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Possibilistic logic: a retrospective and prospective view☆

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Abstract

Possibilistic logic is a weighted logic introduced and developed since the mid-1980s, in the setting of artificial intelligence, with a view to develop a simple and rigorous approach to automated reasoning from uncertain or prioritized incomplete information. Standard possibilistic logic expressions are classical logic formulas associated with weights, interpreted in the framework of possibility theory as lower bounds of necessity degrees. Possibilistic logic handles partial inconsistency since an inconsistency level can be computed for each possibilistic logic base. Logical formulas with a weight strictly greater than this level are immune to inconsistency and can be safely used in deductive reasoning. This paper first recalls the basic features of possibilistic logic, including information fusion operations. Then, several extensions that mainly deal with the nature and the handling of the weights attached to formulas, are suggested or surveyed: the leximin-based comparison of proofs, the use of partially ordered scales for the weights, or the management of fuzzily restricted variables. Inference principles that are more powerful than the basic possibilistic inference in case of inconsistency are also briefly considered. The interest of a companion logic, based on the notion of guaranteed possibility functions, and working in a way opposite to the one of usual logic, is also emphasized. Its joint use with standard possibilistic logic is briefly discussed. This position paper stresses the main ideas only and refers to previous published literature for technical details.

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1. Introduction

The handling of uncertainty in a logical setting has a long history already. Recently, the logical treatment of probabilistic information has been studied in Artificial Intelligence by different authors

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(e.g., [5,74,75,98,78,34,69]. However, there are at least three worth noticing difficulties when casting the probability calculus into a logical framework.

First, probabilities do not fit very well with logical entailment, in the sense that a set of propositions true with a probability greater or equal to some threshold, say α, is not deductively closed. Indeed from the constraints $\text{Prob}(\neg p \vee q) \geqslant \alpha$ and $\text{Prob}(p) \geqslant \alpha$, one can only deduce in general that $\text{Prob}(q) \geqslant \max(0, 2\alpha - 1)$, where $\max(0, 2\alpha - 1) < \alpha$ (except if $\alpha = 1$ or 0). This means that probabilistic reasoning does not maintain probability bounds across inference steps.

Second, the probabilistic counterpart of the resolution rule is not sufficient as a local computation tool for computing the best lower bound of the probability of a formula from a set of probabilistic constraints [45].

Lastly, there is a strong discrepancy between the probability of a material conditional $\text{Prob}(\neg p \vee q)$ and a conditional probability $\text{Prob}(q|p)$, which raises the question of the proper modelling of an uncertain rule 'if p then q' [94]. Moreover, in a logical setting, probabilistic conditional constraints cannot be expressed by means of a weight attached to a classical formula, since the conditioning bar '|' is not a Boolean logical connective.

Nevertheless, the idea of reasoning from sets of (classical) logic formulas stratified in layers corresponding to different levels of confidence is very old. Rescher [101] proposed a deductive machinery on the basis of the principle that the strength of a conclusion is the strength of the weakest argument used in its proof, pointing out that this idea dates back to Theophrastus (372-287 BC), a disciple of Aristotle. However, Rescher [101] did not provide any semantics for his proposal. The contribution of the possibilistic logic setting is to relate this idea (measuring the validity of an inference chain by its weakest link) to fuzzy set-based necessity measures in the framework of Zadeh [107] 's possibility theory, since the following pattern, first pointed out by Prade [100], then holds:

$$N(\neg p \vee q) \geqslant \alpha \text{ and } N(p) \geqslant \beta \text{ imply } N(q) \geqslant \min(\alpha, \beta),$$

where N is a necessity measure; see also Refs. [48,49]. This interpretive setting provides a semantic justification to the claim that the weight attached to a conclusion should be the weakest among the weights attached to the formulas involved in a derivation.

The possibilistic logic framework can be used for modelling pieces of knowledge pervaded with uncertainty, when uncertainty is represented in the setting of possibility theory. The weight α in the constraint $N(p) \geqslant \alpha$ estimates the extent to which it is certain that p is true considering the available, possibly incomplete information about the world. Possibilistic logic is also useful when representing preferences expressed as sets of prioritized goals, as first pointed out by Lang [84]. In this case, the weight α is to be understood as the priority level of goal p. Violating the goal, i.e. choosing an interpretation violating p, will result in downgrading the satisfaction level of this choice at least to level $1 - \alpha$. This is due to the fact that the associated possibility measure of $\neg p$, $\Pi(\neg p) = 1 - N(p)$ is then less than or equal to $1 - \alpha$. The function $1 - (\cdot)$ is thus used as an order-reversing map for the scale to which the weights belong.

This position paper intends to provide an overview of possibilistic logic, emphasizing its main features in a knowledge representation perspective, and referring to the literature for technical details. The paper is organized in three main sections. Section 1 summarizes the basics of possibilistic logic. Section 2 briefly surveys various extensions, while the third one introduces bipolar possibilistic logic, an extended framework that offers a more powerful representation setting both for uncertainty and preference modelling.

2. Basics of possibilistic logic

After a short refresher on possibility theory, syntactic and semantic aspects are summarized, as well as the handling of information fusion in possibilistic logic.

2.1. Background on possibility theory

A necessity measure N on formulas is a function from the set of logical formulas of a language to a totally ordered bounded scale (with 0 and 1 as bottom and top elements), which is characterized by the axioms

(i) $N(\top)=1$,
(ii) $N(\bot)=0$ where $\top$ and $\bot$ stand for tautology and contradiction respectively,
(iii) $N(p\wedge q)=\min(N(p),N(q))$,
(iv) $p\equiv q \Rightarrow N(p)=N(q)$ (syntax-independence) where $\equiv$ denotes equivalence in classical logic.

We use the real interval $[0,1]$ as the range of necessity measures in the following (except in Section 3), but this is not compulsory. A finite or not totally ordered scale bounded by a bottom and a top element is enough. A possibility measure Π is associated with N by duality, namely

$$\Pi(p) = 1 - N(\neg p),$$

where $1-(\cdot)$ is the order-reversing map of the scale. It expresses that the absence of certainty in favour of $\neg p$ makes p possible. Thus, possibility measures Π are such that $\Pi(p\vee q)=\max(\Pi(p),\Pi(q))$, for all propositions p and q. Due to syntax-equivalence, possibility and necessity measures can be defined on sets of interpretations. Possibility (and necessity) measures on finite sets (in the case of a finitely generated language) are associated with possibility distributions in the sense that

$$\Pi(p) = \max\{\pi(\omega) \text{ with } \omega \models p\}$$

where $\omega\models p$ means that p is a model of p, i.e. an interpretation where p is true.

Possibility measures Π (resp. necessity measures N) are the only set-functions compatible with comparative possibility (resp. necessity) relations, as shown in [36]. A comparative possibility relation $\geqslant_\Pi$ is a complete non-trivial transitive relation on propositions such that for all p, q, r,

$$p \geqslant_\Pi q \Rightarrow p\vee r \geqslant_\Pi q \vee r,$$

where $p \geqslant_\Pi q$ expresses that p is at least as possible as q [93]. The dual comparative necessity relation $\geqslant_N$ is defined by $p \geqslant_N q \Leftrightarrow \neg q \geqslant_\Pi \neg p$. Comparative necessity relations are closely related to epistemic entrenchment relations, which are the basis of belief revision theory [67]. Namely, let K be a belief set defined as a closed set of formulas in propositional logic (i.e., K contains all its logical consequences), and let K_p^* be another belief set representing the result of revising K by the input information p. The postulates of revision theory, among other requirements, demand that $p \in K_p^*$. It can be shown [52] that if the revision operation satisfies these postulates, then

$$q \in K_p^* \quad \text{if and only if} \quad p \wedge q >_{\Pi} p \wedge \neg q,$$

for some comparative possibility measure. In other words, q is in the revised belief set if $\neg q$ is more inconsistent with p than q (where $>_{\Pi}$ is the strict part of the possibility relation $\geqslant_{\Pi}$).

This form of belief revision can be connected with the definition of conditional possibility measures. Indeed, in the qualitative setting, conditioning obeys a Bayesian-like property [76]:

$$\Pi(p \wedge q) = \min(\Pi(q|p), \Pi(p)), \tag{1}$$

By virtue of the minimal specificity principle, looking for the maximal degree of possibility (compatible with the above equation), we obtain, assuming $\Pi(p) > 0$ [50]:

$$\begin{aligned} \Pi(q|p) &= 1 \quad \text{if } \Pi(p) = \Pi(p \wedge q), \\ &= \Pi(p \wedge q) < 1 \quad \text{if } \Pi(p) = \Pi(p \wedge \neg q) > \Pi(p \wedge q). \end{aligned}$$

This definition makes sense in a finite setting only. [1] Since by definition $N(q \mid p) = 1 - \Pi(\neg q \mid p)$, it is clear that $N(q \mid p) = N(\neg p \vee q)$ if $N(\neg p \vee q) > N(\neg p)$, and 0 otherwise; hence

$$\Pi(p \wedge q) > \Pi(p \wedge \neg q) \;\Leftrightarrow\; N(q \mid p) > 0.$$

Note that conditional necessity cannot be defined directly by an equation similar to (1) where N would replace Π, since $N(q \mid p)$ would be non-trivially defined only if p is somewhat certain (i.e. if $N(p) > 0$). Worse, such an equation becomes $\min(N(q \mid p), N(p)) = \min(N(q), N(p))$ whose least solution is always $N(q \mid p) = N(p \wedge q)$, which comes down to using $\Pi(q \mid p) = \Pi(\neg p \vee q)$, and does not enrich the possibility calculus at all. Nevertheless, defined as the dual to conditional possibility induced by Eq. (1), conditional necessity satisfies the equation $\min(N(q \mid p), N(p)) = \min(N(q), N(p))$.

In the possibility theory setting, observe that under the definition induced by Eq. (1), there is a closer agreement between the conditioning bar and the material implication than with probability. Moreover the constraint $\Pi(p \wedge q) > \Pi(p \wedge \neg q)$, which expresses that in the context where p is true, having q true is strictly more possible than q false, has been shown to be a proper encoding of a default rule "generally, if p then q" in a nonmonotonic reasoning perspective [14,15,64,54]. Possibilistic conditioning thus significantly differs from the possibility of a material implication. The price paid for this qualitative definition of conditioning is that one cannot have $0 < N(q \mid p) < N(q)$, which induces serious limitations from a learning point of view. This can be remedied in a numerical setting by using the product instead of the minimum operation in (1) for defining conditional possibility.

2.2. Syntactic aspects

A first-order possibilistic logic formula is essentially a pair made of a classical first order logic formula and a weight expressing certainty or priority. As already said, in possibilistic logic [44,45,49],

[1] Moreover, in an infinite setting, the maxitivity axiom $\Pi(p \vee q) = \max(\Pi(p), \Pi(q))$ is extended to a supremum over infinite unions of sets, but sup-decomposability is generally not preserved via ordinal conditioning.

weights of formulas p are interpreted in terms of lower bounds $\alpha \in (0,1]$ of necessity measures, i.e., the possibilistic logic expression (p,α) is understood as $N(p) \geqslant \alpha$, where N is a necessity measure. Constraints of the form $\Pi(p) \geqslant \alpha$ could be also handled in the logic but they correspond to very poor pieces of information [51,87], while $N(p) \geqslant \alpha \Leftrightarrow \Pi(\neg p) \leqslant 1-\alpha$ expresses that $\neg p$ is somewhat impossible and is much more informative.

An axiomatisation of first order possibilistic logic is provided by Lang [85]; see also [44]. In the propositional case, the axioms consist of all propositional axioms with weight 1. The inference rules are:

- $(\neg p \vee q, \alpha)$; $(p,\beta) \vdash (q, \min(\alpha,\beta))$ (modus ponens)
- for $\beta \leqslant \alpha$, $(p,\alpha) \vdash (p,\beta)$ (weight weakening),

where $\vdash$ denotes the syntactic inference of possibilistic logic. The min-decomposability of necessity measures allows us to work with weighted *clauses* without lack of generality, since $N(\bigwedge_{i=1,n} p_i) \geqslant \alpha$ iff $\forall i$, $N(p_i) \geqslant \alpha$. It means that possibilistic logic expressions of the form $(\bigwedge_{i=1,n} p_i, \alpha)$ can be interpreted as a set of n formulas (p_i, α). In other words, any weighted logical formula put in conjunctive normal form is equivalent to a set of weighted clauses. This feature considerably simplifies the proof theory of possibilistic logic. The basic inference rule in possibilistic logic put in clausal form is the resolution rule:

$$(\neg p \vee q, \alpha); (p \vee r, \beta) \vdash (q \vee r, \min(\alpha, \beta)).$$

Classical resolution is retrieved when all the weights are equal to 1. Other valid inference rules are for instance:

- if p entails q classically, $(p,\alpha) \vdash (q,\alpha)$ (formula weakening)
- $(\forall x\, p(x), \alpha) \vdash (p(s), \alpha)$ (particularization)
- $(p,\alpha); (p,\beta) \vdash (p, \max(\alpha,\beta))$ (weight fusion).

Observe that since $(\neg p \vee p, 1)$ is an axiom, formula weakening is a particular case of the resolution rule (indeed (p,α); $(\neg p \vee p \vee r, 1) \vdash (p \vee r, \alpha)$). Formulas of the form $(p, 0)$ that do not contain any information ($\forall p, N(p) \geqslant 0$ always holds), are not part of the possibilistic language.

Refutation can be easily extended to possibilistic logic. Let K be a knowledge base made of possibilistic formulas, i.e., $K = \{(p_i, \alpha_i)\}_{i=1,n}$. Proving (p,α) from K amounts to adding $(\neg p, 1)$, put in clausal form, to K, and using the above rules repeatedly until getting $K \cup \{(\neg p, 1)\} \vdash (\bot, \alpha)$. Clearly, we are interested here in getting the empty clause with the greatest possible weight [38]. It holds that $K \vdash (p,\alpha)$ if and only if $K_\alpha \vdash p$ (in the classical sense), where $K_\alpha = \{p \mid (p,\beta) \in K$ and $\beta \geqslant \alpha\}$. See Lang [86] for algorithms and complexity issues.

An important feature of possibilistic logic is its ability to deal with inconsistency. The level of inconsistency of a possibilistic logic base is defined as

$$\mathrm{Inc}(K) = \max\{\alpha | K \vdash (\bot, \alpha)\} \text{ (by convention } \max \emptyset = 0).$$

More generally, $\mathrm{Inc}(K) = 0$ if and only if $K^* = \{p_i \mid (p_i, \alpha_i) \in K)\}$ is consistent in the usual sense. This would not be true in case α_i did represent a lower bound of the probability of p_i in a probabilistically weighted logic.

A nonmonotonic inference notion can be defined as $K \vdash_{\text{pref}} p$ if and only if $K \vdash (p, \alpha)$ with $\alpha > \text{Inc}(K)$. It can be rewritten as $K^{\text{cons}} \vdash (p, \alpha)$, where $K^{\text{cons}} = K - \{(p_i, \alpha_i) \text{ with } \alpha_i \leqslant \text{Inc}(K)\}$ is the set of weighted formulas whose weights are above the level of inconsistency (they are thus not involved in the inconsistency). Indeed, $\text{Inc}(K^{\text{cons}}) = 0$. This inference is nonmonotonic because due to the non-decreasingness of the inconsistency level when K is augmented, $K \vdash_{\text{pref}} p$ may fail to imply $K \cup \{(q, 1)\} \vdash_{\text{pref}} p$.

2.3. Semantics

Semantic aspects of possibilistic logic, including soundness and completeness results with respect to the above syntactic inference machinery, are presented in [85,87,45,44]. From a semantic point of view, a possibilistic knowledge base $K = \{(p_i, \alpha_i)\}_{i=1,n}$ is understood as the possibility distribution π_K representing the fuzzy set of models of K:

$$\pi_K(\omega) = \min_{i=1,n} \max(\mu_{[p_i]}(\omega), 1 - \alpha_i), \tag{2}$$

where $[p_i]$ denotes the sets of models of p_i so that $\mu_{[p_i]}(\omega) = 1$ if $\omega \in [p_i]$ (i.e. $\omega \models p_i$), and $\mu_{[p_i]}(\omega) = 0$ otherwise. The degree of possibility of ω according to (2) is computed as the complement to 1 of the largest weight of a formula falsified by ω. Thus, ω is all the less possible as it falsifies formulas of higher degrees. In particular, if ω is a counter-model of a formula with weight 1, then ω is impossible, i.e. $\pi_K(\omega) = 0$. It can be shown that π_K is the largest possibility distribution such that $N_K(p_i) \geqslant \alpha_i$, $\forall i = 1, n$, i.e., the possibility distribution which allocates the greatest possible possibility degree to each interpretation in agreement with the constraints induced by K (where N_K is the necessity measure associated with π_K, namely $N_K(p) = \min_{v \in [\neg p]}(1 - \pi_K(v))$). It may be that $N_K(p_i) > \alpha_i$, for some i, due to logical constraints between formulas in K.

Moreover, it can be shown that $\pi_K = \pi_{K'}$ if and only if, for any level α, K_α and K'_α are logically equivalent in the classical sense. K and K' are then said to be semantically equivalent. The semantic entailment is then defined by $K \models (p, \alpha)$ if and only if $N_K(p) \geqslant \alpha$, i.e., if and only if $\forall \omega, \pi_K(\omega) \leqslant \max(\mu_{[p]}(\omega), 1 - \alpha)$. Besides, it can be shown that $\text{Inc}(K) = 1 - \max_u \pi_K(\omega)$. Soundness and completeness are expressed by

$$K \vdash (p, \alpha) \quad \Leftrightarrow \quad K \models (p, \alpha).$$

This form of possibilistic entailment attaches to all formulas weights that are at least equal to the inconsistency level of the base. The inconsistency-free formulas, which are above this level, entail propositions that have higher weights. Biacino and Gerla [26] provide an algebraic analysis of possibility and necessity measures generated by this form of inference.

Thus, a possibilistic logic base is associated with a *fuzzy set* of models. This fuzzy set is understood as either the set of more or less plausible states of the world (given the available information), or as the set of more or less satisfactory states, according as we are dealing with uncertainty or with preference modelling.

Conversely, consider a fuzzy set F representing a fuzzy piece of knowledge, with a membership function μ_F defined from a *finite* set Ω of states to some valuation scale L. It models either a piece of uncertain or vague knowledge, or a utility function associated with some criterion. The fuzzy set F can be equivalently seen as a finite family of nested level cuts $F_{\alpha_i} = \{\omega : \mu_F(\omega) \geqslant \alpha_i\}$ where μ_F is ranging on the finite scale $L' = \{\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_n = 1\} \subseteq L$. F can be equivalently represented by the set of constraints $N(F_{\alpha_i}) \geqslant 1 - \alpha_{i-1}$ for $i = 1, \ldots, n$, where N is the necessity measure defined from $\pi = \mu_F$. This gives birth to a possibilistic logic base of the form $K = \{(p_i, 1 - \alpha_{i-1})\}_{i=1,n}$, where p_i denotes a proposition whose set of models is F_{α_i}. A fuzzy statement is thus semantically equivalent to a possibilistic logic base.

The semantic counterpart of the nonmonotonic inference $K \vdash_{\text{pref}} p$ (that is, $K \vdash (p, \alpha)$ with $\alpha >$ Inc(K)) is defined as $K \models_{\text{pref}} p$ if and only if $N_K(p) > \text{Inc}(K)$. The set $\{\omega, \pi_K(\omega)$ maximal$\}$ forms the set of best models $B(K)$ of K. It turns out that $K \models_{\text{pref}} p$ if and only if $B(K) \subseteq [p]$ if and only if $K \vdash_{\text{pref}} p$.

2.4. Other forms of a possibilistic logic base

The possibilistic logic framework can be made fully qualitative by referring only to a complete preordering of formulas inducing a stratification of the set of formulas (then the possibility distribution is replaced by a well-ordered partition [58]). One may use a symbolic discrete linearly ordered scale. If L is not a numerical scale, $1 - (\cdot)$ is to be replaced by the order-reversing map of the scale. But such a scale may as well be mapped to the unit interval.

A possibilistic logic base can be also changed into a possibilistic directed acyclic graph [8,10]. Such a graph [68] exhibits a conditional independence structure just like for Bayesian nets [35,103].

A set of comparative constraints expressing that some propositions are more possible than others can also be encoded as a possibilistic logic base. Namely, a comparative possibilistic base is of the form $C = \{p_i > q_i,\ i = 1, \ldots, n\}$ where p_i and q_i are mutually exclusive propositions and $p_i > q_i$ means $\Pi(p_i) > \Pi(q_i)$. For instance, this type of knowledge can be obtained as $\Pi(p \wedge q) > \Pi(p \wedge \neg q)$ from default rules of the form "if p then usually q" as pointed out in Section 2.1. Two types of inference of a rule "if p then usually q" from C can be defined: a cautious one according to which C implies that $\Pi(p \wedge q) > \Pi(p \wedge \neg q)$ [64,54]; another one whereby $\Pi^*(p \wedge q) > \Pi^*(p \wedge \neg q)$ for the least specific possibility distribution in agreement with C [14,15]. An algorithm computes Π^*, which comes down to turning a set of default rules into a possibilistic knowledge base and checking whether $K \cup \{(p, 1)\} \vdash_{\text{pref}} q$ [14,17]. These two types of inference are closely related to the preferential approach to nonmonotonic reasoning [90]. See [15].

All these representations prove to be equivalent to a possibility distribution that rank-orders the possible worlds according to their level of plausibility or priority. Moreover there exist procedures for the direct translation of one representation into another [11].

2.5. Information fusion in possibilistic logic

Since possibilistic logic bases are semantically equivalent to fuzzy sets of interpretations, it makes sense to use fuzzy set aggregation operations for merging the bases. Pointwise aggregation operations applying to fuzzy sets can be also directly performed at the syntactic level. Let ⓒ be a non-decreasing fuzzy set aggregation operator such that $1 ⓒ 1 = 1$. The aggregation of (p, α) and (q, β) is expressed at the semantic level by:

$$\mu_{(p,\alpha) \oplus (q,\beta)}(\omega) = \max(\mu_{[p]}(\omega), 1 - \alpha) \text{ ⓒ } \max(\mu_{[q]}(\omega), 1 - \beta), \tag{3}$$

denoting by $\oplus$ the associated aggregation at the syntactic level. The fuzzy set of interpretations defined by (3) is equivalent to the following possibilistic logic base:

$$\{(p, 1-((1-\alpha)\copyright 1)), (q, 1-(1\copyright(1-\beta))), (p \vee q, 1-(1-\alpha)\copyright(1-\beta))\}.$$

This idea was first pointed out by Boldrin [28]; see also [29]. Note that the combination amounts to adding the formula $p \vee q$ with a level of certainty or priority possibly higher than the one of p, and the one of q. Indeed, since © is a non-decreasing operation, $1-((1-\alpha)\copyright(1-\beta))$ is greater than or equal to $1-((1-\alpha)\copyright 1)$ as well as $1-(1\copyright(1-\beta))$. If $\copyright = \min$, this third weighted clause is redundant with respect to the two others, as expected since (3) is then a particular case of (2).

This result can be easily generalized [18] to two possibilistic bases $K_1 = \{(p_i, \alpha_i) \mid i \in I\}$ and $K_2 = \{(q_j, \beta_j) \mid j \in J\}$. It can be, for instance, applied to triangular norm and triangular co-norm operations. Triangular norms are associative non decreasing symmetric operations tn such that $\mathrm{tn}(1,\alpha) = \alpha$ and $\mathrm{tn}(0,0) = 0$. The main triangular norms are min, product and $\max(0, \alpha+\beta-1)$. Let π_{tn} and π_{ct} be the result of the combination of π_{K_1} and π_{K_2} based on the triangular norm operation tn, and the dual triangular conorm operation $\mathrm{ct}(\alpha,\beta) = 1 - \mathrm{tn}(1-\alpha, 1-\beta)$ respectively. Then, π_{tn} and π_{ct} are, respectively, associated with the following possibilistic logic bases:

- $K_{\mathrm{tn}} = K_1 \cup K_2 \cup \{(p_i \vee q_j, \mathrm{ct}(\alpha_i, \beta_j)) \mid (p_i, \alpha_i) \in K_1 \text{ and } (q_j, \beta_j) \in K_2\}$,
- $K_{\mathrm{ct}} = \{(p_i \vee q_j, \ \mathrm{tn}(\alpha_i, \beta_j)) \mid (p_i, \alpha_i) \in K_1 \text{ and } (q_j, \beta_j) \in K_2\}$.

With $\mathrm{tn} = \min$ and $\mathrm{ct} = \max$, we get

- $K_{\min} = K_1 \cup K_2 \cup \{(p_i \vee q_j, \ \max(\alpha_i, \beta_j)) \mid (p_i, \alpha_i) \in K_1 \text{ and } (q_j, \beta_j) \in K_2\}$ semantically equivalent to $K_1 \cup K_2$,
- $K_{\max} = \{(p_i \vee q_j, \min(\alpha_i, \beta_j)) \mid (p_i, \alpha_i) \in K_1 \text{ and } (q_j, \beta_j) \in K_2\}$.

As can be seen, $K_{\min} = K_1 \cup K_2$ agrees with (2), while the disjunction of two possibilistic logic bases amounts to taking the disjunction of all pairs of formulas made from the two bases associated with the smallest of the two weights. This method also provides a framework where symbolic approaches for fusing classical logic bases [82] can be recovered by making the priorities implicitly induced from Hamming distances between sets of models, explicit [9,81].

3. Extensions of possibilistic logic

Possibilistic logic can be viewed as a special case of a labelled deductive system [66]. As such, different types of extensions of possibilistic logic can be considered.

A first type of extension preserves classical logic formulas and weights interpreted in terms of necessity measures, but exploits refined or generalized scales, or yet allows weights to have unknown, or variable values. The latter trick enables a form of hypothetical reasoning to be captured, as well as some kinds of fuzzy rules. Another extension consists in embedding possibilistic logic in a wider object language adding new connectives between possibilistic formulas. On the contrary, one may consider counterparts to possibilistic logic for non-classical logics, such as many-valued logics. This type of generalization sheds light on the difference between many-valued and possibilistic logics. A last kind of extension consists in keeping the language and the semantics of possibilistic logics, while altering the inference relation with a view to make it more productive. Such inference relations that tolerate inconsistency can be defined at the syntactic level. Besides, proof-paths leading to conclusions can be evaluated by more refined strategies than just their weakest links. We briefly outline these extensions in the following.

3.1. Weights in partially ordered scales

It has been observed for a long time that the totally ordered scale used in possibilistic logic could be replaced by a complete distributive lattice with zero and unit elements. At least three examples of the interest of such a construct can be found in the literature:

- A multiple-source possibilistic logic [43] where the weight attached to each formula is replaced by the fuzzy set of sources that more or less certainly support the truth of the formula;
- A timed possibilistic logic [40] where the weight attached to each formula is replaced by the fuzzy set of time points where the formula is known as being true with some certainty level;
- A logic of supporters [83] where the weight attached to each formula is replaced by the set of irredundant subsets of assumptions that support the formula.

A formal study of logics where formulas are associated with general 'weights' in a complete lattice has been carried out by Lehmke [92]. Necessity values attached to formulas can be encoded as a particular case of such '*weights*'. More generally, a partially ordered extension of possibilistic logic whose semantic counterpart consists of partially ordered models has been recently proposed by Benferhat et al. [24] (this issue).

When information comes from different sources, it may be not easy to assess the levels of certainty (or priority) of the formulas, using the same linearly ordered scale. One way to solve this difficulty, is to associate each formula p_i with an unknown symbolic level λ_i, assumed to belong to some linearly ordered scale Λ. The fact that some formulas are known to be more certain than others, or equally certain as others, will be represented by constraints C_j on the λ_i's. For instance, suppose that we have inferred $(q, \max(\min(\lambda_i, \lambda_j, \lambda_k), \min(\lambda_m, \lambda_n))$ and that $\lambda_i \leqslant \lambda_j$ and $\lambda_i < \lambda_k$ are known constraints, then we have established (q, λ_i), i.e. that $N(q) \geqslant \lambda_i > 0$, since formulas are associated with lower bounds of necessity degrees. However, we may thus infer formulas with certainty levels that are not comparable, e.g. (q, λ_i) and (p, λ_m) if no (in)equality relation between λ_i and λ_m can be deduced from the set of constraints. Observe also that an inferred formula is a meaningful result only if it can be established from the constraints that its certainty level is above the inconsistency level of the base. For instance, from $\{(q, \lambda_k), (p, \lambda_i), (\neg p, \lambda_j)\}$ we can safely infer (q, λ_k) only if it can be shown from the constraints that $\lambda_k > \min(\lambda_i, \lambda_j)$. This idea has been formalized and extended by Benferhat et al. [13] where sets of sources are attached to formulas and a set of constraints under the form of a partial order on the set of sources is given.

Other similar extensions may be of interest. For instance, we can perform a sensitivity analysis on standard possibilistic logic by using the following extension of the resolution rule:

$$(\neg p \vee q, [\alpha, \alpha']); (p \vee r, [\beta, \beta']) \vdash (q \vee r, [\min(\alpha, \beta), \min(\alpha', \beta')]).$$

3.2. Hypothetical reasoning and fuzzily restricted variables

It has been noticed that subparts of classical logic formulas may be 'moved' to the weight part of a possibilistic logic formula; For instance, the possibilistic formula

$$(\neg p(x) \vee q(x), \alpha)$$

is semantically equivalent to

$$(q(x), \min(\mu_P(x), \alpha)),$$

where $\mu_P(x) = 1$ if $p(x)$ is true and $\mu_P(x) = 0$ if $p(x)$ is false. It expresses that $q(x)$ is α-certainly true given the proviso that $p(x)$ is true. This is the basis of the use of possibilistic logic in hypothetical reasoning [39], which enables us to compute under what conditions a conclusion could be at least somewhat certain, when information is missing for establishing it unconditionally. This equivalence makes sense for the following reason. Since $(p \wedge q, \alpha)$ is semantically equivalent to the base $\{(p, \alpha), (q, \alpha)\}$, the (implicitly) universally quantified formula $(\neg p(x) \vee q(x), \alpha)$ stands indifferently for $((\forall x)\ (\neg p(x) \vee q(x), \alpha))$ (where the scope of the quantifier does not include the weight), and $(\forall x)\ (\neg p(x) \vee q(x), \alpha)$ (where the scope of the quantifier includes the weight). However, rewriting $(\neg p(x) \vee q(x), \alpha)$ as $(q(x), \min(\mu_P(x), \alpha))$ is only compatible with the latter understanding.

Hypothetical reasoning can be also combined with the handling of inconsistency in possibilistic reasoning. For instance, consider the following possibilistic logic base

$$K = \{(\neg h \vee p, \alpha), (\neg q, \beta), (\neg p \vee q, 1)\}.$$

Observe that it entails $(\neg h, \min(\alpha, \beta))$, as well as $(\neg p, \beta)$ and $(\neg q, \beta)$. However, regarding h as an hypothesis that may be true or false, i.e., rewriting the first formula as $(p, \min(v(h), \alpha))$ with $v(h) = 1$ if h is true and $v(h)$ if h is false, we can no longer conclude that $\neg h$ is somewhat certain.

Now, we can reason case by case. Namely, if h is false, we still conclude $(\neg p, \beta)$ and $(\neg q, \beta)$ as expected. But, under the hypothesis that h is true, the base $K_h = \{(p, \min(v(h), \alpha)), (\neg q, \beta), (\neg p \vee q, 1)\}$ has a non-zero level of inconsistency $\mathrm{Inc}(K_h) = \min(v(h), \alpha, \beta) = \min(\alpha, \beta)$. Assume $\alpha > \beta$, then the conclusions (p, α) and (q, α) are now valid since their level of certainty is above the level of inconsistency $\mathrm{Inc}(K_h) = \beta$. Thus, we have established in our example that if h is true then $p \wedge q$ is somewhat certain, while if h is false then $\neg p \wedge \neg q$ is somewhat certain. This method has been applied to updating problems where possibilistic logic formulas describe the way a system is plausibly evolving [55].

Variable weights can be also useful for fuzzifying the scope of a universal quantifier. Namely, an expression such that $(\neg p(x) \vee q(x), \alpha)$ can be read $\forall x \in P(q(x), \alpha)$ where the set $P = \{x \mid p(x) \text{ is true}\}$. Making one step further, P can be allowed to be fuzzy [46]. The formula $(q(x), \mu_P(x))$ then expresses a piece of information of the form 'the more x is P, the more certain $q(x)$ is true'.

The fuzzy restriction on the scope of an existential quantifier was introduced in the following way [60]. From the two premises $\forall x \in A,\ \neg p(x, y) \vee q(x, y)$, and $\exists x \in B,\ p(x, s_0)$, we can conclude that $\exists x \in B,\ q(x, s_0)$ *provided that* $B \subseteq A$. Let $p(B, s_0)$ stand for that $\exists x \in B,\ q(x, s_0)$. Letting A and B be fuzzy sets, the following pattern can be established:

$$\frac{(\neg p(x, y) \vee q(x, y), \min(\mu_A(x), \alpha));\ \ (p(B, s_0), \beta)}{(q(B, s_0), \min(N_B(A), \alpha, \beta)},$$

where $N_B(A) = \inf_t \max(\mu_A(t), 1 - \mu_B(t))$ is the necessity measure of the fuzzy event A based on fuzzy information B. See [1,3] for the development of similar ideas in a logic programming perspective. In particular the above pattern can be strengthened, replacing B by the cut B_β in $N_B(A)$, and extended to a sound resolution rule [1,3,4].

3.3. Putting possibilistic and many-valued logics together

There is a major difference between possibilistic logic and weighted many-valued logics of Pavelka style [99,71], fuzzy Prolog languages like Lee's fuzzy clausal logic [89], although they look alike syntactically. Namely, in the latter, a weight t attached to a (many-valued) formula p often acts as a truth-value threshold, and (p, t) in a fuzzy knowledge base expresses the requirement that the truth-value of p should be at least equal to t for (p, t) to be valid. So in such fuzzy logics, while truth is many-valued, the validity of a weighted formula is two-valued. [2] On the contrary, in possibilistic logic, truth is two valued (since p is Boolean), but the validity of (p, α) with respect to classical interpretations is many-valued (Dubois and Prade, [59]) as per Eq. (2). In some sense, weights may defuzzify many-valued logics, while they fuzzify Boolean formulas in possibilistic logic. Moreover inferring (p, α) in possibilistic logic can be viewed as inferring p with some certainty, quantified by the weight α, while in standard many valued logics, a formula can be either inferred or not [71].

However, it is possible to cast possibilistic logic inside a (regular) many-valued logic such as Gödel or Lukasiewicz logic. The idea is to consider many-valued atomic sentences ϕ of the form (p, α) where p is a formula in classical logic. Then, one can define well-formed formulas of the form $\phi \vee \psi$, $\phi \wedge \psi$, $\phi \to \psi$, etc. where the "external" connectives linking ϕ and ψ are those of the chosen many-valued logic. From this point of view, possibilistic logic can be viewed as a fragment of Gödel or Lukasiewicz logic that uses only one external connective: conjunction $\wedge$ interpreted as minimum. This approach involving a Boolean algebra embedded in a non-classical one has been proposed by Boldrin and Sossai [30,31] with a view to augment possibilistic logic with fusion modes cast at the object level. Hajek et al. [72] use this method for both probability and possibility theories, thus understanding the probability or the necessity of a classical formula as the truth degree of another formula.

Liau and Lin [97] have augmented possibilistic logic with weighted conditionals of the form $p - (c) \to q$ and $p - \langle c \rangle \to q$ that encode Dempster rule of conditioning, and correspond to constraints $\Pi(p \wedge q) = c \cdot \Pi(p)$ and $\Pi(p \wedge q) > c \cdot \Pi(p)$ and c is a coefficient in the unit interval. Liau [95] considers more general conditionals where a t-norm is used instead of the product. Note that if $p = T$ (tautology), then $T - (c) \to q$ stands for $\Pi(q) \geqslant c$, and $\neg(T - \langle c \rangle \to \neg q)$ for $N(q) \geqslant c$. This augmented possibilistic logic enables various forms of reasoning to be captured such as similarity-based and default reasoning.

An alternative approach is to cast a many-valued logic in the setting of possibility theory by changing the classical logic formula p present in the possibilistic logic formula (p, α) into a many-valued formula, in Gödel or Lukasiewicz logic, for instance. Now (p, α) is still interpreted as $N(p) \geqslant \alpha$, but $N(p)$ is the degree of necessity of a fuzzy event as proposed by Dubois and Prade [51]. A many-

[2] In Pavelka-like languages (p, α) can be encoded as $\alpha \to p$ adding a truth-constant to the language. Using Rescher–Gaines implication ($\alpha \to p$ has validity 1 if p has truth-value at least α, and 0 otherwise), (p, α) is Boolean. Of course, using another many-valued implication, (p, α) remains many-valued.

valued formula can be obtained in first-order logic by making a fuzzy restriction of the scope of an existential quantifier. Alsinet [1,2] cast Gödel many-valued logic in the framework of possibilistic logic. Besnard and Lang [25] have proposed a possibilistic extension of paraconsistent logic in the same spirit.

Lehmke [91,92] has tried to cast Pavelka-style fuzzy logics and possibilistic logic inside the same framework, considering weighted many-valued formulas of the form (p,τ), where p is a many-valued formula with truth set T, and τ is a "label" defined as a monotone mapping from the truth-set T to a validity set L. T and L are supposed to be complete lattices, and the set of labels has properties that make it a fuzzy extension of a filter in L^{T}. Labels encompass what Zadeh [106] called "fuzzy truth-values" of the form "very true", "more or less true". Then they are continuous increasing mappings from $T=[0,1]$ to $L=[0,1]$ such that $\tau(1)=1$. A (many-valued) interpretation Val, associating a truth-value $\theta\in T$ to a formula p, satisfies (p,τ), to degree $\lambda\in L$, whenever $\tau(\theta)=\lambda$. When $T=[0,1]$, $L=\{0,1\}$, $\tau(\theta)=1$ for $\theta\geqslant t$, and 0 otherwise, then (p,τ) can be viewed as a weighted formula in some Pavelka-style logic. When $T=\{0,1\}, L=[0,1], \tau(\theta)=1-\alpha$ for $\theta=0$, and 1 for $\theta=1$, then (p,τ) can be viewed as a weighted formula in possibilistic logic. Lehmke [91] has laid the foundations for developing such labelled fuzzy logics, which can express uncertainty about (many-valued) truth in a graded way. It encompasses proposals of Esteva et al. [62] who suggested that attaching a certainty weight α to a fuzzy proposition p can be modelled by means of a labelled formula (p,τ), where $t(q)=\max(1-\alpha,\theta)$, in agreement with semantic intuitions formalized in [51].

Lastly, possibilistic logic can be cast in the framework of modal logic. Modal accounts of qualitative possibility theory involving conditional statements were already proposed by Lewis [93] (this is called the VN conditional logic, see [58,63]). Other embeddings of possibilistic logic in modal logic are described in [32,70,73].

3.4. Inconsistency tolerant inference relations in possibilistic logic

The handling of inconsistency in possibilistic logic is rather coarse: all formulas in K whose certainty level is equal to or less than the inconsistency level, are inhibited. However the useful subset K^{cons} of K (formulas above the inconsistency level) is not one of its maximal consistent subsets. In classical logic, several inconsistency-tolerant inference relations have been proposed [102,16].

The first idea is to delete from K all minimal subsets of inconsistent formulas. It only remains the subset of formulas not involved in any conflict. Applying this idea to possibilistic logic, leads to the subset K^{free} of possibilistic formulas containing K^{cons}, as well as all other possibilistic formulas (called free) below the inconsistency level and not involved in any conflict. Another idea is to consider that K implies p in the classical case if and only if p is implied by all its inclusion-maximal consistent subsets. Let $K=K_1\cup K_2\cup\cdots\cup K_m$ be the possibilistic knowledge base, partitioned into subsets of formulas of equal weight, and $\mathrm{MC}(K)$ be a maximal consistent subset of K. The extension of the notion of maximal consistent subsets to such layered bases was proposed by Brewka [33]: the so-called preferred subsets of K are of the form

$$K_j^{\mathrm{pref}}=\mathrm{MC}(K_1)\cup\mathrm{MC}(K_1\cup K_2)\cup\cdots\cup\mathrm{MC}(K_1\cup K_2\cup\cdots\cup K_j).$$

Of course $K^{\text{free}} \subseteq K_j^{\text{pref}}\ \forall j$. A more drastic selection of consistent subsets consists in considering those ones, $M\#(K)$ of maximal cardinality. Lexicographically preferred subsets [42] of a possibilistic knowledge base are K_j^{lex}, built like K_j^{pref}, replacing MC by $M\#$. Then the inconsistency-tolerant inference of (p,α) from a possibilistic base K means, with increasing inferential power: $K^{\text{cons}} \vdash (p,\alpha), K^{\text{free}} \vdash (p,\alpha), K_j^{\text{pref}} \vdash (p,\alpha)\ \forall j$ or $K_j^{\text{lex}} \vdash (p,\alpha)\ \forall j$. See [6] for a semantic study of these extensions of possibilistic inference. They collapse down to standard possibilistic inference if $\text{Inc}(K)=0$.

A more local treatment of inconsistency in classical logic consists in inferring p from an inconsistent base K provided that there is a consistent subset of K inferring p but no consistent subset inferring $\neg p$. This is called argued inference [16]. In the possibilistic case a consistent subset A of K with $\text{Inc}(K)=0$ is called a reason for p of strength α, if $A \vdash (p,\alpha)$. Then p is an argued consequence of K if there is a stronger reason for p than for $\neg p$ in K [19]. Again it is more productive than possibilistic inference.

Another extension of possibilistic inference has been proposed for handling paraconsistent information [19,20]. It is defined as follows. First, for each formula p_i such that (p_i,α_i) is in a possibilistic logic base K, compute (p_i,γ_i,δ_i) where γ_i (resp. δ_i) is the highest degree to which p_i (resp. $\neg p_i$) is supported in K. More precisely p_i is said to be supported in K at least at degree ρ_i if there is a reason for p_i of strength ρ_i. Let K° be the set of bi-weighted formulas that is thus obtained. A formula (p_i,γ_i,δ_i) in K° is said to have a paraconsistency degree equal to $\min(\gamma_i,\delta_i)$. K° contains both (p_i,γ_i,δ_i) and $(\neg p_i,\delta_i,\gamma_i)$ if $\min(\gamma_i,\delta_i)>0$.

For instance, take $K=\{(p,\alpha_1), (\neg p\vee q,\alpha_2), (\neg p,\alpha_3), (\neg r,\alpha_4), (r,\alpha_5), (\neg r\vee q,\alpha_6)\}$ with $\alpha_1>\alpha_2>\alpha_3>\alpha_4>\alpha_5>\alpha_6>0$. Then $K^\circ=\{(p,\alpha_1,\alpha_3), (\neg p\vee q,\alpha_2,0), (\neg p,\alpha_3,\alpha_1), (\neg r,\alpha_4,\alpha_5), (r,\alpha_5,\alpha_4), (\neg r\vee q,\alpha_2,0)\}$.

The weights γ_i and δ_i can be extended to consistent subsets A of bi-weighted formulas in K°:

- The defeasibility degree of A accounts for the most weakly supported formulas in A:

$$\text{DEF}(A) = -\text{Log}(\min\{\gamma_i \mid (p_i,\gamma_i,\delta_i)\in K^\circ \text{ and } p_i\in A\})^3;$$

- The safety degree of A accounts for the most severely attacked formulas in A:

$$\text{SAFE}(A) = -\text{Log}(\max\{\delta_i \mid (p_i,\gamma_i,\delta_i)\in K^\circ \text{ and } p_i\in A\}).$$

Let $\text{label}(q)=\{A\subseteq K^\circ \mid A \text{ is a reason for } q, \text{ and } \text{DEF}(A) \text{ is minimal}\}$. Let $A^*\in\text{Label}(q)$ such that $\text{SAFE}(A^*)$ is maximal. Then q is said to be a DS-consequence of K° (or K), denoted, $K^\circ\vdash_{\text{DS}} q$ if $\text{DEF}(A^*)<\text{SAFE}(A^*)$.[4] It can be checked that $\vdash_{\text{DS}}$ extends the entailment in possibilistic logic.

In the above example, $\text{Label}(q)=\{A,B\}$ with $A=\{(p,\alpha_1,\alpha_3), (\neg p\vee q,\alpha_2,0)\}$ and $B=\{(r,\alpha_5,\alpha_4), (\neg r\vee q,\alpha_2,0)\}$. So, $\text{DEF}(A)=-\text{Log}\,\alpha_2$, $\text{SAFE}(A)=-\text{Log}\,\alpha_3$, and $\text{DEF}(B)=-\text{Log}\,\alpha_5$, $\text{SAFE}(B)=-\text{Log}\,\alpha_4$. Then, $K^\circ\vdash_{\text{DS}} q$. If we first maximize SAFE and then minimize DEF, the entailment would not extend possibilistic entailment. Indeed in the above example, we would select B, but $\text{SAFE}(B)>\text{DEF}(B)$ does not hold, while $K\vdash(q,\alpha_2)$ since $\alpha_2>\text{Inc}(K)=\alpha_3$.

[3] Benferhat et al. [19,20] use an integer scale for weights attached to formulas, with the convention that larger weights account for weaker support. In order to keep the terminology consistent with this paper we use a logarithmic scale reversal function for measuring defeasibility and safety here.

[4] DS stands for defeasibility first, then safety.

Another entailment relation denoted $\vdash_{\mathrm{ss}}$, named *safely supported* consequence, a less demanding one than $\vdash_{\mathrm{DS}}$, is defined by $K^{\circ} \vdash_{\mathrm{ss}} q$ iff $\exists A \in \mathrm{label}(q)$ such that $\mathrm{DEF}(A) < \mathrm{SAFE}(A)$. In the case of classical knowledge bases, $\vdash_{\mathrm{ss}}$ is equivalent to $\vdash_{\mathrm{DS}}$ and comes down to the classical entailment from K^{free}. Moreover, $K^{\circ} \vdash_{\mathrm{ss}} q$ implies that q is an argued consequence of K, not the converse. It can be shown that the set $\{q \mid K^{\circ} \vdash_{\mathrm{ss}} q\}$ is consistent, while the set of argued consequences may be not. The reason for this difference is that in the argued consequence, there is no constraint on the subsets serving as reasons for a conclusion. However, if a formula p_i appearing in K is involved in a conflict, and belongs to the reason A for p, this reason becomes unsafe. It may prevent a subset A from being used as a reason for p. Formulas in K° are basically inhibited if $\gamma_i = \delta_i$. Indeed, if $(p_i, \gamma_i, \gamma_i) \in K^{\circ}$ and $p_i \in A$, then $\mathrm{SAFE}(A) = \mathrm{DEF}(A)$. So, even if A is used as a reason for p in proving that p is an argued consequence of K, it cannot be used for proving that p is a safely supported consequence of K. More details on this topic can be found in [19,20].

3.5. Leximin-based comparison of proofs

An obvious consequence of the possibilistic logic resolution rule is that only the smallest weight of the formulas used in the proof is retained. Thus no difference is made between, for instance getting (q, β) from $(\neg p \vee q, 1)$ and (p, β), or getting it from $(\neg s \vee r, 1)$, $(\neg r \vee q, \alpha)$ and (s, β) assuming $\alpha \geqslant \beta$, although we may find the first proof stronger. This idea an be captured by using a new resolution rule

$$(\neg p \vee q, \boldsymbol{\alpha});\ (p \vee r, \boldsymbol{\beta});\ (q \vee r, \boldsymbol{\alpha\beta}),$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are lists of weights, and $\boldsymbol{\alpha\beta}$ is the list obtained as the concatenation of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. In the above example, the first proof yields $(q, (1, \beta))$, while the second ones leads to $(q, (1, \alpha, \beta))$. We have then to rank-order the proofs according to their strength.

Let $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be two lists of weights attached, respectively, to two proofs of the same proposition, say q. Then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ can be ordered using an extended leximin defined as follows. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$. Note that we may have several times the same value in $\boldsymbol{\alpha}$ or $\boldsymbol{\beta}$ and $n \neq m$ generally. First, assume $m \leqslant n$ without loss of generality, and then add $(n - m)$ times 1 to the ordered list $\boldsymbol{\alpha}$. Let $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_m, \ldots, 1)$ be the resulting list of length n. Next, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ must be increasingly reordered, i.e. $\alpha_{\sigma(i)} \leqslant \alpha_{\sigma(i+1)}$ and $\beta_{\tau(i)} \leqslant \beta_{\tau(+1)}$. Let these reordered lists be $(\gamma_1, \ldots, \gamma_n)$ and $(\delta_1, \ldots, \delta_n)$. Then the leximin ordering of two lists $(\gamma_1, \ldots, \gamma_n)$ and $(\delta_1, \ldots, \delta_n)$ of equal length, being increasingly ordered, is defined by

$$\boldsymbol{\alpha} >_{\mathrm{leximin}} \boldsymbol{\beta} \quad \text{iff } \gamma_1 > \delta_1 \text{ or } \exists i \text{ such that } \forall j = i,\ \gamma_j = \delta_j \text{ and } \gamma_{i+1} > \delta_{i+1}.$$

In the above example, we have $(\beta, 1, 1) >_{\mathrm{leximin}} (\beta, \alpha, 1)$. As we can see, no difference is made between a short proof and a longer proof involving fully certain pieces of information, when both involve the same weight(s) smaller than 1; for instance, the proofs of (q, β) from $(\neg p \vee q, 1)$ and (p, β), and from $(\neg s \vee r, 1)$, $(\neg r \vee q, \beta)$ and $(s, 1)$ will be ranked in the same way. This framework may be of interest in argumentation systems for comparing sets of arguments supporting conclusions. It is possible to state the above definitions in terms of multisets of weights instead of lists.

4. Bipolar possibilistic logic

Variants of possibilistic logic may be obtained by no longer interpreting weights as lower bounds of necessity measures, but as constraints on other set functions in possibility theory, such as $\Pi(p) \geqslant \alpha$. Yet another set function expressing *guaranteed* possibility is considered here. In Section 2, it has been recalled how a possibility measure Π and a necessity measure N can be defined from a possibility distribution π. However, given a (non-contradictory, non-tautological) proposition p, the qualitative information conveyed by π pertaining to p can be assessed not only in terms of possibility and necessity measures, but also in terms of two other functions. Namely, $\Delta(p) = \min_{\omega \in [p]} \pi(\omega)$ and $\nabla(p) = 1 - \Delta(\neg p)$. Δ is called a 'guaranteed possibility' function [53].

Starting with a set of constraints of the form $\Delta(p_j) \geqslant \beta_j$ for $j = 1, \ldots, n$, expressing that (all) the models of p_j are guaranteed to be possible at least at level β_j, and applying a principle of *maximal* specificity that minimizes possibility degrees, the most informative possibility distribution π_* such that the constraints are satisfied is obtained. Note that this principle is the converse of the one used for defining π_K in (2), and is in the spirit of a closed-world assumption: only what is said to be (somewhat) guaranteed possible is considered as so. Namely

$$\pi_*(\omega) = \max_{j=1,n} \min(\mu_{[p_j]}(\omega), \beta_j). \tag{4}$$

By contrast with Π and N, the function Δ is nonincreasing (rather than nondecreasing) w.r.t. logical entailment. Fusion of such possibility-pieces of information is disjunctive rather than conjunctive (as expressed by (4) by contrast with (2)). Δ satisfies the characteristic axiom $\Delta(p \vee q) = \min(\Delta(p), \Delta(q))$, and the basic inference rules, in the propositional case, associated with Δ are

- $[\neg p \wedge q, \alpha]$, $[p \wedge r, \beta] \vdash [q \wedge r, \min(\alpha, \beta)]$ (resolution rule)
- if p entails q classically, $[q, \alpha] \vdash [p, \alpha]$ (formula weakening)
- for $\beta \leqslant \alpha, [p, \alpha] \vdash [p, \beta]$ (weight weakening)
- $[p, \alpha]$; $[p, \beta] \vdash [p, \max(\alpha, \beta)]$ (weight fusion).

where $[p, \alpha]$ stands for $\Delta(p) \geqslant \alpha$. The first two properties show the reversed behaviour of Δ-based formulas w.r.t. usual entailment. Indeed, if all the models of q are guaranteed to be possible, then it applies as well to any subset of models, e.g. the models of p, knowing that p entails q. Besides, observe that the formula $[p \wedge q, \alpha]$ is semantically equivalent to $[q, \min(v(p), \alpha)]$, where $v(p) = 1$ if p is true and $v(p) = 0$ if p is false. This means that $p \wedge q$ is guaranteed to be possible at least at level α, if q is guaranteed to be possible at this level when p is true. This remark can be used in hypothetical reasoning, as in the case of standard possibilistic formulas. So, Δ-based formulas behave in a way that is very different and in some sense opposite to the one of standard (N-based) formulas (since the function Δ is nonincreasing). Indeed Δ-based formulas are useful for encoding positive information. They guarantee the fact that interpretations are indeed feasible, while N-based formulas reflect negative information (in the sense that $N(p) \geqslant \alpha \Leftrightarrow \Pi(\neg p) \leqslant 1 - \alpha$ expresses that the models of $\neg p$ are somewhat impossible). See [37].

When dealing with uncertainty, this leads to a twofold representation setting distinguishing between what is not impossible because not ruled out by our beliefs (this is captured by constraints of the form $N(p_i) \geqslant \alpha_i$ associated with π^* in the sense of (2)), and what is known as feasible because it has been observed (expressed by constraints of the form $\Delta(q_j) \geqslant \beta_j$ associated with π_* in the sense of (4)). In other words, it offers a framework for reasoning with rules and cases (or examples) in a joint manner.

Possibilistic logic can be used as a framework for qualitative reasoning about preference [96,22,47]. When modelling preferences, bipolarity enables us to distinguish between positive desires encoded using Δ, and negative desires (states that are rejected) where N-based constraints describe states that are not unacceptable [12]. Deontic reasoning can also be captured by possibilistic logic as shown by Liau [96]. Namely, necessity measures encode obligation and possibility measures model implicit permission. Dubois et al. [37] have pointed out that Δ functions may account for explicit permission.

In all cases, some consistency between the two types of information (what is guaranteed possible cannot be ruled out as impossible) should prevail, namely

$$\forall \omega, \pi_*(\omega) \leqslant \pi^*(\omega)$$

and should be maintained through fusion and revision processes [61].

5. Conclusion

The basic notions and ideas underlying the current state of development of possibilistic logic have been recalled. Clearly, possibilistic logic enjoys properties that keep it close to classical logic, although the introduction of weights (of different kinds) substantially increases its representation capabilities, especially for inconsistency handling.

Applications of this framework can be found in default reasoning [14,15,17,95,97], in description logics [77], flexible constraint satisfaction problems (Schiex, [104]; see also [27]), decision under uncertainty and preference modelling [96,7,47]. Based on conditioning, the revision of a possibilistic knowledge base by some input information (possibly uncertain), which can be viewed as an asymmetrical fusion process, can be also handled directly at the possibilistic level [56,23]. A possibilistic logic counterpart of Kalman filtering, involving a prediction step, followed by a revision step has been proposed recently for handling updating in dynamic worlds [21].

Generally speaking, possibilistic logic provides a rich qualitative framework that augments the classical logic setting with capabilities for modelling plausibility or preference orderings in a flexible way, without a dramatic increase of computational complexity [86]. Possibilistic logic programming can be envisaged [41] and some programming environments based on possibility theory have been developed [1].

Possibilistic logic has been used in some applications to information systems. An early application to scenario identification under multiple-source uncertain information was carried out by Fulvio Monai and Chehire [65]. The representation framework of possibilistic logic can also be used for expressing to what extent it is certain that a database contains only valid, or complete information about a given topic [57]. Jahnke and Heitbreber [79] combine fuzzy Petri nets and possibilistic logic in an application to database reverse engineering. Wong and Lau [105] apply possibilistic logic to modelling agent behaviour in electronic commerce transactions. Lau et al. [88] exploit its capacity to handle inconsistency and belief revision in the modelling of information-filtering agents. Jahnke and Walenstein [80] have found it appropriate for use in the problem of modernizing poorly documented information systems.

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