TESTING MONO-VERSUS MULTIFRACTAL WITH BOOTSTRAPPED WAVELET LEADERS

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ABSTRACT

In many applications where data possess scaling properties, it is of importance to decide whether the data are better modelled with mono- or multifractal processes. However, so far no appropriate test is available. For this purpose, we propose here to test, using a bootstrap procedure, whether the second cumulant of the log of the wavelet coefficients or wavelet Leaders of the data is zero. We study the p-value and the power of the tests through numerical simulation, using synthetic multifractal processes, and end up with a powerful procedure for practically discriminating mono- vs. multifractal processes.

1. MULTIFRACTAL PROCESS AND LEADERS AND CUMULANTS

Wavelet Leaders. Let \(X\) be the process under investigation, and \(n\) its observation duration. Let us denote by \(d_X(j,k) = \langle \psi_{j,k} | X \rangle\) its wavelet coefficients at scales \(2^j\) and time positions \(2^k\). Let us introduce the indexing \(\lambda_{j,k} = [2^2j, (k + 1)2^j]\) and the union \(\lambda_{j,k} = \lambda_{j,k-1} \cup \lambda_{j,k} \cup \lambda_{j,k+1}\). The wavelet leaders \(L_X(j,k)\) are defined as \(L_X(j,k) = \sup_{X \in \lambda_{j,k}} |d_X|\), where the supremum is taken on the \(d_X(\cdot, \cdot)\) in the time neighborhood \(\lambda_{j,k}\) over all finer scales \(2^j < 2^j\) (cf. [1]).

Scaling and Multifractal. A processes \(X\) is said to possess scaling properties if, for some \(q \in [q, q^*]\), the time averages of \(|L_X(j,k)|^q\) taken at fixed scales display power law behaviors with respect to scales, \(|L_X(j,k)|^q = F_q |2^j|^\zeta(q)\), over a wide range of scales (equiv. for \(|d_X(j,k)|^q\)). The \(\zeta(q)\) are referred to as the scaling exponents of \(X\) and are closely related to its multifractal spectrum. The process \(X\) is said to be monofractal when \(\zeta(q)\) is linear in \(q\), i.e., \(\zeta(q) = qH\), and multifractal when \(\zeta(q) \neq qH\) (cf. [1] and references therein).

Log-Cumulants. Through a second characteristic function type expansion argument, scaling implies that \(\zeta(q) = \sum_{p=1}^{\infty} c_p q^p\), where \(c_p\) stands for the cumulants of order \(p \geq 1\) of the random variable \(\ln |L_X(j,k)|\) (or equiv. \(\ln |d_X(j,k)|\)) [3]. Thus, the measurements of the scaling exponents \(\zeta(q)\) can be replaced by those of the log-cumulants \(c_p\) emphasizing the difference between monofractal \((\forall p \geq 2 : c_p = 0)\) and multifractal processes. Estimates \(\hat{c}_p\) of the log-cumulants \(c_p\) are obtained by linear regression of \(\hat{C}_p^j\) vs. \(j\).

2. BOOTSTRAP SIGNIFICANCE TEST

We want to test \(H_{null}: c_{2, null} \equiv 0\) using a simple test statistic \(T = c_2 - c_{2, null}\) with the observed value denoted by \(t = \hat{c}_2 - c_{2, null}\). At each scale \(j\), \(R\) simulated samples \(\{L_X(j,k)\}\) \(\{\{d_X(j,k)\}\}\) are generated from the original sample \(\{L_X(j,k)\}\) \(\{\{d_X(j,k)\}\}\) of length \(n_j\) using a moving blocks bootstrap. From these resamples, the \(R\) bootstrap replica \(\hat{C}_p^j\) and \(\hat{c}_2\) are estimated, and the simulated values \(t^* = \hat{c}_2 - \hat{c}_2\) are calculated and used to estimate the cdf

![Image](https://via.placeholder.com/150)

Fig. 1. Empirical p-value (top left) and power \(\beta\) (top right) vs. nominal \(p = 1 - \alpha\). Power \(\beta(c_2)\) for nominal \(\alpha = 0.9\) (bottom left) and \(\alpha = 0.95\) (bottom right). The symbols (Δ: □: ×: +) correspond to \((L_X, n = 2^{12}; L_X, n = 2^{15}; d_X, n = 2^{12}; d_X, n = 2^{15})\).

\[
F_{null}(\tau) = \frac{\hat{F}^{\alpha}(|\tau|)}{\hat{F}^{\alpha}}
\]

The p-value, \(p = Pr(\tau > \tau_{null})\), is now approximated by the bootstrap p-value, \(p^* = Pr^*(\tau^* > \tau_{null})\) (cf. [2]).

3. RESULTS

We use a large number of realizations of (monofractal) fractional Brownian motion (FBM) to determine the relevance of \(p^*\), and of Multifractal random walk (MRW) to evaluate the power \(\beta\) of the test. The results are summarized in Fig. 1. Whereas the \(dx\) based tests have \(p^*\) closer to nominal \(p\) than on \(L_X\) based tests (top left), the latter have significantly larger power than the former (top right and bottom row). This is partly due to the fact that \(c_2\) based on \(L_X\) possess smaller variance. Most important, the tests involving \(L_X\) maintain large power \(\beta > 0.8\) for \(n = 2^{15}\) over a wide range of values for \(c_2\), including \(c_2\) close to zero, for usual \(\alpha\), whereas the power of tests using \(dx\) is, in comparison, poor (bottom row). We conclude that with the Leaders-based procedure described here, a powerful test is available for practically discriminating monofractal vs. multifractal processes.

4. REFERENCES

