Multifractal analysis

**Abstract** — Multifractal analysis is a reference tool for the analysis of data based on local regularity. Historically fundamentally univariate, recent contributions have explored a first multifractal theoretical ground for multifractal analysis and shown that it can capture transient higher-order dependence beyond correlation between time series. This work studies the use of a quadratic model for the joint multifractal spectrum of bivariate time series and provides expressions for the Pearson correlation in terms of random walk and multifractal cascade dependence parameters that quantify such non-linear, higher-order dependencies.

**Multifractal Spectrum**
- **Local Regularity:**
  - Regularity of function \( X(t) \) at \( t \): regularity exponent \( \lambda \)
  - Holder exponent \( \lambda(t) \): \( |X(t+h)-X(t)| \leq h^\lambda(t) \)
- **Multifractal Spectrum:**
  \[ D(\lambda) = \lim_{h \to 0} \left( \frac{\lambda(t)}{2} \right) \]
- **Property:** “Amount” of points with given regularity
- **Bivariate Multifractal Spectrum:**
  - Bivariate signal \( X = (X_1, X_2) \)
  - Holder exponents \( \lambda_1(t), \lambda_2(t) \)
  \[ D(\lambda_1, \lambda_2) = \lim_{h \to 0} \left( \frac{\lambda_1(t-1) + \lambda_2(t-1) - \lambda_1(t) - \lambda_2(t)}{2} \right) \]
- **Problem:** Can not be computed in practice

**Wavelet leaders**
- **Discrete wavelet transform**
  \[ X(t) \to d_k(x,k) = d_k(x) \]
- **Wavelet leaders**
  (Wendt, Abry & Jaffard, 2007)

**Multifractal Formalism**
1. Structure functions \( S \) and scaling exponents \( \zeta \)
\[ S(q_1, q_2, j) = \sum_{j=0} L(j,k)^{q_1} L(j,k)^{q_2} \sim \left( \frac{b}{c} \right)^{j\theta(q_1, q_2)} \quad \text{for } b, c \to 0 \]
2. Legendre spectrum \( L \). Upper bound estimate for \( D \)
\[ L(h_1, h_2) = \int_{h_1}^{h_2} \left( \frac{1}{2} q_1 h_1^2 + q_2 h_2^2 - q_1 q_2 \right) \geq D(q_1, q_2) \]
- Robust and easy to compute in practice

**Quadratic approximation of \( L(h_1, h_2) \)**
- Bivariate cumulants of \( (L(j,k)^{q_1}, L(j,k)^{q_2}) \)
\[ C_{q_1 q_2}(j) = \mathbb{E} \left[ \left( L(j,k)^{q_1} \right)^{q_1} \left( L(j,k)^{q_2} \right)^{q_2} \right] \quad \text{for } q_1, q_2 \geq 1 \]
- Cumulant expansion of bivariate spectrum
\[ L(h_1, h_2) \approx 1 + \frac{c_{q_1 q_2} h_1^2 + c_{q_2 q_3} h_2^2 + c_{q_1 q_2} h_1 h_2}{6} - \frac{c_{q_1 q_2} c_{q_2 q_3} h_1 h_2}{6} \]
- Interpretation:
  - \( c_{q_1 q_2} \): average regularity on each component
  - \( c_{q_1 q_2} \): width of regularity fluctuations on each component
  - \( c_{q_1 q_2} \): leading-order joint regularity fluctuation
- Estimation → linear regressions \( C_{q_1 q_2}(j) \) vs \( \ln h \)

**Bivariate Multifractal Random Walk**
- **Synthetic process with bivariate multifractal behavior**
- **Definition**
  - Use two pairs of stochastic processes
  - **Pair 1:** bivariate fractional Gaussian noise \( G_{1}(t), G_{2}(t) \)
    - Self-similarity parameters: \( \nu_1, \nu_2 \)
    - Covariance matrix:
      \[ \Sigma_{G} = \begin{pmatrix} 1 & \rho_{G} \nu_1 \nu_2 \ 0 & 1 \end{pmatrix} \]
  - **Pair 2:** Gaussian processes \( u(t), u(t) \)
    - Multifractality parameters: \( \lambda_1, \lambda_2 \)
    - Covariance function: \( \Sigma_{u} \) such that
      \[ \rho_{u} = \frac{1}{1 + \rho_{G} \nu_1 \nu_2} \]
  - Logarithmic covariance to induce multifractals
  - \( G_{1}(t), u(t) \) synthesized following (Hegelson, Pipiras & Abry, 2011)
  - **Final process:** Bivariate MW
    \[ X_0 = \sum_{k=1}^{\infty} G_{k}(t)u(k) \quad \text{for } k \leq 2 \]
  - **Multifractal Properties**
    - Cumulants:
      \[ c_{1} = H_1 + \frac{H_2}{2} \quad \text{and} \quad c_{2} = H_2 + \frac{H_1}{2} \]
      \[ c_{3} = -\frac{H_1}{2} \quad \text{and} \quad c_{4} = -\frac{H_2}{2} \]
      \[ c_{5} = \frac{H_1}{2} \quad \text{and} \quad c_{6} = \frac{H_2}{2} \]
  - **Correlation and dependence**
    - Correlation coefficient of bivariate MW:
      \[ \rho_{XY} = \rho_{G} \rho_{G} \rho_{u} = \frac{1}{1 + \rho_{G} \nu_1 \nu_2} \]
    - From elementary properties of log-normal random variables:
      \[ \rho_{XY} = \rho_{G} \rho_{G} \rho_{u} \left( \frac{1}{e} \right) \left( 1 + \rho_{G} \nu_1 \nu_2 \right) \]
  - Expansion coefficients for joint dependence of \( X \)
  - Natural estimators:
    \[ \rho_{X} \approx 1 + c_{1,1} \quad \text{and} \quad \rho_{X} \approx 1 + c_{1,1} \]

**Dependence parameters**
- **Set** \( \rho_{X} = 0 \)
  \[ \rho_{MBW} = 0 \]

**Natural bivariate multifractal parameters**
- **Set** \( \rho_{X} = 0 \)
  \[ \rho_{MBW} = 0 \]

**Estimation performance**
- \( \rho_{X} \approx 0 \)
  \[ \rho_{MBW} \to \left( \begin{array}{c} \text{uncorrelated, independent} \\ \text{uncorrelated, positive dependence} \\ \text{uncorrelated, negative dependence} \end{array} \right) \]
  \[ \rho_{X} \approx 0 \]
  \[ \rho_{MBW} \to \left( \begin{array}{c} \text{correlated, independent} \\ \text{correlated, positive dependence} \\ \text{correlated, negative dependence} \end{array} \right) \]
  \[ \rho_{X} \approx 0 \]
  \[ \rho_{MBW} \to \left( \begin{array}{c} \text{univariate, independent} \\ \text{univariate, positive dependence} \\ \text{univariate, negative dependence} \end{array} \right) \]

**References**