ESTIMATION - DETECTION

TD 2 — Detection

Exercise 1

A radar system repetitively sends out trains of n pulses. The pulse trains are of sufficiently short duration and the rotation of the antenna during the each emission of a sequence of pulses can be neglected. The reflected pulses are preprocessed and collected in a vector of observations $(x_1, \cdots, x_n)$. They can be modeled as $x_i = \tau + \varepsilon_i$ if a target is present (H_1), and as $x_i = \varepsilon_i$ when no target is present (H_0). Here, $\varepsilon_i \sim \mathcal{N}(0, \sigma^2)$ models all the noise in the radar system and environment. The noise level σ^2 is assumed to be known.

1. Suppose that the value of τ is known. Define the Neyman-Pearson test for this problem.
   (a) Derive the test statistic T. What is the critical region of the test?
   (b) Calculate the critical value t_α of the test for significance α. Derive the expression for the receiver operating characteristic. How does it behave as a function of α, σ^2, τ, n?
   (c) Derive the expression for the value of τ for which the test rejects H_0 and H_1 with equal probability when H_1 is true.
2. In practice, τ will not be known since targets will be at different distances, have different size, geometry, reflection coefficients, etc. The problem can be formulated with the hypothesis $H_0:\ \tau = 0$ and $H_1:\ \tau \neq 0$.
   (a) Define the generalized likelihood ratio test for this problem, derive its test statistic and critical value.
   (b) Derive the probability of rejection of H_0 when $\tau = \tau^*$.
3. Suppose now that we want to formulate the test using the constraints $H_1:\ \tau \sim \mathcal{N}(0, \nu^2)$ and $H_0:\ \tau = 0$. Define the Bayesian test and derive its test statistic. How do you interpret the result?

Exercise 2

Suppose that the data in an imaging problem are complex-valued and corrupted by background noise $\underline{X} = X_R + \mathrm{i}X_I$ with independent Gaussian real and imaginary parts, $X_R \sim \mathcal{N}(0, \sigma^2)$ and $X_I \sim \mathcal{N}(0, \sigma^2)$, respectively. It follows that the magnitude of the noise, $X = \sqrt{X_R^2 + X_I^2}$, is distributed according to a Rayleigh distribution, $X \sim \mathcal{R}(\sigma^2)$. The noise level in the imaging problem equals either σ_0^2 or σ_1^2. Image segmentation is supposed to be performed based on the magnitude of the registered images and two different algorithms are available for this task: Algorithm A0 is optimized for low noise level $\sigma^2 = \sigma_0^2$ but is slow at higher noise levels, and Algorithm A1 is more complex but has good performance at noise level $\sigma^2 = \sigma_1^2 > \sigma_0^2$. A sample $(X_1, \cdots, X_n)$ of n pixels of an image of the background is available.

1. Neyman-Pearson test.

   (a) Define the test for $H_0 : \ \sigma^2 = \sigma_0^2$ and $H_1 : \ \sigma^2 = \sigma_1^2$ and derive its test statistic.

   (b) Derive the expression for the significance α and for the power π of the test.
   (Note that if $x_i \overset{i.i.d.}{\sim} \mathcal{R}(\sigma^2)$, then $\sum_{i=1}^n x_i^2 \sim \mathcal{G}(n, 2\sigma^2)$.)

2. Suppose the probabilities for H_0 and H_1 to be true are $P(H_0)$ and $P(H_1)$, respectively.

   (a) Define the Bayesian detector and derive its test statistic.

   (b) Design appropriate costs c_{ij} such that the expected execution time is minimized when using the Bayesian detector. The execution time of algorithms A0 and A1 for one image is given by:

	A0	A1
H_0	2s	6s
H_1	4s	3s

   What is the expected execution time when $P(H_0) = 3/4$ and $P(H_1) = 1/4$, $n = 16 \times 16$, $\sigma_0^2 = 0.04$ and $\sigma_1^2 = 0.045$?

t	22.1	22.2	22.3	22.4	22.5	22.6	22.7	22.8	22.9	23
$\int_t^\infty f(x\|H_0)dx$	0.895	0.908	0.92	0.93	0.939	0.948	0.955	0.962	0.967	0.972
$\int_t^\infty f(x\|H_1)dx$	0.261	0.284	0.309	0.334	0.361	0.387	0.414	0.442	0.469	0.497
t	23.1	23.2	23.3	23.4	23.5	23.6	23.7	23.8	23.9	24
$\int_t^\infty f(x\|H_0)dx$	0.976	0.98	0.983	0.986	0.988	0.99	0.992	0.994	0.995	0.996
$\int_t^\infty f(x\|H_1)dx$	0.525	0.552	0.58	0.606	0.632	0.658	0.683	0.706	0.729	0.751

   (c) What is the expected execution time when the Neyman-Pearson test with $\alpha = 0.045$ is used?

3. When n is large, $\mathcal{G}(n, 2\sigma^2)$ can be approximated by a Gaussian distribution. In an attempt to simplify the decision process, we want to determine whether we can use this approximation. We have observed the test statistic T_n for $N = 20$ background images of size 64×64,

   $$T_n = \{79.8,\ 80,\ 80.2,\ 80.9,\ 81.2,\ 81.2,\ 81.5,\ 81.6,\ 81.6,\ 82,\\ 82.2,\ 82.2,\ 82.2,\ 82.3,\ 83.1,\ 83.1,\ 83.2,\ 83.7,\ 83.8,\ 84.3\}.$$

   (a) Suppose that the images have been generated under controlled conditions and that the noise level is known to be $\sigma^2 = 0.01$. Perform the χ^2 test with $K = 4$ equi-probable classes for this purpose (note: $F^{-1}(0.75) = 0.675$, where F is the cdf of the standard

Normal distribution). The significance of the test is fixed to $\alpha = 0.1$. The quantiles of the chi-square distribution are given by:

n	1	2	3	4	5
$\left(\chi_n^2\right)^{-1}(0.9)$	2.71	4.61	6.25	7.78	9.24

(b) Now suppose that the noise level is constant but unknown. Perform the χ^2 test.

(c) Suppose again that $\sigma^2 = 0.01$ and perform the Kolmogorov test for $\alpha = 0.1$ (the quantile of the Kolmogorov distribution is $t_{\alpha=0.1} = 0.2316$). The values of the Normal cumulative distribution function evaluated at the observations T_n are given by

$$F_0(T_n) = \{0.05,\ 0.07,\ 0.09,\ 0.21,\ 0.29,\ 0.29,\ 0.37,\ 0.4,\ 0.4,\ 0.52,\ 0.59,\ 0.59,\ 0.59,\ 0.62,\ 0.82,\ 0.82,\ 0.84,\ 0.92,\ 0.93,\ 0.97\}.$$

Exercise 3

Let $(X_1, \cdots, X_n)$ be a sample of Gaussian distribution $\mathcal{N}(\mu, \sigma^2)$ with mean μ and variance σ^2.

1. Let μ be known. We want to decide on the hypothesis $H_0 :\ \sigma^2 = \sigma_0^2$ and $H_1 :\ \sigma^2 = \sigma_1^2 > \sigma_0^2$.

   (a) Define the Neyman-Pearson test and derive its test statistic. What law does the test statistic follow?

   (b) Derive the expressions for the critical value of the test. Determine its power. Discuss the behavior of the receiver operating characteristic as a function of n, σ_0^2 and σ_1^2.

2. Now let μ be unknown. Define the generalized likelihood ratio test for the above hypotheses. and derive its test statistic. Express the test statistic in terms of the sample variance. What law does the test statistic follow?

3. Let σ^2 be unknown. We want to decide between the hypothesis $H_0 :\ \mu = m_0$ and $H_1 :\ \mu \neq m_0$.

   (a) Recall the maximum likelihood estimator $\tilde{\sigma}_{ML}^2$ for the variance when the mean μ is known.

   (b) Recall the maximum likelihood estimators $\hat{m}_{ML}$ and $\hat{\sigma}_{ML}^2$ for the mean and the variance when the mean μ is unknown.

   (c) Define the generalized likelihood ratio test and derive its test statistic t_n.

   (d) Show that an equivalent test can be obtained with the test statistic $T_n = \frac{\overline{X} - m_0}{\sqrt{\sum_{i=1}^n \left(X_i - \overline{X}\right)^2}}$ by decomposing $\sum_{i=1}^n (x_i - m_0)^2$. What law does this test statistic follow?

Exercise 4

Radioactivity is governed by the Poisson distribution. In a detection system for abnormal radioactivity, a measuring device delivers one observation x_i of a Poisson random variable $X_i \sim \mathcal{P}(\lambda)$ per unit time interval. The problem consists in deciding between the hypotheses $H_0 : \lambda = \lambda_0$ (normal) and $H_1 : \lambda = \lambda_1 > \lambda_0$ (abnormal: alarm).

1. Neyman-Pearson test.
   (a) Define the test, derive its test statistic and state its critical region.
   (b) Derive an expression for the significance α.
   (c) Now let $n = 3$, $\lambda_0 = 2/3$, $\lambda_1 = 2$, and $\alpha = 0.01$. Compute the critical value t_α and give the true significance of the test for this critical value. Compute the power of the test. The values of the Poisson distribution for $\lambda = 2$ are given by:

k	0	1	2	3	4	5	6	7
$P[X = k]$	0.1353	0.2707	0.2707	0.1804	0.0902	0.0361	0.0120	0.0034

2. Suppose the probabilities for H_0 and H_1 to be true are $P(H_0)$ and $P(H_1)$, respectively.
   (a) Define the Bayesian detector and derive its test statistic.
   (b) Let $c_{00} = c_{11} = 0$. How does the critical value of the test depend on c_{01} and c_{10}?

3. We want to decide whether the approximation of the law of the test statistic T when $\lambda = \lambda_0$ by a Normal distribution $\mathcal{N}(\lambda, \lambda)$ is appropriate or not, i.e., $H_0 : \quad T \sim \mathcal{N}(\lambda, \lambda)$ and $H_1 :$ not H_0. Let $\lambda_0 = 1$.
   (a) Which test is appropriate for this problem and why?
   (b) Suppose that we observe $N = 30$ independent realizations of the test statistic T_n, where the number of observations per test statistic realizations is $n = 3$:

value of t observed	0	1	2	3	4	5	6	7
number of times observed	1	3	6	7	4	4	2	3

   Define the classes for the test with $K = 4$ equi-probable classes (note: $F^{-1}(0.75) = 0.675$, where F is the cdf of the standard Normal distribution). Perform the test for $\alpha = 0.1$. The quantiles of the chi-square distribution are given by:

n	1	2	3	4	5
$\left(\chi^2_n\right)^{-1}(0.9)$	2.71	4.61	6.25	7.78	9.24

   (c) Now let $n = 20$ and let the $N = 30$ observations of T_n be given by

value of t observed	13	14	15	16	17	18	19	20	21
number of times observed	1	1	3	2	4	3	1	3	3
value of t observed	22	23	24	25	26	27	28	29	30
number of times observed	1	2	0	1	2	2	0	0	1

   As above, define the classes of the test ($K = 4$ equi-probable classes) and perform the test for $\alpha = 0.1$.

Exercise 5

The distribution of the size of files in internet traffic (TCP protocol) is often modeled by a Pareto distribution $\Pi(x_m, \alpha)$ with scale and shape parameter $x_m > 0$ and $\alpha > 0$, respectively.

1. Suppose that x_m is known. We want to decide between the hypotheses $H_0: \ \alpha = \alpha_0$ (normal traffic) and $H_1: \ \alpha = \alpha_1 > \alpha_0$ (abnormal traffic - attack).

   (a) Define the Neyman-Pearson test for this problem. Derive its test statistic as a function of $z_i = \frac{x_i}{x_m}$. State the critical region of the test.

   (b) Derive the integral equation for the critical value t_α and for the receiver operating characteristic $\pi(\alpha)$.
   Hint: $x_i \sim \Pi(x_m, \alpha) \implies \ln\left(\frac{x_i}{x_m}\right) \sim Exponential(\alpha) \implies \sum_{i=1}^n \ln\left(\frac{x_i}{x_m}\right) \sim Erlang(n, \alpha) = \mathcal{G}(n, \frac{1}{\alpha})$.

   (c) Let $n = 1$ and compute the critical value t_α and the receiver operating characteristic.

2. We want to use the Kolmogorov test to decide whether a specific set of observations follows a Pareto law with parameters x_m and α, i.e., $H_0: \ x_i \sim \Pi(x_m, \alpha)$ and $H_1:$ not H_0.

   (a) Express the cumulative distribution function of $\Pi(x_m, \alpha)$ as a function of $\frac{x_i}{x_m}$.

   (b) Let $x_m = \alpha = 1$. The values that have been observed are $(x_1, \cdots, x_5) = (7, 2, 3, 6, 5)$. Perform the Kolmogorov test for $\alpha = 0.05$ and $\alpha = 0.01$ (critical values (quantiles of the Kolmogorov distribution): $t_{\alpha=0.05} = 0.5095$ and $t_{\alpha=0.01} = 0.6272$).

- *Pareto distribution* $\Pi(x_m, \alpha)$*:* $\quad x_m > 0,\ \alpha > 0,\ x \geq x_m$
  - *density* $f(x) = \frac{\alpha x_m^\alpha}{x^{\alpha+1}}$
- *Poisson distribution* $\mathcal{P}(\lambda)$ *:* $\quad \lambda > 0,\ x \in \mathbb{N}$
  - *density* $f(x) = \frac{\lambda^x}{x!}\exp(-\lambda)$
    note: $x_i \sim \mathcal{P}(\tilde{\lambda}) \implies \sum_{i=1}^n x_i \sim \mathcal{P}(\lambda = n\tilde{\lambda})$
- χ_k^2 *distribution with* $k > 0$ *degrees of freedom:* $\quad x \geq 0$
  $x_i \sim \mathcal{N}(0,1) \implies \sum_{i=1}^k x_i^2 \sim \chi_k^2$
- *Rayleigh distribution* $\mathcal{R}(\sigma^2)$*:* $\quad \sigma > 0,\ x \geq 0$
  - *density* $f(x;\sigma^2) = \frac{x}{\sigma^2}\exp\left(-\frac{x^2}{2\sigma^2}\right)$
  - *mean* $\mu = \sigma\sqrt{\frac{\pi}{2}}$
  - *variance* $\nu^2 = \frac{4-\pi}{2}\sigma^2$
- *Gamma distribution* $\mathcal{G}(k, \theta)$*:* $\quad k > 0,\ \theta > 0,\ x > 0$
  - *density* $f(x;k,\theta) = \frac{1}{\Gamma(k)\theta^k}x^{k-1}\exp\left(-\frac{x}{\theta}\right)$
  - *mean* $\mu = k\theta$
  - *variance* $\nu^2 = k\theta^2$
- χ_k^2 *distribution with* $k > 0$ *degrees of freedom:* $\quad x \geq 0$
  $x_i \sim \mathcal{N}(0,1) \implies \sum_{i=1}^k x_i^2 \sim \chi_k^2$
- *Student's t-distribution* $\mathcal{T}_k$ *with* $k > 0$ *degrees of freedom:*
  $U \sim \mathcal{N}(0,1), \quad V \sim \chi_n^2 \implies \frac{U}{\sqrt{V/n}} \sim \mathcal{T}_n$