Qualitative Reasoning Based on Fuzzy Relative Orders of Magnitude

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***Abstract*—This paper proposes a fuzzy set-based approach for handling relative orders of magnitude stated in terms of closeness and negligibility relations. At the semantic level, these relations are represented by means of fuzzy relations controlled by tolerance parameters. A set of sound inference rules, involving the tolerance parameters, is provided, in full accordance with the combination/projection principle underlying the approximate reasoning method of Zadeh. These rules ensure a local propagation of fuzzy closeness and negligibility relations. A numerical semantics is then attached to the symbolic computation process. Required properties of the tolerance parameter are investigated, in order to preserve the validity of the produced conclusions. The effect of the chaining of rules in the inference process can be controlled through the gradual deterioration of closeness and negligibility relations involved in the produced conclusions. Lastly, qualitative reasoning based on fuzzy closeness and negligibility relations is used for simplifying equations and solving them in an approximate way, as often done by engineers who reason about a mathematical model. The problem of handling qualitative probabilities in reasoning under uncertainty is also investigated in this perspective.**

***Index Terms*—Closeness, fuzzy numbers, fuzzy relations, negligibility, qualitative probabilities, relative orders of magnitude.**

I. INTRODUCTION

Qualitative Reasoning about physical systems has been a very active subfield of research in Artificial Intelligence [21][33][26][32]. Its main aim is both to address the need to deal with physical systems where some magnitudes are not easy to quantify precisely (i.e., numerical data is not available), and to be able to reason at a qualitative or symbolic level (for example, reasoning directly in terms of orders of magnitude). Significant progress towards the development of formal methods for qualitative reasoning about the behavior of physical systems has been made. The simplest formalism used in qualitative reasoning is based on the sign algebra (–, 0, +); see for instance [26][32]. Such models are sufficient to represent the sign of quantities, and how the increase or the decrease of quantities can affect other quantities. Information about magnitudes, or even relative orders of magnitude is not represented. As a consequence, the sign−based approach has too limited an expressive power in some practical cases to be widely applicable.

A major limitation often lies in the fact that the sign of the result of an operation can be determined only if the orders of magnitude of the involved parameters are known. This is why some authors (Shen and Leitch [31]; Kim and Fishwick [19,20]) have developed fuzzy set-based approaches to qualitative reasoning. Shen and Leitch [31] use a fuzzy quantity space such as {zero, small, medium, large} (for positive values). This allows for a more detailed description of the values of the variables. Such an approach relies on an approximation principle in order to express the results of calculations in terms of the fuzzy sets of the fuzzy quantity space. Kim and Fishwick [19,20] rather use a qualitative description of the system in terms of fuzzy if-then rules involving arithmetic operations. Both works are oriented towards qualitative simulation and extend ideas first proposed in qualitative reasoning systems such as Qsim [21], where the constraints are used for checking the consistency of a set of values for a set of variables, rather than propagating them for finding unknown values. Most of these works are concerned with the analysis of dynamic systems.

There exists another line of research in qualitative reasoning [30] which focuses on reasoning with *relative* orders of magnitude. The ultimate aim of the proposed approach is to build automated reasoning systems that mimic the process of simplifying and approximately solving equations from the knowledge of relative orders of magnitude of involved parameters. This type of activity corresponds to a particular form of commonsense reasoning where the ideas of closeness, comparability and negligibility are involved. The first attempt to formalize and automate order of magnitude reasoning appeared with the formal system FOG, proposed by Raiman [29,30]. FOG is based on three basic relations: "negligible with respect to" (*Ne*), "close to" (*Cl*), and "has the same sign and order of magnitude as", i.e., "is comparable to" (*Co*). FOG includes one axiom and 31 inference rules, which allow for deduction from pieces of knowledge expressed in terms of relative orders of magnitude. This set of rules was proved to be consistent by giving an interpretation to the three relations in the framework of Non-Standard Analysis. FOG handles relative orders of magnitude through a purely symbolic computation process. However, no numerical interpretation is provided for the symbolic computation of orders of magnitude in the FOG's approach. Neither does it allow for the use of numerical values, as further discussed in section 2.

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This drawback is remedied in the system O(M) proposed by Mavrovouniotis and Stephanopoulos [23,24], where the primitive relations ("much smaller than", "moderately smaller than", "slightly smaller than", "exactly equal", "slightly larger than", "moderately larger than", "much larger than") are interpreted as locating the quotient of the two compared quantities within some prescribed intervals. All these intervals are disjoint, and defined via a unique parameter which is determined by the application context. The fact that the value of this parameter is fixed creates some problems since the result of the (repeated) composition of the primitive relations cannot always be expressed by one of the primitive relations, in the sense of the interval semantics. Indeed for instance, if $x / y \in [a, b]$ and $y / z \in [a, b]$, with $a > 0$, then $x / z \in [a^2, b^2]$, and $[a^2, b^2]$, or more generally $[a^n, b^n]$, are not necessarily included into one of the prescribed intervals corresponding to the semantics of one of the primitive relations. Dague [5] also provides a similar attempt to give a numerical interpretation to FOG-like relations manipulated in a formal way.

Neither FOG nor O(M) take into account the fact that the satisfaction of relations such as *Ne*, *Cl*, or "*much larger than*", by two numerical values is often a matter of degree. Moreover FOG does not acknowledge either that a relation of closeness (resp. negligibility) is usually transitive in a weak (resp. strong) sense (to be made precise in the following), rather than just transitive [11]. The same type of limitations applies to O(M) as well. Modeling relations *Ne*, *Cl* and *Co* by means of fuzzy relations provides an appropriate framework for addressing these issues. In this study, "*x is close to y*" is interpreted as "*x/y is close to 1*", and "*x is negligible w.r.t. y*" can be either interpreted as "*y + x is close to y*", or by "*y − x is close to y*". An extended approach to reasoning with relative orders of magnitude expressed in terms of negligibility and closeness relations is proposed, based on fuzzy relations. Fuzzy numbers play the role of tolerance parameters controlling the semantics of *Ne* and *Cl*. They are manipulated at the symbolic level but can be interpreted in numerical terms. This approach maintains a numerical semantics for the symbolic calculus performed by the FOG's rules. The obtained conclusions are sound approximations of the actual results and do not resort to linguistic approximation as in [31]. The validity of the inference process is checked by quantifying the errors made at each step.

The paper unifies, discusses and substantially develops the contents of previous papers [6,7,8]. More particularly, it provides a thorough analysis of the properties required for the fuzzy relations modeling closeness and negligibility, a more systematic and structured study of the inference rules involving these relations and an extensive illustration of the approach on various examples. The paper is organized as follows. The next Section gives the necessary background on the formal system FOG. This Section also serves as a motivation for the use of fuzzy sets in qualitative reasoning about relative orders of magnitude. In Section 3, we address the problem of modeling the basic relations in FOG by means of fuzzy relations expressed in terms of ratios of values. Then, we show that the composition of the fuzzy relations representing the order of magnitude relations *Ne* and *Cl*, reduces to simple arithmetic computations on the fuzzy numbers which express the semantics of the relations. A minimal set of inference rules, based on relations *Ne* and *Cl*, is established in Section 4. Simplified rules using symmetric fuzzy numbers as tolerance parameters for the relations appearing in conclusion parts, are discussed in Section 5. These rules are still valid ones, but may provide (slightly) more imprecise conclusions. Section 6 shows how qualitative reasoning based on fuzzy relations expressing closeness and negligibility can be used for simplifying equations, with a view to solving them approximately. Section 7 deals with incoherence detection in the set of equations and relations, if any. Lastly, an application to qualitative probabilistic reasoning is outlined; namely, a probabilistic-like default logic with an infinitesimal semantics due to Adams [1] is reconsidered using fuzzy closeness relations modeling probability values "close to 1" in the usual sense. Due to space limitations, not all the proofs of the results are included. They are available in a report [17]. Some examples of proofs are given to show the underlying ideas. Missing proofs are similar to them.

II. The Formal System FOG

The FOG system [29] is based on three relations, which capture closeness, negligibility and comparability and which are respectively denoted by *Cl*, *Ne* and *Co*. These three operators are:

x Ne y, which stands for "*x is negligible with respect to y*";
x Cl y, which stands for "*x is close to y*", and understood as "*(x− y) is negligible with respect to y*";
x Co y, which stands for "*x has the same sign and order of magnitude as y*". The underlying idea is that if *x Ne z* and *x Co y* then *y Ne z*.

FOG has one axiom (*A1: x Cl x*) and 31 inference rules. Hereafter, we give some of these rules (*[x]* denotes the sign (+, −, *0)* of the quantity *x*):

1) $x\ Cl\ y \Rightarrow y\ Cl\ x$	2) $x\ Cl\ y \Rightarrow x\ Co\ y$
3) $x\ Cl\ y,\ y\ Cl\ z \Rightarrow x\ Cl\ z$	4) $x\ Cl\ y,\ y\ Ne\ z \Rightarrow x\ Ne\ z$
5) $x\ Co\ y,\ y\ Co\ z \Rightarrow x\ Co\ z$	6) $x\ Ne\ y,\ y\ Ne\ z \Rightarrow x\ Ne\ z$
7) $x\ Co\ y,\ y\ Cl\ z \Rightarrow x\ Co\ z$	8) $x\ Ne\ y \Rightarrow (x+y)\ Cl\ y$
9) $(x+y)\ Cl\ z,\ y\ Ne\ x \Rightarrow x\ Cl\ z$	10) $x\ Ne\ y,\ z\ Cl\ t \Rightarrow x\cdot z\ Ne\ y\cdot t$
11) $x\ Ne\ y,\ z\ Ne\ t \Rightarrow x\cdot z\ Ne\ y\cdot t$	12) $x\cdot y\ Cl\ z\cdot t,\ x\ Cl\ z,\ [x]\neq 0 \Rightarrow y\ Cl\ t$
13) $x\cdot y\ Cl\ z\cdot t,\ x\ Ne\ z,\ [z]\neq 0,\ [x]\neq 0 \Rightarrow t\ Ne\ y$	14) $x\cdot y\ Ne\ z\cdot t,\ z\ Ne\ x, \Rightarrow y\ Ne\ t.$

Note that both *Cl* and *Co* are equivalence relations (i.e., reflexive, symmetric and transitive), *Co* being coarser than *Cl*. The FOG rules can be justified from the point of view of Non-Standard Analysis. They describe how the three relations work together and allow for the propagation of initial qualitative information stated in terms of relations *Ne, Cl* and *Co*. Most of the rules have a clear intuitive interpretation and need no explanation. Note that they are not independent (e.g., rules 2 and 5 $\Rightarrow$ rule 7). FOG has been used successfully in the DEDALE system of analog circuit diagnosis [4] for reasoning about current intensities which are not directly observable, and in macroeconomics [3]. Nevertheless, FOG has several limitations, as discussed in the introduction:

• No standard numerical interpretation of the three relations *Ne, Cl* and *Co* is provided. It may create interface problems with genuine numerical values. Namely, quantitative information cannot be exploited by the method. However, such information is often available and useful when solving engineering problems. FOG cannot use it because it does not relate orders of magnitude relations to actual orders of magnitude.

• FOG does not take into account the fact that the extent to which two numerical values satisfy the relations *Ne, Cl* and *Co* is often a matter of degree and is also context-dependent (for example, the relation "negligible" does not mean the same thing in a preliminary-design context and a detailed-design context). FOG is not capable of expressing a gradual change from one order of magnitude to another in the computation process, due to the absence of gradual transitions between orders of magnitude [5].

• FOG cannot express that some relations are "weakly", or "strongly", transitive rather than just transitive in the usual sense. For instance, if "*x is close to y*" and "*y is close to z*", it is not so true that "*x is close to y*" in any case (at least with a too restrictive meaning of "close"). This exemplifies the weakening of transitivity. In order to avoid undesirable transitivity effects and obtain results at least approximately valid with respect to a numerical semantics, Raiman [29] is obliged to introduce arbitrary limitations on the chaining of rules (by means of control techniques).

The use of fuzzy relations for modeling relations *Ne, Cl* and *Co* can address the above problems. Especially, the symbolic calculus of orders of magnitude in FOG can be equipped with flexible, but standard numerical semantics.

III. Handling Relative Orders of Magnitude by Means of Fuzzy Relations

Two points of view can be considered when comparing numbers, hence orders of magnitude *a* and *b* on the real line. We can evaluate to what extent the *difference* $a - b$ is large or small (in absolute value); this is the absolute comparative approach. Or, we may use relative orders of magnitude, as often done in engineering calculations (see for example [23,24]), i.e., we evaluate to what extent the *ratio* a/b is close to *1* or not. In the former view, approximate equality relations as well as more or less strong inequality relations, can be modeled by fuzzy relations *R* of the form $\mu_R(x, y) = \mu_P(x - y)$, which only depends on the value of the difference $x - y$, and where *P* is an appropriate fuzzy interval restricting $x - y$. This kind of fuzzy relations can offer a useful setting for representing and processing temporal knowledge [12], since temporal landmarks are absolute notions. This paper rather focuses on the modeling of closeness and negligibility based on ratios. Then approximate equality E is modeled by fuzzy relations $\mu_E(x, y)=\mu_M(x/y)$ where *M* is a fuzzy number representing "*close to 1*".

A. Modeling Closeness and Negligibility

The idea of relative closeness (*Cl*) expresses the approximate equality between two real numbers *x* and *y* and can be captured by the following [11]:

Definition 3.1 (Closeness): The closeness relation (*Cl*) is a reflexive and symmetric fuzzy relation such that

$$\mu_{Cl}(x, y) = \mu_M(x/y), \tag{1}$$

where μ_M is the characteristic function of a fuzzy number close to *1*, such that $\mu_M(1) = 1$ (since *x* is close to *x*), and $\mu_M(t) = 0$ if $t \le 0$. *M* is called a *tolerance parameter.*

The closeness relation (*Cl*) parameterized by *M* is denoted *Cl[M]*. The last condition assumes that two numbers that are close should have the same sign. This is natural in a ratio-based view since division by 0 is undefined. It does not lead to serious limitations in the expressiveness of qualitative relative orders of magnitude. Indeed, the statement that a variable is close to 0 (with an undefined sign) is an *absolute* statement; moreover, difference-based relations would enable us to express that *x* and *y* are close without having the same sign necessarily, but would have other limitations. Closeness is naturally symmetric i.e., $\mu_{Cl}(x, y) = \mu_{Cl}(y, x)$. It implies that *M* should satisfy:

$$\mu_M(t) = \mu_M(1/t). \tag{2}$$

So, the support $\underline{M} = \{t, \mu_M(t) > 0\}$ of the fuzzy number *M* is of the form $[1-\varepsilon,\ 1/(1-\varepsilon)]$, with $\varepsilon \in [0, 1[$. Closeness becomes strict equality for $M = \mathbf{1}$, i.e., $\mu_{\mathbf{1}}(x/y) = 1$ if $x = y$ and $\mu_{\mathbf{1}}(x/y) = 0$ otherwise. Symmetry is preserved under the idempotent fuzzy set intersection:

Proposition 3.1: If two fuzzy numbers *M* and *N* verify the symmetry property (2) then $M \cap N$ is symmetric.

Proof: Let us use α-cuts (i.e., $M_\alpha = \{t,\ \mu_M(t) \ge \alpha\}$). The symmetry of M and N is equivalent to $M_\alpha = [m_\alpha\ ,\ 1/m_\alpha]$ and $N_\alpha = [n_\alpha\ ,\ 1/n_\alpha]$ for all $\alpha > 0$. We have

$$\begin{aligned}(M \cap N)_\alpha = M_\alpha \cap N_\alpha &= [m_\alpha\ ,\ 1/m_\alpha] \cap [n_\alpha\ ,\ 1/n_\alpha] \\ &= [\max(m_\alpha\ ,\ n_\alpha),\ \ \min(1/m_\alpha\ ,\ 1/n_\alpha)] \\ &= [\max(m_\alpha\ ,\ n_\alpha),\ \ 1/\max(m_\alpha\ ,\ n_\alpha)].\end{aligned}$$

This means that $M \cap N$ is symmetric. ♦

Note that if the membership functions μ_M and μ_N obey (2) and are linear-shaped on $[0, 1]$ or on $[1, +\infty)$, then either $M \subseteq N$ or $N \subseteq M$. Then, $M \cap N = M$ or N.

Let A be a fuzzy number with modal value 1 ($\mu_A(1) = 1$), and $[1-\lambda, 1/(1-\rho)]$ its support, λ and ρ lying in [0, 1[. The imprecision of a fuzzy number can be roughly evaluated by the width of the support, i.e., $(\lambda+\rho-\lambda\rho)/(1-\rho)$. However this absolute evaluation is not appropriate when the fuzzy number is used to model relative closeness. Here, we are interested in the *relative* imprecision (with respect to 1), rather than in absolute imprecision. Then, the level of imprecision of the fuzzy number A below 1 should be the same as the level of imprecision of the fuzzy number $1/A$ above 1. The relative imprecision of A is evaluated as follows. First, the relative imprecision of the left-hand side $[1-\lambda, 1]$ of the support is equal to λ. Moreover, the relative imprecision of the right hand side $[1, 1/(1-\rho)]$ of the support is the same as the relative imprecision of $[1-\rho, 1]$ and is ρ. The evaluations of $[1-\lambda, 1]$ and $[1, 1/(1-\rho)]$ thus belong to the same scale $[0, 1]$. Now, the relative imprecision of A can be defined as $max(\lambda, \rho)$.

Now, we define negligibility from closeness. Consistently with FOG [29], they can be related in the following way: "*x is negligible with respect to y*" if and only if "*x + y is close to y*" (see Rule 8 in section 2). It leads to define a fuzzy negligibility relation as follows,

Definition 3.2 (Negligibility): Negligibility is a fuzzy relation defined from a closeness relation *Cl* based on a fuzzy number *M* via the following:

$$\mu_{Ne}(x, y) = \mu_{Cl}(x+y, y) = \mu_M((x+y) / y). \qquad (3)$$

The use of a ratio seems to be more natural than a difference for modeling the idea of negligibility. An approximate equality of the form $\mu_E(x, y) = \mu_P(x-y)$ for modeling closeness would lead to a definition of "*negligible*" which would not depend on y (since $(x + y) - y = x$). In view of definition 3.2, 0 is always negligible with respect to any $y \neq 0$; indeed $\mu_{Ne}(x, y) = \mu_M(1 + x/y) = 1$ if $x = 0$, $\forall y$.

In the following, $(x, y) \in Cl[M]$ expresses that the pairs of values (x, y) are flexibly restricted by the fuzzy set $Cl[M]$, and likewise for $(x, y) \in Ne[M]$.

Proposition 3.2: Let $Ne[M]$ be the negligibility relation parameterized by M and defined in (3). Then, the following equivalence holds

$$(a, b) \in Ne[M] \Leftrightarrow (-a, b) \in Ne[2ÁM].$$

Proof: We have

$$\mu_{Ne[M]}(x, y) = \mu_M(1 + x/y)$$
$$= \mu_M(2 - (1 - x/y)) = \mu_{2ÁM}(1 + (-x)/y),$$

since $\mu_{f(M)}(t) = \mu_M(f^{-1}(t))$ using the extension principle [34] for a one-to-one mapping f. The symbol Á denotes the extended subtraction of fuzzy numbers [10]. Hence $\mu_{Ne[M]}(x, y) = \mu_{Ne[2ÁM]}(-x, y)$. This means that $(-a, b) \in Ne[2ÁM]$. ♦

Closeness and negligibility could thus as well be related in the following manner: "*x is negligible w.r.t. y*" if and only if "*y − x is close to y*". This leads to the following representation of negligibility:

$$\mu_{NE}(x, y) = \mu_{Cl}(y - x, y) = \mu_M((y - x)/y). \qquad (4)$$

Denote by *NE* (resp. *Ne*) the negligibility relation defined according to (4) (resp. (3)). Obviously, from definitions (3) and (4), we obtain:

Proposition 3.3: $(a, b) \in Ne[M]$ if and only if $(a, b) \in NE[2ÁM]$.

Let us compare the fuzzy numbers M and $2ÁM$ from the point of view of their imprecision. The support of M (resp. $2ÁM$) is $[1-\varepsilon, 1/(1-\varepsilon)]$ (resp. $[(1-2\varepsilon)/(1-\varepsilon), 1+\varepsilon]$). It is easy to check that $2ÁM$ is relatively more imprecise than M, although they have the same level of absolute imprecision. Even if M is globally less imprecise than $2ÁM$, observe that if we only focus on values in $[1, +\infty)$, $M \cap [1, +\infty) \supseteq (2ÁM) \cap [1, +\infty)$, while the other inclusion $M \cap [0, 1] \subseteq (2ÁM) \cap [0, 1]$ holds on $[0, 1]$. Then, M is more precise than $2ÁM$ on one side of 1, while the converse holds on the other side. But although $M = 1/M$ (i.e., M is symmetric), $2ÁM$ does not preserves this symmetry. So, the equality between M and $2ÁM$ can never hold (except for $M = \mathbf{1}$, which corresponds to the strict equality). Thus, it is impossible to semantically interpret *Ne* and *NE* by the same fuzzy number. However *Ne* and *NE* are not conceptually very different, and, in practice, it is not always clear if the user refers to *Ne* or to *NE* when expressing negligibility statements. Our approach will try to cope with this situation, especially in section 5 with the use of inference rules having symmetric fuzzy parameters, for which there is no longer any significant difference between *Ne* and *NE*.

Besides, the following monotonicity property, useful in the following, holds as a direct consequence of the definition of fuzzy sets inclusion and intersection:

Proposition 3.4: Let M and N be two fuzzy numbers, and R be relation *Cl* or *Ne*. Then

i) If $M \subseteq N$ and $(a, b) \in R[M]$ then $(a, b) \in R[N]$.
ii) If $(a, b) \in R[M]$ and $(a, b) \in R[N]$
then $(a, b) \in R[M \cap N]$.

Finally, the fuzzy counterpart of the comparability relation *Co* can be defined by

$$\mu_{Co}(x, y) = 1 - max\,(\mu_{Ne}(x, y), \mu_{Ne}(y, x)),$$

since $(a, b) \in Co \Leftrightarrow \neg((a, b) \in Ne \text{ or } (b, a) \in Ne))$, see [29].

However, we shall not explicitly use *Co* in this paper since it is not a basic relation like closeness (*Cl*) and negligibility (*Ne*).

B. Additional Requirements on the Tolerance Parameter

The fuzzy number *M* which parameterizes closeness and negligibility should satisfy more constraints, not only those in (2) and definition 3.1:

- $\mu_{Ne}(x, y) = 0$ for $x = y$, since *x* is not negligible w.r.t. itself. This is equivalent to *enforcing* $\mu_M(2) = 0$, according to (3). Since *M* is symmetric, $\mu_M(1/2) = 0$ also holds. Thus, the support of *M* should be included in $[1/2, 2]$. In other terms, any α-cut of *M* is of the form $[1-\varepsilon, 1/(1-\varepsilon)]$ with $\varepsilon \in [0, 1/2]$.
- It is also natural to require that if *x* is close to *y*, then neither is *x* negligible w.r.t. *y*, nor is *y* negligible w.r.t. *x*. This can be expressed by the following inequality

$$\mu_{Cl}(x, y) \le 1 - \max(\mu_{Ne}(x, y), \mu_{Ne}(y, x)), \quad (5)$$

which corresponds to the fuzzy set inclusion $Cl \subseteq Co$ (where *Co* is the fuzzy comparability relation defined above). The requirement in (5) leads to

Proposition 3.5: Condition (5) forces the support of *M* to lie in the interval $[(\sqrt{5}-1)/2, (\sqrt{5}+1)/2]$.

See the proof in the appendix. It means that each α-level cut of M is of the form $[1-\varepsilon, 1/(1-\varepsilon)]$ with $\varepsilon \in [0, (3-\sqrt{5})/2]$ $(\cong [0, 0.38])$. The interval $[(\sqrt{5}-1)/2, (\sqrt{5}+1)/2]$ $(\cong [0.61, 1.61])$ will be called the *validity interval* and denoted *V*. Later in section 3.4, we shall see another reason to have the support of the tolerance parameter in this interval.

C. Composition of the Basic Relations and Transitivity Issues

First recall that if *R* is a fuzzy relation defined on $X\times Y$ and *S* a fuzzy relation on $Y\times Z$, then the composition of *R* and *S* is a fuzzy relation on $X\times Z$ denoted by $R\circ S$ and defined by [35]:

$$\mu_{R\circ S}(x, z) = \sup_{y\in Y} \min(\mu_R(x, y), \mu_S(y, z)).$$

It has been pointed out in [7] that performing the composition of fuzzy relations whose membership functions depend only on *x/y*, usually reduces to fuzzy arithmetic operations on fuzzy numbers. For instance, the composition of fuzzy relations Cl reads

$$\begin{aligned}\mu_{Cl[M]\circ Cl[N]}(x, z) &= \sup_y \min(\mu_{Cl[M]}(x, y), \mu_{Cl[N]}(y, z))\\ &= \sup_y \min(\mu_M(x/y), \mu_N(y/z))\\ &= \mu_{M\otimes N}(x/z), \text{ observing that } (x/y)\cdot(y/z) = x/z\\ &= \mu_{Cl[M\otimes N]}(x, z),\end{aligned}$$

where ⊗ denotes the product extended to fuzzy numbers, according to extension principle [34][10]:

$$\mu_{M\otimes N}(u) = \sup_{s,t:\ u=s\cdot t} \min(\mu_M(s), \mu_N(t)).$$

Thus $Cl[M]\circ Cl[N] = Cl[M\otimes N]$. In the following, we shall omit '⊗' when writing products.

This type of computation is similar to what is done in approximate reasoning [36]. Note that the '*sup-min*' method is also called "*combination/projection*" since, the pieces of information are combined with '*min*' operation and the result of the combination is projected via the '*sup*' in agreement with possibility theory [10]. It yields the most restrictive fuzzy relation (in the sense of fuzzy set inclusion) between the variables on which the projection takes place, as deduced from the available information.

Proposition 3.6: Using the principle of fuzzy relation composition, the following entailments can be semantically justified

$$(a, b) \in Cl[M] \text{ and } (b, c) \in Cl[N \Rightarrow (a, c) \in Cl[MN] \quad (6)$$

$$(a, b) \in Ne[M] \text{ and } (b, c) \in Ne[N] \Rightarrow (a, c) \in Ne[(MÁ1)(NÁ1)\oplus 1] \quad (7)$$

$$(a, b) \in Cl[M] \text{ and } (b, c) \in Ne[N] \Rightarrow (a, c) \in Ne[M(NÁ1)\oplus 1]. \quad (8)$$

Proof: See above the proof of (6). Proofs of (7) and (8) are similar. Let us only consider the proof of (8). The composition of fuzzy relations Cl and Ne reads

$$\begin{aligned}&\sup_y \min(\mu_{Cl[M]}(x, y), \mu_{Ne[N]}(y, z))\\ &= \sup_y \min(\mu_M(x/y), \mu_N(1 + y/z))\\ &= \sup_{u,v:\ u(v-1)+1=1+x/z} \min(\mu_M(u), \mu_N(v)),\\ &= \mu_{M(NÁ1)\oplus 1}(1 + x/z) = \mu_{Ne[M(NÁ1)\oplus 1]}(x, z),\end{aligned}$$

where $u = x/y$ and $v = 1 + y/z$, and ⊕ denotes the addition of fuzzy numbers [10]. So, $(a, c) \in Ne[M(NÁ1)\oplus 1]$.

The entailments (6) to (8) are the fuzzy counterparts of the following inference rules proposed by Raiman in FOG [29]:

$$(a, b) \in Cl \text{ and } (b, c) \in Cl \Rightarrow (a, c) \in Cl \quad (9)$$

$$(a, b) \in Ne \text{ and } (b, c) \in Ne \Rightarrow (a, c) \in Ne \quad (10)$$

$$(a, b) \in Cl \text{ and } (b, c) \in Ne \Rightarrow (a, c) \in Ne. \quad (11)$$

However, the rules (9) to (11) are only syntactic approximations of (6) to (8). Indeed we have:

Proposition 3.7: Assume that $M \subseteq N$ without loss of generality

$$Cl[M] \subseteq Cl[N] \subseteq Cl[MN], \quad (12)$$

$$Ne[N] \supseteq Ne[M] \supseteq Ne[(MÁ1)(NÁ1)\oplus 1], \quad (13)$$

$$Ne[N] \subseteq Ne[M(NÁ1)\oplus 1]. \quad (14)$$

Proof: See the appendix.

Due to (12), (6) expresses that the relation *Cl* is "*weakly transitive*" rather than just transitive in the usual sense, i.e., *a* may be less close to *c* than *a* is close to *b* and than *b* is close to *c*. By contrast, as shown by (7) and (13), the relation *Ne* is "*strongly transitive*" rather than just transitive in the usual sense, i.e., *a* is more negligible w.r.t. *c* than *a* w.r.t. *b*, and *b* w.r.t. *c*.

The symbolic rules established in proposition 3.6 lead to a formal manipulation of relations *Cl* and *Ne*, in agreement with the fuzzy semantics. The tolerance parameters of the deduced relations are syntactically computed from those underlying the relations appearing in condition parts. However, the computed tolerance parameters can be numerically interpreted in a convenient way. Unfortunately, the produced relations are not always parameterized by symmetric fuzzy numbers (e.g., in rules (7) and (8), the fuzzy numbers $(MÁ1)(NÁ1)\oplus 1$ and $M(NÁ1)\oplus 1$ are not symmetric in the sense of (2)). The problem of the approximating such relations using symmetric fuzzy numbers, will be addressed in section 5.

D. Controlling the Iteration of the Inference Process

It is worth noticing that some of inference rules (in proposition 3.6) lead to relations in which the tolerance parameters are less tight than the original ones. For instance, rule in (6) enables to deduce that $(a, c) \in Cl[MN]$ where $M \subseteq MN$ and $N \subseteq MN$. Then, when iterating the inference process, it may happen that some of the provided results become too imprecise and then the semantics in terms of closeness and negligibility of the symbolically computed relations are lost. Indeed, the semantic requirements of section 3.2 for the supports of the parameters (i.e., they should lie in a validity interval such as $[(\sqrt{5}-1)/2, (\sqrt{5}+1)/2])$, are not automatically preserved. In practice, in order to cope with this problem, we must start with sufficiently tight closeness relations so that the support of the parameter remains in the validity interval across inference steps. The index of relative imprecision, introduced in section 3.1, can be used to measure the possible degradation of the idea of closeness (and negligibility) in the inference process. When the semantic requirements are violated, the inference process must stop. More generally, we can compute the α-level cuts (of the tolerance parameter of a result) that remain included in the validity interval; then this may be the basis for estimating the level of validity of this result: The higher the level of the largest valid α-level cut, the less valid the result (the result being fully valid when the support is valid).

IV. A Fuzzy Order of Magnitude Inference System

As shown by examples (6) to (8), it is possible to develop inference rules where the closeness and negligibility relations are manipulated at a symbolic level and the fuzzy numbers which parameterize them are numerically computed. A set of symbolic inference rules, based on the order of magnitude relations *Cl* and *Ne*, has been proposed and justified in [7]. This rule set has been simplified by deleting redundant rules [16][6,8]. The following basic set of rules can be obtained:

• Remarkable properties of *Cl*

R_1 : $a \le b$ and $c \le d$ and $a/b \le c/d$ and $(a, b) \in Cl[M]$
$\Rightarrow (c, d) \in Cl[M]$ *(Cl-Weakening)*

R_2 : $(a, b) \in Cl[M] \Leftrightarrow (b, a) \in Cl[M]$ *(Symmetry)*

R_3 : $(a, b) \in Cl[M]$ and $(c, d) \in Cl[N]$
$\Rightarrow (a\cdot c, b\cdot d) \in Cl[MN]$ *(Cl-Product Stability)*

• Relationships between *Ne* and *Cl*

R_4 : $(a, b) \in Ne[M] \Leftrightarrow (a+b, b) \in Cl[M]$ *(Interdefinition)*

R_5 : $(a, b) \in Cl[M]$
$\Leftrightarrow (max(a, b) - min(a, b), min(a, b)) \in Ne[M]$ *(Interdefinition)*

• Remarkable properties of *Ne*

R_6 : $(a, b) \in Ne[M] \Leftrightarrow (-a, b) \in Ne[2ÁM]$ *(Sign Change)*

R_7 : if $0 \le a/b \le 1$ and $(b, c) \in Ne[M]$
$\Rightarrow (a, c) \in Ne[M]$ *(Ne-Left Reinforcement)*

R_8 : if $(a, b) \in Ne[M]$ and $0 \le b/c \le 1$
$\Rightarrow (a, c) \in Ne[M]$ *(Ne-Right Reinforcemen)*

R_9 : $(a, a+b) \in Ne[M] \Rightarrow (a, b) \in Ne[M^2]$ *(Right Ne-Simplification)*

R_{10} : $(a, b) \in Ne[M]$ and $(c, d) \in Ne[N]$
$\Rightarrow (a\cdot c, b\cdot d) \in Ne[(MÁ1)(NÁ1)\oplus 1]$ *(Ne-Product Stability)*

• Composition of relations *Cl* and *Ne*

R_{11} : $(a, b) \in Cl[M]$ and $(b, c) \in Ne[N]$
$\Rightarrow (a, c) \in Ne[M(NÁ1)\oplus 1]$ *(Cl-Ne Cut)*

R_{12} : $(a, b) \in Cl[M]$ and $(c, a) \in Ne[N]$
$\Rightarrow (c, b) \in Ne[M(NÁ1)\oplus 1]$ *(Ne-Cl Cut)*

• Properties of *Cl* and *Ne* with respect to addition

R_{13} : $(a, c) \in Cl[M]$ and $(b, c) \in Ne[N]$
$\Rightarrow (a+b, c) \in Cl[M\oplus NÁ1]$ *(Stability w.r.t. Summation)*

R_{14} : $(a+b, c) \in Cl[M]$ and $(b, c) \in Cl[N]$
$\Rightarrow (a, c) \in Ne[MÁN\oplus 1]$ *(Summation Cut)*

R_{15} : $(a, c) \in Ne[M]$ and $(b, a) \in Ne[N]$
$\Rightarrow (a+b, c) \in Ne[(MÁ1)N\oplus 1]$ *(Ne-Sum Stability)*

• Properties of Cl and Ne with respect to product

R_{16} : $(a, b) \in Cl[M]$ and $(c, d) \in Ne[N]$
$\Rightarrow (a\cdot c, b\cdot d) \in Ne[M(NÁ1)\oplus 1]$. *(Cl-Ne Product Stability)*

In the appendix the proof of rule R_9 is given as an illustration. Proofs of all other rules can be found in [17]. Except for rule R_9, all rules enable to provide the most restrictive relations since they can be justified by a direct application of the combination/projection method [36]. Some of these rules lead to relations parameterized by non-symmetric fuzzy numbers, e.g., rules R_6 and R_{10}. By contrast, rules R_3 and R_9 preserve the symmetry property of the original tolerance parameters. Besides, a partial "converse" for rule R_9, namely rule $(a, b) \in Ne[M] \Rightarrow (a, a+b) \in Ne[M]$, can be established only if *a* and *b* are of the same sign, since we have $0 \le b/(a+b) \le 1$. Then, by rule R_8 we deduce that $(a, a+b)$

$\in Ne[M]$. Otherwise if a and b have different signs, we have $1 > 1+a/b > 1+a/(a+b)$ and then $\mu_M(1+a/b) > \mu_M(1+a/(a+b))$; the rule does not hold in the converse direction.

It should be emphasized that these rules are the fuzzy counterparts of the FOG's rules. Moreover, they provide precise numerical semantics for the symbolic calculus of orders of magnitude performed by FOG system, since enabling the local propagation of fuzzy numerical constraints expressed in terms of *Cl* and *Ne*. The latter point enables the conclusions of the symbolic inference system to be validated. Reasoning carried out by these rules is characterized by the separation of the symbolic processing of the labels of the inferred relations from expressions of their semantics in terms of membership functions.

Some semantic counterparts of rules of FOG are not available in our rule base. For instance, rule 9 of FOG (see section 2), does not correspond to any rule in this base. This is because this FOG rule is redundant (since FOG rules are not independent, as already mentioned), while our rules R_1 to R_{16} are independent. But it is possible to obtain the fuzzy counterpart of rule 9, namely $(a+b, c) \in Cl[M]$ and $(b, a) \in Ne[N] \Rightarrow (a, c) \in Cl[MN]$, by chaining rules R_4, R_2 and R_3, as shown hereafter. Now, several rules (i.e., rules R_1, R_7 to R_9 and R_{12} to R_{15}) have no syntactic counterparts in FOG. It means that our rule base is more complete than the one of FOG. In practice, when building an efficient symbolic reasoning system, it may be interesting to exploit derived rules even if they are redundant. Useful rules can indeed be syntactically derived by the inference system.

Examples of Derived Rules:

• R_{17}: *If* $c \neq 0$, $(a\cdot c, b\cdot c) \in Ne[M] \Leftrightarrow (a, b) \in Ne[M]$

(multiplicative cancellation for Ne).

It is a direct consequence of R_4 and R_3 noticing that $c = d$ can be written $(c, d) \in Cl[\mathbf{1}]$ where $\mu_{\mathbf{1}}(x) = 0$ if $x \neq 1$ and $\mu_{\mathbf{1}}(1) = 1$.

• R_{18}: *If* $c \neq 0$, $(a\cdot c, b\cdot c) \in Cl[M] \Leftrightarrow (a, b) \in Cl[M]$

(multiplicative cancellation for Cl).

It is a direct consequence of R_3 noticing also that $(c, c) \in Cl[\mathbf{1}]$.

• The weak transitivity of *Cl*

R_{19}: $(a, b) \in Cl[M]$ *and* $(b, d) \in Cl[N]$
$\Rightarrow (a, d) \in Cl[MN]$

is derived from R_3. Namely, apply R_3 for $c = b$. Then, apply the cancellation rule R_{18}.

• The strong transitivity of *Ne*

R_{20}: $(a, b) \in Ne[M]$ *and* $(b, d) \in Ne[N]$
$\Rightarrow (a, d) \in Ne[(M\text{Á}1)(N\text{Á}1)\oplus 1]$

is derived from R_{10}. Namely apply R_{10} for $c = b$. Then, apply cancellation using R_{17}.

• The summation rule for the *negligibility* relation

R_{21}: $(a, c) \in Ne[M]$ *and* $(b, c) \in Ne[N]$
$\Rightarrow (a+b, c) \in Ne[M\oplus N\text{Á}1]$

is a consequence of R_{13}, substituting $a + c$ to a in the first condition, and then R_4.

• The simplification rule for *closeness* with respect to addition

R_{22}: $(a+b, c) \in Cl[M]$ *and* $(b, a) \in Ne[N]$
$\Rightarrow (a, c) \in Cl[MN]$

is a consequence of R_4, R_2 and R_{19}.

Remark: It is worth noticing that the behavior of the negligibility relation *NE* (defined in (4)) essentially parallels the one of *Ne*. Inference rules based on *NE* and *Cl* can be syntactically derived from rules R_1 to R_{16} using proposition 3.3. A detailed study of *NE* and the complete inference machinery which underlies the symbolic reasoning based on *NE* and *Cl* can be found in [17].

Starting with a knowledge base made of pieces of information expressed in terms of *Cl* and/or *Ne* relations, as follows

$$\left(f^j(a_1, \mathbf{L}, a_n), g^j(a_1, \mathbf{L}, a_n)\right) \in R_j[M_j], \quad j = 1, m$$

where f^j and g^j are functions involving arithmetic operations, R_j represents relation *Cl* or *Ne*, and M_j is the tolerance parameter underlying R_j. Assume that we are interested in finding the resulting relation between two parameters a_k and a_l with $k \neq l$. Answering this question can be envisaged by two types of computation: *local* and *global* methods.

• Local method: it corresponds to separately applying the inference rules on the available pieces of information in order to deduce the relation pertaining to the question (as done in rule-based expert systems). The procedure in this case is computationally efficient but no guarantee exists usually about the optimality in terms of precision of the provided result (as shown in the example below). Generally, different results can be obtained for the same question from different chainings of the inference rules. The conjunctive combination of partial results restricting the possible values of ratios of interest enables us to refine the conclusion. One merit of the local method is its explanation capabilities since rules triggered in the reasoning steps can be retrieved.

• Global method: It consists in applying the combination/projection technique [36] to the set of fuzzy constraints defining the problem. As already explained in section 3.3, this technique consists first in performing the combination of all pieces of information and then projecting the result on the domains of the parameters we are interested in. Thus, considering the above problem, the relation between a_k and a_l can be obtained by projecting the relation $\cap_{j=1,\ldots m} R_j$ on the cartesian product of the domains of a_k and a_l. The result provided by this method is optimally tight (in terms of precision) since all the existing constraints are exploited in the computation. Nevertheless, this approach is often computationally intractable in practice, and the explanation capabilities of the local method are lost.

Example: Let us consider the following constraints

(i) $(a+b, c) \in Cl[M]$, (ii) $(b, a) \in Ne[M]$.

The question is to compare the orders of magnitude of b and c. The two types of computation lead to the following results:

- Local computation: Only two rules need to be chained to get the results, but there are two paths.

Chaining 1:

$\{(i), (ii)\} \Rightarrow_{R_{22}} (a, c) \in Cl[M^2]$ (15)

$\{(15), (ii)\} \Rightarrow_{R_{12}} (b, c) \in Ne[M^2(MÁ1)\oplus 1].$ (16)

Chaining 2:

$\{(i), (ii)\} \Rightarrow_{R_{22}} (a, c) \in Cl[M^2]$ (17)

$\{(i), (17)\} \Rightarrow_{R_{14}} (b, c) \in Ne[MÁM^2\oplus 1].$ (18)

Now by combining (16) and (18), we obtain the following relation (due to proposition 3.4):

$(b, c) \in Ne[K],$ (19)

with $K = M^2(MÁ1)\oplus 1 \cap MÁM^2\oplus 1.$

⇔ Global computation: For this example, the combination/projection method writes

$sup_x \min(\mu_{Cl[M]}(x+y, z), \mu_{Ne[M]}(y, x))$
$= sup_x \min(\mu_M((x+y)/z), \mu_M(x/(x+y))),$ (due to (2))
$= sup_x \min(\mu_M((x+y)/z), \mu_M(1 - y/(x+y)))$
$= sup_x \min(\mu_M((x+y)/z), \mu_{(1ÁM)}(y/(x+y)))$
$= \mu_{Ne[M(1ÁM)\oplus 1]}(y, z),$

observing that $((x+y)/z)\cdot(y/(x+y)) + 1 = y/z + 1$. Then, we obtain

$(b, c) \in Ne[M(1ÁM)\oplus 1].$ (20)

The result provided by (19) is less restrictive than the one obtained by (20) since $K \supseteq M(1ÁM)\oplus 1$. The proof of this fuzzy set inclusion can be easily checked on the α-cuts. See [17] for more details. This example shows that local computations do not guarantee the optimal precision of results. However, it may happen that the results provided by the two types of computation are the same (as it is the case for example 6.1 in section 6, see also [17]).

V. Symmetric Approximations of Inference Rules

In the inference rules of section 4, some tolerance parameters underlying the relations appearing in conclusion parts are expressed in complex, hence unwieldy, symbolic forms. Moreover, as stressed in sections 3.3 and 4, obtained parameters are not symmetric fuzzy numbers (in the sense of (2)). For instance, even if M and N are both symmetric in rules R_6, R_{10}, R_{11}, R_{13} and R_{14}, the symmetry property is no longer verified for the parameters $2ÁM$, $(MÁ1)(NÁ1)\oplus 1$, $M(NÁ1)\oplus 1$, $M\oplus NÁ1$ and $MÁN\oplus 1$ respectively. Then, some difficulties arise when reasoning symbolically with these rules. For instance, we cannot use rule R_2 if the parameter of relation Cl is not symmetric. Moreover, inference rules which exploit the symmetry of Cl, may lead to undesirable and unexpected results when chaining them.

In practice, to make the chaining of the inference rules more valid, symmetric approximations of rules R_6, and R_{10} to R_{16} are defined. These new rules are less precise but the tolerance parameters underlying the relations appearing in conclusion parts remain symmetric. The results provided by these rules are sound, but not locally complete. In the following, the symmetric approximation of a rule R_n is denoted (R_n). For this purpose, we need the following lemma:

Lemma 5.1: The following fuzzy sets inclusions hold:

$2ÁM \subseteq M^2,$
$(MÁ1)(NÁ1)\oplus 1 \subseteq M\cap N,$
$M(NÁ1)\oplus 1 \subseteq MN,$
$(MÁ1)N\oplus 1 \subseteq MN.$

Proof: see [17].

Now by proposition 3.4 and the above lemma, the following fuzzy inclusions hold:

$Ne[2ÁM] \subseteq Ne[M^2],$ (21)
$Ne[(MÁ1)(NÁ1)\oplus 1] \subseteq Ne[M\cap N],$ (22)
$Ne[M(NÁ1)\oplus 1] \subseteq Ne[MN],$ (23)
$Ne[(MÁ1)N\oplus 1] \subseteq Ne[MN].$ (24)

Clearly, if M and N are symmetric then, the parameter MN is symmetric too (since the product operation preserves this property).

Theorem 5.1: The rules R_6, R_{10} to R_{12} and R_{15} to R_{16} are soundly approximated by the following symmetric variants respectively:

(R_6) : $(a, b) \in Ne[M] \Rightarrow (-a, b) \in Ne[M^2],$
(R_{10}) : $(a, b) \in Ne[M]$ and $(c, d) \in Ne[N]$
$\Rightarrow (a\cdot c, b\cdot d) \in Ne[M\cap N],$
(R_{11}) : $(a, b) \in Cl[M]$ and $(b, c) \in Ne[N]$
$\Rightarrow (a, c) \in Ne[MN],$
(R_{12}) : $(a, b) \in Cl[M]$ and $(c, a) \in Ne[N]$
$\Rightarrow (c, b) \in Ne[MN],$
(R_{15}) : $(a, c) \in Ne[M]$ and $(b, a) \in Ne[N]$
$\Rightarrow (a+b, c) \in Ne[MN],$
(R_{16}) : $(a, b) \in Cl[M]$ and $(c, d) \in Ne[N]$
$\Rightarrow (a\cdot c, b\cdot d) \in Ne[MN].$

Proof: Obvious from (21-24).

Now when looking for approximations of rules R_{13} and R_{14}, it is easy to check (using α-cuts) that none the fuzzy set inclusions $M\oplus NÁ1 \subseteq MN$ and $MÁN\oplus 1 \subseteq MN$ holds. However, approximations of R_{13} and R_{14} can be obtained as follows (see [17]):

• Symmetric approximation of R_{13} : In condition part of R_{13} where $(b, c) \in Ne[N]$ appears, b and c can either have the same or different signs. By considering these two cases, we have:

If $b/c \geq 0$ then R_{13} can be approximated by

(R_{13a}) : $(a, c) \in Cl[M]$ *and* $(b, c) \in Ne[N]$
$\Rightarrow (a+b, c) \in Cl[MN]$.

If $b/c < 0$ then R_{13} can be approximated by

(R_{13b}) : $(a, c) \in Cl[M]$ *and* $(b, c) \in Ne[N]$
$\Rightarrow (a+b, c) \in Cl[M^2N]$.

• Lastly, the symmetric approximation of R_{14} can be established taking also into account the signs of *a* and *c*, in $(a+b, c) \in Cl[M]$ which appears in the condition part. Then

If $a/c \geq 0$ (resp. $a/c < 0$ and $b/c \leq 1$) then R_{14} can be approximated by

(R_{14a}) : $(a+b, c) \in Cl[M]$ *and* $(b, c) \in Cl[N]$
$\Rightarrow (a, c) \in Ne[MN]$.

If $a/c < 0$ and $b/c > 1$ then R_{14} can be approximated by

(R_{14b}) : $(a+b, c) \in Cl[M]$ *and* $(b, c) \in Cl[N]$
$\Rightarrow (a, c) \in Ne[(MN)^2]$.

Remark: The precision of the results using locally complete rules may slightly differ according to whether *Ne* or *NE* is used. However, once we consider the symmetric approximations of the rules, almost the same results are obtained. Indeed, rules expressed in terms of *NE* and *Cl* can be approximated using the same symmetric fuzzy numbers as rules expressed in terms of *Ne* and *Cl*, except for rule R_{13} and its counterpart [17]. Even in this case, the difference is slight since, in approximation of R_{13} we obtain the symmetric tolerance parameter M^2N when $b/c < 0$ (see (R_{13a})); while for its counterpart in terms of *NE* we obtain $(MN)^2$. Now, substituting $(MN)^2$ for M^2N in rule (R_{13a}) (since $(MN)^2 \supseteq M^2N$) then, we get the same approximate rules based on *Ne* and *Cl*, as those based on *NE* and *Cl*.

The use of symmetric variants of R_6 and R_{10} to R_{16} enables the inference rules to be chained without restriction or caution since their conclusion parts are negligibility or closeness expressions involving only symmetric fuzzy numbers. Nevertheless, the imprecision of the results provided by these variants, may increase a little faster than when using the rules in their locally complete forms, since the tolerance parameters computed with the former are less tight than those computed with the latter. So the use of the symmetric approximate inference rules many lead to shorter valid deduction chains (in the sense of section 3.4), hence a somewhat less powerful order of magnitude reasoning tool.

VI. Approximate Solving of a Set of Equations

In this section, we demonstrate, through a series of simple examples, how the proposed inference system can be used for simplifying complex equations involving only arithmetic operations, with a view to solving them approximately, as done in physics, or in chemistry. Such equations, involving arithmetic expressions are often supplemented by assumptions expressed in terms of negligibility and closeness relations. The validity of a deduction can be measured in the fuzzy setting. Namely suppose a statement $(x, y) \in Cl[f(M)]$ is obtained with some tolerance parameter $f(M)$ derived using symbolic computations. Then, the statement is no longer completely valid if the support of $f(M)$ no longer lies in the validity interval *V*. The validity degree of the conclusion can be measured as $1 - \mu_{f(M)}((\sqrt{5}-1)/2)$. Indeed only the α–cuts of $f(M)$ for $\alpha \geq \mu_{f(M)}((\sqrt{5}-1)/2)$ can be used as valid tolerance parameters for the statement $(x, y) \in Cl[f(M)]$. Note that Mavrovouniotis [22] makes use of an external belief function for assessing the validity degree of a conclusion; here such degree is a direct by-product of the fuzzy semantics.

Example 6.1: It consists of three constraints:

(i) $(a - b)/(c - d) = 1$, (ii) $(d, b) \in Cl[M]$, (iii) $(b, a) \in Ne[N]$.

The problem is to compare the orders of magnitude of quantities *a* and *c*. The applying of inference rules on the available pieces of information, leads to the following derivations:

$(ii) \Rightarrow_{R_4} (d-b, b) \in Ne[M]$
$\Rightarrow (c-a, b) \in Ne[M]$, (due to (i)) (25)
$\{(25), (iii)\} \Rightarrow_{(R_{20})} (c-a, a) \in Ne[M \cap N]$
$\Rightarrow_{R_4} (c, a) \in Cl[M \cap N]$. (26)

Then, *c* is *close to a* in the sense of fuzzy set $M \cap N$. Note that for $M = N$, the relation (26) reads $(c, a) \in Cl[M]$. Now, if $a = 100$, then we can come up with an estimate of the quantity *c* under the form of a fuzzy number *F*, namely:

$$\mu_F(z) = \mu_{Cl[M]}(z, 100) = \mu_M(z/100) = \mu_{(100M)}(z),$$

i.e., $F = 100M$, which can be easily computed. It points out that fuzzy relations can provide a natural framework for interfacing symbolic information and numerical data. It is worth noticing that this example can be solved using another chaining of rules. For instance, chaining (R_{11}), (R_6), (R_{21}) and R_4 infers $(c, a) \in Cl[MN^3]$ (resp. $(c, a) \in Cl[M^2N^4]$) if $b/a \leq 0$ (resp. $b/a > 0$). In both cases, the obtained results are clearly less precise than (26). Due to the possibility of various reasoning paths leading to the result, one of the main issues, that the forthcoming implemented system [18] must cope with, is to find the best chaining of rules that enables the most precise result to be inferred.

Example 6.2: This is a more elaborate example in biochemical engineering domain, taken from [25]. It discusses the so-called Michaelis-Menten kinetics and the different modes of inhibition of an enzymatic reaction. Note that all parameters manipulated in this example are *positive*. In the case of competitive inhibition, the kinetic analysis of isolated enzymatic reaction is described by the following equation (see [25] for more details):

$$r = (V_{max}\, K_i\, [A]) / (K_i\, [A] + K_m\, (K_i + [A])), \quad (27)$$

where A, K_m, B, K_i, r *and* V_{max} denote the substrate, the Michaelis constant, the inhibitor, the inhibition constant, the rate of biochemical reaction and the maximum enzyme turnover respectively. $[x]$ means the concentration of x. In order to analyze the order of magnitude of the rate of reaction, several assumptions on orders of magnitude relations would be made by the chemist. Let us assume that the following qualitative assumptions are supposed to be hold:

(i) $([B], K_i) \in Ne[M]$, (ii) $([A], K_m) \in Ne[M]$.

Our objective is to compare the orders of magnitude of the rate r and V_{max}. First let us rewrite the equation (27) under the form

$$r / V_{max} = (K_i [A]) / (K_i [A] + K_m (K_i + [A])). \quad (28)$$

Now, the following derivations can be obtained

$$(i) \Rightarrow_{R_4} (K_i +[B], K_i) \in Cl[M] \quad (29)$$
$$(29) \Rightarrow_{R_2} (K_i , K_i +[B]) \in Cl[M] \quad (30)$$
$$\{(30), (ii)\} \Rightarrow_{(R_{16})} (K_i[A], K_m (K_i +[B])) \in Ne[M^2] \quad (31)$$
$$(31) \Rightarrow_{R_8} (K_i[A], K_i[A]+K_m(K_i +[B])) \in Ne[M^2], \quad (32)$$

since $0 \le K_m (K_i +[B]) / (K_i[A]+K_m(K_i +[B])) \le 1$. Now by (28), the pair (r,V_{max}) should satisfy the same fuzzy relation, i.e., $(r, V_{max}) \in Ne[M^2]$. This means that the rate of the biochemical reaction r is *negligible w.r.t.* the maximum enzyme turnover V_{max} , in the sense of the fuzzy set M^2. Note that for this example the solution found in [25] is also $r << V_{max}$ i.e., r is much smaller than V_{max}. The benefit of our approach is to provide a fuzzy numerical estimate for the extent of the negligibility of r w.r.t. V_{max}. Letting $\underline{M} = [1-\varepsilon, 1/(1-\varepsilon)]$, the support of M^2 is $[(1-\varepsilon)^2, 1/(1-\varepsilon)^2]$. Now, if $\varepsilon = 0.1$ then $[(1-\varepsilon)^2, 1/(1-\varepsilon)^2] \cong [0.81, 1.23] \subset V$. Then, the validity of the obtained negligibility relation is preserved. In contrast, if $\varepsilon = 0.25$ then $[(1-\varepsilon)^2, 1/(1-\varepsilon)^2] \cong [0.56, 1.77] \not\subset V$. Then, the result no longer expresses negligibility (see section 3.4).

Example 6.3: Consider the example of a counter–current heat exchanger described in [23,24]. A heat–exchanger is a device that transfers heat from a hot stream to a cold stream. The important parameters in the analysis of the device are the molar heat KH, the molar flowrate FH of the hot stream, the molar heat KC and the molar flowrate FC of the cold stream. Four temperature differences are defined: DTH is the temperature drop of the hot stream, DTC is the temperature rise of the cold stream, DT_1 is the driving force at the left end of the device, and DT_2 is the driving force at the right end of the device. The two following equations hold (see [23] for more details):

$$DTH-DT_1-DTC+DT_2 = 0, \quad (33)$$
$$DTH \cdot KH \cdot FH = DTC \cdot KC \cdot FC. \quad (34)$$

The first one is a consequence of the definition of the temperature differences, and the second one is the energy balance of the device. The following assumptions are made and expressed as order of magnitude relations:

(i) $(DT_2, DT_1) \in Cl[M]$, (ii) $(DT_1, DTH) \in Ne[N]$,
(iii) $(KC, KH) \in Cl[M]$.

The problem is to find a relation between the parameters FC and FH. From the available pieces of information, we can derive the following results:

$$(i) \Rightarrow (DT_1+DTC-DTH, DT_1) \in Cl[M], \text{ (due to (33))} \quad (35)$$
$$(35) \Rightarrow_{R_4} (DTC-DTH, DT_1) \in Ne[M] \quad (36)$$
$$\{(36), (ii)\} \Rightarrow_{(R_{20})} (DTC-DTH, DTH) \in Ne[M \cap N] \quad (37)$$
$$(37) \Rightarrow_{R_4} (DTC, DTH) \in Cl[M \cap N] \quad (38)$$
$$\{(38), (iii)\} \Rightarrow_{R_3} (DTC \cdot KC, DTH \cdot KH) \in Cl[M(M \cap N)], \quad (39)$$

and finally since $DTC \cdot KC/DTH \cdot KH = FH/FC$, see equation (34), the pair (FH, FC) should satisfy the same fuzzy relation, i.e.,

$$(FH, FC) \in Cl[M(M \cap N)]. \quad (40)$$

It means that the molar flowrate FH of the hot stream is *close* to the molar flowrate FC of the cold stream in the sense of the fuzzy set $M(M \cap N)$. Note that with approximately the same assumptions, the relation between FH and FC provided in [23,24] is FC ~<··>~ FH which means that FC is approximately equal to FH. Again, the benefit of our approach is to give a fuzzy numerical estimate for the closeness obtained in (40).

For $M = N$, we have $M(M \cap N) = M^2$. Then, the support of $M(M \cap N)$ is of the form $[(1-\varepsilon)^2, 1/(1-\varepsilon)^2]$ if $\underline{M} = [1-\varepsilon, 1/(1-\varepsilon)]$. We can check that for $\varepsilon = 0.05$ the validity of the closeness relation obtained in (40), is preserved (since $[(1-\varepsilon)^2, 1/(1-\varepsilon)^2] \cong [0.90, 1.10] \subset V$). However, for $\varepsilon = 0.24$ we have $[(1-\varepsilon)^2, 1/(1-\varepsilon)^2] \cong [0.57, 1,73] \not\subset V$ and the result is no longer valid.

VII. Incoherence Detection in a Set of Equations

In this section, we show how to detect a possible incoherence in a set of equations and closeness and negligibility statements by means of the order of magnitude reasoning system, and reduction ab absurdo. The idea is that, if the statements $(x, y) \in Cl[M]$ and $(x, y) \in Ne[M]$ can be inferred from the set of equations and the order of magnitude assumptions, then an inconsistency has been detected. It can useful for invalidating qualitative assumptions made by experts if these assumptions can be proved contradictory. In section 3.2, it has been proposed as a natural property of a well-behaved system that statements $(x, y) \in Cl[M]$ and $(x, y) \in Ne[M]$ should be contradictory. However since they depend on the choice of the tolerance parameter, it enforces a constraint expressed by inequality (5). However, this condition only ensures that $\mu_{Cl[M]}(x, y)+ \mu_{Ne[M]}(x, y) \le 1$.

An apparently more stringent condition can actually be enforced, namely that $(a, b) \in Ne[M]$ and $(a, b) \in Cl[M]$

should be *absolutely* contradictory. In other words $min(\mu_{Cl[M]}(x, y), \mu_{Ne[M]}(x, y)) = 0, \forall x, y$. The following result is obtained:

Proposition 7.1: $(a, b) \in Ne[M]$ and $(a, b) \in Cl[M]$ are absolutely contradictory if and only if the support $\underline{M}$ lies in $V = [(\sqrt{5}-1)/2, (\sqrt{5}+1)/2]$.

Proof: Let $\underline{M} = [1-\varepsilon, 1/(1-\varepsilon)]$. $(x, y) \in Cl[M] \Leftrightarrow 1-\varepsilon \le x/y \le 1/(1-\varepsilon)$. $(x, y) \in Ne[M] \Leftrightarrow 1-\varepsilon \le 1 + x/y \le 1/(1-\varepsilon)$. So, $min(\mu_{Cl[M]}(x, y), \mu_{Ne[M]}(x, y)) = 0$ if and only if $(1 - \varepsilon) > 1/(1-\varepsilon) - 1 = \varepsilon/(1-\varepsilon)$. This is equivalent to $\varepsilon^2 - 3\varepsilon + 1 > 0$. This inequality is precisely the one obtained from condition (5). ♦

So, condition (5) maintains a strict contradiction between $(a, b) \in Ne[M]$ *and* $(a, b) \in Cl[M]$. As a consequence, if $(a, b) \in Ne[M]$ *and* $(a, b) \in Cl[N]$ hold for two tolerance parameters in the validity interval, they are also inconsistent. So the inference system can detect inconsistencies from negligibility and closeness statements involving different tolerance parameters. The validity of the incoherence detection can be measured as $1 - max(\mu_M((\sqrt{5}-1)/2), \mu_N((\sqrt{5}-1)/2))$.

Remark: Proposition 7.1 is also valid with the relation *NE* [17].

Example 7.1: Consider the constraints

(i) $(b/c - a/b, 1) \in Cl[M]$, (ii) $(a, b) \in Ne[M]$,
(iii) $(b, c) \in Ne[M]$.

Is the system incoherent? For the sake of simplicity, let us assume that all the parameters are positive. Then, the following new relations can be deduced:

$$(i) \Rightarrow_{R_{18}} (b^2/c - a, b) \in Cl[M] \quad (41)$$
$$\{(41), (ii)\} \Rightarrow_{(R_{13a})} (b^2/c, b) \in Cl[M^2] \quad (42)$$
$$(42) \Rightarrow_{R_{18}} (b, c) \in Cl[M^2]. \quad (43)$$

Now, relations (iii) and (43) can lead to an incoherence in the system. Indeed the level cuts of M and M^2 are of the form $[1-\varepsilon, 1/(1-\varepsilon)]$ and $[(1-\varepsilon)^2, 1/(1-\varepsilon)^2]$ respectively. In order to semantically establish the contradiction we need $(1-\varepsilon)^2 > (\sqrt{5}-1)/2$. It means that $0 < \varepsilon < 0.22$. Thus, the initial system is incoherent if the tolerance parameter obeys this condition. The degree of validity of the conclusion stating this incoherence is $1 - \mu_{M^2}((\sqrt{5}-1)/2) = 1 - \mu_M(0.78)$.

Example 7.2 (Inspired from [2] and [27]):
This is a practical problem from the domain of acid-base chemistry. An important task in this domain is to find the concentration of H^+ ions in a solution. The concentration of ions in solution depends on the dynamic equilibrium resulting from competing chemical reactions. Consider dissolving an acid, *AH*, in water. The two reversible reactions that occur, corresponding to the ionization of *AH* and H_2O, are given by (see [27] for more details):

$$AH \rightleftharpoons H^+ + A^-$$

$$H_2O \rightleftharpoons H^+ + OH^-.$$

The equilibrium concentrations of the three ions (H^+, OH^-, A^-) and the acid (AH) are determined by the following set of equations:

Charge balance: $[H^+] = [A^-] + [OH^-]$ (44)
Mass balance: $C_a = [A^-] + [AH]$ (45)
Acid ionization equilibrium: $K_a\ [AH] = [A^-]\ [H^+]$ (46)
Water ionization equilibrium: $K_w = [OH^-]\ [H^+]$. (47)

Square brackets denote concentrations; $C_a = 10^{-5}$ is the initial concentration of the acid; $K_w = 10^{-14}$ is the ion product of water; and $K_a = 10^{-2}$ is the ionization constant of the acid.

Let S be the set of equations (44) to (47). It has been pointed out in [2][27][30] that solving S *analytically* for $[H^+]$ results in a cubic equation which is difficult to solve. An alternative to this approach is to use the order of magnitude reasoning. For example, a chemist might guess that the acid is strong, so that $[OH^-]$ and $[AH]$ are negligible w.r.t. $[A^-]$ (these qualitative assumptions can be used in simplifying and solving S). Assume other assumptions are made such as:

(i) $([OH^-], [A^-]) \in Cl[M]$, (ii) $([AH], [A^-]) \in Cl[M]$.

Now, the system to solve is formed of S and relations (i), (ii). The rules applied on the available pieces of information, enable us to obtain:

$(ii) \Rightarrow ([H^+], K_a) \in Cl[M]$ (by (46)) (48)
$(ii) \Rightarrow_{R_2} ([A^-], [AH]) \in Cl[M]$ (49)
$\{(i), (49)\} \Rightarrow_{R_{19}} ([OH^-], [AH]) \in Cl[M^2]$ (50)
$(50) \Rightarrow_{R_1} ([OH^-]+[A^-], [AH]+[A^-]) \in Cl[M^2]$ (51)
$(51) \Rightarrow ([H^+], C_a) \in Cl[M^2]$ (by (44) and (45)) (52)
$(48) \Rightarrow ([H^+], K_a) \in Cl[M^2]$. (due to proposition 3.4) (53)

Thus, the system proves that concentration $[H^+]$ is at the same time *close* to 10^{-5} and 10^{-2} (in the sense of the fuzzy set M^2), according to (52) and (53). This is impossible in the chemical context. Therefore, the system formed of S and relations (i) to (ii), is incoherent. This situation occurs because the assumptions (i) and (ii) that the chemist has made are wrong, and should not be used for simplifying S in order to compare the relative orders of magnitude of ionic concentrations or exogenous quantities like C_a, K_w *and* K_a.

Other assumptions can be made in order to solve S. For instance, the assumptions $([OH^-], [A^-]) \in Ne[M]$ and $([AH], [A^-]) \in Ne[M]$ do not lead to a contradiction, and the solution to S can be obtained as follows: since $[H^+] / K_a = [AH] / [A^-]$ then $([H^+], K_a)$ and $([AH], [A^-])$ should satisfy the same fuzzy relation. Then, we obtain $([H^+], K_a) \in Ne[M]$ which means that the concentration of H^+ ions is *negligible w.r.t.* K_a in the sense of the fuzzy set *M*.

The above examples show that qualitative reasoning on fuzzy order of magnitude relations *Cl* and *Ne* can mimic the lines of reasoning of an individual who tries to simplify complex equations and approximatively solve them. The proposed inference system can provide a useful validation tool in such tasks.

VIII. Application to Reasoning with Qualitative Probabilities

After considering a series of examples for illustrating the applicability of the approach, we consider a generic application which deals with the handling of small probability values in an exception-tolerant reasoning. Adams [1] has proposed a probabilistic inference system based on the following three inference rules:

Bayes rule: $A \to B\ ,\ (A \cap B) \to C \Rightarrow A \to C$
Triangularity: $A \to B\ ,\ A \to C \Rightarrow (A \cap B) \to C$
Disjunction: $A \to C\ ,\ B \to C \Rightarrow (A \cup B) \to C.$

They are sound and complete when $A \to B$ is understood as "the conditional probability $P(B|A)$ is infinitely close to *1*". These rules have been used in [28] to build a probabilistic-inference-like default logic. In this section, we study the counterparts of these rules with a fuzzy, non-infinitesimal, semantics (in order to cope with the graduality of the idea of closeness). In this respect $A \to B$ will be interpreted as "almost all *A*s are *B*s"; in other words $P(B|A)$ is close to *1* in the sense of closeness relation *Cl*. Note that an interval-based semantics has been already proposed in [13]. "$P(B|A)$ close to *1*" can then be written $(P(B|A),1) \in Cl[M]$. As $P(B|A) = P(A \cap B) / P(A)$, we have (by R_{18}):

$$(P(B|A),1) \in Cl[M] \Leftrightarrow (P(A \cap B), P(A)) \in Cl[M]. \quad (54)$$

A. Analysis of Bayes Rule

The rule reads

$$A \to B,\ (A \cap B) \to C \Rightarrow A \to C.$$

It has been shown in [9] that:

$$P(C|A) \geq P(B|A) \cdot P(C|A \cap B) = P(C \cap A \cap B) / P(A). \quad (55)$$

By hypothesis, $A \to B$ and $(A \cap B) \to C$ respectively read:

$$(P(A \cap B), P(A)) \in Cl[M], \quad (56)$$
$$(P(C \cap A \cap B), P(A \cap B)) \in Cl[N]. \quad (57)$$

Now applying the inference machinery of section 4 to the above assumptions, we obtain the following derivations:

$$\{(57), (56)\} \Rightarrow_{R_{19}} (P(C \cap A \cap B), P(A)) \in Cl[MN] \quad (58)$$
$$(58) \Rightarrow_{R_{18}} (P(C \cap A \cap B) / P(A), 1) \in Cl[MN] \quad (59)$$
$$(59) \Rightarrow_{R_1} (P(C|A), 1) \in Cl[MN]. \quad \text{(using (55))} \quad (60)$$

In the last derivation, we have applied a simplified form of rule R_1, i.e.,

$$1 \geq y \geq x > 0 \text{ and } (x, 1) \in Cl[M] \Rightarrow (y, 1) \in Cl[M]. \quad (61)$$

In terms of intervals [13], obtained as α–cuts of *M* (resp. *N*), (60) expresses that $P(C|A) \in [(1-\varepsilon)(1-\eta),\ 1] = [1-(\varepsilon+\eta-\varepsilon\eta),\ 1]$. The latter result coincides with the one obtained by Gilio in [15] for probabilities restricted by intervals.

B. Analysis of the Triangularity Rule

The rule reads

$$A \to B,\ A \to C \Rightarrow (A \cap B) \to C.$$

The best lower bound of $P(C|A \cap B)$ knowing $P(B|A)$ and $P(C|A)$, is given in [9]:

$$P(C|A \cap B) \geq \max\,[0, 1-((1-P(C|A)) / P(B|A))]. \quad (62)$$

Note that:

$$1-((1-P(C|A)) / P(B|A)) = (P(A \cap B) + P(C \cap A) - P(A)) / P(A \cap B). \quad (63)$$

By hypothesis, we have:

$$(P(A \cap B), P(A)) \in Cl[M], \quad (64)$$
$$(P(C \cap A), P(A)) \in Cl[N]. \quad (65)$$

From the above hypothesis, we can also apply the inference rules as follows:

$$(64) \Rightarrow_{R_2} (P(A), P(A \cap B)) \in Cl[M] \quad (66)$$
$$(65) \Rightarrow_{R_4} (P(C \cap A) - P(A), P(A)) \in Ne[N] \quad (67)$$
$$\{(66), (67)\} \Rightarrow_{R_{12}} (P(C \cap A) - P(A), P(A \cap B)) \in Ne[M(N\text{Á}1) \oplus 1] \quad (68)$$
$$(68) \Rightarrow_{R_4} (P(C \cap A) - P(A) + P(A \cap B), P(A \cap B)) \in Cl[M(N\text{Á}1) \oplus 1] \quad (69)$$
$$(69) \Rightarrow_{R_{18}} ((P(C \cap A) + P(A \cap B) - P(A)) / P(A \cap B)\ ,\ 1) \in Cl[M(N\text{Á}1) \oplus 1] \quad (70)$$
$$(70) \Rightarrow (P(C|A \cap B), 1) \in Cl[M(N\text{Á}1) \oplus 1]. \quad (71)$$

(using (61), (62) and (63))

Using an α–cut of $M = [1-\varepsilon\ ,\ 1/(1-\varepsilon)]$ (resp. $N = [1-\eta,\ 1/(1-\eta)]$), (71) expresses that $P(C|A \cap B) \in [1-\varepsilon\ ,\ 1/(1-\varepsilon)] \cdot [-\eta,\ \eta/(1-\eta)] + 1$, from which we deduce that

$$P(C|A \cap B) \in [1-\eta/(1-\varepsilon),\ 1],$$

which corresponds to a result in [13]. Also, the latter result coincides with the one of Gilio in [15].

C. Analysis of the Disjunction Rule

The rule reads

$$A \to C,\, B \to C \Rightarrow (A{\cup}B) \to C.$$

The best lower bound of $P(C|A{\cup}B)$ in terms of $P(C|A)$ and $P(C|B)$ is given by [14]:

$$P(C|A{\cup}B) \geq P(C|A) \cdot P(C|B) \,/\, (P(C|A) + P(C|B) - P(C|A) \cdot P(C|B))$$
$$= 1 \,/\, (1/P(C|B) + 1/P(C|A) - 1). \qquad (72)$$

This bound improves the one used in [9]:

$$P(C|A{\cup}B) \geq max(0, P(C|A) + P(C|B) - 1).$$

Indeed on [0, 1], ab / (a+b−ab) ≥ max(0, a+b−1) holds. The following relations are assumed:

$$(P(C|A), 1) \in Cl[M], \qquad (73)$$
$$(P(C|B), 1) \in Cl[N]. \qquad (74)$$

Now, from (73) and (74), we can derive the following relations:

$$(73) \Rightarrow_{R_{18},R_2} (1 / P(C|A), 1) \in Cl[M] \qquad (75)$$
$$(74) \Rightarrow_{R_{18},R_2} (1 / P(C|B), 1) \in Cl[N] \qquad (76)$$
$$(76) \Rightarrow_{R_4} (1 / P(C|B) - 1, 1) \in Ne[N] \qquad (77)$$
$$\{(75), (77)\} \Rightarrow_{R_{13}} (1 / P(C|A) + 1 / P(C|B) - 1, 1) \in Cl[M{\oplus}NÁ1] \qquad (78)$$
$$(78) \Rightarrow_{R_{18}} (1, 1 / (1/P(C|A) + 1 / P(C|B) - 1)) \in Cl[M{\oplus}NÁ1] \qquad (79)$$
$$(79) \Rightarrow (1, P(C|A{\cup}B)) \in Cl[M{\oplus}NÁ1]. \qquad (80)$$

(using (61) and (72))

Since the fuzzy number $M{\oplus}NÁ1$ is not symmetric, then (80) implies that

$$(P(C|A{\cup}B), 1) \in Cl[1/(M{\oplus}NÁ1)]. \qquad (81)$$

Now, in terms of α-cuts, (81) writes $P(C|A{\cup}B) \in [(1-\varepsilon)(1-\eta)/(1-\varepsilon\eta), 1/(1-\varepsilon-\eta)]$, from which we deduce that

$$P(C|A{\cup}B) \in [(1-\varepsilon)(1-\eta)/(1-\varepsilon\eta), 1]$$
$$= [1 - (\varepsilon+\eta-2\varepsilon\eta) / (1-\varepsilon\eta), 1],$$

which exactly coincides with the optimal result obtained by Gilio in [15].

Thus, using our inference system, we can derive the following three rules:

$$A{\to}_M B, (A{\cap}B){\to}_N C \Rightarrow A{\to}_{MN} C$$
$$A{\to}_M B, A{\to}_N C \Rightarrow (A{\cap}B){\to}_{M(NÁ1)\oplus 1} C$$
$$A{\to}_M B, B{\to}_N C \Rightarrow (A{\cup}B){\to}_{1/(M\oplus NÁ1)} C,$$

where $A{\to}_M B$ reads $(P(B|A),1) \in Cl[M]$. These rules enable to give a fuzzy semantics to Adams' qualitative probabilistic logic, and to numerically assess the validity of conclusions. Chaining deteriorates the validity of his axioms, since $Cl[M] \subseteq Cl[M^2]$ and $Cl[M] \subseteq Cl[M(MÁ1){\oplus}1]$.

Remark: The use of the negligibility variant *NE* leads to the same fuzzy interpretation of Adams' axioms, as above [17].

Note that if we use symmetric approximations of rules R_{12} and R_{13}, namely rules (R_{12}) and (R_{13a}) respectively, we obtain suboptimal similar derivations, with tolerance parameter *MN* for the three conclusions.

IX. Conclusion

In this paper, we have shown that fuzzy relations can provide an appropriate semantics for inference rules for reasoning about relative orders of magnitude. Modeling closeness and negligibility relations in this fuzzy semantics captures in a rigorous way some attenuation of transitivity for the closeness relation, as well as its reinforcement for the negligibility relation. It also provides a natural interface between numbers and qualitative terms. The consistency between the symbolic and the numerical level is maintained by a fuzzy arithmetical semantics of inference with fuzzy relative comparators, in full accordance with the combination/projection principle.

Besides, the semantic requirements on the tolerance parameter ensure the semantic meaningfulness of the produced conclusions (e.g., we can check if the obtained relation of closeness is not too permissive). Indeed even if we start from semantically valid relations, it may happen that after several reasoning steps, the result is no longer semantically valid (i.e., the tolerance parameter is not included in the validity interval), in spite of its being syntactically inferred. One way to address this problem is to pursue the inference process, restricted to the valid α-cuts of the tolerance parameter. The use of fuzzy numbers also make it possible to evaluate the degree validity of conclusions. Another way to use the rules consists in delaying the computation of the numerical semantics of the tolerance parameter associated to a conclusion, to the end of the reasoning step. Then, we can figure out for which initial interval(s) tolerance the provided relation is still semantically meaningful.

Relative orders of magnitude using fuzzy relations, is a promising approach for solving some ambiguity problems in qualitative reasoning. This approach can also mechanize the commonsense reasoning of engineers simplifying complex equations and computing approximate solutions. Moreover, this approach can be applied to provide a fuzzy finistic semantics to plausible reasoning with qualitative probabilities. An implementation of the developed inference machinery has been undertaken [18]. The software realized so far can solve linear equations under closeness and negligibility assumptions. The current work concerns the reasoning strategy of the inference machinery for the handling of larger and more complex examples.

Lastly, our approach to order of magnitude reasoning, may be viewed as a simple illustration of the idea of computing with words ("*negligible*" and "*close to*") recently emphasized by Zadeh [37]. It is done in full agreement with the numerical semantics underlying these words [7]. The symbolic side of the computation emphasizes the granular nature of closeness and negligibility [8]. Indeed, sets of close, or negligible values (w.r.t. another value) are manipulated as a whole. It is a form of fuzzy granulation, which plays a central role in human cognition and reasoning, as recently emphasized by Zadeh [38].

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Appendix

Proof of proposition 3.5 (Section 3.2): The condition $Cl \subseteq Co$ is equivalent to

$$\mu_M(t) \leq 1 - max\,(\mu_M(1 + t), \mu_M(1 + 1/t))$$

$$\Leftrightarrow max\,(\mu_{MÁ1}(t), \mu_{1/(MÁ1)}(t)) \leq 1 - \mu_M(t),$$

$$\text{(since } \mu_{f(M)}(t) = \mu_M(f^{-1}(t)))$$

$$\Leftrightarrow MÁ1 \subseteq \overline{M} \text{ and } 1/(MÁ1) \subseteq \overline{M},$$

where the overbar denotes the fuzzy set complementation. Let us use α-cuts, and the fact that $F \subseteq G$ is equivalent to $F_\alpha \subseteq G_\alpha$ for all $\alpha > 0$. Any α-cut of M is of the form $[1-\varepsilon, 1/(1-\varepsilon)]$ where $\varepsilon \in [0, 1]$. Hence for α-cuts the two inclusions write:

$[-\varepsilon, \varepsilon/(1-\varepsilon)] \subseteq]-\infty, 1-\varepsilon[\cup]1/(1-\varepsilon), +\infty[$ and

$]-\infty, -1/\varepsilon] \cup [(1-\varepsilon)/\varepsilon, +\infty[\subseteq]-\infty, 1-\varepsilon[\cup]1/(1-\varepsilon), +\infty[$.

Note that $-1/\varepsilon < 1-\varepsilon$ is always true for $\varepsilon \in [0, 1[$. The two inclusions hold if and only if $\varepsilon/(1-\varepsilon) \leq 1-\varepsilon$ and $(1-\varepsilon)/\varepsilon \geq 1/(1-\varepsilon)$. They are both equivalent to $\varepsilon^2-3\varepsilon+1 \geq 0$, which is true if and only if $\varepsilon \in [0, (3-\sqrt{5})/2]$.

Proof of proposition 3.7 (Section 3.3):
The fuzzy sets inclusions (12) to (14) hold if and only if

i) $M \subseteq N \subseteq MN$,
ii) $N \supseteq M \supseteq (MÁ1)(NÁ1)\oplus 1$,
iii) $N \subseteq M(NÁ1)\oplus 1$,

are verified respectively. Again, we use α-cuts of M (resp. N) of the form $[1-\varepsilon, 1/(1-\varepsilon)]$ (resp. $[1-\eta, 1/(1-\eta)]$ where ε (resp. η) lies in $[0, (3-\sqrt{5})/2]$. The corresponding α-cuts of the fuzzy sets MN, $(MÁ1)(NÁ1)\oplus 1$ and $M(NÁ1)\oplus 1$ are of the form $[(1-\varepsilon)(1-\eta), 1/((1-\varepsilon)(1-\eta))]$, $[1-(\varepsilon\eta/(1-\eta)), 1+(\varepsilon\eta/((1-\varepsilon)(1-\eta)))]$, and $[1-(\eta/(1-\varepsilon)), 1+(\eta/((1-\varepsilon)(1-\eta)))]$ respectively.

i) Since $M \subseteq N$, we just show that $N \subseteq MN$. This is equivalent to $[1-\eta, 1/(1-\eta)] \subseteq [(1-\varepsilon)(1-\eta), 1/((1-\varepsilon)(1-\eta))]$. The inequalities $(1-\varepsilon)(1-\eta) \leq 1-\eta$ and $1/(1-\eta) \leq 1/((1-\varepsilon)(1-\eta))$ are obvious.

ii) We must prove that $M \supseteq (MÁ1)(NÁ1)\oplus 1$. This is equivalent to $[1-\varepsilon, 1/(1-\varepsilon)] \supseteq [1-(\varepsilon\eta/(1-\eta)), 1+(\varepsilon\eta/((1-\varepsilon)(1-\eta)))]$. The inequalities $1-\varepsilon \leq 1-(\varepsilon\eta/(1-\eta))$ and $1+(\varepsilon\eta/((1-\varepsilon)(1-\eta))) \leq 1/(1-\varepsilon)$ are both equivalent to $\eta/(1-\eta) \leq 1 \Leftrightarrow \eta \leq 1/2$ which is true since $0 \leq \eta \leq (3-\sqrt{5})/2 \leq 1/2$.

iii) $N \subseteq M(NÁ1)\oplus 1$ holds if and only if $[1-\eta, 1/(1-\eta)] \subseteq [1-(\eta/(1-\varepsilon)), 1+(\eta/((1-\varepsilon)(1-\eta)))]$. The inequalities $1-(\eta/(1-\varepsilon)) \leq 1-\eta$ and $1/(1-\eta) \leq 1+(\eta/((1-\varepsilon)(1-\eta)))$ are equivalent to $1/(1-\varepsilon) \geq 1$ and $\eta/(1-\eta) \leq \eta/(1-\varepsilon)(1-\eta)$ respectively, which are both equivalent to $1-\varepsilon \leq 1$ which is true.

Proof of rule R_9 (Section 4): Note that

$\mu_{Ne[M]}(x, x+y) = \mu_M(1 + x/(x+y)) = \mu_M(2 - y/(x+y))$
$= \mu_{2ÁM}(y/(x+y))$, since $\mu_M(f^{-1}(t)) = \mu_{f(M)}(t)$
$\neq \mu_{2ÁM}((x+y)/y)$,

due to the fact that $2ÁM$ is not symmetric (in the sense of (2)). Hence we cannot get that $\mu_{Ne[M]}(x, x+y) = \mu_{Ne[2ÁM]}(x, y)$. The equality approximately hold when ε is small. Indeed any α-cut of a symmetric M is of the form $[1-\varepsilon, 1/(1-\varepsilon)]$ and the corresponding level cut of $2ÁM$ is $[(1-2\varepsilon)/(1-\varepsilon), 1+\varepsilon]$; observe that $(1-2\varepsilon)/(1-\varepsilon) \cong 1/(1+\varepsilon)$ neglecting ε^2.
To get a sound rule, $2ÁM$ is changed into a less restrictive symmetric tolerance parameter M^2, a symmetric fuzzy number which contains $2ÁM$ if and only if $\varepsilon \in [0, (3-\sqrt{5})/2]$. Indeed, the corresponding level cut of M^2 is $[(1-\varepsilon)^2, 1/(1-\varepsilon)^2]$, it is easy to see that the inequality $(1-\varepsilon)^2 \leq (1-2\varepsilon)/(1-\varepsilon)$ is equivalent to $\varepsilon^2-3\varepsilon+1 \geq 0$ the characteristic validity equation of proposition 3.5. Now, we show that $1/(1-\varepsilon)^2 \geq 1+\varepsilon$. Since $1/(1-\varepsilon)^2 \geq 1/(1-\varepsilon)$, it suffices to show that $1/(1-\varepsilon) \geq 1+\varepsilon$. This is equivalent to $1-\varepsilon^2 \leq 1$ which is true. So, $[(1-2\varepsilon)/(1-\varepsilon), 1+\varepsilon] \subseteq [(1-\varepsilon)^2, 1/(1-\varepsilon)^2]$ holds, which implies that $2ÁM \subseteq M^2$ provided that when M lies in the validity interval V. Then, we can infer, since M^2 is symmetric:
$\mu_{Ne[M]}(x, x+y) = \mu_{2ÁM}(y/(x+y)) \leq \mu_{M^2}(y/(x+y)) = \mu_{Ne[M^2]}(x, y)$.
This means that $(a, b) \in Ne[M^2]$.

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