New exponents for pointwise singularities classification

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October 14, 2017

Abstract

We introduce new tools for pointwise singularity classification: We investigate the properties of the two variable function which is defined at every point as the p-exponent of a fractional integral of order t; new exponents are derived which are not of regularity type but give a more precise description of the behavior of the function near a singularity. We revisit several classical examples (deterministic and random) of multifractal functions for which the additional information supplied by this classification is derived. Finally, a new example of multifractal function is studied, where these exponents prove pertinent.

Keywords: multifractal analysis, wavelets, chirps, Hausdorff dimension, p-exponent, oscillation exponent, lacunarity exponent, cancellation exponent.

1 Introduction

A long-standing problem in the 19th century was to determine if a continuous function necessarily has points of differentiability. In 1895, K. Weierstrass finally settled this issue by introducing the functions

$$W_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x),$$

and proving that, if $a \in (0,1)$, b is a positive odd integer and $ab > 1+3\pi/2$, then $W_{a,b}$ is continuous and nowhere differentiable. In 1916, G. Hardy sharpened this result in several ways. First, he improved the last requirement, by showing that this result holds under the natural (sharp) condition $ab > 1$; second, he showed that, when this condition is fulfilled, $W_{a,b}$ satisfies

$$\forall x, y \in \mathbb{R}, \qquad |W_{a,b}(x) - W_{a,b}(y)| \le C|x-y|^\alpha \quad \text{where } \alpha = -\frac{\log a}{\log b}$$

and that, for every x_0, $|W_{a,b}(x) - W_{a,b}(x_0)|$ is nowhere a $o(|x-x_0|^\alpha)$. Hardy's result can be restated using the following definition.

Definition 1 *Let $x_0 \in \mathbb{R}$ and $\alpha \ge 0$. A locally bounded function $f : \mathbb{R} \to \mathbb{R}$ belongs to $C^\alpha(x_0)$ if there exist $C > 0$ and a polynomial P_{x_0} with deg$(P_{x_0}) < \alpha$ and such that on a neighborhood of x_0,*

$$|f(x) - P_{x_0}(x)| \le C|x-x_0|^\alpha. \tag{1}$$

The pointwise Hölder exponent of f at x_0 is $h_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}$.

Note that, in all this paper, we only consider functions of one variable only; most definitions and results extend without difficulty to the several variable setting. However, most examples require arguments involving primitives that could not be reproduced as such in several dimensions.

Thus, the pointwise Hölder exponent of Weierstrass functions is constant and equal to α. This seminal result opened the way to the study of the regularity of functions by using pointwise Hölder conditions. A milestone was the determination of the Hölder exponent of Brownian motion. Then a key development occurred in 1961, when Calderón and Zygmund realized that pointwise Hölder regularity does not possess natural continuity properties under the action of singular integral operators: For instance, the space $C^\alpha(x_0)$ is not invariant under the Hilbert transform. As a substitute for Hölder regularity without this drawback, they introduced the following notion in [6].

Definition 2 *Let $p \ge 1$ and assume that $f \in L^p_{loc}(\mathbb{R})$. Let $\alpha \in \mathbb{R}$; f belongs to $T^p_\alpha(x_0)$ if there exists a constant C and a polynomial P_{x_0} of degree less than α such that, for r small enough,*

$$\left(\frac{1}{r}\int_{x_0-r}^{x_0+r} |f(x) - P_{x_0}(x)|^p dx\right)^{1/p} \le Cr^\alpha. \tag{2}$$

The p**-exponent** *of f at x_0 is*

$$h^p_f(x_0) = \sup\{\alpha : f \in T^p_\alpha(x_0)\}. \tag{3}$$

The *Taylor polynomial* P_{x_0} of f at x_0 is unique for a given α, and is independant of p; but its degree depends on $[\alpha]$ [17]; however, we introduce no such dependency in the notations, which will lead to no ambiguity. The condition $f \in L^p_{loc}$ implies that (2) holds for $\alpha = -1/p$, so that $h^p_f(x_0) \geq -1/p$.

An additional advantage is that this notion only requires that $f \in L^p_{loc}$ in order to be well defined, whereas Hölder regularity requires that f is locally bounded. This issue is important in modeling, since large classes of experimental signals cannot be modeled by locally bounded functions [13, 19, 21]. Despite their early definition, p-exponents were not used in signal and image processing until recently; a reason is that numerically efficient (wavelet based) methods for their estimation were only proposed in 2005 [16] and used in practice in 2015 [13, 17, 20]. Instead, the problem of estimating regularity exponents for data that are not locally bounded was implicitly resolved through a different technique. At the end of the 1980s, A. Arneodo and his collaborators used the *wavelet transform maxima method* to study the singularities of signals [25]. Such maxima are bounded if the function under investigation is locally bounded. Typically, this is not the case for quantities modeled by singular measures, such as the energy dissipation in turbulent fields (which motivated the early developments of multifractal analysis [5, 26]). If these maxima were found to diverge in the limit of small scales ($a \to 0$), then an extra convergence factor a^t was applied to the continuous wavelet transform $C(a, b) = a^{-t} \int f(u)\psi\left(\frac{u-b}{a}\right) du$, with t large enough. This "renormalization" of the wavelet transform can be interpreted as performing a *fractional integration* of order t on the data [2], and thus as a regularization of the signal.

Definition 3 *Let $t > 0$ and let ϕ be a C^∞ compactly supported function satisfying $\phi(x_0) = 1$. Let $(Id - \Delta)^{-t/2}$ be the convolution operator which amounts to multiplying the Fourier transform of the function with $(1 + |\xi|^2)^{-t/2}$. The fractional integral of order t of f is the function $f^{(-t)} = (Id - \Delta)^{-t/2}(\phi f)$.*

Note that, though this definition depends on the function ϕ, the pointwise regularity properties of $f^{(-t)}$ do not [2]. A drawback of using p-exponents or fractional integration in the definition of pointwise regularity is that this notion may depend on p or t. It is therefore important to understand this dependency. The heuristic forged by considering the simplest type of pointwise singularities, e.g. *cusps* $|x - x_0|^\alpha$ at x_0 (for $\alpha \notin 2\mathbb{N}$), makes one expect that they are invariant under a change of p and shifted by t under a fractional integration. This heuristic, however, does not hold in full generality, as shown by the following example. Let $\alpha, \beta > 0$. The *chirp* $\mathcal{C}_{\alpha,\beta}$ is

$$\mathcal{C}_{\alpha,\beta}(x) = |x|^\alpha \sin\left(\frac{1}{|x|^\beta}\right). \tag{4}$$

One easily checks that the Taylor polynomial of a chirp vanishes, so that its Hölder exponent is α. Additionally, an integration by parts yields that the Hölder exponent at 0 of $\mathcal{C}^{(-1)}$ is $\alpha + \beta + 1$, so that it is increased by $1 + \beta$ after one integration; and it is the same for a fractional integration of order 1, see [2].

Therefore, different types of singularities behave differently under the action of a fractional integration. A new perspective, introduced in [2], is that, far from being a drawback, this fact can be used as a way to probe into the difference of nature between singularities that "behave like" cusps or chirps in the neighborhood of x_0. To that end, the following definition was proposed.

Definition 4 *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a locally bounded function. If $h_f(x_0) \neq +\infty$, then the oscillation exponent of f at x_0 is*

$$\mathcal{O}_f(x_0) = \left(\frac{\partial}{\partial t} h_{f^{(-t)}}(x_0)\right)_{t=0^+} - 1. \tag{5}$$

The choice of taking the derivative at $t = 0^+$ is motivated by a perturbation argument: The exponent should not be perturbed when adding to f a term that would be a $o(|x - x_0|^h)$ for an $h > h_f(x_0)$.

The oscillation exponent takes the value β for a chirp; it is the first of *second generation exponents* that do not measure a regularity, but yield additional information, paving the way to a richer description of singularities. Our purpose in this paper is to discuss this classification based on the Hölder and oscillation exponents, show its limitations, and propose a richer description where the oscillation exponent actually splits into two new exponents, which in turn yield additional informations of different natures, which we will investigate.

This paper is organized as follows. In Section 2.1, we discuss several examples of toy singularities in order to put into light the limitations of using the oscillation exponent only. In Section 2.2, we introduce the notion of *fractional exponent*, which encapsulates all the available pointwise regularity information, and show how to derive from it two relevant parameters: the *lacunarity* and the *cancellation* exponents. At the beginning of Section 3 the properties of the fractional exponent are derived. In Sections 3.4, 3.5 and 3.6, we revisit several deterministic and random models of multifractal functions where lacunarity exponents are relevant. Then, we turn to the new cancellation exponent. In

Section 5 we show how to construct pointwise singularities with given lacunarity and cancellation exponents, and in Section 6 we construct new examples of deterministic multifractal functions where a multifractal analysis using cancellation exponents can be performed.

The authors thank the referee for a careful reading of a first version of this text and for many suggestions of improvement.

2 Motivation and definitions

To motivate the need for a more accurate classification of singularities, we compare examples of oscillating singularities (i.e. singularities with a non-vanishing oscillation exponent).

2.1 Two kinds of oscillating behaviors

We start by considering again the example of the chirp (4). An integration by part yields also that the Hölder exponent at 0 of $\mathcal{C}^{(-2)}$ is $\alpha + 2(\beta + 1)$. Since the mapping $t \to h_{(f)^{(-t)}}(0)$ is concave [2], the fact that the Hölder exponents of $\mathcal{C}_{\alpha,\beta}$, $(\mathcal{C}_{\alpha,\beta})^{(-1)}$ and $(\mathcal{C}_{\alpha,\beta})^{(-2)}$ at 0 are in arithmetic progression implies that it is necessarily an affine function for $t \in [0, 2]$; it follows that $\mathcal{O}_{\mathcal{C}_{\alpha,\beta}}(0) = \beta$.

The second example (already considered in [13]) is the *lacunary comb*. Let ψ be the Haar wavelet: $\psi = \mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}$ and

$$\theta(x) = \psi(2x) - \psi(2x - 1) \tag{6}$$

(so that θ is supported by $[0, 1]$ and its two first moments vanish).

Definition 5 *Let $\alpha \in \mathbb{R}$ and $\gamma > \omega > 0$. The lacunary comb $F^{\alpha}_{\omega,\gamma}$, is*

$$F^{\alpha}_{\omega,\gamma}(x) = \sum_{j=1}^{\infty} 2^{-\alpha j} \theta\left(2^{\gamma j}(x - 2^{-\omega j})\right). \tag{7}$$

An illustration is provided in Figure 1. We consider its singularity at $x_0 = 0$: $F^{\alpha}_{\omega,\gamma}$ is locally bounded if and only if $\alpha \geq 0$, which we now assume. Denote by $\theta^{(-1)}$ the primitive of θ which has support on $[0, 1]$ and by $\theta^{(-2)}$ the primitive of $\theta^{(-1)}$ which has support on $[0, 1]$. Then the primitive of $F^{\alpha}_{\omega,\gamma}$ is

$${F^{\alpha}_{\omega,\gamma}}^{(-1)}(x) = \sum_{j=1}^{\infty} 2^{-(\alpha+\gamma)j} \theta^{(-1)}\left(2^{\gamma j}(x - 2^{-\omega j})\right),$$

and its second primitive is

$$F_{\omega,\gamma}^{\alpha}{}^{(-2)}(x) = \sum_{j=1}^{\infty} 2^{-(\alpha+2\gamma)j} \theta^{(-2)} \left(2^{\gamma j}(x - 2^{-\omega j}) \right).$$

Note that the Taylor polynomials of $F_{\omega,\gamma}^{\alpha}{}^{(-1)}$ and $F_{\omega,\gamma}^{\alpha}{}^{(-2)}$ vanish at 0 because these functions vanish on $\mathbb{R}^-$; it follows that

$$h_{F_{\omega,\gamma}^{\alpha}}(0) = \frac{\alpha}{\omega}, \quad h_{F_{\omega,\gamma}^{\alpha}{}^{(-1)}}(0) = \frac{\alpha+\gamma}{\omega} \quad \text{and} \quad h_{F_{\omega,\gamma}^{\alpha}{}^{(-2)}}(0) = \frac{\alpha+2\gamma}{\omega}.$$

Since the mapping $t \to h_{(f)^{(-t)}}(0)$ is concave, the same argument as in the chirp case implies that $\mathcal{O}_{F_{\omega,\gamma}^{\alpha}}(0) = \frac{\gamma}{\omega} - 1$.

We conclude that chirps and lacunary combs are two examples of oscillating singularities. They are, however, of different nature: In the comb case, oscillation is due to the fact that this function vanishes on larger and larger proportions of small balls centered at the origin (this is detailed in [13], where this phenomenon is precisely quantified through the use of *accessibility exponent* of a set at a point). We will also see that $|F_{\omega,\gamma}^{\alpha}|$ also displays an oscillating singularity at the origin, with the same oscillation exponent as $F_{\omega,\gamma}^{\alpha}$ (see the remark after Definition 11 below). On the other hand, chirps are oscillating singularities for a very different reason: It is due to very fast oscillations, and compensations of signs. This can be checked by verifying that the oscillation exponent of the absolute value of $\mathcal{C}_{\alpha,\beta}$ at 0 vanishes. Thus $\mathcal{C}_{\alpha,\beta}$ and $F_{\omega,\gamma}^{\alpha}$ display oscillating behaviors of very different natures. We will introduce new exponents that will allow to draw a difference between these two different behaviors.

2.2 The fractional exponent

Comparing the p-exponents of chirps and lacunary combs allows to draw a distinction between their singularities; indeed, for $p \geq 1$, see [17],

$$h_{F_{\omega,\gamma}^{\alpha}}^{p}(0) = \alpha + \frac{1}{p}\left(\frac{\gamma}{\omega} - 1\right) \tag{8}$$

whereas a straightforward computation yields that $h_{\mathcal{C}_{\alpha,\beta}}^{p}(0) = \alpha$. We conclude that the p-exponent of $F_{\omega,\gamma}^{\alpha}$ varies with p, whereas the one of $\mathcal{C}_{\alpha,\beta}$ does not. Therefore, a natural idea is to consider the whole pointwise regularity information available, i.e. **the p-exponent of a fractional integration of the data**, and investigate which information on the nature of the singularities can be derived from it. We can infer from (8) that the "right" variable when considering p-exponents is $q = 1/p$.

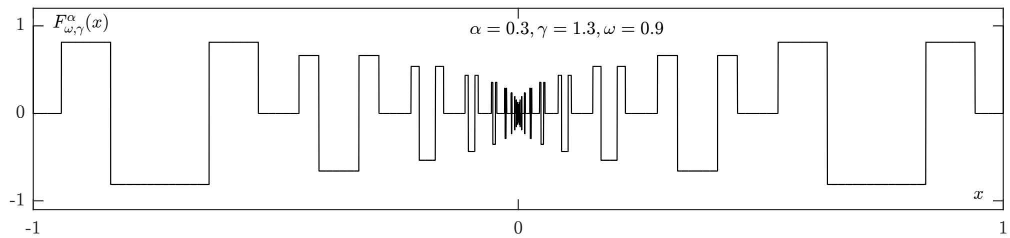

Figure 1: Illustration of a lacunary comb $F^{\alpha}_{\omega,\gamma}$ with $\alpha = 0.3$, $\gamma = 1.3$ and $\omega = 0.9$.

Definition 6 *Let f be a tempered distribution. The fractional exponent of f at x_0 is the two variable function*

$$\mathcal{H}_{f,x_0}(q,t) = h^{1/q}_{f^{(-t)}}(x_0) - t \tag{9}$$

(where $h^{\infty}_f(x_0)$ denotes the Hölder exponent at x_0).

Properties of the mapping $t \to h_{f^{(-t)}}(x_0)$ (called the *two-microlocal domain*) have been investigated by J. Lévy-Véhel and S. Seuret [22] and applications to stochastic processes have been worked out by E. Herbin and P. Balança [3, 4]. The reason for substracting t in (9) is that $\mathcal{H}_{f,x_0}$ measures the "excess" of the increase of regularity in a fractional integral of order t when compared with what is "expected" in general (and is verified for cusps), i.e. t, and several properties will be easier to state in terms of this "excess". The domain of definition of $\mathcal{H}_{f,x_0}$ is a subset of $\mathbb{R}^+ \times \mathbb{R}$ studied in Section 3.2; we introduce now a notion which allows to make precise this domain of definition for $t = 0$.

Definition 7 *If $f \in L^p_{loc}$ in a neighborhood of x_0 for $p > 1$, the critical Lebesgue index of f at x_0 is*

$$p_f(x_0) = \sup\{p : f \in L^p_{loc}(\mathbb{R}) \text{ in a neighborhood of } x_0\}. \tag{10}$$

The p-exponent at x_0 is defined on the interval $[1, p_f(x_0)]$ or $[1, p_f(x_0))$. We denote: $q_f(x_0) = 1/p_f(x_0)$.

Note that $p_f(x_0)$ can take the value $+\infty$. Keeping at every point x_0 a two-variable function is excessive for classification purposes. So the next goal is to extract a pertinent information that can be encapsulated into a few parameters. On top of a regularity exponent, we will derive two additional exponents. The first one is the *lacunarity exponent*, already introduced in [13].

Definition 8 *Let $f \in L^p_{loc}$ in a neighborhood of x_0 for a $p > 1$, and assume that the p-exponent of f is finite (i.e $< \infty$) in a left neighborhood of $p_f(x_0)$. The lacunarity exponent of f at x_0 is*

$$\mathcal{L}_f(x_0) = \frac{\partial}{\partial q} \left(\mathcal{H}_{f,x_0}(q,0)\right)_{q=q_f(x_0)^+}. \tag{11}$$

This quantity may have to be understood as a limit when $q \to q_f(x_0)$, since $h_f^{1/q}(x_0)$ is not necessarily defined for $q = q_f(x_0)$. This limit always exists as a consequence of the concavity of the mapping $q \to h_f^{1/q}(x_0)$, and it is nonnegative (because this mapping is increasing). We now compute the fractional exponent and derive the lacunarity exponent for the examples already introduced.

Lemma 2.1 *Let $\alpha > -\gamma$. The fractional exponent of the Lacunary comb $F^\alpha_{\omega,\gamma}$ for $t \le 2$ and $q \in [\max(0, -\alpha/\gamma), 1]$ is*

$$\mathcal{H}_{F^\alpha_{\omega,\gamma},0}(q,t) = \frac{\alpha}{\omega} + \left(\frac{\gamma}{\omega} - 1\right)(q+t). \tag{12}$$

Note that we no longer assume that $\alpha > 0$. The proof of this lemma is straightforward: The computation of the p-exponent is similar as the one done in [13] for $\alpha > 0$. And it is also the case for the primitive and the second primitive. The fractional exponent follows from the usual concavity argument.

Thus, the lacunarity exponent of $F^\alpha_{\omega,\gamma}$ at 0 is $\frac{\gamma}{\omega} - 1$, which puts into light the fact that this exponent allows to measure how $F^\alpha_{\omega,\gamma}$ vanishes on "large sets" in the neighborhood of 0 (see [13] for a precise statement). Furthermore the oscillation exponent of $F^\alpha_{\omega,\gamma}$ at 0 is $\frac{\gamma}{\omega} - 1$, so that it coincides with the lacunarity exponent.

Chirps: We only assume that $\alpha > -1$ in (4), so that $\mathcal{C}_{\alpha,\beta}$ can be unbounded. If $\alpha \ge 0$, it is a bounded function so that $p_{\mathcal{C}_{\alpha,\beta}}(0) = +\infty$, whereas, if $\alpha \in (-1, 0)$, $p_{\mathcal{C}_{\alpha,\beta}}(0) = -1/\alpha$; $\mathcal{C}_{\alpha,\beta}$ clearly satisfies $\forall p < p_{\mathcal{C}_{\alpha,\beta}}(0)$, $h^p_{\mathcal{C}_{\alpha,\beta}}(0) = \alpha$, and the integration by parts argument already mentioned shows that, $\forall p \ge 1$ the p-exponent of $\mathcal{C}_{\alpha,\beta}$ is increased by $1 + \beta$ after each integration. The usual concavity argument yield that, after a fractional integration of order t, it is increased by $(1+\beta)t$, so that $\mathcal{H}_{\mathcal{C}_{\alpha,\beta},x_0}(q,t) = \alpha + \beta t$. Therefore chirps are another example of functions with a vanishing lacunarity exponent, which reflects the fact that, on the average, chirps do not vanish on a "large set" near 0.

Comparing lacunary combs and chirps, we see that the oscillation exponent takes into account two quantities of different natures: the lacunarity and the "cancellation" which encapsulates compensations between positive and negative values which are cancelled by a local averaging (such as taking a fractional integral); in order to singularize this quantity, a new exponent is required. We now consider another toy-examples of pointwise singularities, which stand between lacunary combs and chirps: the *fat combs* $F^\alpha_{\omega,\gamma,\delta}$, see [13].

Fat combs: Let

$$\theta_N(x) = \sum_{k=0}^{N-1} \theta(x-k)$$

(where θ was defined by (6)); the support of θ_N is $[0,N]$ and, on this interval, it coincides with the 1-periodic periodization of θ). Let ω, γ, δ be such that

$$0 < \omega < \gamma < \delta; \tag{13}$$

we define

$$F^\alpha_{\omega,\gamma,\delta}(x) = \sum_{j\geq 0} 2^{-\alpha j} \theta_{[2^{(\delta-\gamma)j}]} \left(2^{\delta j}(x - 2^{-\omega j}) \right). \tag{14}$$

The function $F^\alpha_{\omega,\gamma,\delta}$ is illustrated in Figure 2.

Lemma 2.2 *If $\alpha > -\gamma$, $t \leq 2$ and $q \in [\max(0, -\alpha/\gamma), 1]$, then*

$$\mathcal{H}_{F^\alpha_{\omega,\gamma,\delta},0}(q,t) = \frac{\alpha}{\omega} + \left(\frac{\gamma}{\omega} - 1\right) q + \left(\frac{\delta}{\omega} - 1\right) t. \tag{15}$$

Let us sketch the proof. First (13) implies that the different components in the series (14) have disjoint support; so that its p-exponent at 0 is

$$h^p_{F^\alpha_{\omega,\gamma,\delta}}(x_0) = \frac{\alpha}{\omega} + \left(\frac{\gamma}{\omega} - 1\right) \frac{1}{p},$$

so that (15) holds for $t = 0$. A computation of the two first primitives yields that (15) also holds for $t = 1$ and 2, and the usual concavity arguments yield the property for intermediate orders of integration.

It follows that the lacunarity exponent of $F^\alpha_{\omega,\gamma,\delta}$ at 0 is $\frac{\gamma}{\omega} - 1$, and its oscillation exponent is $\frac{\delta}{\omega} - 1$, which is larger. We infer that a new **cancellation exponent** should be the difference of theses two quantities, and would take the value $(\delta - \gamma)/\omega$ for fat combs; this motivates the following definition.

Definition 9 *Let $f \in L^p_{loc}$ in a neighborhood of x_0 for a $p > 1$, and assume that the $p_f(x_0)$-exponent of $f^{(-t)}$ is finite for small enough t. The cancellation exponent of f at x_0 is*

$$\mathcal{C}_f(x_0) = \frac{\partial}{\partial t} \left(\mathcal{H}_{f,x_0}(q_f(x_0) - t, t) \right)_{t=0^+}. \tag{16}$$

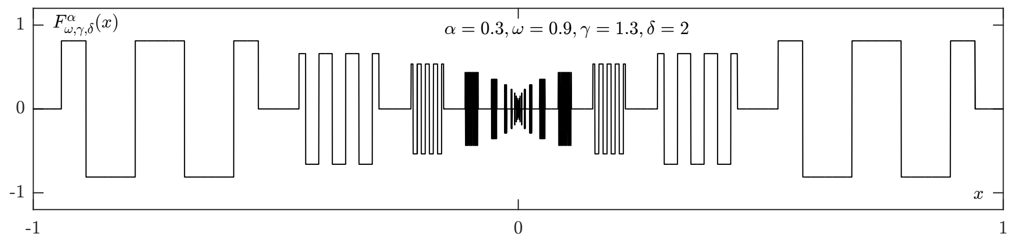

Figure 2: Illustration of a fat comb $F^{\alpha}_{\omega,\gamma,\delta}$ with $\alpha = 0.3$, $\gamma = 1.3$, $\omega = 0.9$ and $\delta = 2$. Note that the number of pulses in each block increases as $x \to 0$.

Remarks:

• For consistency, the derivative is computed at the same point as for the lacunarity exponent. Note however that it needs not always exist, either because $\mathcal{H}_{f,x_0}$ is not defined at $(q_f(x_0), 0)$ or because $q_f(x_0) = 0$ (which is the case if f is locally bounded). In such cases $\mathcal{C}_f(x_0)$ should be understood as the limsup of the quantity $\frac{\partial}{\partial t}\left(\mathcal{H}_{f,x_0}(q-t,t)\right)$ when $q \to q_f(x_0)^+$.

• $\mathcal{C}_f(x_0)$ is non-negative, as a consequence of Theorem 1 below.

• If $\mathcal{H}_{f,x_0}(q,t)$ has a differentiable extension in a neighborhood of $(q_f(x_0), 0)$,

$$\mathcal{O}_f(x_0) = \mathcal{C}_f(x_0) + \mathcal{L}_f(x_0) \tag{17}$$

(as a consequence of the relationships between partial derivatives). Equality will hold in several examples; however, it needs not hold in all cases (cf. the end of Section 5). In order to describe the properties of the cancellation exponent, we first need to investigate the properties of the fractional exponent.

3 Properties of the fractional exponent

In signal and image processing, one often meets data that cannot be modeled by functions in L^1_{loc}, see [13, 19]. It is therefore necessary to set the analysis in a wider functional setting. One possibility is to consider real Hardy spaces H^p (with $p < 1$), instead of L^p spaces, see [12]. We will therefore consider the whole collection of p-exponents (for $p > 0$) of fractional integrals of f of arbitrary order. First, we need to extend definitions to the range $p \in (0, 1]$ (i.e. $q \geq 1$). A key point is that the wavelet characterization of L^p remains unchanged for H^p; it follows that T^p_α regularity can be extended to this setting while retaining the same wavelet characterization, see [12]. Therefore, all definitions and wavelet characterizations introduced previously extend to this setting, and it is in particular the case of the fractional exponent (see Definition 6). We start by recalling classical properties of orthonormal wavelet expansions.

3.1 Wavelet characterizations

3.1.1 Wavelet bases

An **orthonormal wavelet basis** is generated by a couple of functions $(\varphi,\ \psi)$, which will either be in the Schwartz class, or compactly supported and smooth enough (the required smoothness depends on the considered space, and will always be assumed to be "large enough"). The functions $\varphi(x-k)$, (for $k \in \mathbb{Z}$) together with the $2^{j/2}\psi(2^j x - k)$, (for $j \geq 0$, and $k \in \mathbb{Z}$) form an orthonormal basis of $L^2(\mathbb{R})$. Thus any function $f \in L^2(\mathbb{R})$ can be written

$$f(x) = \sum_{k \in \mathbb{Z}} c_k\ \varphi(x-k) + \sum_{j \geq 0} \sum_{k \in \mathbb{Z}} c_{j,k}\ \psi(2^j x - k),$$

where the wavelet coefficients of f are given by

$$c_k = \int \varphi(t-k) f(t) dt \quad \text{and} \quad c_{j,k} = 2^j \int \psi(2^j t - k) f(t) dt.$$

These formulas also hold in many different functional settings (such as the Besov or Sobolev spaces), if the selected wavelets are smooth enough.

Instead of using the indices (j,k), we will use dyadic intervals: Let

$$\lambda\ (= \lambda(j,k))\ = \left[\frac{k}{2^j}, \frac{k+1}{2^j}\right) \tag{18}$$

and, accordingly, $c_\lambda = c_{j,k}$ and $\psi_\lambda(x) = \psi(2^j x - k)$. Indexing by dyadic intervals will be useful because λ indicates the localization of the corresponding wavelet.

3.1.2 Wavelet characterization of L^p spaces

We denote indifferently by $\chi_{j,k}$ or χ_λ the characteristic function of the interval λ defined by (18). The **wavelet square function** of f is

$$\mathcal{W}_f(x) = \left(\sum_{(j,k) \in \mathbb{Z}^2} |c_{j,k}|^2 \chi_{j,k}(x) \right)^{1/2}.$$

Then $f \in L^p(\mathbb{R})$ when $p > 1$ (resp. $f \in H^p(\mathbb{R})$ when $p \leq 1$) if and only if $\int (\mathcal{W}_f(x))^p\, dx < \infty$. One associates a norm (resp. a semi-norm) to L^p (resp.

H^p): $\| f \|_p = \| \mathcal{W}_f \|_p$, see [24]. The elements of H^p are no more functions but can be distributions; therefore the restriction of f to an interval cannot be done directly. If I is an open interval, one defines $\| f \|_{H^p(I)} = \inf \| g \|_p$ where the infimum is taken on the $g \in H^p$ such that $f = g$ on I. The T^p_α condition for $p \leq 1$ is then defined by $f \in T^p_\alpha(x_0)$ if $\| f \|_{H^p(B(x_0,r))} \leq C \cdot r^{\alpha+1/p}$. This extension of the p-exponent still takes values in $[-1/p, +\infty]$, as shown below.

In order to define the lacunarity, oscillation and cancellation exponent, we assumed that $f \in L^1$. We can now replace this assumption by $f \in H^p$ for a $p > 0$, and these definitions remain unchanged. From now on, we will often use a slight abuse of notation and denote by L^p the space H^p when $p < 1$.

Examples of distributions for which the p-exponent is constant (see Proposition 3.2 below) and equal to a given $\alpha < -1$ are supplied by the cusps, which are defined for $\alpha \leq -1$ as follows. First, note that cusps cannot be defined directly (as distributions) for $\alpha \leq -1$ by $C_\alpha(x) = |x|^\alpha$ because they do not belong to L^1_{loc} so that their integral against a C^∞ compactly supported function φ is not well defined; instead, we note that, if $\alpha > 1$, then $\mathcal{C}''_\alpha = \alpha(\alpha-1)\mathcal{C}_{\alpha-2}$. Thus we can define $\mathcal{C}_\alpha$ by recursion, when $\alpha < -1$ and $\alpha \notin \mathbb{Z}$, by

$$\text{if} \quad \alpha < 0, \qquad \mathcal{C}_\alpha = \frac{1}{(\alpha+1)(\alpha+2)} \mathcal{C}^{(2)}_{\alpha+2},$$

where the derivative is taken in the sense of distributions. The $\mathcal{C}_\alpha$ are thus defined as distributions when α is not a negative integer. It can also be done when α is an integer, by taking $\mathcal{C}_0 = \log(|x|)$ and $\mathcal{C}_{-1} = \mathcal{C}'_0 = P.V.(1/x)$.

The following result will prove useful for the characterization of the two-variable functions that are fractional exponents.

Proposition 3.1 *Let $p, q \in (0, +\infty]$, and suppose that $f \in T^p_\alpha(x_0) \cap T^q_\beta(x_0)$; let $\theta \in [0,1]$. Then $f \in T^r_\gamma(x_0)$, where*

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{q} \quad \text{and} \quad \gamma = \theta\alpha + (1-\theta)\beta.$$

Proof: When $p, q < \infty$, the result will be a consequence of the wavelet characterization of $T^p_\alpha(x_0)$, see [12]. Let λ be a dyadic interval; 3λ will denote the interval of same center and three times wider (it is the union of λ and its two closest neighbors). For $x_0 \in \mathbb{R}^d$, denote by $\lambda_j(x_0)$ the dyadic cube of width 2^{-j} which contains x_0. The **local square functions** at x_0 are

$$\mathcal{W}^j_f(x) = \left(\sum_{\lambda \subset 3\lambda_j(x_0)} | c_\lambda|^2 \chi_\lambda(x) \right)^{1/2}.$$

Recall, see [12], that

$$f \in T^p_\alpha(x_0) \quad \text{if and only if} \qquad \left\| \mathcal{W}^j_f \right\|_p \leq 2^{-(\alpha+1/p)j}. \tag{19}$$

By interpolation, $\left\| \mathcal{W}^j_f \right\|_r \leq \left\| \mathcal{W}^j_f \right\|_p^{\theta/p} \left\| \mathcal{W}^j_f \right\|_q^{(1-\theta)/q}$, hence the result holds for $p, q < \infty$. The case when p or $q = +\infty$ does not follow, because there exist no wavelet characterization of $C^\alpha(x_0) = T^\infty_\alpha(x_0)$; however, when $p, q > 1$, one can use the initial definition of $T^p_\alpha(x_0)$ and $C^\alpha(x_0)$ through local L^p and L^∞ norms and the result follows from Hölder's inequality; hence Proposition 3.1 holds.

If $f \in H^p$, then $\| \mathcal{W}_f \|_p \leq C$. Since $\mathcal{W}^j_f \leq \mathcal{W}_f$, $\| \mathcal{W}^j_f \|_p \leq C$, so that (19) holds with $\alpha = -1/p$. Thus p-exponents are always larger that $-1/p$. The following result shows that they can take values down to $-1/p$ (its proof follows from the estimation of wavelet coefficients, using the selfsimilarity of cusps).

Proposition 3.2 *If $\alpha \geq 0$, the cusp $\mathcal{C}_\alpha$ belongs to L^∞_{loc} and its p-exponent is α. If $\alpha < 0$, the cusp $\mathcal{C}_\alpha$ belongs to L^p_{loc} for $p < -1/\alpha$ and its p-exponent is α.*

3.1.3 p-leaders

We will derive T^p_α regularity from simpler quantities than the local square functions. The p**-leaders** of f are defined by

$$d^p_\lambda = \left(\sum_{\lambda' \subset 3\lambda} |c_{\lambda'}|^p 2^{-(j'-j)} \right)^{1/p} \tag{20}$$

(they are finite if $f \in L^p_{loc}(\mathbb{R}^d)$, see [16]). Note that, if $p = +\infty$, the corresponding quantity (called the **wavelet leaders**) is

$$d_\lambda := d^\infty_\lambda = \sup_{\lambda' \subset 3\lambda} |c_{\lambda'}|. \tag{21}$$

The notion of T^p_α regularity can be related to p-leaders (which are local l^p norms of wavelet coefficients) as follows (see [14, 16]) :

$$\text{If} \quad \eta_f(p) > 0, \quad \text{then} \qquad h^p_f(x_0) = \liminf_{j \to +\infty} \frac{\log\left(d^p_{\lambda_j(x_0)} \right)}{\log(2^{-j})}. \tag{22}$$

3.2 The fractional exponent domain

Let $\mathcal{D}_{f,x_0}$ denote the **domain of definition** of $\mathcal{H}_{f,x_0}(q,s)$; it is the set of points $(q,t) \in \mathbb{R}^+ \times \mathbb{R}$ such that $f^{(-t)}$ locally belongs to $L^{1/q}$ (resp. $H^{1/q}$ if $q \geq 1$) in a neighborhood of x_0. Note that we allow the fractional integration parameter t to take positive *and* negative values (i.e. we consider both fractional integrals and fractional derivatives of f). If f is a tempered distribution, it has a finite order, so that for t large enough $f^{(-t)} \in L^\infty_{loc}$. It follows that $\mathcal{D}_{f,x_0}$ is never empty. In order to investigate the properties of $\mathcal{D}_{f,x_0}$, we recall the definition of the **local scaling function at** x_0. The Sobolev space $L^{p,s}$ is defined by

$$\forall s \in \mathbb{R},\ \forall p > 0, \qquad f \in L^{p,s} \quad \Longleftrightarrow \quad f^{(s)} \in L^p$$

The *local scaling function* at x_0, $\eta_{f,x_0}(p)$, is

$$\eta_{f,x_0}(p) = p \cdot \sup\{s : f \in L^{p,s}_{loc} \text{ in a neighborhood of } x_0\}. \tag{23}$$

The **uniform Hölder exponent** of a tempered distribution f is

$$H_f^{min} = \sup\{s : \ f \in C^s_{loc}(\mathbb{R}^d)\}.$$

Proposition 3.3 *The boundary of $\mathcal{D}_{f,x_0}$ is the graph of the function*

$$\mathcal{B}_{f,x_0}(q) := -q\ \eta_{f,x_0}\left(\frac{1}{q}\right).$$

Furthermore:

1. $\mathcal{B}_{f,x_0}$ *is convex;*
2. $\forall q \geq 0$, $\mathcal{B}'_{f,x_0}(q) \leq -1$.
3. $\mathcal{B}_{f,x_0}(q_f(x_0)) = 0$ *and* $\mathcal{B}_{f,x_0}(0) = -H_f^{min}$.

We will refer to the function $\mathcal{B}_{f,x_0}$ as the p-**boundary** of f at x_0. Note that most examples of multifractal functions that have been studied are **homogeneous**, i.e. their local scaling function does not depend on x_0. In such situations, though the fractional exponent may strongly differ from point to point, its domain of definition does not depend on x_0.

Proof of Proposition 3.3: For a given p, $\mathcal{H}_{f,x_0}$ is well defined at $(1/p, t)$ if $t > -\eta_{f,x_0}(p)/p$. Therefore, the first assertion of the proposition holds.

As regards Point 1, recall that the function η_{f,x_0} is concave; and, for functions defined on $\mathbb{R}^+$, the mapping $\mathcal{I}$ is defined for concave functions on $\mathbb{R}^+$ by $(\mathcal{I}\eta)(q) = q\eta\,(1/q)$ which maps concave functions to concave functions [10].

Point 2 follows from the Sobolev interpretation of the p-boundary. Indeed, the Sobolev embeddings state that (see [28], Section 2.7.1 which covers $p < 1$), if $f \in L^{p,s}$ and if $p < \tilde{p}$, then $f \in L^{\tilde{p},t}$ where t is such that $\frac{1}{p} - \frac{1}{\tilde{p}} = s - t$. It follows that, if $(q,t) \in \mathcal{D}_{f,x_0}$, then the segment $\{(q-s, t+s): \quad 0 \leq s \leq q\}$ is included in $\mathcal{D}_{f,x_0}$, i.e. the p-boundary satisfies $\mathcal{B}'_{f,x_0}(q) \leq -1$.

The p-boundary does not necessarily cross the q axis; assume that it does, so that $f \notin L^\infty_{loc}$, but $f \in L^p_{loc}$ for a $p > 0$. Coming back to the definition of $p_f(x_0)$ given by (10), we see that if $p < p_f(x_0)$, then $f \in L^p_{loc}$, so that $\eta_f(p) > 0$, and if $p > p_f(x_0)$, then $f \notin L^p_{loc}$, so that $\eta_f(p) < 0$; it follows that $\eta_f(p_f(x_0)) = 0$, so that $\mathcal{B}_{f,x_0}(q_f(x_0)) = 0$. Another important point is the initial value, at $q = 0$, of the p-boundary: Recall that the function $\mathcal{H}_{f,x_0}$ is well defined at (q,t) if and only if $f^{(-t)}$ locally belongs to $L^{1/q}$; for $q = 0$ this means that f locally belongs to the Hölder space C^{-t}. It follows from the definition of the uniform Hölder exponent that $\mathcal{B}_{f,x_0}(0) = -H_f^{min}$. These properties imply that the fractional exponent is well defined on the half-line $(q = q_f(x_0), t > 0)$, a property used in the definition of the oscillation exponent. Proposition 3.3 is completely proved.

3.3 Fractional exponent characterization

After investigating the properties of the domain of definition of $\mathcal{H}_{f,x_0}$, we now turn to the properties of this function itself.

Theorem 1 *Let f be a tempered distribution, and $x_0 \in \mathbb{R}$. The mapping $(q,t) \to \mathcal{H}_{f,x_0}(q,t)$ has the following properties:*

1. *It is concave on its domain of definition;*
2. *it is increasing in the first variable;*
3. *it is increasing in the direction of the second bissector, i.e., $\forall (q,t)$ where $\mathcal{H}_{f,x_0}$ is defined, the mapping $s \to \mathcal{H}_{f,x_0}(q-s, t+s)$ is increasing;*
4. *$\forall (q,t)$ where $\mathcal{H}_{f,x_0}$ is defined, $\mathcal{H}_{f,x_0}(q,t) \geq -q-t$.*

Furthermore, these conditions are optimal, i.e. if $\mathcal{H}$ is any function defined on a convex subset of $\mathbb{R}^+ \times \mathbb{R}$ of the form $t > B(q)$ with B convex and satisfying $\forall q \geq 0, B'(q) \leq -1$, and if $\mathcal{H}$ satisfies the above conditions; then $\mathcal{H}$ is the fractional exponent at x_0 of a tempered distribution.

Note that the statement of the third result requires the property asserted in Point 2 of Proposition 3.3.

Proof of Theorem 1: The first statement follows from Proposition 3.1. The second statement holds because, locally, $L^p \subset L^{\tilde{p}}$ if $p > \tilde{p}$; thus, if $f^{(-t)} \in T^p_\alpha(x_0)$, then $\forall \tilde{p} < p$, $f^{(-t)} \in T^{\tilde{p}}_\alpha(x_0)$. When $p \geq 1$, the third statement is a reformulation of Theorem 4 in [6], which we now recall: Assume that $f \in T^p_\alpha(x_0)$, and that $t \leq 1/p$; then $f^{(-t)} \in T^{\tilde{p}}_{\alpha+t}(x_0)$, provided that $\frac{1}{p} - \frac{1}{\tilde{p}} = t$ (this is a pointwise equivalent of the Sobolev embeddings). This is extended in Annex 7, covering the cases $p < 1$ and $q < 1$. This result, applied to $f^{(-t)}$, exactly means that the mapping $s \to \mathcal{H}_{f,x_0}(q-s, t+s)$ is increasing. The fourth statement is a reformulation of the fact that the p-exponent is larger than $-1/p$. The optimality requires the construction of new toy-examples defined through their wavelet expansion, and will be proved in Section 5.2.

Theorem 1 leaves room for a large variety of possible functions $\mathcal{H}_{f,x_0}$. A natural question is to find sufficient conditions under which it is constant, thus yielding cases where the regularity exponent is canonically defined. In this respect the following notion plays an important role.

Definition 10 *Let f be a tempered distribution on $\mathbb{R}$; f has a* **canonical singularity** *of index $(q_f(x_0), t_0)$ at x_0 if $(q_f(x_0), t_0) \in \mathcal{D}_{f,x_0}$ and*

$$\frac{\partial}{\partial t} \left(\mathcal{H}_{f,x_0}\right)_{q=q_f(x_0), t=t_0} = 0.$$

This definition is motivated by the following result.

Proposition 3.4 *Let f be a tempered distribution with a canonical singularity of index $(q_f(x_0), t_0)$ at x_0; then $\mathcal{H}_{f,x_0}$ is constant in the domain defined by the conditions: $q \geq 0$, $t \geq t_0$, and $q + t \geq q_f(x_0) + t_0$.*

Proof of Proposition 3.4: First, note that the function

$$t \to \mathcal{H}_{f,x_0}(q_f(x_0), t) \tag{24}$$

is concave so that its derivative is decreasing. Since this derivative vanishes at t_0, it is nonpositive for $t > t_0$, so that (24) is decreasing. Since, on other hand, it is increasing, we obtain that it is constant (note that, strictly speaking, the considered function may not be differentiable everywhere; however, as a concave function, it has everywhere right and left derivatives, and the argument is correct using this slightly more general setting). Let $q > q_f(x_0)$; $\mathcal{H}_{f,x_0}$ is increasing on the segment of ends $(q_f(x_0), t_0)$ and (q, t_0), and it is also increasing on the segment of ends (q, t_0) and $(q_f(x_0), t_0 + (q - q_f(x_0)))$. Since it is constant on the vertical axis, it cannot have increased, and it follows that it is constant on the two mentioned segments. Since $q > q_f(x_0)$ is arbitrary, it follows that $\mathcal{H}_{f,x_0}$ is constant in the first quadrant $(q \geq q_f(x_0), t \geq t_0)$. The same argument can be reproduced starting from $(q_f(x_0), t_0)$ and going first in the direction of the second bissector, and then in the direction of the q axis, and it follows that $\mathcal{H}_{f,x_0}$ is also constant in the next half quadrant issued from $(q_f(x_0), t_0)$ which corresponds to the directions $\pi/2 \leq \theta \leq 3\pi/4$. Hence Proposition 3.4 holds.

A first important class of singularities are **canonical singularities**, for which the oscillation exponent vanishes; the key example being cusps. Having now at our disposal two new exponents, it is therefore natural to introduce two other kinds of singularities, by requiring that one of these exponents vanishes.

Definition 11 *Let f be a tempered distribution on $\mathbb{R}$:*

- *f has a* **balanced singularity** *at x_0 if $\mathcal{L}_f(x_0) = 0$ and $\mathcal{C}_f(x_0) \neq 0$.*
- *f has a* **lacunary singularity** *at x_0 if $\mathcal{C}_f(x_0) = 0$ and $\mathcal{L}_f(x_0) \neq 0$.*

Chirps are typical examples of balanced singularities and lacunary combs are typical examples of lacunary singularities.

Remarks: Proposition 3.4 implies that, if f has a canonical singularity at x_0, then $\mathcal{L}_f(x_0) = \mathcal{C}_f(x_0) = 0$. More precisely, Properties 2 and 3 of Theorem 1 imply that, though (17) needs not always hold, one has: $\mathcal{O}_f(x_0) \geq \mathcal{L}_f(x_0)$ and $\mathcal{O}_f(x_0) \geq \mathcal{C}_f(x_0)$. Let us now come back to the lacunary comb (7). Since its Taylor polynomial vanishes, the p-exponent of $F^\alpha_{\omega,\gamma}$ and $|F^\alpha_{\omega,\gamma}|$ coincide, so that it is also the case for their lacunarity exponents. It follows that the cancellation exponent of $|F^\alpha_{\omega,\gamma}|$ is larger that the one of $F^\alpha_{\omega,\gamma}$. One easily checks that it cannot be larger, so they necessarily coincide, as mentioned in Section 2.1.

We now revisit some classical multifractal functions and investigate what this new classification allows to say about their singularities.

3.4 The Brjuno function

Let x be an irrational number in $]0,1[$, and let $x = [0; a_1, \cdots a_n, \cdots]$ denote its continued fraction expansion. The convergents p_n/q_n of x are $[0; a_1, \cdots a_n]$ with $p_n \wedge q_n = 1$. The Brjuno function at x is

$$B(x) = \sum_{n=0}^{\infty} |p_{n-1} - q_{n-1}x| \log \left(\frac{p_{n-1} - x\ q_{n-1}}{q_n x - p_n} \right), \tag{25}$$

where, by convention, $(p_{-1}, q_{-1}) = (1, 0)$, $(p_0, q_0) = (0, 1)$, and $(p_1, q_1) = (1, a_1)$, so that the first term in (25) is $\log(1/x)$. The Brjuno function is extended by periodicity on $\mathbb{R} - \mathbb{Q}$.

The Brjuno function is nowhere locally bounded; however, since it belongs to BMO, see [23], it follows that it is locally in L^p for any $p < \infty$, and one can consider its p-exponent at any point x_0, and for all values of $p < \infty$. It is related with the *(Diophantine) irrationality exponent* of x_0.

Definition 12 *Let $x_0 \notin \mathbb{Q}$, and p_n/q_n the sequence of convergents of the continued fraction expansion of x_0. Let $\tau_n(x_0)$ be defined by*

$$\left| x_0 - \frac{p_n}{q_n} \right| = \frac{1}{q_n^{\tau_n(x_0)}}. \tag{26}$$

The irrationality exponent of x_0 is $\tau(x_0) = \limsup_{n\to+\infty} \tau_n(x_0)$.

If x_0 is irrational, then $|x_0 - \frac{p_n}{q_n}| < \frac{1}{q_n^2}$, so that $\tau_n(x_0) > 2$, and $\tau(x_0) \geq 2$. The following result is proved in [15].

Theorem 2 *Let $p \in [1, +\infty)$. If $x_0 \in \mathbb{Q}$, then $h^p_B(x_0) = 0$. Otherwise, $h^p_B(x_0) = 1/\tau(x_0)$. Additionally, the Hölder exponent of the primitive of B is given by $h_{B^{(-1)}}(x_0) = 1 + 1/\tau(x_0)$.*

It follows that the fractional exponent of the Brjuno function is $\forall (q,t) \in \mathbb{R}^+ \times \mathbb{R}^+ - (0,0)$, $\mathcal{H}_{B,x_0}(q,t) = \frac{1}{\tau(x_0)}$. This is an example where the fractional exponent is not defined at $(q_B(0), 0) = (0, 0)$. Nonetheless, the "second generation" exponents are well defined as limits and satisfy: $\mathcal{O}_B(x_0) = \mathcal{L}_B(x_0) = \mathcal{C}_B(x_0) = 0$. Therefore B **has a canonical singularity at every point.**

An open problem concerns the fractional derivatives of B: For which values of s and p does $B^{(s)}$ locally belong to L^p? And, when such is the case, what is the corresponding p-exponent? A natural conjecture is that the fractional exponent of B is constant where it is defined (i.e. also for negative values of t).

3.5 The Riemann function

According to the tradition, Riemann would have proposed the function

$$\mathcal{R}(x) = \sum_1^\infty \frac{1}{n^2} \sin(\pi n^2 x)$$

as an example of a continuous nowhere differentiable function. Unlike lacunary series, the regularity of this function varies strongly from point to point. Let $x_0 \notin \mathbb{Q}$, p_n/q_n be its continued fraction expansion and

$$\rho(x_0) = \sup \left\{ \tau : \left| x_0 - \frac{p_m}{q_m} \right| \leq \frac{1}{q_m^\rho} \right\}$$

for infinitely values of m **such that p_m and q_m are not both odd** (note that $\rho(x_0)$ usually differs from $\tau(x_0)$ because of the additional parity constraint that we impose here). The Hölder exponent of $\mathcal{R}$ is $h_\mathcal{R}(x_0) = \frac{1}{2} + \frac{1}{2\rho(x_0)}$, see [8] and at such points, the Hölder exponent is shifted by s under a fractional integral of order s, if $s \in (-1/2, +\infty)$ (Corollary 2 of [8]). It follows that these points are **canonical singularities** of $\mathcal{R}$. At rational points explicit local expansions yield that these rationals are **balanced singularities** [18]. It follows that the Riemann function **has no lacunary singularities**.

It also follows from Corollary 2 of [8] that, if $s < 1/2$, then the Hölder exponent of $\mathcal{R}^{(s)}$ at a point x_0 which is not of the form $(2p+1)/(2q+1)$ is $\frac{1}{2} + \frac{1}{2\rho(x)} - s$; thus the domain where $\mathcal{H}_{\mathcal{R},x_0}$ is constant includes $[0, +\infty) \times (-1/2, +\infty)$. The regularity of $\mathcal{R}^{(s)}$ for $s > 1/2$ is much more difficult to handle since $\mathcal{R}^{(s)}$ is no more locally bounded. S. Seuret and A. Ubis proved that, at these points, $h^2_{\mathcal{R}^{(s)}}(x) = \frac{1}{2} + \frac{1}{2\rho(x)} - s$, thus providing an additional extension of the domain where $\mathcal{H}_{\mathcal{R},x_0}$ is constant, see [27]. A natural conjecture is that it takes the constant value $\frac{1}{2} + \frac{1}{2\rho(x)}$ at every couple (q, t) where it is defined.

3.6 Lacunary wavelet series

In this section, we revisit the model of lacunary wavelet series introduced in [9] and extended to a p-exponents setting in [13], and we prove that such models only display canonical or lacunary singularities.

We assume that ψ is a wavelet in the Schwartz class. Lacunary wavelet series depend on a **lacunarity parameter** $\eta \in (0, 1)$ and a **regularity parameter** $\alpha \in \mathbb{R}$. The stochastic process $X_{\alpha,\eta}$ is of the form

$$\sum_{j=0}^{\infty} \sum_{\frac{k}{2^j} \in K_j} 2^{-\alpha j} \psi_{j,k}(x), \tag{27}$$

where the K_j are random sets defined as follows:

$$\forall l \in \mathbb{Z}, \qquad Card(K_j) \cap [l, l+1) = [2^{\eta j}]$$

and the locations of the points in K_j are picked at random: In each interval $[l, l+1)$ ($l \in \mathbb{Z}$), all drawings of $[2^{\eta j}]$ among the 2^j possibilities $\frac{k}{2^j} \in [l, l+1)$ have the same probability. Such a series is called a **lacunary wavelet series** of parameters (α, η). The sample paths of $X_{\alpha,\eta}$ are locally bounded if and only if $\alpha > 0$. The case considered in [9] dealt with $\alpha > 0$, and was restricted to the computation of Hölder exponents. Considering p-exponents in [13] allowed to extend the model to negative values of α, and also to see how the global sparsity of the wavelet expansion is related with the pointwise lacunarity of the sample paths. A sample path of this process is illustrated in Figure 3 (top row).

We first recall the global regularity of the sample paths [13]: a.s., $\forall x_0$, $p_{X_{\alpha,\eta}}(x_0) = \eta - 1/\alpha$ if $\alpha < 0$ and $p_{X_{\alpha,\eta}}(x_0) = +\infty$ if $\alpha > 0$. Note that $p_{X_{\alpha,\eta}}(x_0)$ always exists and is positive, even if α takes arbitrarily large negative values. We recover the fact that p-exponents allow to deal with singularities of arbitrarily large negative order. Let now $p \in (0, \eta - 1/\alpha)$, so that the sample paths of $X_{\alpha,\eta}$ belong to L^p_{loc} and the p-exponent of $X_{\alpha,\eta}$ is well-defined everywhere. The following result is proved in [13].

Theorem 3 *Let $\alpha \in \mathbb{R}$, $\eta \in (0,1)$ and let $X_{\alpha,\eta}$ be a lacunary wavelet series of parameters (α, η); the p-spectrum of almost every sample path of $X_{\alpha,\eta}$ (i.e. the multifractal spectrum associated to the p-exponent) is supported by the interval $[\alpha, H_{max}]$ where $H_{max} = (\alpha + 1/p)/\eta - 1/p$, and, on this interval,*

$$a.s. \quad \forall p < p_{X_{\alpha,\eta}}(x_0), \;\; \forall H, \qquad d^p(H) = \eta \frac{H + 1/p}{\alpha + 1/p}.$$

Furthermore, its lacunarity spectrum is given by

$$a.s. \quad \forall L \in [0, 1/\eta - 1], \qquad d^{\mathcal{L}}(L) = \eta(L+1).$$

We recall how the pointwise regularity of $X_{\alpha,\eta}$ is determined. For each j, let E^j_ω denote the subset of $[0,1]$ composed of intervals 3λ ($\lambda \in \Lambda_j$) inside which the first nonvanishing wavelet coefficient is attained at a scale $l \leq [\omega j]$; let

$$E_\omega = \limsup E^j_\omega \quad \text{and} \qquad H_\omega = \bigcap_{\omega' > \omega} E_{\omega'} - \bigcup_{\omega' < \omega} E_{\omega'}.$$

It is shown in [13] that the sets H_ω are the sets with a given p-exponent, and,

$$\text{if } x_0 \in H_\omega, \text{ then} \qquad h^p_{X_{\alpha,\eta}}(x_0) = \alpha\omega + \frac{\omega - 1}{p}. \tag{28}$$

Additionally, the Hausdorff dimension of H_ω is $\eta\omega$.

Let us now consider the fractional integral $X^{(-t)}_{\alpha,\eta}$; it is of the form

$$\sum_{j=0}^{\infty} \sum_{\frac{k}{2^j} \in K_j} 2^{-(\alpha+t)j} \psi^{(-t)}_{j,k}(x); \tag{29}$$

it is therefore another lacunary wavelet series, using the "pseudo-wavelet" $\psi^{(-t)}$. Pointwise regularity results are the same as for orthonormal wavelet bases [2]. Thus, if $x_0 \in H_\omega$, then $\forall (q,t) \in \mathbb{R}^+ \times \mathbb{R}^+$, $\mathcal{H}_{X_{\alpha,\eta},x_0}(q,t) = \alpha\omega + (\omega-1)(q+t)$. Thus, **the cancellation exponent vanishes everywhere** and $X_{\alpha,\eta}$ only displays canonical singularities (case $\omega = 1$) and lacunary singularities.

4 Heuristic derivation of new multifractal formalisms

4.1 Multiscale quantities for lacunarity and cancellation exponents

We now derive new multiresolution quantities based on p-leaders and suitable for the estimation and characterization of the second generation exponents that we introduced. We build on arguments developed in [13, 14] for the lacunarity and oscillation exponents. In this section, let us denote with $d^p_{\lambda,f}$ the p-leaders of f. Note that these exponents can also be defined for functions that do not belong to L^1, but to an H^p space for $p < 1$; indeed Definitions 8 and 9 immediately extend without modification to this setting (which we assume from now on).

Lacunarity exponent. We use the same method as in finite difference schemes for the numerical resolution of PDEs. Using (22) and a discrete approximation for the derivative in (11) we approximate $\frac{\partial}{\partial q}\left(\mathcal{H}_{f,x_0}(q,0)\right)_{q=q_f(x_0)^+}$ by $\left(h_f^{1/(q_f(x_0)+\Delta q}(x_0) - h_f^{1/q_f(x_0)}(x_0)\right)/\Delta q$, where Δq is a fixed (small enough) quantity. Using the properties of the p-leader defined by (21),

$$\frac{h_f^{1/(q_f(x_0)+\Delta q}(x_0) - h_f^{1/q_f(x_0)}(x_0)}{\Delta q} = \liminf_{j\to\infty} \frac{1}{j\Delta q} \log_2 \left(\frac{d^{1/(q_f(x_0)+\Delta q)}_{\lambda(x_0),f}}{d^{1/(q_f(x_0))}_{\lambda(x_0),f}} \right).$$

Thus, we define the numerical $\mathcal{L}$-leaders $d^{\mathcal{L}}_\lambda$ as

$$d^{\mathcal{L}}_\lambda = \left(\frac{d^{1/(q_f(x_0)+\Delta q)}_{\lambda(x_0),f}}{d^{1/(q_f(x_0))}_{\lambda(x_0),f}} \right)^{\frac{1}{\Delta q}}. \tag{30}$$

We expect $d_\lambda^{\mathcal{L}}$ to be of the order of magnitude of $2^{-j\mathcal{L}_f(x_0)}$ (when $j \to +\infty$).

Oscillation exponent. Again using (22) and a discretization of the derivative in (5), we pick Δt small enough, and approximate $\left(\frac{\partial}{\partial t} h_{f^{(-t)}}^{1/q_f(x_0)}(x_0)\right)_{t=0^+} - 1$ by $\frac{h_{f^{(-\Delta t)}}^{1/q_f(x_0)}(x_0) - h_f^{1/q_f(x_0)}(x_0)}{\Delta t} - 1$. We have

$$\frac{h_{f^{(-\Delta t)}}^{1/q_f(x_0)}(x_0) - h_f^{1/q_f(x_0)}(x_0)}{\Delta t} - 1 = \liminf_{j\to\infty} \frac{1}{j\Delta t} \log_2 \left(2^{-j} \frac{d_{\lambda(x_0), f^{(-\Delta t)}}^{1/q_f(x_0)}}{d_{\lambda(x_0), f}^{1/q_f(x_0)}} \right).$$

Thus, we define the $\mathcal{O}$-leaders $d_\lambda^{\mathcal{O}}$ as

$$d_\lambda^{\mathcal{O}} = \left(\frac{d_{\lambda(x_0), f^{(-\Delta t)}}^{1/q_f(x_0)}}{d_{\lambda(x_0), f}^{1/q_f(x_0)}} \right)^{\frac{1}{\Delta t}}, \tag{31}$$

and we expect $d_\lambda^{\mathcal{O}}$ to be of the order of magnitude of $2^{-j\mathcal{O}_f(x_0)}$ (when $j \to +\infty$).

Cancellation exponent. Using (22) and a discrete approximation for the derivative in (16) yields

$$\frac{h_{f^{(-\Delta t)}}^{1/(q_f(x_0)-\Delta t)}(x_0) - h_f^{1/q_f(x_0)}(x_0)}{\Delta t} - 1 = \liminf_{j\to\infty} \frac{1}{j\Delta t} \log_2 \left(2^{-j} \frac{d_{\lambda(x_0), f^{(-\Delta t)}}^{1/(q_f(x_0)-\Delta t}}{d_{\lambda(x_0), f}^{1/q_f(x_0)}} \right)$$

Then, we define the $\mathcal{C}$-leaders $d_\lambda^{\mathcal{C}}$ as

$$d_\lambda^{\mathcal{C}} = \left(\frac{d_{\lambda(x_0), f^{(-\Delta t)}}^{1/(q_f(x_0)-\Delta t)}}{d_{\lambda(x_0), f}^{1/q_f(x_0)}} \right)^{\frac{1}{\Delta t}} \tag{32}$$

We expect that $d_\lambda^{\mathcal{C}} \sim 2^{-j\mathcal{C}_f(x_0)}$ (when $j \to +\infty$).

Thus, the rates of decay of the quantities $d_\lambda^{\mathcal{L}}$, $d_\lambda^{\mathcal{O}}$ and $d_\lambda^{\mathcal{C}}$ are controlled by the lacunarity, oscillation and cancellation exponents. Moreover, (30), (31) and (32) indicate that $d_\lambda^{\mathcal{L}}$, $d_\lambda^{\mathcal{O}}$ and $d_\lambda^{\mathcal{C}}$ fulfill the basic requirement for a multifractal analysis ([11], see Section 4.2).

4.2 Multifractal formalism formulas

We consider the general setting where a signal is characterized by several pointwise exponents $h^m(x_0)$ for $m = 1, \cdots M$, which can be for instance a p-exponent, a lacunarity exponent, a cancellation exponent, ... and we additionally assume that we dispose of multiresolution quantities d_λ^m associated with each of these quantities and such that a log-log plot regression property such as (22) holds for each of them. The **Grand-canonical spectrum** associated with these quantities is the function $d(H^1, \cdots, H^M)$ defined as the Hausdorff dimension of the set of points x such that $h^1(x) = H^1, \cdots, h^M(x) = H^M$. The *structure function* associated to the whole vector of these multiresolution quantities is:

$$\forall r = (r_1, \cdots r_M) \in \mathbb{R}^M, \quad S(r,j) = 2^{-j} \sum_{\lambda \in \Lambda_j} \prod_{m=1}^{M} (d_\lambda^m)^{r_m} \sim 2^{-j\zeta(r)}; \tag{33}$$

the function ζ thus defined is called the *scaling function*. In practice, scaling functions are determined through log-log plot regressions (one estimates the slope of the $\log_2$ of the structure function as a function of j), see Figs. 3 and 7.

The derivation of an upper bound for the spectrum from the scaling function is referred to as the multifractal formalism; it was initially proposed in the seminal work of G. Parisi and U. Frisch [26]), reformulated using the wavelet maxima method in [25], and then reinterpreted in the wavelet leader setting in [11], and later extended to several exponents in [14]. It is based on a (multidimensional) Legendre transform of the scaling function, according to :

$$d(H) \leq \inf_{r \in \mathbb{R}^M} (1 - \zeta(r) + r \cdot H). \tag{34}$$

The multifractal formalism conjecture is that (34) is actually an equality, which is used to derive the spectrum $d(H)$ numerically.

Results on lacunary series are shown. Theorem 3 is illustrated numerically in Figure 3. Numerical estimations were performed using the multifractal formalism described just above. The left column shows the logscale diagrams for order $r = 2$ and all four exponents. The scaling behavior at coarse scales ($j < 13$) is remarkable, and allows for an efficient computation of the estimates through linear regressions. Further, Figure 3 (right column) shows that estimations of d^p and $d^{\mathcal{L}}$ (second and third row, respectively), are in good agreement with those predicted by Theorem 3. Moreover, Figure 3 also shows that $d^{\mathcal{C}}$ (fifth row) collapses at the point $(0, 1)$, as expected for a process that has no cancellating singularities and, thus, $d^{\mathcal{O}}$ (fourth row) is remarkably similar to $d^{\mathcal{L}}$.

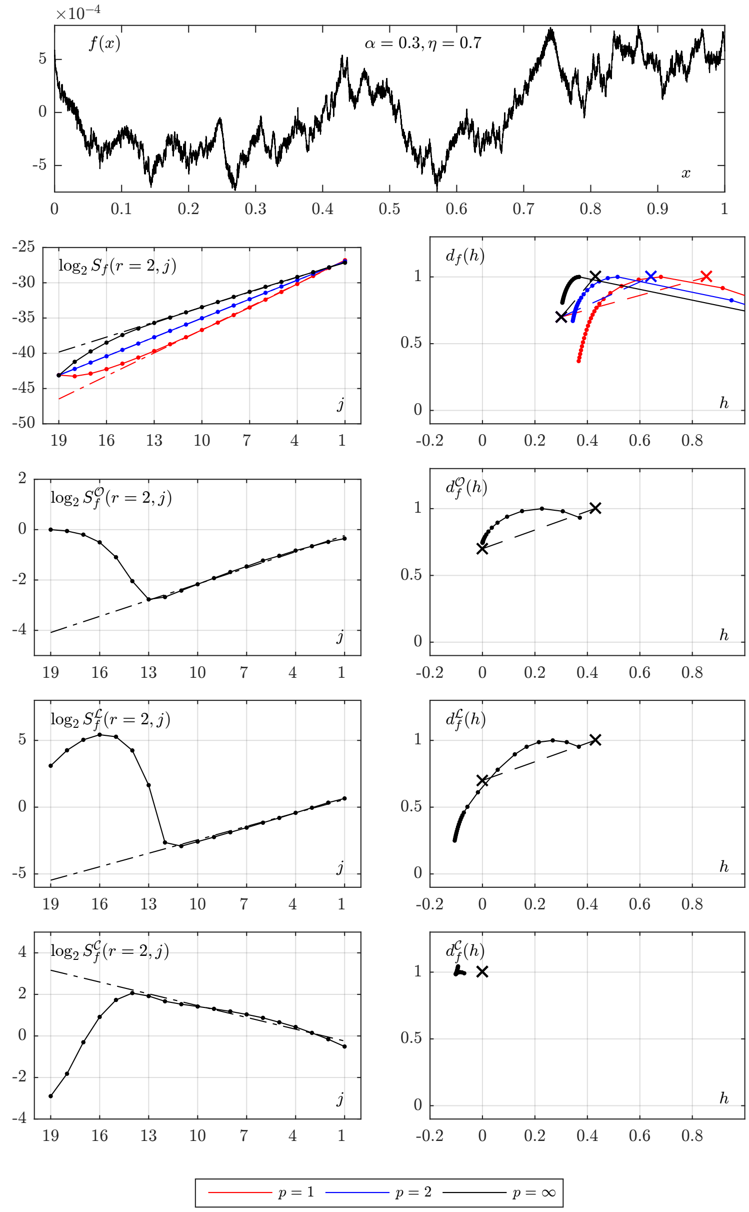

Figure 3: **Lacunary wavelet series.** A typical sample path of a lacunary wavelet series ($\alpha = 0.3$, $\eta = 0.7$, top row), estimated structure functions (left column), and estimated spectra (right column), for the p, oscillation, lacunarity and cancellation exponents (from second to fifth rows, respectively). The dashed lines with crosses indicate the theoretical spectra. The dash-dotted lines indicate the regression lines.

5 Functions with prescribed exponents at a point

We introduce new examples of pointwise singularities for the following purposes:

- In contradistinction with the previous examples, both their lacunarity and cancellation exponents will be allowed to take arbitrary values.
- They will be the building blocks that will allow to construct examples showing the optimality of Theorem 1.

5.1 Thin chirps

Since the definition that we use of H^p spaces is based on wavelet coefficients, it is easier to work with examples that are defined directly by their wavelet coefficients on a smooth wavelet basis. The **thin chirp** $\mathcal{T}_{a,b,\alpha}$ is defined by its wavelet coefficients as follows (we assume that the wavelet ψ belongs to $\mathcal{S}(\mathbb{R})$).

Definition 13 *Let $a, b \in (0,1)$ satisfying $0 < b < 1-a$, and let $\alpha \in \mathbb{R}$. At a given scale j, all wavelet coefficients of $\mathcal{T}_{a,b,\alpha}$ vanish, except for $k \in [2^{(1-a)j}, 2^{(1-a)j} + 2^{bj}]$, in which case $c_{j,k} = 2^{-\alpha j}$.*

Proposition 5.1 *The thin chirp $\mathcal{T}_{a,b,\alpha}$ is bounded if and only if $\alpha \geq 0$. If $\alpha < 0$, its critical Lebesgue index at $x_0 = 0$ is given by $p_{\mathcal{T}_{a,b,\alpha}}(0) = (1-b)/-\alpha$. Furthermore, at $x_0 = 0$, $\mathcal{B}_{\mathcal{T}_{a,b,\alpha},\,0}(q) = -\alpha - q(1-b)$ and*

$$\mathcal{H}_{\mathcal{T}_{a,b,\alpha},0}(q,t) = \frac{1-a-b}{a} q + \frac{\alpha}{a} + \frac{1-a}{a} t, \tag{35}$$

so that: $\mathcal{L}_{\mathcal{T}_{a,b,\alpha}}(0) = \frac{1-a-b}{a}$, $\mathcal{C}_{\mathcal{T}_{a,b,\alpha}}(0) = \frac{b}{a}$, *and* $\mathcal{O}_{\mathcal{T}_{a,b,\alpha}}(0) = \frac{1-a}{a}$.

Proof of Proposition 5.1: If $\alpha \neq 0$, the first result follows directly from the wavelet characterization of the $C^\alpha(\mathbb{R})$ spaces. If $\alpha = 0$, then we note that, for a given j, each block $\sum_k c_{j,k}\psi_{j,k}$ is bounded by a constant independent of j. If the wavelets are compactly supported, then the result follows because, for a given x, the number of blocks that do not vanish at x is finite, and bounded by a constant which depends only on a and b. The result also holds if wavelets have fast decay, using the corresponding decay estimates.

We denote by $\mathcal{T}^j_{a,b,\alpha}$ the wavelet series of $\mathcal{T}_{a,b,\alpha}$ restricted to the scale j. Then

$$\int |\mathcal{T}^j_{a,b,\alpha}(x)|^p dx \sim 2^{bj} 2^{-\alpha p j} 2^{-j}, \tag{36}$$

and it follows that $\mathcal{T}_{a,b,\alpha}$ belongs to L^p if and only if $b - \alpha p - 1 < 0$; the value of the critical Lebesgue index follows.

In order to determine the p-boundary of $\mathcal{T}_{a,b,\alpha}$, we use the fact that a fractional integral of order t amounts to replacing the wavelet coefficients $2^{-\alpha j}$ by $2^{-(\alpha+t)j}$, and to replacing the orthonormal wavelet basis by a bi-orthogonal wavelet basis, for which the same characterizations hold, see [2]. Therefore, the previous result implies that the condition for $\mathcal{T}^{(-t)}_{a,b,\alpha}$ to belong to L^p is $b-(\alpha+t)p-1<0$; and the value of p-boundary follows.

Finally, it follows from (36) that the integral of $|\mathcal{T}^{(-t)}_{a,b,\alpha}|^p$ on a ball of radius $r \sim 2^{-aj}$ is of the order of magnitude of $2^{bj}2^{-(\alpha+t)pj}2^{-j}$, so that the p-exponent h of $\mathcal{T}^{(-t)}_{a,b,\alpha}$ satisfies $a + b(\alpha+t) - 1 = -ahp$ and (35) follows. The values of the other exponents follow immediately.

The numerical estimation of the pointwise exponents of a thin chirp is illustrated in Figure 4. The top row shows the sample path of the function. The second and third rows display the decay with the scales of the multiresolution quantities corresponding to each exponent, as defined in Section 4.1. They all show an excellent scaling behavior at coarse scales ($j < 13$) that enables an accurate estimation of the exponent by means of linear regressions.

5.2 Functions with prescribed exponents

We now turn to the last statement of Theorem 1, i.e. the conditions enumerated in this theorem characterize the functions $\mathcal{H}$ that are fractional exponents at x_0 of a tempered distribution f. We now describe a first method to generate functions with a general fractional exponent at 0. Denote by $\mathcal{C}^j_{(a,b,\alpha)}$ the mapping which associates to $(j,(a,b,\alpha))$ the sequence $(c_{j,k})_{k\in\mathbb{Z}}$ defined by Definition 13. To a given sequence (a_n, b_n, α_n) we will associate a whole collection of wavelet coefficients such that, for each n, the $\mathcal{C}^j_{(a_n,b_n,\alpha_n)}$ show up for an infinite number of values of j. This can be obtained by the classical diagonal procedure as follows: Recall that each $j \geq 1$ can be written in a unique way as $j = 2^m(2n-1)$ for an $m \geq 0$ and $n \geq 1$. We pick the coefficients $(c_{j,k})_{k\in\mathbb{Z}}$ such that

$$\text{for} \quad j = 2^m(2n-1), \qquad (c_{j,k})_{k\in\mathbb{Z}} = \mathcal{C}^j_{(a_n,b_n,\alpha_n)}.$$

The collection of wavelet coefficients thus constructed yields a function f such that the fractional exponent of its the p-boundary is

$$\mathcal{B}_{f,\,0}(q) = \sup_n \left(-\alpha_n - q(1-b_n)\right),$$

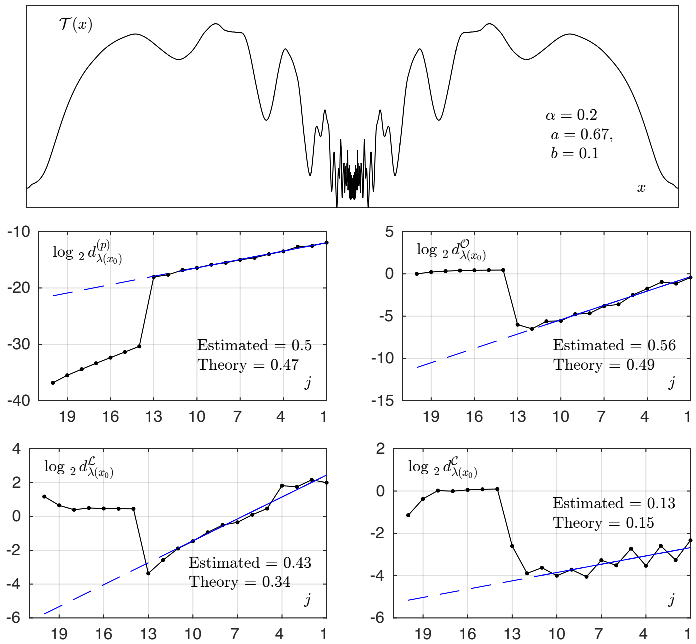

Figure 4: **Thin chirp.** Graph (top row), and estimation of p-exponent (middle row, left), oscillation exponent (middle row, right), lacunarity exponent (bottom row, left) and cancellation exponent (bottom row, right).

and which, on its domain of definition, is given by

$$\mathcal{H}_{f,0} = \inf_n \left(\frac{1 - a_n - b_n}{a_n} q + \frac{\alpha_n}{a_n} + t \frac{1 - a_n}{a_n} \right).$$

Picking for (a_n, b_n, α_n) a sequence that satisfies the conditions of Definition 13 yields a rather general construction, but does not allow to reach the level of generality stated in Theorem 1. Indeed, all the fractional exponents of the $\mathcal{T}_{a,b,\alpha}$ take the minimal possible value $-q - t$ at the boundary of its domain of definition, so that this construction also yields a fractional exponent satisfying the same restriction. In order to perform a more general construction, we now introduce new examples, which are a degenerate case of thin chirp.

Degenerate thin chirps. In the above construction of thin chirps supplied by Definition 13, we fix $b \in (0, 1)$, and $\alpha \in \mathbb{R}$, but the parameter a now depends on the scale j and tends slowly to 0; we can pick for instance the sequence $a(j) = 1/\log j$. We thus obtain a **degenerate thin chirp** $\mathcal{D}_{b,\alpha}$: At a given scale j, all wavelet coefficients of $\mathcal{D}_{b,\alpha}$ vanish, except for $k \in [2^{(1-a(j))j}, 2^{(1-a(j))j} + 2^{bj}]$, in which case $c_{j,k} = 2^{-\alpha j}$. The proof of the following proposition is similar to the one of Proposition 5.1.

Proposition 5.2 *The degenerate thin chirp $\mathcal{D}_{b,\alpha}$ is bounded if and only if $\alpha \geq 0$. If $\alpha < 0$, then $p_{\mathcal{D}_{b,\alpha}}(0) = (b-1)/\alpha$. The p-boundary of $\mathcal{D}_{b,\alpha}$ at the origin is $\mathcal{B}_{\mathcal{D}_{b,\alpha},\,0}(q) = -\alpha - q(1-b)$, and, if $t > -\alpha - q(1-b)$, the fractional exponent of $\mathcal{D}_{b,\alpha}$ at the origin is $\mathcal{H}_{\mathcal{D}_{b,\alpha},0}(q,t) = +\infty$.*

We now come back to the proof of the last statement of Theorem 1. First, using the diagonal trick already mentioned, we can alternate at different scales the wavelet coefficients of degenerate thin chirps, thus obtaining a new pointwise singularity, the p-exponent of which will have any arbitrary convex p-boundary (provided that it satisfies the conditions of Proposition 3.3), and the value taken by the p-exponent inside the domain of definition being $+\infty$.

Let $\mathcal{H}$ be a function defined on a convex subset of $\mathbb{R}^+ \times \mathbb{R}$ of the form $t > B(q)$ with B convex and satisfying $\forall q \geq 0, B'(q) \leq -1$; and assume furthermore that $\mathcal{H}$ satisfies the four conditions of Theorem 1. First, we construct a degenerate thin chirp $\mathcal{D}$ whose p-boundary is the function B; we will actually use a slight modification, namely, the function whose wavelet coefficients $d_{j,k}$ are defined by $d_{j,k} = c_{j,-k}$ (where the $c_{j,k}$ are the wavelet coefficients of the thin chirp); of course this modification does not modify the pointwise properties of $\mathcal{D}$. Next, since the function $\mathcal{H}$ satisfies $\mathcal{H}(q,t) \geq -q-t$, we extend it outside of its domain of definition into a function which still satisfies the three first conditions of Theorem 1, but which will be defined on a larger domain, where, at its boundary, $\mathcal{H}(q,t) = -q-t$. Because of this condition, we know that there exists a thin chirp $\mathcal{T}$ the p-exponent of which is precisely this extended function $\mathcal{H}(q,t)$. It suffices now to consider the function which has the wavelet coefficients of $\mathcal{D}$ for $k < 0$ and those of $\mathcal{T}$ for $k > 0$: The domain of definition of its p-exponent at 0 is the intersection of the domains of $\mathcal{D}$ and $\mathcal{T}$. Since $\mathcal{D}$ has the smallest one, its p-boundary therefore is the function B; and, inside its domain of definition, the value taken by the p-exponent is the infimum of the p-exponents of $\mathcal{D}$ and $\mathcal{T}$; but since the one taken by $\mathcal{T}$ is $+\infty$, it follows that it is exactly the function $\mathcal{H}$; and the last statement of Theorem 1 follows.

Remark: A way to construct examples for which (17) fails is to derive them from the construction of functions with prescribed exponents. It suffices to notice that a function of two variables can satisfy the conditions enumerated in Theorem 1 and, nonetheless, its partial derivatives at a "corner-point" where the exponents are computed do not satisfy (17).

6 Examples of multifractality for the cancellation exponent

In this section we will study new multifractal functions whose regularity and cancellation exponents change from point to point.

6.1 Definition of the model

We define a family of functions f on $[0,1]$ as a modification of a model in [7]. The functions in this family depend on three parameters η, β and γ, with $\beta \geq 1$ integer, $0 \leq b \leq \beta - 1$, $\eta = b/\beta$ and $\gamma \in \mathbb{R}$ a non integer. We set

$$f(x) = \sum_{\lambda \in \Lambda(\beta)} 2^{-\gamma j} \psi_\lambda(x). \tag{37}$$

In (37), $\Lambda(\beta) = \bigcup_{m \geq 1} \Lambda_m^{(\beta)}$, where $\Lambda_m^{(\beta)}$ is the set of $\lambda = (j,k)$ such that $j = \beta m$, $m > 1$, $2^{-j}k = \frac{K}{2^m} + \frac{n}{2^j}$, K odd, $n \leq 2^{\eta j}$.

We denote by c_λ the wavelet coefficients of f. Each $m > 1$ generates $2^{\eta j} 2^{m-1}$ non vanishing coefficients identified by the dyadics $\frac{K}{2^m} + \frac{n}{2^j}$, K odd, $n \leq 2^{\eta j}$, and their scale βm. Their values are all equal to $2^{-\gamma \beta m}$.

If $\beta = 1$, then the $c_{j,k} \neq 0$ appear on dyadics which are irreducible at scale $j-1$. If $\beta > 1$, then the fraction of type $K/2^m$ is no more irreducible at scale $j = \beta m$. The coefficient will appear at a finer scale than scale m. Figure 5 gives an insight of this situation for $\beta = 3$ and $\eta = 0$, whereas the case $\beta = 3$ and $\eta = 1/\beta$ is presented in Figure 6 .

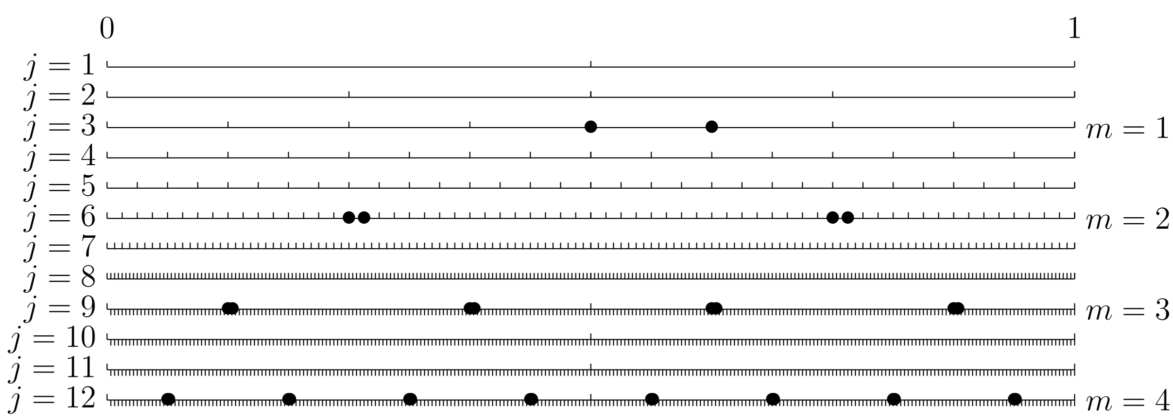

Figure 5: Case $\beta = 3$ and $\eta = 0$. The two non vanishing wavelet coefficients • appear on dyadic points $\frac{k}{2^j} = \frac{K}{2^m}$ and $\frac{k}{2^j} = \frac{K}{2^m} + \frac{1}{2^j}$ with K odd and $j = \beta m$.

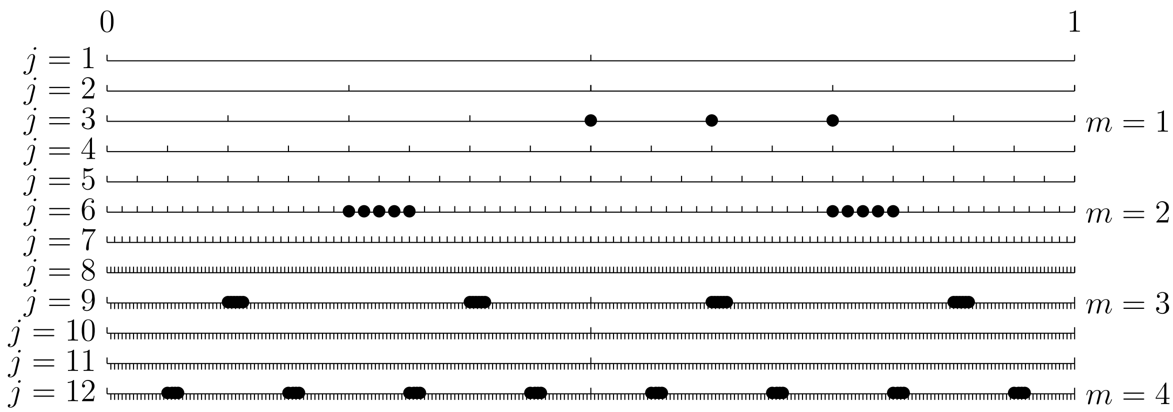

Figure 6: Case $\beta = 3$ and $\eta = 1/\beta$. For $j = \beta m$ exactly $2^{m-1}(2^m + 1)$ non vanishing wavelet coefficients $\bullet$ appear on dyadic points $\frac{k}{2^j} = \frac{K}{2^m} + \frac{n}{2^j}$ with K odd and $0 \leq n \leq 2^m$.

The wavelet characterization of $C^\gamma(\mathbb{R})$ implies that $f \in C^\gamma(\mathbb{R})$ (for any $\gamma \in \mathbb{R} - \mathbb{Q}$). If $\gamma < 0$, we now derive $p_f(x_0)$ from the local scaling function; it is clearly independent of x_0 and given by

$$\eta_f(p) = \liminf_{m \to +\infty} \frac{\log\left(2^{-\beta m}(2^{\eta\beta m}+1)2^{m-1}2^{-p\gamma\beta m}\right)}{\log(2^{-\beta m})} = p\gamma - \frac{1}{\beta} - \eta + 1 \qquad (38)$$

and Proposition 3.3 implies that $p_f(x_0) = \frac{1}{\gamma}\left(\eta - 1 + \frac{1}{\beta}\right)$.

A similar computation for fractional integrals and derivatives yields that the fractional exponent domain at any point is $\mathcal{D}_{f,x_0} = \left\{(q,t) : > \gamma - q\left(\eta - 1 + \frac{1}{\beta}\right)\right\}$.

We will suppose in the following that $(q,t) \in \mathcal{D}_{f,x_0}$.

6.2 Regularity exponents of f

Recall that the rate of approximation by dyadics of a real number $x_0 \in [0,1]$ is

$$r(x_0) = \limsup_{j \to \infty} \frac{\log(|K_j(x_0)2^{-j} - x_0|)}{\log(2^{-j})}, \qquad (39)$$

with $K_j(x_0) = \mathrm{argmin}_{k \in \{0,..2^j-1\}}(|x_0 - k2^{-j}|)$. Clearly, $r(x_0) \geq 1$.

Theorem 4 *Suppose $x_0 \in [0,1]$, and let η, β and γ such that $\beta \geq 1$ integer, $0 \leq b \leq \beta - 1$, $\eta = \frac{b}{\beta}$ and $\gamma \in \mathbb{R}$*

1. *If $r(x_0) \leq (1-\eta)\beta$, then $\mathcal{H}_{f,x_0}(q,t)) = \frac{(\gamma+t)\beta}{r(x_0)} - t + q\left(\frac{\beta(1-\eta)}{r(x_0)} - 1\right)$ so that $\mathcal{O}_f(x_0) = \frac{\beta}{r(x_0)} - 1$, $\mathcal{L}_f(x_0) = \frac{\beta(1-\eta)}{r(x_0)} - 1$, and $\mathcal{C}_f(x_0) = \frac{\beta\eta}{r(x_0)}$.*

2. *If* $(1-\eta)\beta < r(x_0) \le \beta$, *then* $\mathcal{H}_{f,x_0}(q,t)) = \frac{(\gamma+t)\beta}{r(x_0)} - t$ *so that* $\mathcal{O}_f(x_0) = \frac{\beta}{r(x_0)} - 1, \quad \mathcal{L}_f(x_0) = 0, \quad$ *and* $\quad \mathcal{C}_f(x_0) = \frac{\beta}{r(x_0)} - 1.$

3. *If* $r(x_0) > \beta$, *then* $\mathcal{H}_{f,x_0}(q,t)) = (\gamma+t)\beta - t + q((1-\eta)\beta - 1)$ *so that* $\mathcal{O}_f(x_0) = \beta - 1, \quad \mathcal{L}_f(x_0) = \beta(1-\eta) - 1 \quad$ *and* $\quad \mathcal{C}_f(x_0) = \eta\beta.$

Corollary 6.1 *If* $\gamma > 0$, *then:*

- *the Hölder spectrum of* f *is defined on* $[\gamma, \gamma\beta]$, *where* $d_f(h) = h/\beta\gamma$.
- *The p-spectrum is defined on* $[\gamma, \beta\gamma + q(\beta(1-\eta)-1)]$ *where*
  - *If* $\gamma \le u \le \gamma/(1-\eta)$, *then* $d_{f,p}(u) = \frac{u}{\beta\gamma}$.
  - *If* $\gamma/(1-\eta) \le u \le \gamma\beta + q(\beta(1-\eta)-1)$, *then* $d_{f,p}(u) = \frac{u+q}{\gamma\beta+q(\beta(1-\eta))}$.

If $\gamma < 0$ *the p spectrum is the function* $d_{f,p}$ *defined on the interval* $[\frac{\gamma}{1-\eta}, \beta\gamma + q(\beta(1-\eta)-1)]$ *where* $d_{f,p}(u) = \frac{u+q}{\gamma\beta+q\beta(1-\eta)}$.

Remark: One may wonder why the two p-spectra in case $\gamma > 0$ and $\gamma < 0$ are different even if the computation of the p exponent is the same. This is because, if $\gamma < 0$, then $-\frac{1}{p} < \frac{\gamma}{1-\eta} < \gamma < \beta\gamma + \frac{\beta(1-\eta)-1}{p}$. Thus the cases $(1-\eta)\beta < r(x_0)$ yield the same range $\frac{\gamma}{1-\eta} \le h_f^p(x_0) \le \beta\gamma + \frac{\beta(1-\eta)-1}{p}$ for the p-exponents as in case $r(x_0) \le (1-\eta)\beta$. The true dimension is thus derived from the formula $\frac{1}{r(x_0)} = \frac{h_f^p(x_0)+\frac{1}{p}}{\gamma\beta+\frac{(1-\eta)\beta}{p}}$ which corresponds to the case $r(x_0) \le (1-\eta)\beta$.

As above we can define the s-wavelet leader for $s > 0$ by

$$d_j^s(x_0) = \sup_{\lambda' \subset 3\lambda_j(x_0)} |2^{-sj'} c_{\lambda'}|, \; (s - \text{leader}). \tag{40}$$

The following characterization holds [1].

Proposition 6.2 *Let* f *be in* $C^\varepsilon(\mathbb{R})$ *for* $\epsilon > 0$. *Then* $h_{f^{(-s)}}(x_0) = \liminf_{j\to\infty} \frac{\ln(d_j^s(x_0))}{\ln(2^{-j})}$ *and* $\mathcal{O}_f(x_0) = \frac{\partial}{\partial s}\left(h_{f^{(-s)}(x_0)}\right)_{t=0^+} - 1$.

Corollary 6.3

- *The Oscillation spectrum of* f *is the function* d_f^o *defined on the interval* $[0, \beta-1]$ *such that* $d_f^o(s) = s + 1/\beta$.
- *The Lacunarity spectrum of* f *is the function* d_f^l *defined on the interval* $[0, (1-\eta)\beta - 1]$ *such that* $d_f^l(s) = s + 1/\beta(1-\eta)$.
- *The Cancellation spectrum of* f *is the function* d_f^c *defined on the interval* $[0, \beta\eta]$ *such that*
  - *if* $0 \le s \le \eta/(1-\eta)$ *then* $d_f^c(s) = s + 1/\beta$,

– *if* $\eta/(1-\eta) \le s \le \eta\beta$ *then* $d^c_f(s) = s/\beta\eta$.

Corollaries 6.1 and 6.3 are illustrated numerically in Figure 7. Numerical estimations were performed using the multifractal formalism described in Section 4.2. The left column shows the logscale diagrams for order $r = 2$ and all four exponents. The scaling behavior at coarse scales ($j < 10$) is remarkable, and allows for an efficient computation of the estimates through linear regressions. Note that the oscillations on the structure functions for $S^{\mathcal{O}}$ and for $S^{\mathcal{L}}$ are due to the choice $\beta = 2$, which implies that one every two scales have no nonvanishing wavelet coefficients. Further, Figure 7 (right column) shows that estimations of all multifractal spectra are in remarkable agreement with those predicted by theory. Note that the estimate for $d^{\mathcal{C}}$ is expected to yield only the concave hull of the true non-concave spectrum since it is computed from a Legendre-transform-based multifractal formalism [11].

6.3 Pointwise regularity of f

We now prove Theorem 4. We will need the following quantities:

$$D_{\lambda,p} = \left(\sum_{\lambda' \subset \lambda} |c_{\lambda'}|^p 2^{-(j'-j)} \right)^{1/p}, \quad \text{and} \quad D_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|, \tag{41}$$

Note that the sum in the definition of d^p_λ is taken as in $D_{\lambda,p}$ except that it is extended to the two nearest neighboors. We start by computing the wavelet-leaders and p-leaders at a point x_0. From this information we will be able to compute the Hölder exponent if $\gamma > 0$ and p-exponents in all cases with the restriction that $p < p_0$ if $\gamma < 0$. We will not recall these restrictions which will be implicit in all computations.

6.3.1 Wavelet and p leaders

Let λ be a dyadic interval indexed by (j, k). Let m_0, m_1 be integers such that $\beta(m_0 - 1) \le j < \beta m_0$ and $(1-\eta)\beta(m_1 - 1) \le j < (1-\eta)\beta m_1$. Since $0 \le \eta < 1$ we have always $m_1 \ge m_0$. We have the following cases:

Case 1: $\frac{k}{2^j} = \frac{K}{2^m}$ **with** $m \ge m_1$**.** Thus $m \ge m_0$ and the coefficients associated with the irreducible fraction $\frac{K}{2^m}$ appear at scale $\beta m \ge \beta m_0 \ge j$. These coefficients will be the first non vanishing coefficients and we have the $2^{\eta\beta m} + 1$ of them inside the dyadic cube $\left[\frac{K}{2^m}, \frac{K}{2^m} + \frac{1}{2^j}\right]$ since $\frac{2^{\eta\beta m}}{2^{\beta m}} \le 2^{-j}$.

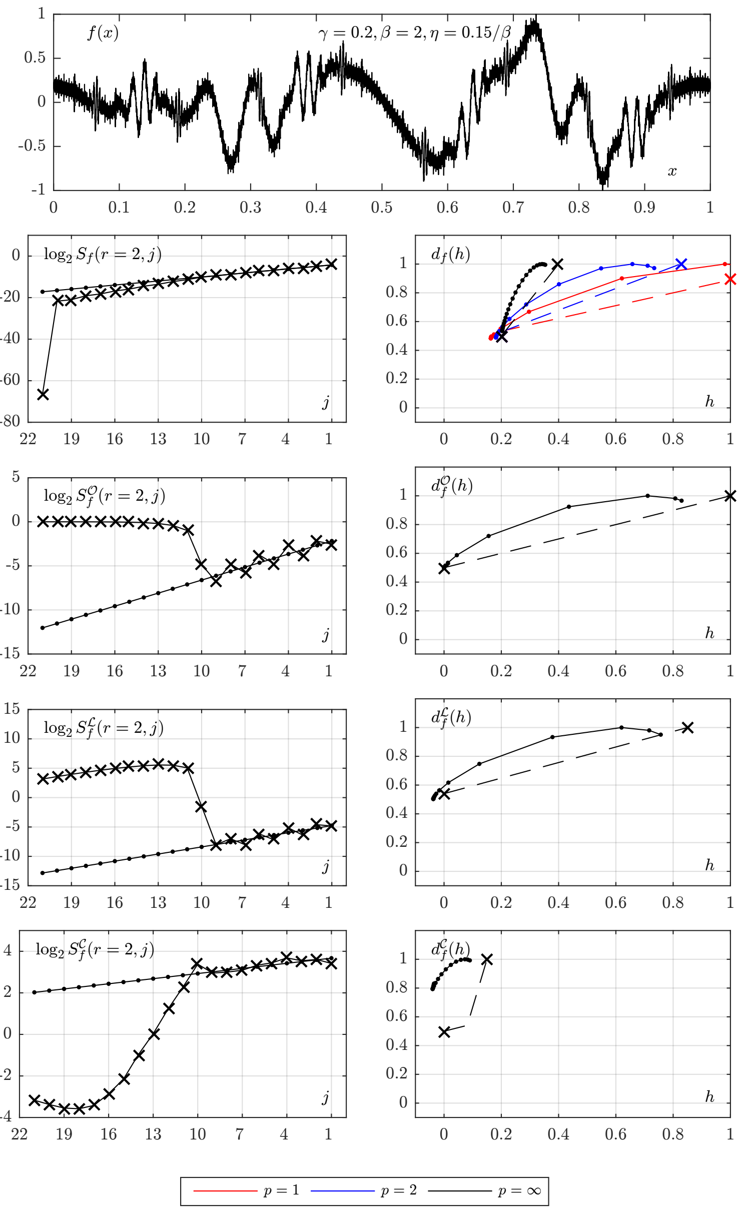

Figure 7: **Multifractal function.** Graph of f given by (37) (top row), structure functions (left column), and estimated spectra (right column), for h^p, $\mathcal{O}$, $\mathcal{L}$ and $\mathcal{C}$ (from second to fifth rows, respectively). The dashed lines with crosses indicate the theoretical spectra.

The first irreducible fraction at scale $j' > m$ is $\frac{k}{2^j} + \frac{1}{2^{j+1}}$. Since $\eta < \frac{\beta-1}{\beta}$ we have $\frac{2^{\eta\beta(j+1)}}{2^{\beta(j+1)}} \leq 2^{-(j+1)}$ and thus we have the all $2^{\eta\beta(j+1)}+1$ of them inside the cube $\left[\frac{K}{2^m}, \frac{K}{2^m} + \frac{1}{2^j}\right]$. At each scale $j' \geq j+1$ we will have $2^{j'-j-1}$ such irreducible fractions which will give coefficients at scale $\beta j'$. Thus: $D_\lambda = \sup_{\lambda' \subset \lambda} |c'_\lambda| = 2^{-\gamma\beta m}$, and the p-leader satisfies

$$2^{\eta\beta m} 2^{-p\gamma\beta m} 2^{-\beta m + j} \leq D^p_{\lambda,p} \leq$$

$$\leq (2^{\eta\beta m} + 1) 2^{-p\gamma\beta m} 2^{-\beta m + j} + \sum_{j' \geq j+1} 2^{j'-j-1} (2^{\eta\beta j'} + 1) 2^{-p\gamma\beta j'} 2^{-\beta j' + j}$$

$$\text{and} \quad C 2^{(\eta\beta - p\gamma\beta - \beta)m} 2^j \leq D^p_{\lambda,p} \leq C" 2^{(\eta\beta - p\gamma\beta - \beta)m} 2^j.$$

indeed $\sum_{j' \geq j+1} 2^{(1+\eta\beta - p\gamma\beta - \beta)j'} < +\infty$ since $\eta_f(p) > 0$ (see (38)). Remark also that since $m < j$ we have $2^{(\eta\beta - p\gamma\beta - \beta)m} 2^j > 2^{(1+\eta\beta - p\gamma\beta - \beta)j}$.

Case 2: $\frac{k}{2^j} = \frac{K}{2^m}$ with $m_1 \geq m \geq m_0$. This means that $\frac{2^{\eta\beta m}}{2^{\beta m}} \geq 2^{-j}$ since $m \leq m_1$. Since $m \geq m_0$ the coefficients associated with the irreducible fraction $\frac{K}{2^m}$ appear at scale $\beta m \geq \beta m_0 \geq j$. These coefficients will be the first non vanishing coefficients in the dyadic cube λ and we have $1 + 2^{-j+\beta m}$ of them. Again the coarsest scale $j' > m$ such that an irreducible fraction appears in the dyadic cube $\left[\frac{K}{2^m}, \frac{K}{2^m} + \frac{1}{2^j}\right]$ is $j+1$. Following the same proof than in Case 1 we will have at each scale $j' \geq j+1$ $2^{j'-j-1}$ such irreducible fractions, which will give $2^{\eta\beta j'} + 1$ wavelet coefficients in the dyadic cube $\left[\frac{K}{2^m}, \frac{K}{2^m} + \frac{1}{2^j}\right]$. Thus, the leader has the same value than in Case 1 $D_\lambda = \sup_{\lambda' \subset \lambda} |c'_\lambda| = 2^{-\gamma\beta m}$ and the p-leader satisfies

$$2^{-j+\beta m} 2^{-p\gamma\beta m} 2^{-\beta m + j} \leq D^p_{\lambda,p}$$

$$\leq 2^{-j+\beta m} 2^{-p\gamma\beta m} (2^{-\beta m + j} + 1) + \sum_{j' \geq j+1} 2^{j'-j-1} (2^{\eta\beta j'} + 1) 2^{-p\gamma\beta j'} 2^{-\beta j' + j}$$

$$\text{and} \quad C 2^{-p\gamma\beta m} \leq D^p_{\lambda,p} \leq C' 2^{-p\gamma\beta m} + C 2^{(1+\eta\beta - p\gamma\beta - \beta)j} \leq C" 2^{-p\gamma\beta m}.$$

Case 3: $\frac{k}{2^j} = \frac{K}{2^m}$ and $m < m_0$. The coefficients associated to this fraction already appeared at the scale $\beta m < j$.

The first ones to be seen inside the cube $\left[\frac{K}{2^m}, \frac{K}{2^m} + \frac{1}{2^j}\right]$ are the ones related to the irreducible fraction $\left[\frac{k}{2^j} + \frac{1}{2^{j+1}}, \frac{k}{2^j} + \frac{1}{2^j}\right]$. Since $\eta \leq \frac{\beta - 1}{\beta}$ we have $2^{\eta\beta(j+1)}$ such coefficients inside the cube $\left[\frac{k}{2^j} + \frac{1}{2^{j+1}}, \frac{k}{2^j} + \frac{1}{2^j}\right]$, thus inside the cube $\left[\frac{K}{2^m}, \frac{K}{2^m} + \frac{1}{2^j}\right]$, the leader is $D_\lambda = \sup_{\lambda' \subset \lambda} |c'_\lambda| = 2^{-\gamma\beta(j+1)}$, and, using similar upper and lower bounds as previously, $D_{\lambda,p}$ is estimated by

$$C 2^{(\eta\beta - p\gamma\beta - \beta)j} 2^j \leq D^p_{\lambda,p} \leq C' 2^{(\eta\beta - p\gamma\beta - \beta)j} 2^j.$$

Computation of the local regularity of f Let $x_0 \in \mathbb{R}$, $p > 0$ or $p < p_0$ in case $\gamma < 0$. Let $\delta > 0$; there exists $m_n \to +\infty$ such that

$$|K_{m_n}(x_0)2^{-m_n} - x_0| \leq 2^{-m_n(r(x_0)-\delta)}. \tag{42}$$

Let $j_n = [m_n(r(x_0) - \delta)]$, k_n be such that $\lambda_{j_n}(x_0) = (j_n, k_n) = \lambda_n$, and let $m_0^{(n)}$ be defined by

$$\beta(m_0^{(n)} - 1) \leq j_n < \beta m_0^{(n)} \quad \text{i.e.} \quad \beta(m_0^{(n)} - 1) \leq m_n(r(x_0) - \delta) < \beta m_0^{(n)}. \tag{43}$$

Let $m_1^{(n)}$ be defined by $(1-\eta)\beta(m_1^{(n)} - 1) \leq j_n < (1-\eta)\beta m_1^{(n)}$, i.e.

$$(1-\eta)\beta(m_1^{(n)} - 1) \leq m_n(r(x_0) - \delta) < (1-\eta)\beta m_1^{(n)}. \tag{44}$$

Let $\lambda_n^l = (j_n, k_n - 1)$, and $\lambda_n^r = (j_n, k_n + 1)$; then $d_{\lambda_n}(x_0) = \sup\{D_{\lambda_n^l}, D_{\lambda_n^r}, D_{\lambda_n}\}$. On other hand for $\varepsilon > 0$ one can find M such that for $m \geq M$

$$|K_m(x_0)2^{-m} - x_0| > 2^{-m(r(x_0)+\varepsilon)} \tag{45}$$

Let us consider $3\lambda_j(x_0) = [(k_j - 1)2^{-j}, (k_j + 2)2^{-j}[$. Let m be the smallest integer such that $\frac{K_m}{2^m}$ belongs to $3\lambda_j(x_0)$. It is always possible to choose j large enough such that $m \geq M$. Thus $\left|\frac{K_m}{2^m} - x_0\right| > 2^{-m(r(x_0)+\varepsilon)}$.

Since $K_m 2^{-m} \in 3\lambda_j(x_0)$, $\frac{3}{2^{j+1}} > 2^{-m(r(x_0)+\varepsilon)}$ so that $\frac{\ln(3)}{\ln(2)} - 1 + m(r(x_0)+\varepsilon) > j$. Thus $j \leq m(r(x_0) + \varepsilon)$. Let m_0 be $\beta(m_0 - 1) \leq j < \beta m_0$, and m_1 such that $(1-\eta)\beta(m_1 - 1) \leq j < (1-\eta)\beta m_1$.

We consider the following cases:

Case 1: $r(x_0) \leq (1-\eta)\beta$**.** Choose $\delta > 0$, and consider the sequences $(m_n)_n$,$(m_0^{(n)})_n$, and $(m_1^{(n)})_n$. Since $(1-\eta)\beta(m_1^{(n)} - 1) < (1-\eta)\beta m^{(n)}$ by (44) this yields $m_n \geq m_1^n$. This falls into Case 1 and by (42) we know that $\frac{K_{m_n}}{2^{m_n}} \in 3\lambda_{j_n}(x_0)$ thus $d_{\lambda_{j_n}} \geq 2^{-\gamma\beta m_n}$. Thus if $\gamma > 0$, then

$$h_f(x_0) = \liminf_{j \to +\infty} \frac{\log d_{\lambda_j}(x_0)}{\log(2^{-j})} \leq \liminf_{j_n \to +\infty} \frac{\log d_{\lambda_{j_n}}(x_0)}{\log(2^{-j_n})} \leq \frac{\gamma\beta m_n}{(r(x_0) - \delta)m_n} \tag{46}$$

We need to separate two cases.

1. Suppose $r(x_0) < (1-\eta)\beta$. Then choose $\varepsilon > 0$ such that $r(x_0) + \varepsilon < (1-\eta)\beta$. Thus we have $(1-\eta)\beta(m_1 - 1) \leq m(r(x_0) + \varepsilon) < (1-\eta)\beta m$. This yields $m \geq m_1$ and we fall again in Case 1. We have $d_\lambda \leq 2^{-\gamma\beta m}$ Thus we get $h_f(x_0) = \frac{\gamma\beta}{r(x_0)}$. Following the same proof we obtain

$$h_f^p(x_0) = \frac{\gamma\beta}{r(x_0)} + \frac{1}{p}\left(\frac{\beta(1-\eta)}{r(x_0)} - 1\right)$$

2. Suppose $r(x_0) = (1-\eta)\beta$. Then we may have $m \le m_1$ but anyway we have $d_\lambda \le 2^{-\gamma\beta m}$. Thus we get

$$\frac{\log(d_\lambda)}{\log(2^{-j})} \ge \frac{\gamma\beta m}{j} \ge \frac{\gamma\beta m}{((1-\eta)\beta+\varepsilon)m} \ge \frac{\gamma\beta}{(1-\eta)\beta+\varepsilon}$$

   Together with (46) this yields $h_f(x_0) = \gamma\beta/(1-\eta)\beta = \gamma\beta/r(x_0)$.

   Similar computations as above yield $h_f^p(x_0) = \gamma\beta/r(x_0)$.

Case 2: $(1-\eta)\beta < r(x_0) \le \beta$. By (43) and (44) together with the fact that $(1-\eta)\beta < r(x_0) - \delta \le \beta$ we have $m_0^{(n)} \le m_n \le m_1^{(n)}$. This falls into Case 2 and yields $d_{\lambda_{j_n}} \ge 2^{-\gamma\beta m_n}$. Thus, as above, we conclude that $h_f(x_0) \le \gamma\beta/r(x_0)$, and the p-exponent satisfies $h_f^p(x_0) \le \gamma\beta/r(x_0)$.

For the lower-bound we need to distinguish two cases:

1. Suppose $(1-\eta)\beta < r(x_0) < \beta$. Thus take $\varepsilon > 0$ such that $(1-\eta)\beta < r(x_0) + \varepsilon < \beta$. This yields $m_0 \le m \le m_1$. Again we have to refer to Case 2 and we have $d_\lambda \le 2^{-\gamma\beta m}$. This yields $h_f(x_0) = \gamma\beta/r(x_0)$. The same technic yields $h_f^p(x_0) = \gamma\beta/r(x_0)$.

2. Suppose that $r(x_0) = \beta$. We have $r(x_0) + \varepsilon > \beta$. We have $m \le m_0$ but we have $d_\lambda \le 2^{-\gamma\beta m}$; and again we use

$$\frac{\log(d_\lambda)}{\log(2^{-j})} \ge \frac{\gamma\beta m}{j} \ge \frac{\gamma\beta m}{(\beta+\varepsilon)m} \ge \frac{\gamma\beta}{\beta+\varepsilon}.$$

   Together with (46) this yields $h_f(x_0) = \gamma = \gamma\beta/r(x_0)$.

   Remark that we always have $2^{-\gamma p\beta m} \ge 2^{(\eta\beta - p\gamma\beta - \beta + 1)j}$. Thus (even if $m \le m_0$) we have $d_\lambda^p(x_0) \le 2^{-\gamma\beta m}$, so that $h_f^p(x_0) = \gamma$.

Case 3: $\beta < r(x_0)$. Let $\delta > 0$ be such that $\beta < r(x_0) - \delta$; (43) yields $m_n \le m_0^{(n)}$. This falls into Case 3; thus $d_{\lambda_n} \ge 2^{-\gamma\beta(j_n+1)}$, so that $h_f(x_0) \le \gamma\beta$.

We have also in the same way $h_f^p(x_0) \le \frac{1}{p}\left((1-\eta)\beta - 1\right) + \gamma\beta$.

The same lower bounds as in the previous cases yield $h_f(x_0) = \gamma\beta$ and $h_f^p(x_0) = \frac{1}{p}\left((1-\eta)\beta - 1\right) + \gamma\beta$.

The computation of the dimensions of the set E_h and E_u^p is standart using the fact that the dimensions of the sets $\{x_0 : r(x_0) = \alpha\}$ is $1/\alpha$ for $\alpha \ge 1$.

6.3.2 Oscillating singularities

Let $s > 0$. Remark that computing $h_{f^{(-s)}}(x_0)$ and $h_f(x_0)$ is similar. Indeed $f^{(-s)} = \sum_{\lambda \in \Lambda(\alpha,\beta)} c_\lambda^s \psi_\lambda^s(x)$ where ψ^s is the fractional integrate of ψ, and $c_\lambda^s =$

$2^{-j(\gamma+s)}$ if $\lambda \in \Lambda(\beta)$ and 0 otherwise. One concludes by using again the "pseudo-wavelet" argument already mentioned.

The location of the non vanishing coefficients is the same in $f^{(-s)}$ and f. Their amplitude at scale j is respectively $2^{-j(\gamma+s)}$ and $2^{-j\gamma}$. Thus we can estimate the wavelet-leaders or p-leaders of $f^{(-s)}$ with the same formula which yield the wavelet leaders or p-leaders of f taking $\gamma + s$ instead of γ. Thus the fractional exponent follows and this yields

1. If $r(x_0) \leq (1-\eta)\beta$, then $\mathcal{H}_{f,x_0}(q,t)) = \frac{(\gamma+t)\beta}{r(x_0)} - t + q\left(\frac{\beta(1-\eta)}{r(x_0)} - 1\right)$
2. If $(1-\eta)\beta < r(x_0) \leq \beta$, then $\mathcal{H}_{f,x_0}(q,t)) = \frac{(\gamma+t)\beta}{r(x_0)} - t$
3. If $r(x_0) > \beta$, then $\mathcal{H}_{f,x_0}(q,t)) = (\gamma+t)\beta - t + q((1-\eta)\beta - 1)$

If $r(x_0) \leq \beta$, then $\mathcal{O}_f(x_0) = \frac{\beta}{r(x_0)} - 1$, and if $r(x_0) > \beta$, then $\mathcal{O}_f(x_0) = \beta - 1$.

6.3.3 Lacunarity exponents

A straightforward computation yields:

1. If $r(x_0) \leq (1-\eta)\beta$, then $\mathcal{L}_f(x_0) = \frac{\beta(1-\eta)}{r(x_0)} - 1$
2. if $(1-\eta)\beta < r(x_0) \leq \beta$, then $\mathcal{L}_f(x_0) = 0$.
3. if $r(x_0) > \beta$, then $\mathcal{L}_f(x_0) = (1-\eta)\beta - 1$.

7 Annex

We will prove the following result.

Theorem 5 *Let $p > 0$ and $0 \leq t \leq \frac{1}{p}$ with $\frac{1}{p} - \frac{1}{q} = t$. Suppose $f \in T_p^\alpha$. Then $f^{(-t)}$ belongs to $T_q^{\alpha+t}$.*

Proof: By hypothesis f satisfies

$$\sum_{\lambda' \subset 3\lambda_j(x_0)} |c_{\lambda'}|^p 2^{-j'} \leq C 2^{-j(\alpha p+1)} \tag{47}$$

We will use the following inequalities whose proof we leave to the reader.

Lemma 7.1 *Let I a set of countable indices and $q > 0$. We have*

$$\sum_{l\in\mathbb{Z}} (2^{lq} - 2^{(l-1)q}) \sharp(k : |a_k| \geq 2^l) \leq \sum_{k\in I} |a_k|^q \leq \sum_{l\in\mathbb{Z}} (2^{(l+1)q} - 2^{lq}) \sharp(\{k : |a_k| \geq 2^l\})$$

A fractional integration of order t amounts to a change of wavelet basis and a multiplication of the coefficients $c_{j,k}$ by 2^{-jt}. We want to compute

$$D_\lambda^{(t),q} = \left(\sum_{\lambda' \subset 3\lambda_j(x_0)} |c_{\lambda'}|^q 2^{-j'tq} 2^{-j'} \right)^{1/q}$$

and prove that $D_\lambda^{(t),q} \leq C2^{-j(\alpha+t+\frac{1}{q})} = C2^{-j(\alpha+\frac{1}{p})}$

Following Lemma 7.1, since $\frac{1}{p} = \frac{1}{q} + t$,

$$(D_\lambda^{(t),q})^q \leq \sum_{l\in\mathbb{Z}} (2^{(l+1)q} - 2^{lq}) \sharp(\{\lambda' \subset 3\Lambda_j(x_0) : |c_{\lambda'}| 2^{-j't} 2^{-\frac{j'}{q}} \geq 2^l\})$$

$$\leq \sum_{l\in\mathbb{Z}} (2^{(l+1)q} - 2^{lq}) \sharp(\{\lambda' \subset 3\Lambda_j(x_0) : |c_{\lambda'}| 2^{-\frac{j'}{p}} \geq 2^l\})$$

Remark that by (47) we have $|c_{\lambda'}|2^{-\frac{j'}{p}} \leq C2^{-j(\alpha+\frac{1}{p})}$. Thus if $2^l \leq |c_{\lambda'}|2^{-\frac{j'}{p}}$ we have $2^l \leq C2^{-j(\alpha+\frac{1}{p})}$, which yields $l \leq -j(\alpha+\frac{1}{p}) + J_0 \leq l_1$ with $J_0 \in \mathbb{Z}$ a constant independant of j and j'. Following (47) we have

$$\sum_{\lambda' \subset 3\lambda_j(x_0)} |c_{\lambda'}|^p 2^{-j'} \leq C2^{-j(\alpha p+1)}, \qquad \text{so that}$$

$$\sum_{l=-\infty}^{\infty} (2^{lp} - 2^{(l-1)p}) \sharp(\{\lambda' \subset 3\Lambda_j(x_0) : |c_{\lambda'}| 2^{-\frac{j'}{p}} \geq 2^l\}) \leq C2^{-j(\alpha p+1)}.$$

Thus for all $l \in \mathbb{Z}$,

$$\begin{aligned}
(1-2^{-p})2^{lp} \sharp(\{\lambda' \subset 3\Lambda_j(x_0) : |c_{\lambda'}| 2^{-\frac{j'}{p}} \geq 2^l\}) &\leq C2^{-j(\alpha p+1)} \\
\sharp(\{\lambda' \subset 3\Lambda_j(x_0) : |c_{\lambda'}| 2^{-\frac{j'}{p}} \geq 2^l\}) &\leq \frac{C}{1-2^{-p}} 2^{-lp} 2^{-j(\alpha p+1)}.
\end{aligned}$$

Since $q - p > 0$, this yields,

$$\begin{aligned}
(D_\lambda^{(t),q})^q &\leq \sum_{l=-\infty}^{l_1} (2^{(l+1)q} - 2^{lq}) \sharp(\{\lambda' \subset 3\Lambda_j(x_0) : |c_{\lambda'}| 2^{-\frac{j'}{p}} \geq 2^l\}) \\
&\leq \frac{C(2^q-1)}{1-2^{-p}} \sum_{l=-\infty}^{l_1} 2^{lq-lp} 2^{-j(\alpha p+1)} \leq C' 2^{-j(\alpha p+1)} 2^{-j(q-p)(\alpha+\frac{1}{p})} \\
&\leq C' 2^{-j(\alpha p+1+\alpha q-p\alpha-1+\frac{q}{p})} = C' 2^{-j(\alpha q+\frac{q}{p})},
\end{aligned}$$

which yields the result.

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