

Bayesian estimation of the multifractality parameter for images via a closed-form Whittle likelihood

S. COMBREXELLE^{1,*}, H. WENDT¹, J.-Y. TOURNERET¹, P.
ABRY² AND S. McLAUGHLIN³

¹ IRIT - ENSEEIHT, Toulouse, France

² Ecole Normale Supérieure, Lyon, France

³ Heriot-Watt University, Edinburgh, Scotland

* Supported by the Direction Générale de l'Armement (DGA)

EUSIPCO 2015, Nice



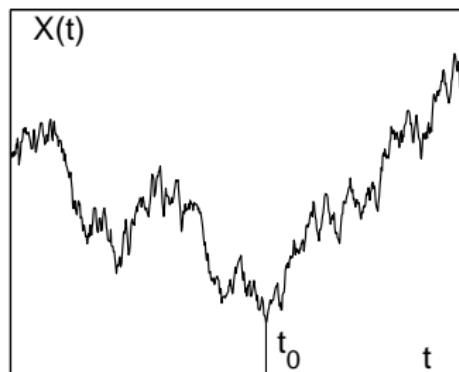
Introduction

- Multifractal analysis
 - a widely used tool in signal/image processing
 - applications in various fields (texture analysis, ...)
 - challenging estimation for images of small sizes
- Recent work
 - statistical estimation procedure based on a Bayesian framework
- Contribution
 - efficient Bayesian estimation with low computational cost

Characterization by local regularity

- Local regularity of $X(t)$ at t_0

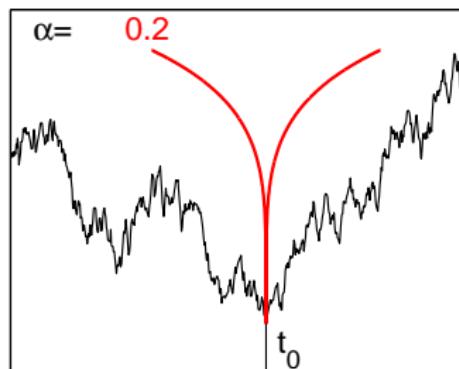
$X \in C^\alpha(t_0)$ if $\exists C, \alpha > 0; P_{t_0}(t); \deg(P_{t_0}) < \alpha :$
 $|X(t) - P_{t_0}(t)| < C|t - t_0|^\alpha$



Characterization by local regularity

- Local regularity of $X(t)$ at t_0

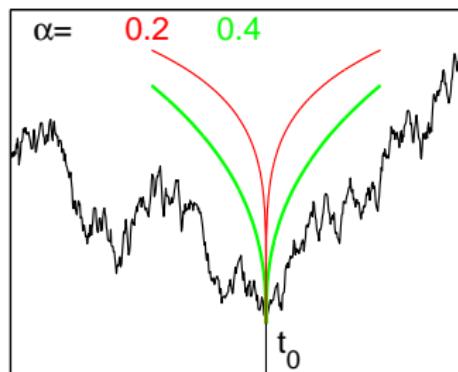
$X \in C^\alpha(t_0) \quad \text{if} \quad \exists C, \alpha > 0; P_{t_0}(t); \deg(P_{t_0}) < \alpha :$
 $|X(t) - P_{t_0}(t)| < C|t - t_0|^\alpha$



Characterization by local regularity

- Local regularity of $X(t)$ at t_0

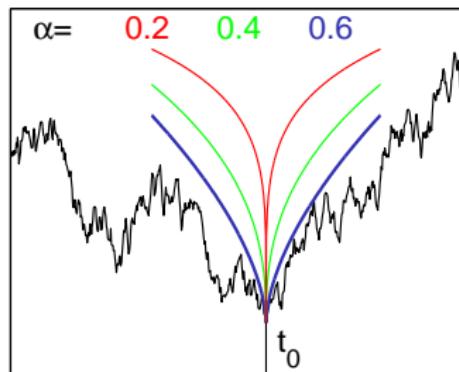
$X \in C^\alpha(t_0) \quad \text{if} \quad \exists C, \alpha > 0; P_{t_0}(t); \deg(P_{t_0}) < \alpha :$
 $|X(t) - P_{t_0}(t)| < C|t - t_0|^\alpha$



Characterization by local regularity

- Local regularity of $X(t)$ at t_0

$X \in C^\alpha(t_0) \quad \text{if} \quad \exists C, \alpha > 0; P_{t_0}(t); \deg(P_{t_0}) < \alpha :$
 $|X(t) - P_{t_0}(t)| < C|t - t_0|^\alpha$



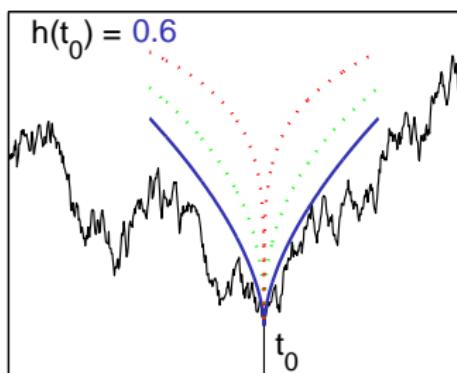
Characterization by local regularity

- Local regularity of $X(t)$ at t_0

$X \in C^\alpha(t_0)$ if $\exists C, \alpha > 0; P_{t_0}(t); \deg(P_{t_0}) < \alpha :$
 $|X(t) - P_{t_0}(t)| < C|t - t_0|^\alpha$

- Hölder exponent:

$$h(t_0) = \sup_\alpha \{\alpha : X \in C^\alpha(t_0)\}$$



Characterization by local regularity

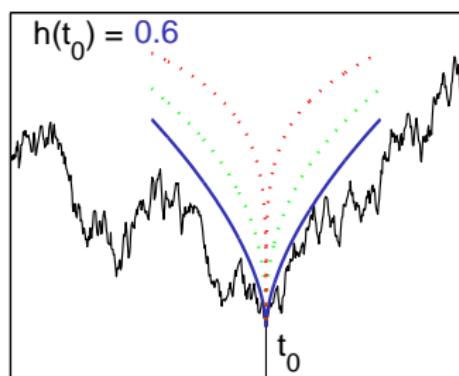
- Local regularity of $X(t)$ at t_0

$$X \in C^\alpha(t_0) \quad \text{if} \quad \exists C, \alpha > 0; P_{t_0}(t); \deg(P_{t_0}) < \alpha : \\ |X(t) - P_{t_0}(t)| < C|t - t_0|^\alpha$$

- Hölder exponent:

$$h(t_0) = \sup_\alpha \{\alpha : X \in C^\alpha(t_0)\}$$

$h(t_0) \rightarrow 1 \Rightarrow$ smooth, very regular,
 $h(t_0) \rightarrow 0 \Rightarrow$ rough, very irregular

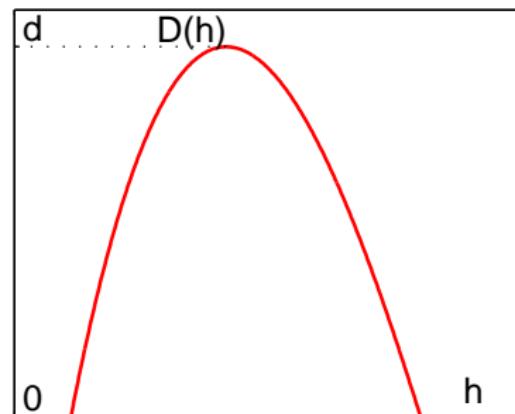
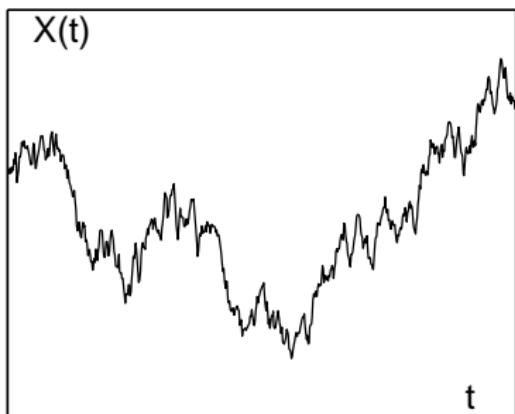


Multifractal spectrum

- Multifractal spectrum $D(h)$:

- fluctuation of the local regularity
- Haussdorf dimension of the sets $\{t_i | h(t_i) = h\}$

$$D(h) = \dim_H \{t : h(t) = h\}$$

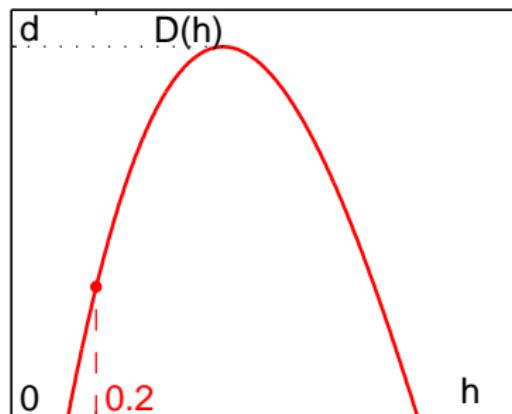
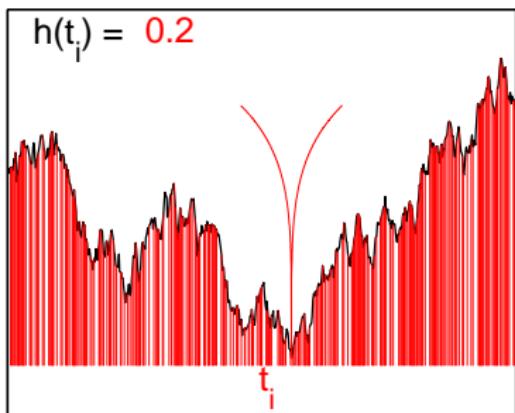


Multifractal spectrum

- Multifractal spectrum $D(h)$:

- fluctuation of the local regularity
- Hausdorff dimension of the sets $\{t_i | h(t_i) = h\}$

$$D(h) = \dim_H\{t : h(t) = h\}$$

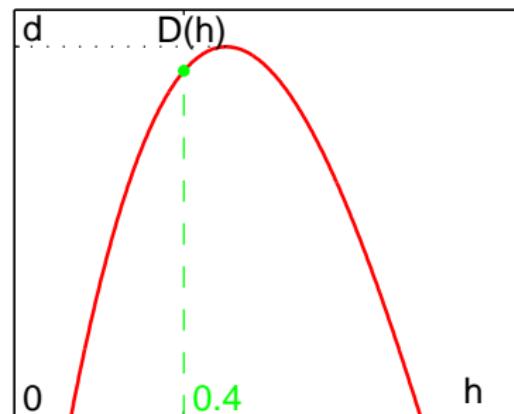
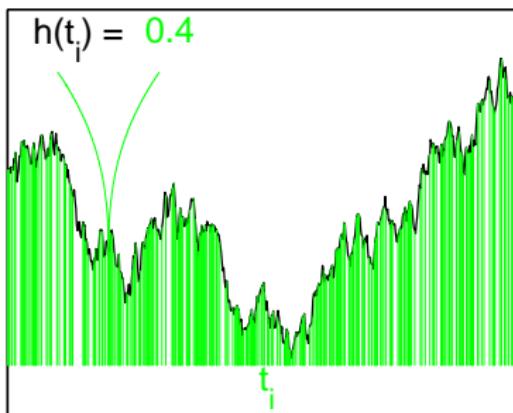


Multifractal spectrum

- Multifractal spectrum $D(h)$:

- fluctuation of the local regularity
- Hausdorff dimension of the sets $\{t_i | h(t_i) = h\}$

$$D(h) = \dim_H\{t : h(t) = h\}$$

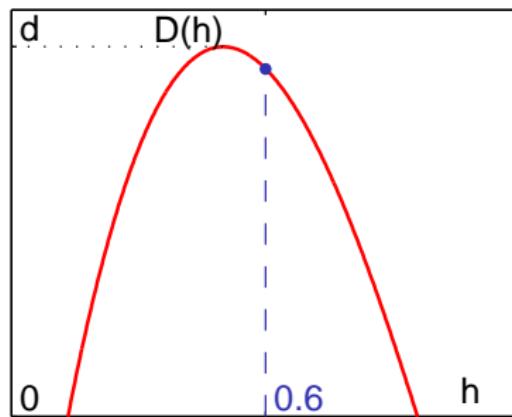
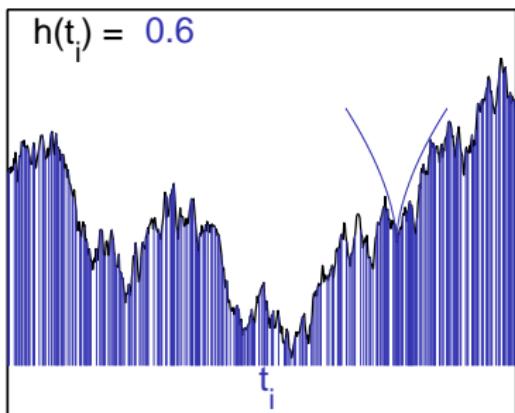


Multifractal spectrum

- Multifractal spectrum $D(h)$:

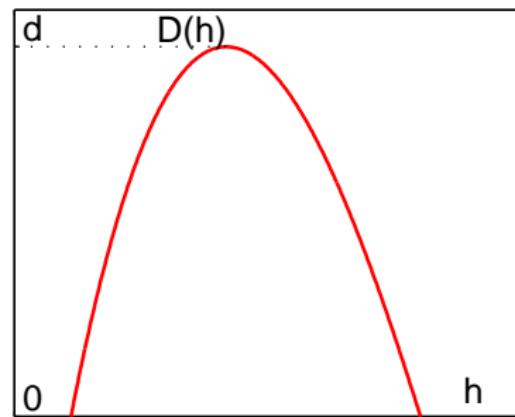
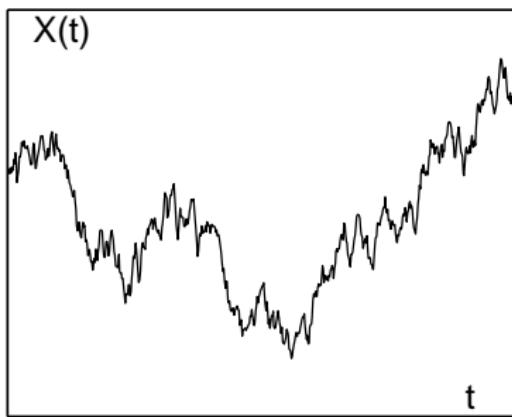
- fluctuation of the local regularity
- Hausdorff dimension of the sets $\{t_i | h(t_i) = h\}$

$$D(h) = \dim_H\{t : h(t) = h\}$$



Multifractal spectrum

- Multifractal spectrum $D(h)$:
 - fluctuation of the local regularity
 - Hausdorff dimension of the sets $\{t_i | h(t_i) = h\}$
$$D(h) = \dim_H \{t : h(t) = h\}$$
- In practice → multifractal formalism [Parisi85]

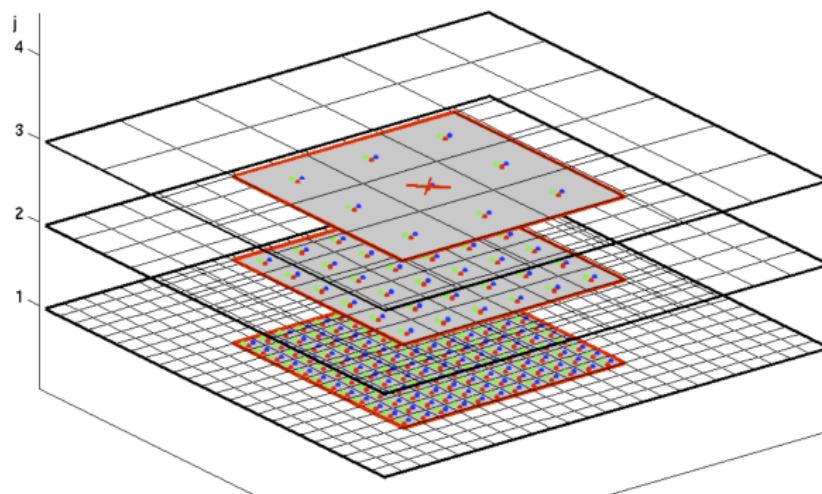


Legendre spectrum and log-cumulant

- Dyadic wavelet transform (DWT) $\rightarrow \{d^{(m)}(j, \cdot, \cdot)\}_{m=1,2,3}$
- Wavelet leaders $\{L(j, \cdot, \cdot)\}$

[Jaffard04]

$$L(j, k_1, k_2) = \sup_{m, \lambda' \subset 3\lambda_{j, k_1, k_2}} |d^{(m)}(\lambda')|$$



Legendre spectrum and log-cumulant

- Dyadic wavelet transform (DWT) $\rightarrow \{d^{(m)}(j, \cdot, \cdot)\}_{m=1,2,3}$
- Wavelet leaders $\{L(j, \cdot, \cdot)\}$

[Jaffard04]

$$L(j, k_1, k_2) = \sup_{m, \lambda' \subset 3\lambda_{j, k_1, k_2}} |d^{(m)}(\lambda')|$$

- Polynomial expansion

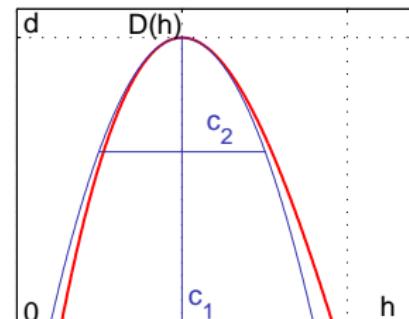
[Castaing93]

$$D^L(h) = 2 + \frac{c_2}{2!} \left(\frac{h - c_1}{c_2} \right)^2 - \frac{c_3}{3!} \left(\frac{h - c_1}{c_2} \right)^3 + \dots \approx D(h)$$

→ Log-cumulants c_m

- c_1 : mode
- c_2 : width
- c_3 : asymmetry

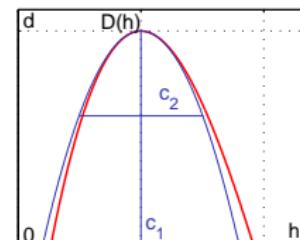
$$C_m(j) = \text{Cum}_m[\ln L(j, \cdot, \cdot)] = c_m^0 + c_m \ln 2^j$$



Multifractality parameter

- Multifractality parameter c_2

- self-similar processes \leftrightarrow multiplicative cascades
- $\text{Var} [\ln L(j, \cdot, \cdot)] = c_2^0 + c_2 \ln 2^j$



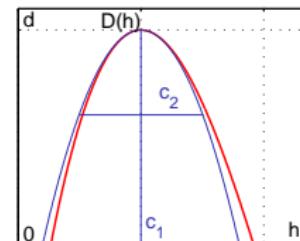
- Procedures for the estimation of c_2

- classical linear regression-based estimation [Castaing93] [Wendt09]
 - ✓ computational cost
 - ✗ estimation performance
- recently proposed Bayesian estimation [Combrexelle15]
 - ✓ estimation performance
 - ✗ computational cost

Multifractality parameter

- Multifractality parameter c_2

- self-similar processes \leftrightarrow multiplicative cascades
- $\text{Var} [\ln L(j, \cdot, \cdot)] = c_2^0 + c_2 \ln 2^j$



- Procedures for the estimation of c_2

- classical linear regression-based estimation [Castaing93] [Wendt09]
 - ✓ computational cost
 - ✗ estimation performance
- recently proposed Bayesian estimation [Combrexelle15]
 - ✗ estimation performance
 - ✓ computational cost

Statistical model of log-leaders: marginal distribution

- Multifractal multiplicative cascade (MMC) based processes
 - strong departures from Gaussian
 - complicated dependence structure (long-memory)

1. Marginal distribution of **log-leaders** approximated by **Gaussian**

$$l(j, \cdot, \cdot) = \log L(j, \cdot, \cdot) \sim \mathcal{N}(\cdot, \text{Var}[l(j, \cdot, \cdot)])$$

$$\text{with } \text{Var}[l(j, \cdot, \cdot)] = c_2^0 + c_2 \ln 2^j$$

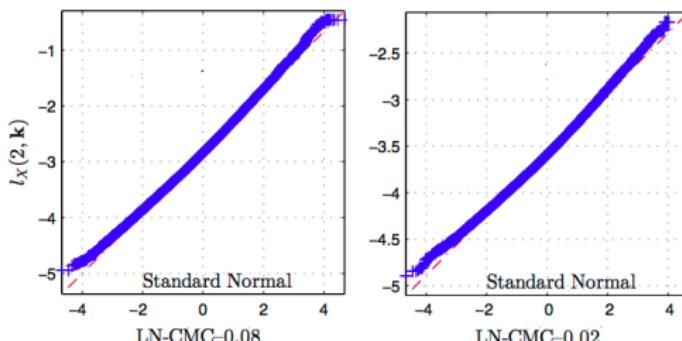
Statistical model of log-leaders: marginal distribution

- Multifractal multiplicative cascade (MMC) based processes
 - strong departures from Gaussian
 - complicated dependence structure (long-memory)

1. Marginal distribution of **log-leaders** approximated by **Gaussian**

$$l(j, \cdot, \cdot) = \log L(j, \cdot, \cdot) \sim \mathcal{N}(\cdot, \text{Var}[l(j, \cdot, \cdot)])$$

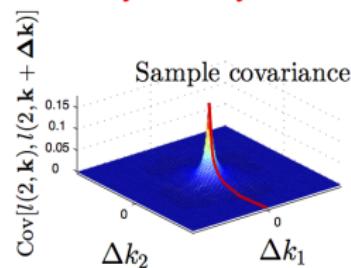
$$\text{with } \text{Var}[l(j, \cdot, \cdot)] = c_2^0 + c_2 \ln 2^j$$



Statistical model of log-leaders: intra-scale covariance

2. Covariance model

- radial symmetry of the covariance

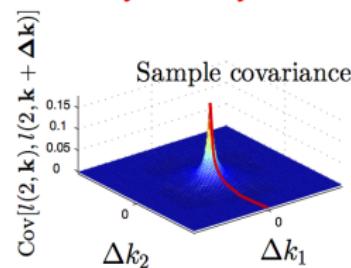


$$\rightarrow \text{Cov}[l(j, \mathbf{k}), l(j, \mathbf{k} + \Delta\mathbf{k})] \stackrel{\Delta r = |\Delta\mathbf{k}|}{\approx} \varrho_j(\Delta r; \theta)$$

Statistical model of log-leaders: intra-scale covariance

2. Covariance model

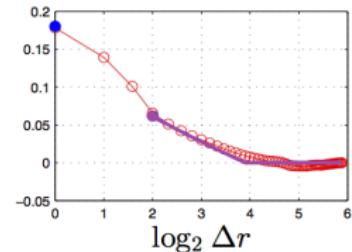
- radial symmetry of the covariance



$$\rightarrow \text{Cov}[l(j, \mathbf{k}), l(j, \mathbf{k} + \Delta\mathbf{k})] \stackrel{\Delta r = |\Delta\mathbf{k}|}{\approx} \varrho_j(\Delta r; \theta)$$

- piece-wise logarithmic model parametrized by $\theta = [c_2, c_2^0]^T$

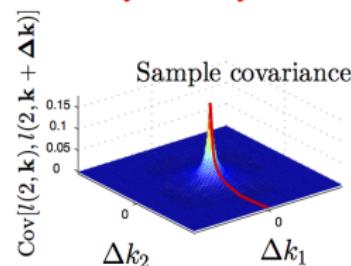
$$\varrho_j(\Delta r; \theta) = \begin{cases} c_2^0 + c_2 \log 2^j & \Delta r = 0 \\ \max(0, \varrho_j^{(1)}(\Delta r; \theta)) & 3 \leq \Delta r \leq \sqrt{2n_j} \end{cases}$$



Statistical model of log-leaders: intra-scale covariance

2. Covariance model

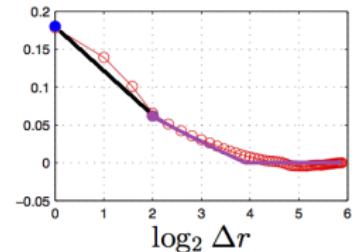
- radial symmetry of the covariance



$$\rightarrow \text{Cov}[l(j, \mathbf{k}), l(j, \mathbf{k} + \Delta\mathbf{k})] \stackrel{\Delta r = |\Delta\mathbf{k}|}{\approx} \varrho_j(\Delta r; \theta)$$

- piece-wise logarithmic model parametrized by $\theta = [c_2, c_2^0]^T$

$$\varrho_j(\Delta r; \theta) = \begin{cases} c_2^0 + c_2 \log 2^j & \Delta r = 0 \\ \varrho_j^{(0)}(\Delta r; \theta) & 0 \leq \Delta r \leq 3 \\ \max(0, \varrho_j^{(1)}(\Delta r; \theta)) & 3 \leq \Delta r \leq \sqrt{2n_j} \end{cases}$$



Bayesian model

- Data: **centered** log-leaders $l_j, j = j_1, \dots, j_2$ stacked in $\mathcal{L} = [l_{j_1}^T, \dots, l_{j_2}^T]^T$
- Likelihood: **scale-wise product** of Gaussian likelihoods

$$p(\mathcal{L} | \theta) = \prod_{j=j_1}^{j_2} p(l_j | \theta), \quad p(l_j | \theta) = \frac{\exp\left(-\frac{1}{2}l_j^T \Sigma_j(\theta)^{-1} l_j\right)}{\sqrt{(2\pi)^{m_j^2} \det \Sigma_j(\theta)}},$$

covariance matrix $\Sigma_j(\theta)$ induced by $\varrho_j(\Delta r; \theta)$

- Prior: **uniform prior** $\pi(\theta)$ on a bounded admissible set
- Posterior:

$$p(\theta | \mathcal{L}) \propto p(\mathcal{L} | \theta) \pi(\theta)$$

Bayesian model

- Data: **centered** log-leaders $l_j, j = j_1, \dots, j_2$ stacked in $\mathcal{L} = [l_{j_1}^T, \dots, l_{j_2}^T]^T$
- Likelihood: **scale-wise product** of Gaussian likelihoods

$$p(\mathcal{L} | \theta) = \prod_{j=j_1}^{j_2} p(l_j | \theta), \quad p(l_j | \theta) = \frac{\exp\left(-\frac{1}{2} l_j^T \Sigma_j(\theta)^{-1} l_j\right)}{\sqrt{(2\pi)^{m_j^2} \det \Sigma_j(\theta)}},$$

covariance matrix $\Sigma_j(\theta)$ induced by $\varrho_j(\Delta r; \theta)$

- Prior: **uniform prior** $\pi(\theta)$ on a bounded admissible set
- Posterior:

$$p(\theta | \mathcal{L}) \propto p(\mathcal{L} | \theta) \pi(\theta)$$

Bayesian estimators

- Bayesian estimator

- minimum mean squared error (MMSE): $\hat{\theta}^{\text{MMSE}} = \mathbb{E}[\theta | \mathcal{L}]$

- Computation by Markov chain Monte Carlo algorithm (MCMC)

- generation of N_{mc} samples $\{\theta^{(t)}\}_{t=1}^{N_{mc}}$ with a Gibbs sampler
- asymptotical distribution according to the posterior $p(\theta | \mathcal{L})$

$$\hat{\theta}^{\text{MMSE}} \approx \frac{1}{N_{mc} - N_{bi}} \sum_{t=N_{bi}+1}^{N_{mc}} \theta^{(t)}$$

- iterative evaluation of the likelihood
- ✗ prohibitive cost / computation of $\Sigma_j(\theta)^{-1}$

Bayesian estimators

- Bayesian estimator
 - minimum mean squared error (MMSE): $\hat{\theta}^{\text{MMSE}} = \mathbb{E}[\theta | \mathcal{L}]$
- Computation by **Markov chain Monte Carlo algorithm (MCMC)**
 - generation of N_{mc} samples $\{\theta^{(t)}\}_{t=1}^{N_{mc}}$ with a **Gibbs sampler**
 - asymptotical distribution according to the posterior $p(\theta | \mathcal{L})$

$$\hat{\theta}^{\text{MMSE}} \approx \frac{1}{N_{mc} - N_{bi}} \sum_{t=N_{bi}+1}^{N_{mc}} \theta^{(t)}$$

- iterative evaluation of the likelihood
- X** prohibitive cost / computation of $\Sigma_j(\theta)^{-1}$

Whittle approximation

- Whittle approximation

- Gaussian likelihood approximated in the spectral domain

$$\ln p(l_j | \theta) \approx \ln p_W(l_j | \theta) = -\frac{1}{2} \sum_{\omega \in D_j} \ln \phi_j(\omega; \theta) + \frac{l_j(\omega)}{\phi_j(\omega; \theta)}$$

- $l_j(\omega)$ periodogram of the log-leaders l_j
- $\phi_j(\omega; \theta)$ parametric spectral density associated with $\varrho_j(\Delta r; \theta)$

Brute force approach → Discrete Fourier Transform

$$\phi_j^{DFT}(\omega; \theta) = \left| \sum \varrho_j(|\Delta k|; \theta) e^{-i(\Delta k^T \omega)} \right|$$

S. Combexelle et al, *Bayesian Estimation of the Multifractality Parameter for Image Texture Using a Whittle Approximation*, IEEE T. Image Proces., vol. 24, no. 8, pp. 2540-2551, Aug. 2015.

Whittle approximation

- Whittle approximation

- Gaussian likelihood approximated in the spectral domain

$$\ln p(l_j | \theta) \approx \ln p_W(l_j | \theta) = -\frac{1}{2} \sum_{\omega \in D_j} \ln \phi_j(\omega; \theta) + \frac{l_j(\omega)}{\phi_j(\omega; \theta)}$$

- $l_j(\omega)$ periodogram of the log-leaders l_j
- $\phi_j(\omega; \theta)$ parametric spectral density associated with $\varrho_j(\Delta r; \theta)$

Brute force approach → Discrete Fourier Transform

$$\phi_j^{DFT}(\omega; \theta) = \left| \sum \varrho_j(|\Delta k|; \theta) e^{-i(\Delta k^T \omega)} \right|$$

S. Combrexelle et al, *Bayesian Estimation of the Multifractality Parameter for Image Texture Using a Whittle Approximation*, IEEE T. Image Proces., vol. 24, no. 8, pp. 2540-2551, Aug. 2015.

Closed-form Whittle approximation

- Close-form expression for the spectral density

- Bochner's theorem → spectral density

$$\phi_j(\omega; \theta) = \int_{\mathbb{R}^2} \varrho_j(|\mathbf{x}|; \theta) e^{-i(\mathbf{x}^T \omega)} d\mathbf{x}$$

- radial symmetry → Hankel transform

$$\phi_j(\omega; \theta) = \phi_j(|\omega|; \theta) = 2\pi \int_0^\infty r \varrho_j(r; \theta) J_0(r|\omega|) dr$$

- piece-wise logarithmic covariance model

→ analytical expression for Hankel transform

$$\phi_j(|\omega|; \theta) = c_2 f_j(|\omega|) + c_2^0 g_j(|\omega|), \quad \theta = [c_2, c_2^0]^T$$

$$f_j(|\omega|) = 2\pi \left(\frac{J_0(r_j|\omega|) - J_0(3|\omega|)}{|\omega|^2} + \frac{3\ln(r_j 2^j / 3) J_1(3|\omega|)}{|\omega|} \right) - \frac{2\pi \ln(r_j 2^j / 3)}{\ln 4} \int_0^3 r \ln(1+r) J_0(r|\omega|) dr$$

$$g_j(|\omega|) = 6\pi \frac{J_1(3|\omega|)}{|\omega|} - 2\pi \int_0^3 r \ln(1+r) J_0(r|\omega|) dr / \ln 4, \quad \text{with } r_j = \sqrt{n_j} / 4$$

Closed-form Whittle approximation

- Close-form expression for the spectral density

- Bochner's theorem → spectral density

$$\phi_j(\omega; \theta) = \int_{\mathbb{R}^2} \varrho_j(|\mathbf{x}|; \theta) e^{-i(\mathbf{x}^T \omega)} d\mathbf{x}$$

- radial symmetry → Hankel transform

$$\phi_j(\omega; \theta) = \phi_j(|\omega|; \theta) = 2\pi \int_0^\infty r \varrho_j(r; \theta) J_0(r|\omega|) dr$$

- piece-wise logarithmic covariance model

→ analytical expression for Hankel transform

$$\phi_j(|\omega|; \theta) = c_2 f_j(|\omega|) + c_2^0 g_j(|\omega|), \quad \theta = [c_2, c_2^0]^T$$

$$f_j(|\omega|) = 2\pi \left(\frac{J_0(r_j|\omega|) - J_0(3|\omega|)}{|\omega|^2} + \frac{3\ln(r_j 2^j / 3) J_1(3|\omega|)}{|\omega|} \right) - \frac{2\pi \ln(r_j 2^j / 3)}{\ln 4} \int_0^3 r \ln(1+r) J_0(r|\omega|) dr$$

$$g_j(|\omega|) = 6\pi \frac{J_1(3|\omega|)}{|\omega|} - 2\pi \int_0^3 r \ln(1+r) J_0(r|\omega|) dr / \ln 4, \quad \text{with } r_j = \sqrt{n_j} / 4$$

Closed-form Whittle approximation

- Close-form expression for the spectral density

- Bochner's theorem → spectral density

$$\phi_j(\omega; \theta) = \int_{\mathbb{R}^2} \varrho_j(|\mathbf{x}|; \theta) e^{-i(\mathbf{x}^T \omega)} d\mathbf{x}$$

- radial symmetry → Hankel transform

$$\phi_j(\omega; \theta) = \phi_j(|\omega|; \theta) = 2\pi \int_0^\infty r \varrho_j(r; \theta) J_0(r|\omega|) dr$$

- piece-wise logarithmic covariance model

→ analytical expression for Hankel transform

$$\phi_j(|\omega|; \theta) = c_2 f_j(|\omega|) + c_2^0 g_j(|\omega|), \quad \theta = [c_2, c_2^0]^T$$

$$f_j(|\omega|) = 2\pi \left(\frac{J_0(r_j|\omega|) - J_0(3|\omega|)}{|\omega|^2} + \frac{3\ln(r_j 2^j / 3) J_1(3|\omega|)}{|\omega|} \right) - \frac{2\pi \ln(r_j 2^j / 3)}{\ln 4} \int_0^3 r \ln(1+r) J_0(r|\omega|) dr$$

$$g_j(|\omega|) = 6\pi \frac{J_1(3|\omega|)}{|\omega|} - 2\pi \int_0^3 r \ln(1+r) J_0(r|\omega|) dr / \ln 4, \quad \text{with } r_j = \sqrt{n_j}/4$$

Numerical simulations

- Synthetic process: Multifractal Random Walk (MRW)
 - non-Gaussian process
 - multifractal properties \sim Mandelbrot's multiplicative cascades
 - $c_2 \in \{-0.01, -0.02, \dots, -0.1\}$, $N \in \{2^7, 2^8\}$
- Estimation setup
 - Daubechies' mother wavelet
 - $j_1 \in \{1, 2\}$, $j_2 = \log_2 N - 4$
 - 2000 iteration in the Gibbs sampler
- Performance assessment
 - $\text{BIAS} = \widehat{\mathbb{E}}[\hat{c}_2] - c_2$, $\text{STD} = \sqrt{\text{Var}[\hat{c}_2]}$, $\text{RMSE} = \sqrt{\text{BIAS}^2 + \text{STD}^2}$
 - computed over 100 independent realizations

Numerical simulations

- Synthetic process: Multifractal Random Walk (MRW)
 - non-Gaussian process
 - multifractal properties \sim Mandelbrot's multiplicative cascades
 - $c_2 \in \{-0.01, -0.02, \dots, -0.1\}$, $N \in \{2^7, 2^8\}$
- Estimation setup
 - Daubechies' mother wavelet
 - $j_1 \in \{1, 2\}$, $j_2 = \log_2 N - 4$
 - 2000 iteration in the Gibbs sampler
- Performance assessment

$$\text{BIAS} = \widehat{\mathbb{E}}[\hat{c}_2] - c_2, \quad \text{STD} = \sqrt{\text{Var}[\hat{c}_2]}, \quad \text{RMSE} = \sqrt{\text{BIAS}^2 + \text{STD}^2}$$

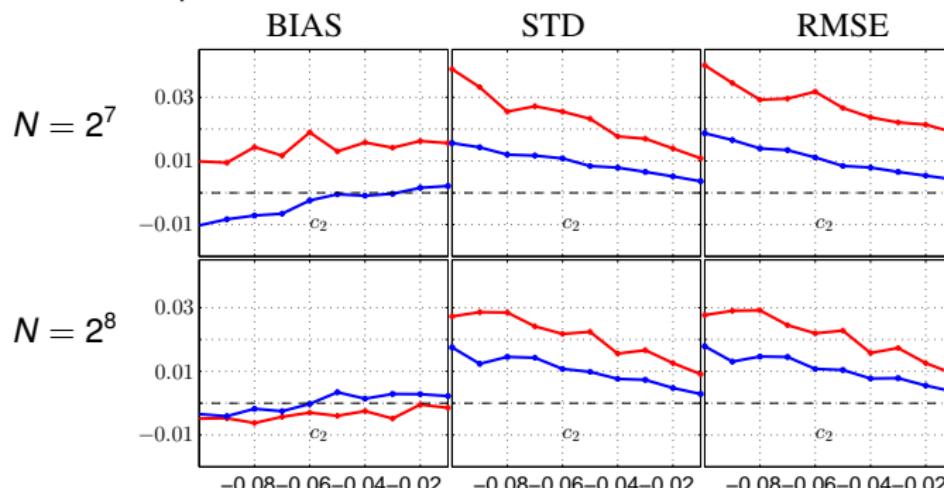
→ computed over 100 independent realizations

Numerical simulations

- Synthetic process: Multifractal Random Walk (MRW)
 - non-Gaussian process
 - multifractal properties \sim Mandelbrot's multiplicative cascades
 - $c_2 \in \{-0.01, -0.02, \dots, -0.1\}$, $N \in \{2^7, 2^8\}$
- Estimation setup
 - Daubechies' mother wavelet
 - $j_1 \in \{1, 2\}$, $j_2 = \log_2 N - 4$
 - 2000 iteration in the Gibbs sampler
- Performance assessment
$$\text{BIAS} = \widehat{\mathbb{E}}[\hat{c}_2] - c_2, \quad \text{STD} = \sqrt{\text{Var}[\hat{c}_2]}, \quad \text{RMSE} = \sqrt{\text{BIAS}^2 + \text{STD}^2}$$
 - computed over 100 independent realizations

Results: estimation performance

- Estimation performance



LF linear regression based estimator

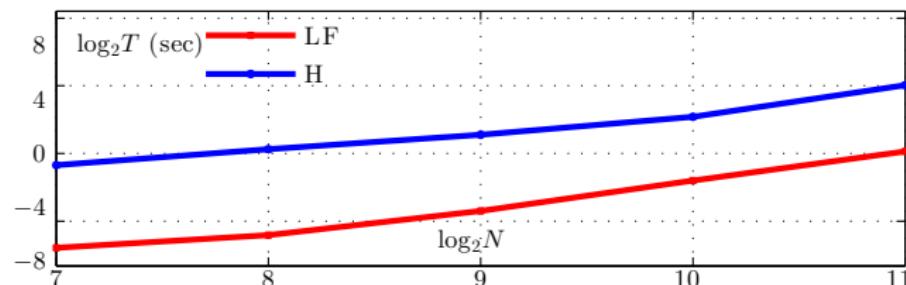
H closed-form Whittle approximation based estimator

→ significant reduction of the bias and standard deviation

→ root mean square error divided by 4

Results: computational cost

- Computational time (DWT + estimation algorithm)

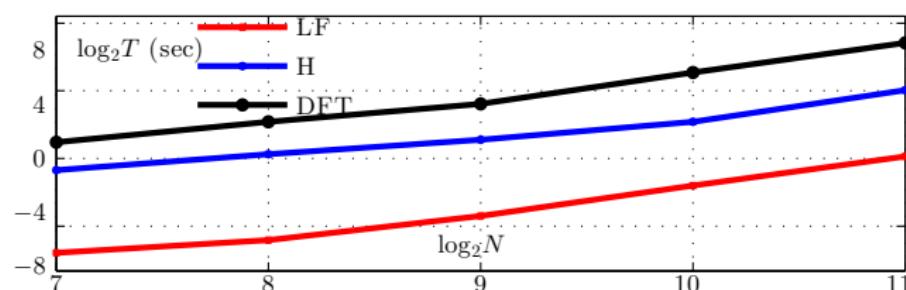


→ linear regression (LF) ≪ Bayesian approach (H)

Ex. $N = 2^{11} \rightarrow$ (LF) 30 times faster than (H)

Results: computational cost

- Computational time (DWT + estimation algorithm)



- linear regression (LF) ≪ Bayesian approach (H)
 - closed-form Whittle (H) ≪ brute force approach (DFT)
- Ex. $N = 2^{11}$ → (LF) 30 times faster than (H)
 → (LF) 400 times faster than (DFT)

Conclusion and future work

● Conclusion

- Bayesian estimation of c_2 in image texture
- generic semi-parametric statistical model for log-leaders
- estimators computed by a sampling scheme
- computational cost reduced (explicit Whittle likelihood)
- operational and efficient for small and large images
- attractive alternative to linear regression based procedures

● Future work

- multivariate multifractal analysis
- multivariate priors (regularization, segmentation/classification)
- adequate sampling schemes (Hamiltonian Monte Carlo)

Thanks for your attention

References

- [Parisi85] U. Frisch, and G. Parisi, *On the singularity structure of fully developed turbulence; appendix to Fully developed turbulence and intermittency, by U. Frisch*, in Proc. Int. Summer School Phys. Enrico Fermi, North-Holland, 1985, pp. 84-88
- [Jaffard04] S. Jaffard, *Wavelet techniques in multifractal analysis*, in Fractal Geometry and Applications: A Jubilee of Benoît Mandelbrot, Proc. Symp. Pure Math., M. Lapidus and M. van Frankenhuysen, Eds. 2004, vol. 72(2), pp. 91-152, AMS
- [Castaing93] B. Castaing, Y. Gagne, and M. Marchand, *Log-similarity for turbulent flows?*, Physica D, vol. 68, no. 34, pp. 387-400, 1993
- [Wendt09] H. Wendt, S. G. Roux, S. Jaffard, and P. Abry, *Wavelet leaders and bootstrap for multifractal analysis of images*, Signal Process., vol. 89, no. 6, pp. 1100-1114, 2009
- [Combrexelle15] S. Combrexelle, H. Wendt, N. Dobigeon, J.-Y. Tourneret, S. McLaughlin, and P. Abry, *Bayesian Estimation of the Multifractality Parameter for Image Texture Using a Whittle Approximation*, IEEE T. Image Proces., vol. 24, no. 8, pp. 2540-2551, Aug. 2015

Multifractal formalism

- Wavelet coefficients $\{d^{(m)}(j, \cdot, \cdot)\}_{m=1,2,3} \rightarrow$ wavelet leaders $\{L(j, \cdot, \cdot)\}$
- Scaling exponents $\zeta^L(q)$ $q \in [q^-, q^+]$

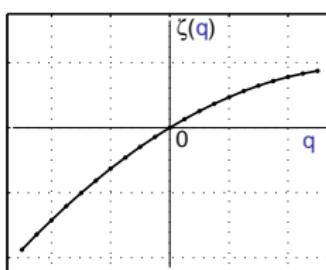
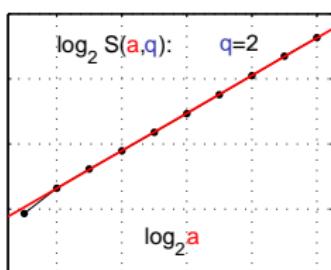
$$S^L(j, q) = \frac{1}{n_j} \sum L(j, \cdot, \cdot)^q \simeq 2^{j\zeta^L(q)}$$

- Legendre spectrum

$$D^L(h) = \min_{q \neq 0} (d + qh - \zeta^L(q)) \geq D(h)$$

- Polynomial expansion

$$\zeta^L(q) = c_1 q + c_2 q^2 / 2 + c_3 q^3 / 6 + \dots$$



Multifractal formalism

- Wavelet coefficients $\{d^{(m)}(j, \cdot, \cdot)\}_{m=1,2,3} \rightarrow$ wavelet leaders $\{L(j, \cdot, \cdot)\}$
- Scaling exponents $\zeta^L(q)$ $q \in [q^-, q^+]$

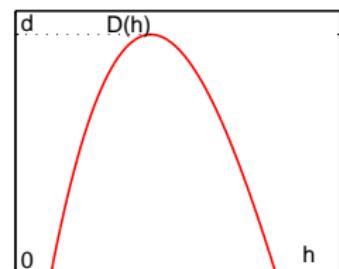
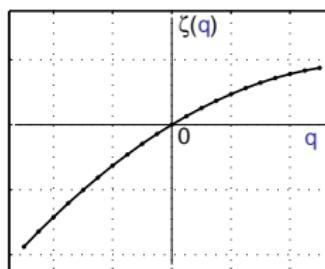
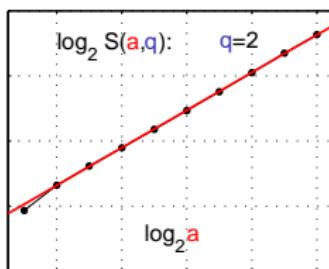
$$S^L(j, q) = \frac{1}{n_j} \sum L(j, \cdot, \cdot)^q \simeq 2^{j\zeta^L(q)}$$

- Legendre spectrum

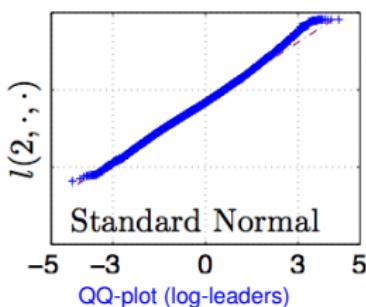
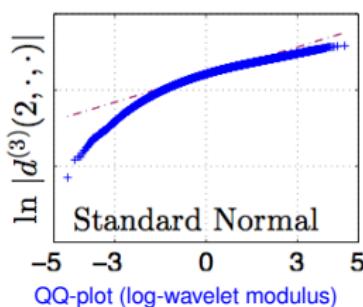
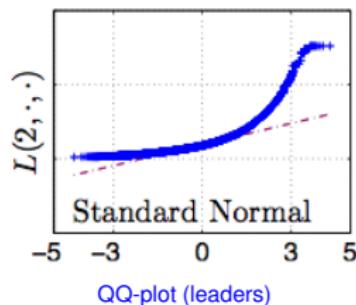
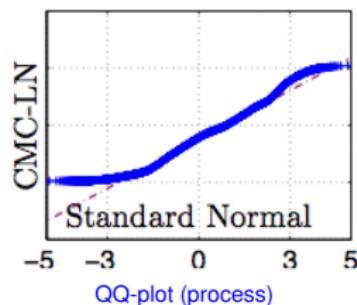
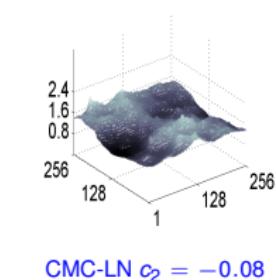
$$D^L(h) = \min_{q \neq 0} (d + qh - \zeta^L(q)) \geq D(h)$$

- Polynomial expansion

$$\zeta^L(q) = c_1 q + c_2 q^2 / 2 + c_3 q^3 / 6 + \dots$$



Gaussian assumption



Spectral density

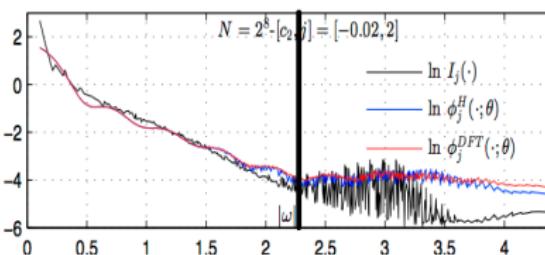
- Closed-form expression of the spectral density

$$\phi_j(|\omega|; \theta) = c_2 f_j(|\omega|) + c_2^0 g_j(|\omega|)$$

$$f_j(|\omega|) = 2\pi \left(\frac{J_0(r_j|\omega|) - J_0(3|\omega|)}{|\omega|^2} + \frac{3\ln(r_j 2^j / 3) J_1(3|\omega|)}{|\omega|} \right) - \frac{2\pi \ln(r_j 2^j / 3)}{\ln 4} \int_0^3 r \ln(1+r) J_0(r|\omega|) dr$$

$$g_j(|\omega|) = 6\pi \frac{J_1(3|\omega|)}{|\omega|} - 2\pi \int_0^3 r \ln(1+r) J_0(r|\omega|) dr / \ln 4, \quad \text{with } r_j = \sqrt{n_j}/4$$

- Model fitting



→ spectral grid restricted to low frequencies $|\omega| \leq \pi\sqrt{\eta}$ with $\eta = 0.25$

MCMC algorithm

- Strategy of Gibbs sampler

- iterative sampling according to conditional laws
- not standard conditional laws → Metropolis-within-Gibbs
- computation of **acceptance ratio** at each iteration

$$r_{c_2} = \sqrt{\frac{\det \boldsymbol{\Sigma}(\boldsymbol{\theta}^{(t)})}{\det \boldsymbol{\Sigma}(\boldsymbol{\theta}^{(*)})}} \times \prod_{j=j_1}^{j_2} \exp \left(-\frac{1}{2} l_j^T (\boldsymbol{\Sigma}_j(\boldsymbol{\theta}^{(*)})^{-1} - \boldsymbol{\Sigma}_j(\boldsymbol{\theta}^{(t)})^{-1}) l_j \right)$$