

A BAYESIAN APPROACH FOR THE JOINT ESTIMATION OF THE MULTIFRACTALITY PARAMETER AND INTEGRAL SCALE BASED ON THE WHITTLE APPROXIMATION



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SUMMARY

Multifractal models are characterised by two parameters, the multifractality parameter c_2 and the integral scale \mathcal{J} (the dyadic scale beyond which the multifractal properties vanish). Joint estimation of c_2 and \mathcal{J} is challenging due to the strong dependence and non-Gaussian properties of multifractal processes. This contribution proposes a Bayesian procedure for the joint estimation of c_2 and \mathcal{J} . Its originality lies in the construction of a generic multivariate model for the statistics of the logarithm of the wavelet leaders for multifractal cascade-based processes and in the use of a Whittle approximation for the likelihood associated with the model. The excellent performance is demonstrated numerically on synthetic multifractal processes and wind-tunnel turbulence data.

MULTIFRACTAL ANALYSIS

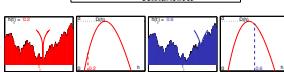
MULTIFRACTAL SPECTRUM

- LOCAL REGULARITY:

- locally bounded function $X(t)$
- local power law behavior
- $|X(t) - X(t')| \leq C|t - t'|^\alpha$, $C > 0$, $\alpha > 0$
- largest such α : Hölder exponent $h(t_0)$

- MULTIFRACTAL SPECTRUM:

- geometric structure of subsets $E_h : h(t_i) = h$
- $D(h) = \dim_{\text{Hausdorff}}(E(h))$

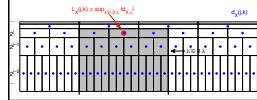


WAVELET LEADERS

$$L_X(j, k) = \sup_{\lambda \in \Delta_{j,k}} |d_X(\lambda)|$$

- $d_X(j, k)$ – dyadic wavelet transform coefficients
- $\lambda_{j,k}$ – dyadic cube $((k-1)2^j, k2^j]$
- $\Delta_{j,k}$ – union with two closest neighbors

→ local supremum of wavelet coefficients



MULTIFRACTAL FORMALISM

- Scaling function and integral scale \mathcal{J}

$$S(2^j, q) = \sum_k L_X(j, k)^q \simeq 2^{qj}, \quad 1 \leq 2^j \leq 2^q$$

$$\zeta(q) = \liminf_{2^j \rightarrow 0} \log_2 S(2^j, q) / \log_2 2^j$$

- Legendre spectrum:

$$\mathcal{L}(h) = \min_q (1 + qh - \zeta(q)) \geq \mathcal{D}(h)$$

→ upper bound for multifractal spectrum

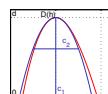
CUMULANT EXPANSION

- Polynomial expansion around $q = 0$:
- $\zeta(q) = \sum_{p \geq 1} \frac{c_p}{p!} \frac{q^p}{p!}$
- $D(h) \leq 1 + (h - c_2)/2(c_2) + \dots$
- c_2 – parameter of multifractality
- $C_p(j) = c_p^0 + c_p \ln 2^j$ – p -th cumulant of $\ln L_X(j, k)$

$$C_2(j) \equiv \text{Var}[\ln L_X(j, k)] = c_2^0 + c_2 \ln 2^j$$

valid for $1 \leq j \leq \mathcal{J}$ with \mathcal{J} – integral scale

- ESTIMATION OF c_p : most common by linear fit (LF)



MODELING WAVELET LEADER STATISTICS

- MARGINAL DISTRIBUTION MODEL

- multifractal processes and their wavelet coefficients / leaders: X strongly non-Gaussian marginals
- logarithm of wavelet leaders:
- numerically shown to be well modeled by a Gaussian

$$\ln L_X(j, k) \sim \mathcal{N}(m, c_2^0 + c_2 \ln 2^j)$$

- INTRA-SCALE COVARIANCE MODEL

- induced by the multiplicative cascade construction
- proposed model:
- asymptotic decay of $\text{Cov}[\ln L_X(j, k), \ln L_X(j, k + \Delta k)]$

$$\Gamma_j(\Delta k; c_2) = \eta + c_2 (\ln \Delta k + \ln 2^j)$$

- piece-wise linear in $\ln \Delta k$ coordinates

$$\text{Cov}[\ln L_X(j, k), \ln L_X(j, k + \Delta k)] = \Sigma_j(\Delta k; \theta) =$$

$$= \begin{cases} c_2^0 + c_2 \ln 2^j & \text{if } \Delta k \equiv 0 \\ \max(0, \Gamma_j(\Delta k; c_2)) & \text{if } 1 \leq \Delta k \leq 2 \\ \max(0, \Gamma_j(\Delta k; c_2)) + c_2^0 + jc_2 \ln 2 & \text{if } 3 \leq \Delta k \leq 2^j \end{cases}$$

- parametrized by $\theta = [c_2, c_2^0, \eta]^T$

- INTEGRAL SCALE

- \mathcal{J} related to the correlation length of the leaders
- $\text{Cov}[\ln L_X(j, k), \ln L_X(j, k + \Delta k)] = 0$ for $\Delta k = 2^{\lceil j/2 \rceil}$

$$\mathcal{J} = -\frac{\eta}{c_2 \ln 2^j}$$

BAYESIAN ESTIMATION

BAYESIAN MODEL

- DATA
- Scale-wise centered log-leaders $L_j = [L_X(j, 1), \dots, L_X(j, n_j)]^T$
- covariance $\Sigma_j(\theta) = \mathbb{E}[L_j L_j^T]$ induced by $\Sigma_j(\Delta k; \theta)$
- multivariate Gaussian distribution

$$f(L_j | \theta) = ((2\pi)^{n_j} \det \Sigma_j(\theta))^{-1} \exp\left(-\frac{1}{2} L_j^T \Sigma_j(\theta)^{-1} L_j\right).$$

LIKELIHOOD

$$f(\mathcal{L}_X | \theta) = \prod_{j=1}^J f(L_j | \theta), \quad \text{with } \mathcal{L}_X = [L_1^T, \dots, L_J^T]^T$$

PRIOR FOR θ

- ensure positivity of variance $C_2(j) = [\Sigma_j(\theta)]_{jj}$
- $f(\theta) \propto 1_{C_2(\theta) > 0}$, with C_2 admissible set

POSTERIOR DISTRIBUTION

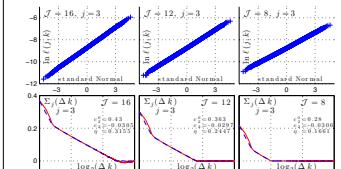
$$f(\theta | \mathcal{L}_X) \propto f(\mathcal{L}_X | \theta) f(\theta).$$

- maximum a posterior (MAP) estimator
- minimum mean squared error (MMSE) estimator

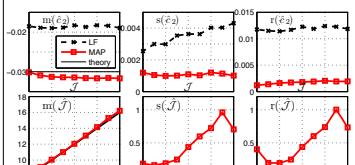
NUMERICAL EXPERIMENTS

- Synthetic data: Multifractal Random Walk (100 realizations)
- multiplicative cascade based multifractal process
- $c_2 \neq 0$, $\eta = \sqrt{2\pi} \geq 3$
- Daubechies' wavelet $N_d = 2$ vanishing moments, $j_1 = 3$, $j_2 = 6$
- $\mathcal{J} = 2^{10} = 262144$, $c_2 = -0.03$ and a wide range of values of \mathcal{J}

MARGINAL DISTRIBUTION AND SAMPLE COVARIANCE



ESTIMATION PERFORMANCE



RESULTS

- c_2 estimation performance gain of up to a factor 4
- reliable estimation of the integral scale \mathcal{J}

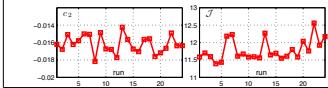
APPLICATION TO TURBULENCE DATA

- Wind-tunnel turbulence data

- $R = 24$ runs of longitudinal Eulerian velocity signals

- Reynolds number $R_A \approx 2000$ and integral scale $\mathcal{J} = 13$

- Estimation with $[j_1, j_2, \mathcal{J}] = [6, 10, 3]$



REFERENCES & SOFTWARE

Related toolbox: Bayesian estimation for image texture



<http://www.irit.fr/~Herwig.Wendl/software.html>

<http://www.irit.fr/~Herwig.Wendl/publications.html>