INVERSE PROBLEM FORMULATION FOR REGULARITY ESTIMATION IN IMAGES

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ABSTRACT
The identification of texture changes is a challenging problem that can be addressed by considering the local regularity in an image. This work develops a procedure for local regularity estimation that combines a convex optimization strategy with wavelet leaders, specific wavelet coefficients recently introduced in the context of multifractal analysis. To this end, this estimation problem is formulated as an inverse problem that jointly involves the local regularity exponents to be estimated and the regression weights required in their estimation, and solved by a convex optimization procedure. Numerical experiments using synthetic texture indicate that the performance of the proposed approach compares favorably against other wavelet based local regularity estimation formulations. The method is also illustrated with an example involving natural texture.

Index Terms— Local regularity, variational approach, convex optimization, wavelet leaders

1. INTRODUCTION
Wavelet decompositions are now well recognized as sparsity-inducing transforms and have been widely used in contexts of image compression and restoration [1]. Another very useful property of wavelet transforms found, so far, less widespread use in the field of image processing: their ability to evidence and measure scale invariance (cf., e.g., [2, 3, 4, 5, 6]). Notably, wavelet transforms have been shown to be relevant tools for the estimation of local regularity [4]. The earliest contribution to this subject traces back to [2], where the skeleton (maxima lines across scales) of the continuous wavelet transform is shown to enable practitioners to identify irregular (i.e., non-smooth) behavior in signals. By imposing an additional monotonicity constraint on the skeleton, the wavelet transform modulus maxima (WTMM) formalism has been further developed in, e.g., [3] to precisely measure Hölder exponents \( h \). These constitute the canonical theoretical quantity for the measurement of local regularity and for performing multifractal analysis. More recently, it has been shown that accurate estimates of local regularity can be obtained using wavelet leaders (WL), specific multiresolution quantities that have been demonstrated to precisely reproduce Hölder exponents theoretically [4]. Wavelet leaders are defined as local suprema of the coefficients of the dyadic wavelet transform and inherit their computational efficiency, see, e.g., [5, 6] and Section 2 below for more precise definition.

Both the WTMM and the WL have been extensively used to perform multifractal analysis of real-world signals and images (see [7, 5, 8, 9] and references therein for examples of successful applications). In contrast to local regularity estimation, multifractal analysis does not aim at the time- or space-resolved estimation of \( h \) but rather provides a global and geometric description of the fluctuations of \( h \) in an image in terms of the so-called multifractal spectrum. However, for certain applications, the central information of interest is precisely the evolution along time or space of the Hölder exponent. In this case, multifractal analysis is not directly relevant. Instead, direct estimation of the local regularity evolution, sometimes referred to as multifractional analysis, needs to be performed. Yet, the estimation of \( h \) at a precise time/space location suffers from poor performance that impairs its actual practical use. Consequently, the estimation of the time/space evolution of local regularity remains barely used in applications (see, a contrario, [10, 11]). In particular, local estimation of regularity suffers from a large variance and common post-processing techniques for variance reduction, such as local smoothing, induce significant bias and inaccuracies in the localization of changes in regularity.

In a previous contribution [12], we have proposed an original two-step procedure that addresses the bias-variance trade-off difficulty in the specific context of images with piece-wise constant regularity: (i) Unbiased estimation of the Hölder exponent for every position in the image using a patch-based wavelet leaders approach, (ii) Extraction of areas with uniform Hölder exponent from these local estimates using a variational procedure relying on total-variation (TV) [13]. In the present contribution, we elaborate on this approach and propose a one-step procedure that directly yields piece-wise constant local regularity estimates. The originality of the approach resides in the use of TV-regularization in a criterion that involves directly the value of the WL coefficients instead of the intermediary local estimates of \( h \). Through this formulation, an efficient local regularity estimation procedure is designed, yielding significantly increased estimation quality for images with piece-wise constant texture.

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The remainder of this work is organized as follows. Section 2 recalls the notion of Hölder exponent, defines wavelet leaders and the state-of-the-art Hölder exponent estimation procedure. Section 3 formalizes the estimation of local regularity explicitly as an inverse problem and details the proposed variational procedure. Section 4 reports experimental results illustrating the performance of the proposed piecewise constant regularity estimation/segmentation procedure.

2. HÖLDER EXPONENT AND WAVELET LEADERS

Loosely speaking, the Hölder exponent \( h(x) \) is a positive quantity that measures the regularity of a function at the location \( x \in \mathbb{R}^2 \) by comparing its amplitude evolution around \( x \) against a local power law behavior (see [4] for a precise definition). Qualitatively, a small value of the Hölder exponent indicates a locally highly irregular behavior, close to discontinuous, of the function, while a large value indicates local smoothness. A simple, efficient and theoretically well grounded solution to practically compute \( h(x) \) relies on the use of wavelet leaders [5], which we define in what follows. Note that it has recently been proven [4] that wavelet leaders enable to measure the Hölder exponent for much more general classes of bi-dimensional functions than wavelet coefficients, and with a significantly improved accuracy [4, 5, 6, 12].

Let \( f \) denote the bi-dimensional function taking bounded values (i.e., the image) to be analyzed. Let \( \phi \) and \( \psi \) denote respectively the scaling function and mother wavelet defining a 1D multiresolution analysis. The corresponding 2D tensor product wavelets are defined, for every \( x = (x_1, x_2) \in \mathbb{R}^2 \), as: 
\[
\psi^{(0)}(x) = \phi(x_1)\phi(x_2), \quad \psi^{(1)}(x) = \psi(x_1)\phi(x_2), \quad \psi^{(2)}(x) = \phi(x_1)\psi(x_2), \quad \text{and} \quad \psi^{(3)}(x) = \psi(x_1)\psi(x_2).
\]

The collection \( \psi^{(m)}(x) = 2^{-j}\psi^{(m)}(2^{-j}x - k) \) of dilated (to scales \( 2^j \)) and translated (to space positions \( 2^j k \) ) templates of \( \psi^{(m)} \) form a basis of \( L^2(\mathbb{R}^2) \) for well chosen functions \( \psi \). The \((L^1)-\)normalized discrete wavelet transform (DWT) coefficient at scale \( j \), location \( k \) and subband \( m \in \{1, 2, 3\} \) is defined as 
\[
d^{(m)}(j, k) = \langle f, 2^{-j}\psi^{(m)}(\cdot - k) \rangle.
\]

The wavelet leader coefficient \( L(j, k) \) is defined, for each scale \( j \) and location \( k \), as the local supremum of all wavelet coefficients taken within a spatial neighborhood across all finer scales \( j' \leq j \),
\[
L(j, k) = \sup_{m=\{1,2,3\}} \sup_{\lambda \in L_{j', k}^j} |d^{(m)}(j', k')|,
\]
where \( \lambda_{j, k} = [2^{j'}, (k+1)2^{j'}] \) and \( L_{j', k}^j = \bigcup_{\lambda \in \mathbb{R}^2} \lambda \).

When \( 2^j \rightarrow 0 \) in order to reduce the variance of (4), one can attempt to optimize the weights \( w(j, k) \) which reflect the confidence granted to the quantities \( L(j, k) \). Indeed, the variance of \( L(j, k) \) depends on the value of the Hölder exponent \( h(x) \) at location \( k \) and the optimal choice of the weights \( w(j, k) \) varies from one location to the other. Therefore, we propose in this paper to formulate the estimation of \( h(x) \) as an inverse problem involving jointly the Hölder exponents \( h(x) \) and the weights \( w(j, k) \) (constrained only by (6)) as parameters to be estimated. The inverse problem formulation and the proposed proximal based minimization procedure for finding its solution are detailed in the following section.

3. INVERSE PROBLEM BASED LOCAL REGULARITY ESTIMATION

“Degradation” model From now on, we make use of a discrete time formalism. Let \( f = (f[n])_{1 \leq n \leq N} \) denote the vector representation of the image to be analyzed, of size \( N = N_1 \times N_2 \). The orthonormal wavelet transform is labeled \( F \in \mathbb{R}^{N \times N} \) and the wavelet coefficients of \( f \) are denoted \( d = (d[n])_{1 \leq n \leq N} = Ff \). At each scale \( j \in \{1, \ldots, J\} \), \( L_j : \mathbb{R}^N \rightarrow \mathbb{R}^{2^{-2j}N} \) denotes the non-linear transform that associates the wavelet coefficients to the wavelet leaders \( L_j = (L_j[k])_{1 \leq k \leq 2^{-2j}N} \) (as defined in (1)) such that \( L_j = L_j(F f) \). A matrix formulation of (4) leads to:
\[
\sum_{j=j_1}^{j_2} W_j D_j \ln L_j = h + \varepsilon
\]
with $1 \leq j_1 < j_2 \leq J$ and where $\varepsilon$ models the uncertainties in the estimation, mostly due to data discretization and thus the finite range of available scales. For each $j \in \{j_1, \ldots, j_2\}$, $W_j \in \mathbb{R}^{N \times N}$ denotes a diagonal matrix whose diagonal values are the regression weights, i.e., $W_j = \text{diag}(w_j)$ with $w_j = (w_j[n])_{1 \leq n \leq N} \in \mathbb{R}^N$, and $D_j \in \mathbb{R}^{N \times 2^{-j}N}$ denotes a matrix that duplicates the signal such that, for every $(u, v) \in \mathbb{R}^{2^{-j}N} \times \mathbb{R}^N$, if we denote $u \in \mathbb{R}^{2^{-j}N_1 \times 2^{-j}N_2}$ (resp. $v \in \mathbb{R}^{N_1 \times N_2}$) the matrix representation of $u$ (resp. $v$), $v = D_j u$ means that, for every $(n_1, n_2) \in \{1, \ldots, N_1\} \times \{1, \ldots, N_2\}$, $v[n_1, n_2] = u[2^{-j}n_1, 2^{-j}n_2]$.

The model in (7) undergoes an inverse problem in which $h = h(n)[1 \leq n \leq N]$ need to be recovered from the logarithm of the wavelet leaders coefficients ($\ln L_j)[1 \leq j \leq J]$. This inverse problem resembles a denoising problem, yet including the additional challenge that a part of the observations (the regression matrices weights ($W_j)[1 \leq j \leq J]$) is unknown and must satisfy constraints (6).

**Variational approach** We propose to estimate the local regularity $h$ and the regression weight matrices ($W_j)[1 \leq j \leq J]$ by solving the following minimization problem:

$$
(\hat{h}, \hat{W}) \in \text{Argmin}_{h, W} \left\{ \sum_{j=1}^{J_2} W_j D_j \ln L_j - h \right\} + \lambda TV(h) + \eta_1 \left( \sum_{n=1}^{N} d_{C_1}(w[n]) + \eta_2 \sum_{n=1}^{N} d_{C_2}(w[n]) \right) \tag{8}
$$

where $W = (W_{j_1}, \ldots, W_{j_2})$ and thus $w[n] = (w_{j_1}[n], \ldots, w_{j_2}[n])$ belongs to $\mathbb{R}^{2^{-j_1}N_j - 1}$. The first term denotes a data fidelity term. Distances to the convex sets $C_1$ and $C_2$, denoted $d_{C_1}$ and $d_{C_2}$, are introduced to provide some flexibility in the hyperplane constraints $C_1$ and $C_2$:

$$
C_1 = \{ (\omega_1, \ldots, \omega_{j_2}) \in \mathbb{R} \times \cdots \times \mathbb{R} | \sum_{n=1}^{N} \omega_j[n] = 0 \}, \quad C_2 = \{ (\omega_1, \ldots, \omega_{j_2}) \in \mathbb{R} \times \cdots \times \mathbb{R} | \sum_{n=1}^{N} j \omega_j[n] = 1 \}.
$$

For every $w \in \mathbb{R}^{(2^{-j_1}N_j - 1)}$, $d_{C_1}(w) = ||w - P_{C_1}(w)||$, with $P_{C_1}(w) = \text{arg min}_{w \in C_1} ||w - v||^2$, denotes the projection onto the convex set $C_1$ (resp. $d_{C_2}$ and $P_{C_2}$). The second term $TV(h)$ acts as a penalization that forces a solution with a minimal total variation [13] that is, for every $h \in \mathbb{R}^N$,

$$
TV(h) = ||Th||_{2,1} = \sum_{n=1}^{N} \sqrt{ ||(Hh)[n]||^2 + ||(Vh)[n]||^2 } \tag{9}
$$

where $T = [HT \, VT^T]^T$ with $H \in \mathbb{R}^{N \times N}$ and $V \in \mathbb{R}^{N \times N}$ are matrix representations of, respectively, the horizontal and vertical first-order discrete differences. The parameters $\lambda, \eta_1$, and $\eta_2$ will impact directly the solution. One could note that the choice of parameters $\lambda = 1/\eta_1 = 1/\eta_2 = 0$ leads to the standard estimation procedure, formulated in (5).

**Proximal algorithm** The minimization problem (8) is convex but non-smooth. In the recent literature dedicated to non-smooth convex optimization, several efficient algorithms have been proposed. For instance, when a Lipschitz data fidelity term is involved, such as a quadratic data fidelity term, as well as several regularization terms, such as TV regularization or distance to convex sets, one suited algorithm is referred to as CV (for Condat-Vũ) [14, 15]. The corresponding iterations tailored to solve the problem in (8) are given in Algo. 1, that ensures convergence of the sequence $(h^{[k]}, W^{[k]})_{k \in \mathbb{N}}$ to a solution of (8).

**Algorithm 1** Algorithm for solving (8)

```plaintext
 Initialization
 Set $\sigma > 0$ and $\tau \in ]0, 1]\}

Set $h^{[0]} \in \mathbb{R}^N$, $w^{[0]} = (w^{[0]}_{j_1}, \ldots, w^{[0]}_{j_2}) \in \mathbb{R}^{(2^{-j_1}N_j - 1)}$, $u^{[0]} = (u^{[0]}_{j_1}, \ldots, u^{[0]}_{j_2}) \in \mathbb{R}^{(2^{-j_1}N_j - 1)}$, and $y^{[0]} \in \mathbb{R}^{2N}$.

For $k = 1, \ldots, J_2$

For $j = j_1, \ldots, j_2$

$u^{[k]}_j = \text{diag}(w^{[k]}_j)$

$W^{[k]}_j = \text{diag}(w^{[k]}_j)$

$\ast$ Gradient descent steps $\ast$

$h^{[k+1]}_j = h^{[k]}_j - 2\tau (h^{[k]}_j - \sum_{j=j_1}^{j_2} W^{[k]}_j D_j \ln L_j) - \tau T^T y^{[k]}$

For $j = j_1, \ldots, j_2$

$W^{[k+1]}_j = W^{[k]}_j - 2\tau \sum_{j=j_1}^{j_2} W^{[k]}_j D_j \ln L_j - (h^{[k]}_j)(D_j \ln L_j)^T - \tau U^{[k]}_j$

$\ast$ Proximity operator step based on $d_{C_1} \ast$

For every $j$, denote $w^{[k]}_j$ the diagonal values of $W^{[k]}_j$

$p^{[k]} = y^{[k]}_j - \sigma T(2h^{[k+1]}_j - h^{[k]}_j)$

$q^{[k]} = q^{[k]} - \sigma \text{prox}_{\sigma d_{C_1}}(w^{[k]}_j[n], \ldots, w^{[k]}_j[n])$

$u^{[k+1]}_j = q^{[k]} - \sigma \text{prox}_{\sigma d_{C_2}}(\sigma^{-1} (u^{[k]}_j[n], \ldots, u^{[k]}_j[n]))$

Algo. 1 requires the computation of the proximity operators associated to the mixed $l_2,1$-pseudo norm and to the distance to convex sets. Let us recall that the proximity operator associated to a convex, lower semi-continuous convex function $\varphi$ from $H$ (where $H$ denotes a real Hilbert space) to $]-\infty, +\infty[$, denoted $\text{prox}_{\varphi}$, is defined as, for every $u \in H$,

$\text{prox}_{\varphi}(u) = \text{arg min}_{v \in H} \frac{1}{2} ||u - v||^2 + \varphi(v)$.

When $\varphi$ denotes the indicator function of a non-empty closed convex set $C \subset H$, that is $\iota_C(x) = 0$ if $x \in C$ and $+\infty$ otherwise, the proximity operator reduces to the projection, denoted $P_C$, onto the convex set.

The proximity operators steps involved in Algo. 1 have a closed-form expression. Indeed, it is shown in [16], that for every $u = (u[n])_{1 \leq n \leq N}$ with $u[n] \in \mathbb{R}$,

$$
\text{prox}_{\frac{\lambda}{\sigma} ||u||^2_2} u = \left( \max(0, 1 - \frac{\lambda}{\sigma ||u[n]||}) u[n] \right)_{1 \leq n \leq N} \tag{10}
$$


Moreover, according to [17, Proposition 2.8], if \( C \) denotes a non-empty closed convex subset of \( \mathbb{R}^{(j_2-j_1+1)} \) and if \( \eta > 0 \), for every \( u \in \mathbb{R}^{(j_2-j_1+1)} \),

\[
\text{prox}_{\eta d_C} u = \begin{cases} 
 u - \frac{d_C(u) - \eta}{d_C(u) > \eta} d_C(u) & \text{if } d_C(u) > \eta, \\
 P_C(u) & \text{if } d_C(u) \leq \eta.
\end{cases}
\]  

(11)

For our purpose, \( C \) models the hyperplane constraints \( C_1 \) and \( C_2 \) and one could note that \( P_{C_1} \) and \( P_{C_2} \) have a closed form expression given in [18].

\section{4. EXPERIMENTAL VALIDATION}

We first evaluate the performance of the proposed estimating strategy on synthetic data, consisting of 2D multifractional Brownian fields [19, 20], whose definition has been slightly modified here to ensure an homogeneous variance across the image (cf. [12] for details). The synthetic process has piece-wise constant regularity according to the mask shown in Fig. 1-b), with \( h = 0.5 \) in the central area and \( h = 0.3 \) for the background. A sample field is displayed in Fig. 1-a).

Analysis is conducted using a standard 2D DWT with orthonomal tensor product Daubechies mother wavelets with 2 vanishing moments. Regularity is estimated using the scaling range \((j_1, j_2) = (2, 4)\). We compare the performance of the proposed solution against two other approaches. First, a standard estimation procedure with a priori fixed weights \( w_j \) chosen to achieve ordinary linear regression, labelled \( \hat{h}_o \), (results are plotted in Fig. 1-c)). Second, we evaluate the performance of the technique proposed in [12] consisting in computing the proximity operator of the total variation of \( \hat{h}_o \), i.e, \( \text{prox}_{\lambda \text{TV}}(\hat{h}_o) \). The solution of this latter method is labelled \( \hat{h}_\lambda \) (cf., Fig. 1-d)) and the parameter \( \lambda \) is empirically tuned to minimize the normalized \textit{mean square error} (MSE) in this second approach. The solution proposed in Section 3, whose result is depicted in Fig. 1-e), achieves a smaller MSE and better evidence of the central area.

A second experiment deals with real data, obtained by mixing textures. The image is generated by inclusion of a distinct ellipse-shaped zone of cloud texture in a snow texture background (cf. Fig. 2-a)). First, it is worthy to note that the unbiased local regularity estimation in Fig. 2-c) allows the edges of the added areas to be identified. However, without these synthetic edges, the local regularity changes are difficult to identify. Similar observations are made for the solution obtained with [12], shown in Fig. 2-d) and the proposed solution. However, the proposed solution provides a better discrimination with sharper edges.

\section{5. CONCLUSION}

An efficient local regularity estimation strategy adapted to a multifractional framework has been devised. It enriches the classical estimation procedure by including the estimation of the weights that enter the linear regressions. These extra degrees of freedom yield local regularity estimates with significantly reduced variance. The proposed estimation procedure is formulated as an inverse problem and is solved using proximal minimization. Numerical experiments indicate that the proposed procedure significantly outperforms the standard estimation procedure and to further improve estimation performance compared to an earlier TV based procedure. Moreover, local regularity is also shown to be efficient to discriminate between visually similar textures such as cloud and snow.
6. REFERENCES


