviously, f is DP.
In addition, $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T_d(\mu(M_\Gamma)) = T_d(M_{T(\mu(M_\Gamma))}) = T_d(f(M_\Gamma))$.
Consequently, by Lemma 32 (0), $\mid\!\sim$ satisfies $(\mid\!\sim 6)$, $(\mid\!\sim 7)$, and $(\mid\!\sim 8)$.
In addition, by Lemma 32 (1), $\forall\, \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma,\mid\sim(\Gamma),H(\Gamma)}$.
We show that $\mid\!\sim$ satisfies $(\mid\!\sim 11)$. Let $\Gamma \subseteq \mathcal{F}$.
Then, $\vdash(\Gamma, \mid\!\sim(\Gamma), H(\Gamma)) = T(M_{\Gamma,\mid\sim(\Gamma),H(\Gamma)}) = T(f(M_\Gamma)) = T(M_{T(\mu_f(M_\Gamma))}) = T(\mu_f(M_\Gamma)) =$
$T(\{v \in M_\Gamma : \forall\, W \in \mathbf{D}$, if $v \in W$, then $v \in f(W)\}) =$

$T(\{v \in M_\Gamma : \forall\, \Delta \subseteq \mathcal{F}$, if $v \in M_\Delta$, then $v \in f(M_\Delta)\}) =$
$T(\{v \in M_\Gamma : \forall\, \Delta \subseteq \mathcal{F}$, if $v \in M_\Delta$, then $v \in M_{\Delta, \mid\!\sim(\Delta), H(\Delta)}\}) =$
$T(\{v \in M_\Gamma : \forall\, \Delta \subseteq \mathcal{F},$ if $v \in M_\Delta,$ then $v \in M_{\mid\!\sim(\Delta), H(\Delta)}\})$.

Direction: "$\leftarrow$".

Suppose $(\mid\!\sim 0)$, $(\mid\!\sim 6)$, $(\mid\!\sim 7)$, $(\mid\!\sim 8)$, and $(\mid\!\sim 11)$ hold.
Let f be the function from $\mathbf{D}$ to $\mathbf{D}$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma, \mid\!\sim(\Gamma), H(\Gamma)}$.
By $(\mid\!\sim 0)$, f is well-defined.
By Lemma 31, $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T_d(M_{\Gamma, \mid\!\sim(\Gamma), H(\Gamma)}) = T_d(f(M_\Gamma))$.
By $(\mid\!\sim 11)$, $\forall\, \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{T(\mu_f(M_\Gamma))}$.
Therefore, $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T_d(f(M_\Gamma)) = T_d(M_{T(\mu_f(M_\Gamma))}) = T_d(\mu_f(M_\Gamma))$.
But, by Lemma 23, μ_f is a SC choice function.

Proof of (3). Direction: "$\rightarrow$".
Verbatim the proof of (2), except that in addition $(A0)$ holds and μ is UC.
We show that $(\mid\!\sim 12)$ holds. As μ is UC, $\mathcal{V} \setminus \mu(\mathcal{V}) \in \mathbf{D}$. But, by Lemma 24 (1), $\mu(\mathcal{V}) = \mu_f(\mathcal{V}) =$
$\{v \in \mathcal{V} : \forall\, W \in \mathbf{D}$, if $v \in W$, then $v \in f(W)\} =$
$\{v \in \mathcal{V} : \forall\, \Delta \subseteq \mathcal{F}$, if $v \in M_\Delta$, then $v \in f(M_\Delta)\} =$
$\{v \in \mathcal{V} : \forall\, \Delta \subseteq \mathcal{F},$ if $v \in M_\Delta,$ then $v \in M_{\Delta, \mid\!\sim(\Delta), H(\Delta)}\} =$
$\{v \in \mathcal{V} : \forall\, \Delta \subseteq \mathcal{F},$ if $v \in M_\Delta,$ then $v \in M_{\mid\!\sim(\Delta), H(\Delta)}\}$.

Direction: "$\leftarrow$".

Verbatim the proof of (2), except that in addition $(A0)$ holds and $\mid\!\sim$ satisfies $(\mid\!\sim 12)$.
But, because of $(\mid\!\sim 12)$, $\mathcal{V} \setminus \mu_f(\mathcal{V}) \in \mathbf{D}$. Therefore μ_f is UC.
Note that $(A0)$ is not used in this direction.

Proof of (0). Direction: "$\rightarrow$".
Verbatim the proof of (2), except that $(A2)$ does no longer hold, whilst μ is now CP.
Note that $(A2)$ was used, in (2), only to apply Lemma 32 (1) to get $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma, \mid\!\sim(\Gamma), H(\Gamma)}$.
But, we will get this equality by another mean.
Indeed, if $V \in \mathbf{D} \cap \mathbf{C}$, then, as μ is CP, $\mu(V) \in \mathbf{C}$, thus $M_{T(\mu(V))} \in \mathbf{C}$, thus $f(V) \in \mathbf{C}$.
Therefore f is CP.
Consequently, by Lemma 32 (2), we get $\forall\, \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma, \mid\!\sim(\Gamma), H(\Gamma)}$.
In addition, by verbatim the proof of (0.1) of Proposition 30, $\mid\!\sim$ satisfies $(\mid\!\sim 10)$.

Direction: "$\leftarrow$".

Verbatim the proof of (2), except that $(A2)$ does no longer hold, whilst $\mid\!\sim$ satisfies now $(\mid\!\sim 10)$.
But, in this direction, $(A2)$ was not used in (2).
It remains to show that μ_f is CP.
By verbatim the proof of (0.3) of Proposition 30, we get that f is CP.
Let $V \in \mathbf{D} \cap \mathbf{C}$. Then, $f(V) \in \mathbf{C}$. Thus, $M_{T(\mu_f(V))} \in \mathbf{C}$. Thus, $\mu_f(V) \in \mathbf{C}$ and we are done.

Proof of (1). Direction: "$\rightarrow$".
Verbatim the proof of (2), except that $(A2)$ does no longer hold, whilst $(A0)$ now holds and μ is now UC and CP.
Note that $(A2)$ was used, in (2), only to apply Lemma 32 (1) to get $\forall \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma, \mid\!\sim(\Gamma), H(\Gamma)}$.
But, by verbatim the proof of (0), we get anyway $\forall\, \Gamma \subseteq \mathcal{F}$, $f(M_\Gamma) = M_{\Gamma, \mid\!\sim(\Gamma), H(\Gamma)}$.
In addition, by verbatim the proof of (0.1) of Proposition 30, $\mid\!\sim$ satisfies $(\mid\!\sim 10)$.
And, by verbatim the proof of (3), $\mid\!\sim$ satisfies $(\mid\!\sim 12)$.

Direction: "$\leftarrow$".

Verbatim the proof of (2), except that $(A2)$ does no longer hold, whilst $(A0)$ now holds and $\mid\!\sim$ satisfies now $(\mid\!\sim 10)$ and $(\mid\!\sim 12)$.
But, in this direction, $(A2)$ was not used in (2).
In addition, by verbatim the proof of (0), μ_f is CP.
And, because of $(\mid\!\sim 12)$, $\mathcal{V} \setminus \mu_f(\mathcal{V}) \in \mathbf{D}$. Therefore μ_f is UC.
Note that $(A0)$ is not used in this direction. ∎

4 Nonexistence of normal characterizations

4.1 Definition

Let $\mathcal{F}$ be a set, $\mathcal{R}$ a set of relations on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$, and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
Approximatively, a characterization of $\mathcal{R}$ will be called "normal" iff it contains only conditions which are universally quantified and "apply" $\mid\!\sim$ at most $|\mathcal{F}|$ times. More formally,

Definition 35 Let $\mathcal{F}$ be a set and $\mathcal{R}$ a set of relations on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
We say that that $\mathcal{C}$ is a *normal characterization* of $\mathcal{R}$ iff $\mathcal{C} = \langle \lambda, \Phi \rangle$, where $\lambda \leq |\mathcal{F}|$ is a (finite or infinite) cardinal and Φ is a relation on $\mathcal{P}(\mathcal{F})^{2\lambda}$ such that for every relation $\mid\!\sim$ on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$,

$$\mid\!\sim \in \mathcal{R} \text{ iff } \forall\, \Gamma_1, \ldots, \Gamma_\lambda \subseteq \mathcal{F},\ (\Gamma_1, \ldots, \Gamma_\lambda, \mid\!\sim(\Gamma_1), \ldots, \mid\!\sim(\Gamma_\lambda)) \in \Phi.$$

Now, suppose there is no normal characterization of $\mathcal{R}$. Here are examples (i.e. $(C1)$, $(C2)$, and $(C3)$ below) that will give the reader (we hope) a good idea which conditions cannot characterize $\mathcal{R}$. This will thus make clearer the range of our impossibility result (Proposition 37 below). To begin, consider the following condition:

$(C1)$ $\forall\, \Gamma, \Delta \in \mathbf{F} \subseteq \mathcal{P}(\mathcal{F})$, $\mid\!\sim(\Gamma \cup \mid\!\sim(\Delta)) = \emptyset$.

Then, $(C1)$ cannot characterize $\mathcal{R}$. Indeed, suppose the contrary, i.e.
suppose $\mid\!\sim \in \mathcal{R}$ iff $\forall\, \Gamma, \Delta \in \mathbf{F}$, $\mid\!\sim(\Gamma \cup \mid\!\sim(\Delta)) = \emptyset$.
Then, take $\lambda = 3$ and the relation Φ such that $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6) \in \Phi$ iff
$(\Gamma_1, \Gamma_2 \in \mathbf{F}$ and $\Gamma_3 = \Gamma_1 \cup \Gamma_5)$ entails $\Gamma_6 = \emptyset$.
Then, $\langle 3, \Phi \rangle$ is a normal characterization of $\mathcal{R}$. We give the easy proof of this, so that the reader can check that a convenient relation Φ can be found quickly for all simple conditions like $(C1)$.

Proof Direction: "$\rightarrow$".
Suppose $\mid\!\sim \in \mathcal{R}$.
Then, $\forall\, \Gamma, \Delta \in \mathbf{F}$, $\mid\!\sim(\Gamma \cup \mid\!\sim(\Delta)) = \emptyset$.
Let $\Gamma_1, \Gamma_2, \Gamma_3 \subseteq \mathcal{F}$.
We show $(\Gamma_1, \Gamma_2, \Gamma_3, \mid\!\sim(\Gamma_1), \mid\!\sim(\Gamma_2), \mid\!\sim(\Gamma_3)) \in \Phi$.
Suppose $\Gamma_1, \Gamma_2 \in \mathbf{F}$ and $\Gamma_3 = \Gamma_1 \cup \mid\!\sim(\Gamma_2)$.
Then, as $\Gamma_1, \Gamma_2 \in \mathbf{F}$, we get $\mid\!\sim(\Gamma_1 \cup \mid\!\sim(\Gamma_2)) = \emptyset$.
But, $\mid\!\sim(\Gamma_1 \cup \mid\!\sim(\Gamma_2)) = \mid\!\sim(\Gamma_3)$. Therefore, $\mid\!\sim(\Gamma_3) = \emptyset$.
Direction: "$\leftarrow$".
Suppose $\forall\, \Gamma_1, \Gamma_2, \Gamma_3 \subseteq \mathcal{F}$, $(\Gamma_1, \Gamma_2, \Gamma_3, \mid\!\sim(\Gamma_1), \mid\!\sim(\Gamma_2), \mid\!\sim(\Gamma_3)) \in \Phi$.
We show $\mid\!\sim \in \mathcal{R}$. Let $\Gamma, \Delta \in \mathbf{F}$.
Take $\Gamma_1 = \Gamma$, $\Gamma_2 = \Delta$, $\Gamma_3 = \Gamma_1 \cup \mid\!\sim(\Gamma_2)$.
Then, we have $\Gamma_1 \in \mathbf{F}$, $\Gamma_2 \in \mathbf{F}$, and $\Gamma_3 = \Gamma_1 \cup \mid\!\sim(\Gamma_2)$.
But, $(\Gamma_1, \Gamma_2, \Gamma_3, \mid\!\sim(\Gamma_1), \mid\!\sim(\Gamma_2), \mid\!\sim(\Gamma_3)) \in \Phi$.

Therefore, by definition of Φ, $\mid\!\sim(\Gamma_3) = \emptyset$.
But, $\mid\!\sim(\Gamma_3) = \mid\!\sim(\Gamma_1 \cup \mid\!\sim(\Gamma_2)) = \mid\!\sim(\Gamma \cup \mid\!\sim(\Delta))$. ∎

But actually, we are not limited to simple operations (like e.g. $\cup$, $\cap$, $\setminus$). More complex conditions than $(C1)$ are also excluded. For instance, let f be any function from $\mathcal{P}(\mathcal{F})$ to $\mathcal{P}(\mathcal{F})$ and consider the following condition:

$(C2)$ $\forall\, \Gamma, \Delta \in \mathbf{F}$, $\mid\!\sim(f(\Gamma) \cup \mid\!\sim(\Delta)) = \emptyset$.

Then, $(C2)$ cannot characterize $\mathcal{R}$. Indeed, suppose it characterizes $\mathcal{R}$.
Then, take the relation Φ such that $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6) \in \Phi$ iff
$(\Gamma_1, \Gamma_2 \in \mathbf{F}$ and $\Gamma_3 = f(\Gamma_1) \cup \Gamma_5)$ entails $\Gamma_6 = \emptyset$.
It can be checked that $\langle 3, \Phi \rangle$ is a normal characterization of $\mathcal{R}$. We leave the easy proof to the reader.

We can even go further combining universal (not existential) quantifiers and functions like f. For instance, let $\mathcal{G}$ be a set of functions from $\mathcal{P}(\mathcal{F})$ to $\mathcal{P}(\mathcal{F})$ and consider the following condition:

$(C3)$ $\forall\, \Gamma, \Delta \in \mathbf{F}, \forall\, f \in \mathcal{G}$, $\mid\!\sim(f(\Gamma) \cup \mid\!\sim(\Delta)) = \emptyset$.

Then, $(C3)$ cannot characterize $\mathcal{R}$. Indeed, suppose it characterizes $\mathcal{R}$.
Then, take the relation Φ such that $(\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4, \Gamma_5, \Gamma_6) \in \Phi$ iff
$\forall\, f \in \mathcal{G}$, if $(\Gamma_1, \Gamma_2 \in \mathbf{F}$ and $\Gamma_3 = f(\Gamma_1) \cup \Gamma_5)$, then $\Gamma_6 = \emptyset$.
It can be checked that $\langle 3, \Phi \rangle$ is a normal characterization of $\mathcal{R}$. The easy proof is left to the reader.

Finally, a good example of a condition which is not excluded is $(\mid\!\sim 4)$. We have seen in Proposition 26 that it characterizes the family of all pivotal consequence relations.

4.2 Impossibility results

In the present section, we will show, in an infinite classical framework, that there is no normal characterization for the family of all pivotal consequence relations (in other words, $(\mid\!\sim 4)$ cannot be replaced by a simpler condition in Proposition 26). In the same vein, in Proposition 5.2.15 of [Sch04], K. Schlechta showed that there does not exist a normal characterization for the family of all preferential consequence relations.

Note that he used the word "normal" in a more restrictive sense (see Section 1.6.2.1 of [Sch04]). Approximatively, a characterization of $\mathcal{R}$ is called normal by Schlechta iff it contains only conditions like $(C1)$, i.e. conditions which are universally quantified, "apply" $\mid\!\sim$ at most $|\mathcal{F}|$ times, and use only elementary operations like e.g. $\cup$, $\cap$, $\setminus$ (complex structures, functions, etc are not allowed). We have been inspired by the techniques of Schlechta. We will need Lemma 5.2.14 of [Sch04]:

Lemma 36 From [Sch04].
Suppose $\mathcal{A}$ is infinite and $\langle \mathcal{F}_c, \mathcal{V}, \models \rangle$ is a classical propositional semantic structure.
Let $\mathbf{V} \subseteq \{V \subseteq \mathcal{V} : |V| \leq |\mathcal{A}|\}$ satisfying the two following conditions:
first, if $V \in \mathbf{V}$ and $W \subseteq V$, then $W \in \mathbf{V}$;
and second, $\forall\, V, W \in \mathbf{V}$, if $|V \cup W| \leq |\mathcal{A}|$, then $V \cup W \in \mathbf{V}$.
Then, $\forall\, \Gamma \subseteq \mathcal{F}_c$, $\exists\, V_\Gamma \in \mathbf{V}$,

(0) $T(\bigcap_{V \in \mathbf{V}} M_{T(M_\Gamma \setminus V)}) = T(M_\Gamma \setminus V_\Gamma)$;

(1) $\forall\, V \in \mathbf{V}$, $T(M_\Gamma \setminus V) \subseteq T(M_\Gamma \setminus V_\Gamma)$.

Recall that $\mathcal{A}$ and $\mathcal{F}_c$ have been introduced in Section 2.1.2. Note that the subscript in V_Γ is written just to keep in mind that V_Γ depends on Γ.

Proposition 37 Suppose $\mathcal{A}$ is infinite and $\langle \mathcal{F}_c, \mathcal{V}, \models\rangle$ is a classical propositional semantic structure. Then, there doesn't exist a normal characterization for the family of all pivotal consequence relations.

Proof Suppose the contrary, i.e. suppose there exist a cardinal $\lambda \leq |\mathcal{F}_c|$ and a relation Φ over $\mathcal{P}(\mathcal{F}_c)^{2\lambda}$ such that for every relation $\mid\!\sim$ on $\mathcal{P}(\mathcal{F}_c) \times \mathcal{F}_c$, $\mid\!\sim$ is a pivotal consequence relation iff $\forall\, \Gamma_1, \ldots, \Gamma_\lambda \subseteq \mathcal{F}_c$, $(\Gamma_1, \ldots, \Gamma_\lambda, \mid\!\sim(\Gamma_1), \ldots, \mid\!\sim(\Gamma_\lambda)) \in \Phi$. Then, define:
$\mathbf{V} := \{V \subseteq \mathcal{V} : |V| \leq |\mathcal{A}|\}$.
In addition, let $\mid\!\sim$ be the relation on $\mathcal{P}(\mathcal{F}_c) \times \mathcal{F}_c$ such that $\forall\, \Gamma \subseteq \mathcal{F}_c$,
$\mid\!\sim(\Gamma) = T(\bigcap_{V \in \mathbf{V}} M_{T(M_\Gamma \setminus V)})$.
We will show:
(0) $\forall\, V \subseteq \mathcal{V}$, if $|V| \leq |\mathcal{A}|$, then $T(\mathcal{V}) = T(\mathcal{V} \setminus V)$;
(1) $\exists\, \Gamma_1, \ldots, \Gamma_\lambda \subseteq \mathcal{F}_c$ such that $(\Gamma_1, \ldots, \Gamma_\lambda, \mid\!\sim(\Gamma_1), \ldots, \mid\!\sim(\Gamma_\lambda)) \notin \Phi$.
Now, by lemma 36, we get:
(2) $\forall\, \Gamma \subseteq \mathcal{F}_c$, $\exists\, V_\Gamma \in \mathbf{V}$, $\mid\!\sim(\Gamma) = T(M_\Gamma \setminus V_\Gamma)$ and $\forall\, V \in \mathbf{V}$, $T(M_\Gamma \setminus V) \subseteq T(M_\Gamma \setminus V_\Gamma)$.
Then, define:
$\mathcal{X} := \bigcup_{\Gamma \in \{\Gamma_1, \ldots, \Gamma_\lambda\}} V_\Gamma$.
Then, we will show:
(3) $\forall\, \Gamma \in \{\Gamma_1, \ldots, \Gamma_\lambda\}$, $\mid\!\sim(\Gamma) = T(M_\Gamma \setminus \mathcal{X})$.
Let μ be the function from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, V \in \mathbf{D}$, $\mu(V) = V \setminus \mathcal{X}$.
We will show:
(4) μ is a SC choice function.
Let $\mid\!\sim'$ be the pivotal consequence relation defined by μ.
We will show the following, which entails a contradiction:
(5) $\mid\!\sim'$ is not a pivotal consequence relation.

Proof of (0). Let $V \subseteq \mathcal{V}$ and suppose $|V| \leq |\mathcal{A}|$.
Obviously, $T(\mathcal{V}) \subseteq T(\mathcal{V} \setminus V)$.
We show $T(\mathcal{V} \setminus V) \subseteq T(\mathcal{V})$.
Suppose the contrary, i.e. suppose $\exists\, \alpha \in T(\mathcal{V} \setminus V)$, $\alpha \notin T(\mathcal{V})$.
Then, $\exists\, v \in \mathcal{V}$, $v \notin M_\alpha$.
Now, define:
$W := \{w \in \mathcal{V}$: for all atom q occurring in α, $w(q) = v(q)\}$.
Then, $\forall\, w \in W$, we have $w(\alpha) = v(\alpha)$ and thus $w \notin M_\alpha$.
As the number of atoms occurring in α is finite and $\mathcal{A}$ is infinite, we get $|W| = 2^{|\mathcal{A}|}$.
Therefore, $|V| \leq |\mathcal{A}| < |W|$. Thus, $\exists\, w \in W \setminus V \subseteq \mathcal{V} \setminus V$.
Thus, $\mathcal{V} \setminus V \not\subseteq M_\alpha$. Therefore, $\alpha \notin T(\mathcal{V} \setminus V)$, which is impossible.

Proof of (1). It suffices to show that $\mid\!\sim$ is not a pivotal consequence relation.
Suppose the contrary, i.e. suppose there exists a SC choice function μ from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, \Gamma \subseteq \mathcal{F}_c$, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma))$.
As $\mathcal{A}$ is infinite, $\exists\, p \in \mathcal{A}$. We show that all cases are impossible.
Case 1: $\exists\, v \in \mu(\mathcal{V})$, $v \notin M_p$.
Let $\Gamma = T(v)$. Then, $M_\Gamma = \{v\}$.
By SC of μ, we have $\mu(M_\Gamma) = \mu(M_\Gamma) \cap \mathcal{V} \subseteq \mu(\mathcal{V})$. Thus, $\mu(M_\Gamma) \subseteq \mu(\mathcal{V}) \cap M_\Gamma$.
On the other hand, again by SC, $\mu(\mathcal{V}) \cap M_\Gamma \subseteq \mu(M_\Gamma)$. Consequently, $\mu(\mathcal{V}) \cap M_\Gamma = \mu(M_\Gamma)$.
Therefore, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma)) = T(\mu(\mathcal{V}) \cap M_\Gamma) = T(\mu(\mathcal{V}) \cap \{v\}) = T(v)$.
But, $p \notin T(v)$. Thus, $p \notin \mid\!\sim(\Gamma)$.
However, $M_\Gamma \in \mathbf{V}$. Therefore, $\bigcap_{V \in \mathbf{V}} M_{T(M_\Gamma \setminus V)} \subseteq M_{T(M_\Gamma \setminus M_\Gamma)} = M_{T(\emptyset)} = M_{\mathcal{F}_c} = \emptyset$.

Therefore, by definition of $\mid\!\sim$, we have $\mid\!\sim(\Gamma) = T(\emptyset) = \mathcal{F}_c$.
Thus, $p \in \mid\!\sim(\Gamma)$, which is impossible.
Case 2: $\mu(\mathcal{V}) \subseteq M_p$.
Then, by (0), $\mid\!\sim(\emptyset) = T(\bigcap_{V \in \mathbf{V}} M_{T(\mathcal{V} \setminus V)}) = T(\bigcap_{V \in \mathbf{V}} M_{T(\mathcal{V})}) = T(M_{T(\mathcal{V})}) = T(\mathcal{V})$.
But, $\mathcal{V} \not\subseteq M_p$. Thus, $p \notin T(\mathcal{V}) = \mid\!\sim(\emptyset)$.
On the other hand, $\mid\!\sim(\emptyset) = T(\mu(M_\emptyset)) = T(\mu(\mathcal{V}))$.
But, $\mu(\mathcal{V}) \subseteq M_p$. Thus, $p \in T(\mu(\mathcal{V})) = \mid\!\sim(\emptyset)$, which is impossible.

Proof of (3). Let $\Gamma \in \{\Gamma_1, \ldots, \Gamma_\lambda\}$. Direction: "$\subseteq$".
We have $V_\Gamma \subseteq \mathcal{X}$. Thus, $M_\Gamma \setminus \mathcal{X} \subseteq M_\Gamma \setminus V_\Gamma$.
Therefore, by (2), $\mid\!\sim(\Gamma) = T(M_\Gamma \setminus V_\Gamma) \subseteq T(M_\Gamma \setminus \mathcal{X})$.
Direction: "$\supseteq$".
As $\mathcal{A}$ is infinite, $|\mathcal{A}| = |\mathcal{F}_c|$. Therefore, $\lambda \leq |\mathcal{A}|$. Thus, $|\mathcal{X}| \leq |\mathcal{A}|^2 = |\mathcal{A}|$.
Thus, $\mathcal{X} \in \mathbf{V}$. Thus, by (2), $T(M_\Gamma \setminus \mathcal{X}) \subseteq T(M_\Gamma \setminus V_\Gamma) = \mid\!\sim(\Gamma)$.

Proof of (4). μ is clearly a choice function. We show that μ satisfies SC. Let $V, W \subseteq \mathcal{V}$.
Then, $\mu(W) \cap V = (W \setminus \mathcal{X}) \cap V = (W \cap V) \setminus \mathcal{X} \subseteq V \setminus \mathcal{X} = \mu(V)$.

Proof of (5). By (3), $\forall\, \Gamma \in \{\Gamma_1, \ldots, \Gamma_\lambda\}$, $\mid\!\sim'(\Gamma) = T(\mu(M_\Gamma)) = T(M_\Gamma \setminus \mathcal{X}) = \mid\!\sim(\Gamma)$.
But, $(\Gamma_1, \ldots, \Gamma_\lambda, \mid\!\sim(\Gamma_1), \ldots, \mid\!\sim(\Gamma_\lambda)) \notin \Phi$. Therefore, $(\Gamma_1, \ldots, \Gamma_\lambda, \mid\!\sim'(\Gamma_1), \ldots, \mid\!\sim'(\Gamma_\lambda)) \notin \Phi$.
Consequently, as $\langle \lambda, \Phi \rangle$ is a normal characterization, $\mid\!\sim'$ is not a pivotal consequence relation. ∎

5 A link with X-logics

In this section, we investigate a link between pivotal consequence relations and pertinence consequence relations (alias X-logics) which were first introduced by Forget, Risch, and Siegel [FRS01]. Suppose some formulas are considered to be the pertinent ones in the absolute sense and collect them in a set $\mathcal{E}$. Then, it is natural to conclude a formula α from a set of formulas Γ iff every pertinent basic consequence of $\Gamma \cup \{\alpha\}$ is a basic consequence of Γ (i.e. the addition of α to Γ does not yield more pertinent formulas than with Γ alone). This constitutes a pertinence consequence relation. More formally,

Definition 38 Let $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ be a semantic structure and $\mid\!\sim$ a relation on $\mathcal{P}(\mathcal{F}) \times \mathcal{F}$.
We say that $\mid\!\sim$ is a *pertinence consequence relation* (alias X-logic) iff there exists $\mathcal{E} \subseteq \mathcal{F}$ such that $\forall\, \Gamma \subseteq \mathcal{F}, \forall\, \alpha \in \mathcal{F}$,

$$\Gamma \mid\!\sim \alpha \text{ iff } \vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma).$$

In addition, if $\vdash(\mathcal{E}) = \mathcal{E}$, we say that $\mid\!\sim$ is *closed*.

We introduce a new assumption about semantic structures (in fact, simply a weak version of $(A3)$):

Definition 39 Suppose $\mathcal{L}$ is a language, $\vee$ a binary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, and $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure. Then, define the following condition:

$(A4)$ $\forall\, \alpha, \beta \in \mathcal{F}$, $M_{\alpha \vee \beta} = M_\alpha \cup M_\beta$.

We will show that when $(A4)$ is assumed, then UC pivotal consequence relations are precisely closed pertinence consequence relations. We need before Notation 40 and the very easy Proposition 41 (which we will use implicitly in the sequel).

Notation 40 Suppose $\mathcal{L}$ is a language, $\vee$ a binary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\Gamma \subseteq \mathcal{F}$ and $\Delta \subseteq \mathcal{F}$. Then:
$\Gamma \vee \Delta := \{\alpha \vee \beta : \alpha \in \Gamma \text{ and } \beta \in \Delta\}$.

Proposition 41 Suppose $\mathcal{L}$ is a language, $\vee$ a binary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying $(A4)$, $\Gamma \subseteq \mathcal{F}$, and $\Delta \subseteq \mathcal{F}$.
Then, $M_\Gamma \cup M_\Delta = M_{\Gamma \vee \Delta}$.

Proof Direction: "$\subseteq$".
Suppose the contrary, i.e. suppose $\exists\, v \in M_\Gamma \cup M_\Delta$, $v \not\in M_{\Gamma \vee \Delta}$.
Then, $\exists\, \alpha \in \Gamma$, $\exists\, \beta \in \Delta$, $v \not\in M_{\alpha \vee \beta}$.
But, by $(A4)$, $v \in M_\Gamma \cup M_\Delta \subseteq M_\alpha \cup M_\beta = M_{\alpha \vee \beta}$, which is impossible.
Direction: "$\supseteq$".
Suppose the contrary, i.e. suppose $\exists\, v \in M_{\Gamma \vee \Delta}$, $v \not\in M_\Gamma \cup M_\Delta$.
Then, $\exists\, \alpha \in \Gamma$, $v \not\in M_\alpha$ and $\exists\, \beta \in \Delta$, $v \not\in M_\beta$.
Therefore, by $(A4)$, $v \not\in M_\alpha \cup M_\beta = M_{\alpha \vee \beta}$.
However $\alpha \vee \beta \in \Gamma \vee \Delta$. Thus, $v \not\in M_{\Gamma \vee \Delta}$ which is impossible. ∎

Proposition 42 Suppose $\mathcal{L}$ is a language, $\vee$ a binary connective of $\mathcal{L}$, $\mathcal{F}$ the set of all wffs of $\mathcal{L}$, and $\langle \mathcal{F}, \mathcal{V}, \models \rangle$ a semantic structure satisfying $(A4)$.
Then, UC pivotal consequence relations are precisely closed pertinence consequence relations.

Proof Direction: "$\subseteq$".
Let $\mid\!\sim$ be an UC pivotal consequence relation.
Then, there is an UC SC choice function from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma))$.
Thus, by Proposition 17, there exists $\mathcal{I} \subseteq \mathcal{V}$ such that $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$ and $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T(M_\Gamma \cap \mathcal{I})$.
Define: $\mathcal{E} := T(\mathcal{V} \setminus \mathcal{I})$.
Then, $\vdash(\mathcal{E}) = T(M_\mathcal{E}) = T(M_{T(\mathcal{V} \setminus \mathcal{I})}) = T(\mathcal{V} \setminus \mathcal{I}) = \mathcal{E}$.
In addition, as $\mathcal{V} \setminus \mathcal{I} \in \mathbf{D}$, we have $M_\mathcal{E} = M_{T(\mathcal{V} \setminus \mathcal{I})} = \mathcal{V} \setminus \mathcal{I}$.
We show:
(0) $\quad \forall\, \Gamma \subseteq \mathcal{F}, \forall\, \alpha \in \mathcal{F}$, $\Gamma \mid\!\sim \alpha$ iff $\vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma)$.
Consequently, $\mid\!\sim$ is a closed pertinence consequence relation.
Direction: "$\supseteq$".
Let $\mid\!\sim$ be a closed pertinence consequence relation.
Then, there is $\mathcal{E} \subseteq \mathcal{F}$ such that $\mathcal{E} = \vdash(\mathcal{E})$ and $\forall\, \Gamma \subseteq \mathcal{F}$, $\forall\, \alpha \in \mathcal{F}$, $\Gamma \mid\!\sim \alpha$ iff $\vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma)$.
Define: $\mathcal{I} := \mathcal{V} \setminus M_\mathcal{E}$.
Then, $\mathcal{V} \setminus \mathcal{I} = M_\mathcal{E} \in \mathbf{D}$.
We will show:
(1) $\quad \forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T(M_\Gamma \cap \mathcal{I})$.
Let μ be the choice function from $\mathbf{D}$ to $\mathcal{P}(\mathcal{V})$ such that $\forall\, V \in \mathbf{D}$, $\mu(V) = V \cap \mathcal{I}$.
Then, $\forall\, \Gamma \subseteq \mathcal{F}$, $\mid\!\sim(\Gamma) = T(\mu(M_\Gamma))$.
In addition, by Proposition 17, μ is a UC SC choice function.
Consequently, $\mid\!\sim$ is an UC pivotal consequence relation.

Proof of (0). Let $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$. Then:
$\Gamma \mid\!\sim \alpha$ iff
$M_\Gamma \cap \mathcal{I} \subseteq M_\alpha$ iff
$M_\Gamma \subseteq M_\alpha \cup (\mathcal{V} \setminus \mathcal{I})$ iff

$M_\Gamma \subseteq M_\alpha \cup M_\mathcal{E}$ iff
$M_\Gamma \subseteq M_{\Gamma \cup \{\alpha\}} \cup M_\mathcal{E}$ iff
$M_\Gamma \subseteq M_{(\Gamma \cup \{\alpha\}) \vee \mathcal{E}}$ iff
$T(M_{(\Gamma \cup \{\alpha\}) \vee \mathcal{E}}) \subseteq T(M_\Gamma)$ iff
$T(M_{\Gamma \cup \{\alpha\}} \cup M_\mathcal{E}) \subseteq T(M_\Gamma)$ iff
$T(M_{\Gamma \cup \{\alpha\}}) \cap T(M_\mathcal{E}) \subseteq T(M_\Gamma)$ iff
$\vdash(\Gamma, \alpha) \cap \vdash(\mathcal{E}) \subseteq \vdash(\Gamma)$ iff
$\vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma)$.

Proof of (1). Let $\Gamma \subseteq \mathcal{F}$ and $\alpha \in \mathcal{F}$. Then:
$\Gamma \mathrel{|\!\sim} \alpha$ iff
$\vdash(\Gamma, \alpha) \cap \mathcal{E} \subseteq \vdash(\Gamma)$ iff
$\vdash(\Gamma, \alpha) \cap \vdash(\mathcal{E}) \subseteq \vdash(\Gamma)$ iff
$T(M_{\Gamma \cup \{\alpha\}}) \cap T(M_\mathcal{E}) \subseteq T(M_\Gamma)$ iff
$T(M_{\Gamma \cup \{\alpha\}} \cup M_\mathcal{E}) \subseteq T(M_\Gamma)$ iff
$T(M_{(\Gamma \cup \{\alpha\}) \vee \mathcal{E})}) \subseteq T(M_\Gamma)$ iff
$M_\Gamma \subseteq M_{(\Gamma \cup \{\alpha\}) \vee \mathcal{E})}$ iff
$M_\Gamma \subseteq M_{\Gamma \cup \{\alpha\}} \cup M_\mathcal{E}$ iff
$M_\Gamma \subseteq M_\alpha \cup M_\mathcal{E}$ iff
$M_\Gamma \cap (\mathcal{V} \setminus M_\mathcal{E}) \subseteq M_\alpha$ iff
$M_\Gamma \cap \mathcal{I} \subseteq M_\alpha$. ∎

6 Conclusion

We provided, in a general framework, characterizations for families of pivotal(-discriminative) consequence relations. We showed, in an infinite classical framework, that there is no normal characterization for the family of all pivotal consequence relations. And, we showed that UC pivotal consequence relations are precisely those X-logics such that X is closed under the basic entailment. Beyond the contributions, an interest of the present paper is to give an example of how the techniques developed in [BN05] (in particular in the discriminative case) can be adapted to new properties (here Strong Coherence in the place of Coherence). So naturally, we turn now to conclusions similar to those of [BN05]. In many cases, our conditions are purely syntactic. In fact, when the choice functions under consideration are not necessarily definability preserving, we provided solutions with semi-syntactic conditions. We managed to do so thanks to Lemmas 23 and 24. An interesting thing is that we used them both in the plain and the discriminative versions. This suggests that they can be used in yet other versions. In addition, Lemmas 31 and 32 have been applied both here and previously in [BN05] to characterize families of consequence relations defined in the discriminative manner by DP choice functions. But, [BN05] is about coherent choice functions, whilst the present paper is about strongly coherent choice functions. This suggests that these lemmas can be applied with yet other properties.

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