An information-based discussion of vagueness[1]

Six scenarios leading to vagueness

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Abstract The issue of understanding and modeling vagueness was already addressed by many authors, especially in the second half of the XXieth century. In this paper, we try to provide an organized discussion of different categories of vagueness, pointing out circumstances where they appear. They all lead to a trichotomy of the universe of discourse, which seems to be the common feature of the different forms of vagueness. Basic representations frameworks are proposed for each case. The paper does not advocate a particular view against others but rather identify the characteristic features of each situation.

Keywords: vagueness; non-classical°logics; likelihood; fuzzy set; rough set; similarity, possibility theory.

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1. Introduction

Vagueness has been discussed for a long time by philosophers and logicians (Peirce, 1878; 1931; Russell, 1923; Hempel, 1939; K. Fine, 1975; Machina, 1976; Williamson, 1994; Haack, 1996; Keefe, 2000; Sorensen, 2001). It is generally considered in relation to the "Sorites paradox" and the failure of the principle of bivalence in logic. As pointed out in Sanford(1995), there are two drastically opposite approaches to vagueness: supervaluations (preserving a form of bivalence but admitting truth-value gaps), and degrees of truth (rejecting the bivalence principle). In the first view (Van Fraassen, 1969), predicates are Boolean, but their extension may be unknown. In the second view, some predicates are intrinsically non-Boolean. The idea that truth is a matter of degree is already advocated in the philosophical works of Bradley (1914) ("All truths and all errors in my view may be called relative, and the difference in the end between them is a matter of degree").

Vagueness is usually viewed as a defect. However, Black (1937) did not see vagueness necessarily as a defect, and distinguished it from both generality (or nonspecificity) and ambiguity (e.g. a word with several interpretations). He first proposed so-called "consistency profiles" in order to "characterize vague symbols", and his view is a premonition of the idea of fuzzy set in the sense of Zadeh (1965). Fuzzy sets embody the notion of gradual predicates for which the idea of a precise boundary between situations where this predicate applies and situations where it does not is meaningless. However Zadeh (1978) considered that vagueness covers both fuzziness and non-specificity. He wrote (foot note on p. 396) "Although the terms *fuzzy* and *vague* are frequently used interchangeably in the literature, there is, in fact, a significant difference between them. Specifically, a proposition, *p*, is f*uzzy* if it contains words which are labels of fuzzy sets; and *p* is *vague* if it is both fuzzy and insufficiently specific for a particular purpose." The introduction of fuzzy sets by Zadeh was not meant to be a contribution to the philosophy of vagueness. It was motivated by the need of a computational representation for linguistic terms appearing in statements which often aim at providing synthetic information about complex situations.

In this paper, we also put the discussion of vagueness in an information processing perspective, by focusing on knowledge representation aspects in the sense of Artificial Intelligence. The paper systematically investigates information scenarios where forms of what could be called ˙°vagueness°¨ appear. We shall refer to the idea of a vague concept (or category) as soon as this concept partitions the universe of discourse (sometimes implicitly) into more than two regions. In the following, we investigate six basic scenarios that we identified as giving birth to situations of this kind. Beforehand, in Section 2, we first introduce the information framework common to all these scenarios, and the corresponding notations. Sections 3 to 8 are devoted to the presentation and discussion of these six scenarios. However, as emphasized in the conclusion, hybrid situations can be encountered where several features of basic situations are found together.

2. The information framework

We use a general information-based framework where objects are described by an agent in terms of attribute values and can be put in categories according to the properties they enjoy. Such properties refer to subsets of attribute domains. Namely, let O be a finite set of objects or entities, and $\mathcal{A}$ be a finite set of attributes applicable to these objects. The possible values of an attribute *a* in $\mathscr{A}$ for the objects in O belong to the attribute domain D_a. Therefore we shall understand each attribute *a* in $\mathscr{A}$ as a mapping *a*: O → D_a. A property A regarding an attribute *a* will refer to the relationship between the objects and some classification of the attribute values in D_a, as explained now.

For a two-valued (or equivalently Boolean) attribute *a*, D_a contains two elements only, say {y, n}. Then one can only speak of a property A and of its opposite ´A with respect to attribute *a.* The property A is true or not for an object *o* in O according to whether *a*(*o*) = y or *a*(*o*) = n, respectively. Then each object either satisfies A or ´A. If the information is complete about all objects in O regarding attribute *a*, then A is not perceived as a vague category. Let Ext(A) be the extension of A in O, as perceived by the agent, i.e.:

$$\text{Ext(A)} = \{o \in \text{O} \mid a(o) = \text{y}\};\quad \text{Ext(´A)} = \{o \in \text{O} \mid a(o) = \text{n}\}.$$

More generally, if the attribute domain contains more than two elements, the property A and its negation ´A respectively refer to a pair of non-empty subsets Y_A and N_A of D_a. For a classical property, Y_A and N_A form a partition of D_a . The extensions Ext(A) = $\{o \in \text{O} \mid a(o) \in Y_A\}$, and Ext(´A) = $\{o \in \text{O} \mid a(o) \in N_A\}$, also form a partition of the set of objects. From now on, we shall identify properties A and ´A with the subsets Y_A and N_A of D_a, respectively when no confusion is possible from the context.

N.B. In daily practice, a property is associated with a label in a natural language. There is a context-dependent use of labels that is important for natural language understanding, but it will not be considered further in the rest of this paper.

In the following, we study several variants of the above information framework°: First A and ´A may no longer make a partition of D_a, because they are gradual properties°; in the next variant the agent may fail to know the extension of a property precisely even if it exists°; another case is when the attribute domain D_a is equipped with a distance, and a notion of conceptual centrality can be introduced accordingly°; in yet another setting, several agents may partially disagree on the extension of the property A, resulting in global uncertainty°; the fifth scenario is when some attribute values may be ill-known for some objects°; finally, the considered attributes may not provide a sufficiently expressive language for characterizing some subsets of objects precisely.

3. Classical vs. gradual properties

A property A referring to an attribute *a* is said to be classical if Y_A and N_A make an ordinary partition of D_a. Then the following properties hold:

Excluded-Middle Law : $Y_A \cup N_A = D_a$ (EM)

Non-Contradiction Law : $Y_A \cap N_A = \emptyset$. (NC)

However, (EM) or (NC) may fail in more general settings. For instance, if only (NC) (resp. (EM)) holds, A is said to be an intuitionistic (resp. paraconsistent) property. When both fail, attribute values may be simultaneously somewhat A and somewhat ´A. This situation is encountered with properties that are inherently gradual. Examples of gradual properties are numerous in natural languages. Clearly, properties such as Ò*young*Ó, Ò*small*Ó, Ò*heavy*Ó É do not lead to a clear-cut binary partition of the domain. A clear sufficiency test for checking whether a property A is gradual or not is to try to prefix it with the hedge ˙°very°¨. If ˙°very A°¨ makes sense, then there are natural situations where the property A is gradual. The above examples are gradual properties according to this test°; while for instance, ˙°*single*°¨ is not.

Graduality and partial pre-orderings. Gradual properties naturally induce a preordering $\succcurlyeq_A$ on the set of objects O in the sense that, for any $o_1, o_2 \in O$, $o_1 \succcurlyeq_A o_2$ means " o_1 is at least as A as o_2". A first example is the case of general categories like ˙°bird°¨, ˙°chair°¨, and the like. The preordering then reflects an idea of typicality so that there are preferred instances in the class. For instance, *robin* $\succ_A$ *penguin* because penguins do not fly. In this sense, ˙°bird°¨ is not a classical category because when the agent claims some animal is a bird, this animal is more likely a robin than a penguin. As for ´*A*, one can reasonably admit $o_1 \succcurlyeq_{\acute{}A} o_2$ if and only if $o_2 \succcurlyeq_A o_1$. When dealing with categories described by multiple attributes or features, as in the ÔbirdÕexample, it is more meaningful to have a small number of intermediary classes of relative membership. This succession of classes may for instance reflect the progressive failure of more and more key features. It makes little sense to try and build a numerical scale of membership in this case (a scale of ÔbirdinessÕin our example). See, e.g. Oden (1979) for a psycho-linguistic discussion of the use of fuzzy sets for modeling cognitive categories.

Another example is the case of predicates referring to a numerical scale. Then $\succcurlyeq_A$ is now defined on D_a, and it induces a partial order on the set of objects O. For instance, for *a* = height, A = *tall*, and D_a = [1.20, 2.20] meters, $\succcurlyeq_A$ is nothing but the natural order in the real interval [1.20, 2.20], since the greater is the height of a person, the taller he/she is; similarly with for *a* = age, A = *young* and D_a = [0, 150] years. This is not always the case. For *a* = age, take A = *middle-aged*, then the ordering on D_a is not in agreement with the natural order of ages. This view of a gradual property A just as an ordered structure (D_a, $\succcurlyeq_A$), advocated in (Finch, 1981; Basu, Deb and Pattanaik, 1992; Trillas and Alsina, 1999; Lee et al., 2002; Lee, 2003), is very elegant, however very difficult to exploit for operational purposes when it comes to building a logic, due to the lack of commensurateness between two such preorderings pertaining to distinct properties.

Membership functions as total pre-orders. A richer representation scheme is to model the extensions of gradual properties by means of fuzzy sets (Zadeh, 1965). In that case, to a structure (D_a, $\succcurlyeq_A$) we attach a membership function μ_A: $D_a \to [0, 1]$ preserving the ordering $\succcurlyeq_A$, that is, verifying $\mu_A(u)$ ‡ $\mu_A(v)$ whenever $u \succcurlyeq_A v$, and mapping to 1 (resp. 0) the maximal (resp. the minimal) elements of (D_a, $\succcurlyeq_A$). Notice that when doing this, we are in fact enriching the knowledge representation setting. Namely we are extending the possibly partial ordering $\succcurlyeq_A$ to a linear (and thus total) pre-order. See (Keefe, 1998) for a negative view on the adequacy for arbitrary vague concepts of the assumption of such a linear extension, and its measurement in a continuous scale like [0, 1]. In the above example, the membership function for *middle-aged* enables every age to be compared with any other, just by comparing their membership values. Moreover, we are also assuming that maximal (resp. minimal) elements of (D_a, $\geqslant_A$) are fully compatible with (resp. incompatible) with A, thus providing landmarks for full (resp. complete lack of) membership. These landmarks or anchor values cannot be expressed by means of a partial ordering alone. The pre-order induced by a fuzzy set on the domain D_a partitions it into (possibly) infinitely-many subsets $\{u \in D_a \mid \mu_A(u) = \alpha\}_{\alpha \in [0, 1]}$, in contrast with the binary partitions of classical properties. In fact here, we could replace [0, 1] by any other linear, bounded, sufficiently discriminating scale. Observe that there is some relation between the nature of the attribute domain D_a and the possible number of levels in the membership scale. For modeling gradual properties, the membership scale needs to be (and naturally becomes) a continuum only if D_a is a continuum. Indeed vagueness (in fact fuzziness) naturally arises when trying to represent a gradual property in a continuous referential D_a. In particular, any classical partition-based representation of such a property leads to a Sorites paradox. See Goguen (1969), Gaines (1977) and Copeland (1997) for fuzzy set-based discussions of this paradox.

Focusing only on the boundaries of the membership scale, fuzzy sets naturally induce a tri-partition on the attribute domain. Indeed, let $Y_A = A_¡$ be the core of A which gathers the elements which undisputedly belong to A, i.e., $A_¡ = \{u, \mu_A(u) = 1\}$. Similarly, $N_A = (\neg A)_¡ = \{u, \mu_A(u) = 0\}$, assuming that fuzzy set complementation agrees with classical complementation for the extreme values in the scale. Then, for a genuine fuzzy set, we have the strict inclusion $Y_A \cup N_A \subset D_a$. Thus, we can define a set of borderline elements as $B_A = D_a - (Y_A \cup N_A)$. Also note that the supports of A and ¬A, namely $A_S = \{u, \mu_A(u) > 0\}$ and $(\neg A)_S = \{u, \mu_A(u) < 1\}$ are not disjoint. Fuzzy sets violate the excluded-middle and contradiction laws (EM) and (NC): the support of a fuzzy set and the support of its complement overlap on the one hand, while the union of their cores does not cover the referential domain. This violation emphasizes the fact that with genuine fuzzy sets, a clear-cut boundary between A and its complement *does not exist*. This view of vagueness is quite similar to the one introduced by Black (1937), and is close to Alston (1964)'s definition of degree of vagueness. The pair (Y_A, $Y_A \cup B_A$), was called Òensemble flouÓ by Gentilhomme (1968), who viewed Y_A as as the set of central elements of A and B_A as the set of peripheral elements. See Lakoff (1987) and Smithson (1987) for more discussions about gradations in categories.

Fuzzy sets and similarity to prototypes. When the agent is able to measure how close or similar is one element of the domain D_a to another with respect to the attribute *a*, one can propose the following computation of membership degrees (Ruspini, 1991). Assume the agent is provided with a closeness relation S: $D_a \times D_a \to [0, 1]$, verifying at least S(u, u) = 1 for all $u \in D_a$, where S(u, v) = 1 means that u and v are indistinguishable, S(u, v) = 0 means that u and v have nothing in common, and if S(u, v) > S(u, vÕ) means that u is more similar to v than to vÕThen given A¡ and (´A)¡, which can be seen as (proto)typical values defining A and ´A respectively, one can define the degree $\mu_A(u)$ in which a value u belongs to A, as the extent to which u is close or similar to some typical value of A¡. In some sense we are identifying A with those values which are close to (or within) A¡. We proceed similarly for $\mu_{´A}(u)$. According to this view, we can define for all u in D_a:

$$\mu_A(u) = \sup\{S(u, v) \mid v \in A_{¡}\};\ \ \mu_{´A}(u) = \sup\{S(u, v) \mid v \in (´A)_{¡}\}.$$

Notice that $\mu_A(u) = 1$ for all $u \in A_{¡}$ and $\mu_{´A}(u) = 1$ for all $u \in (´A)_{¡}$, but in principle nothing prevents from having $\mu_A(u) > 0$ for some $u \in (´A)_{¡}$, or $\mu_{´A}(u) > 0$ for some $u \in A_{¡}$. See Osherson and Smith (1981) for a critical discussion of a fuzzy set-based approach to prototype theory, and ZadehÕs (1982) reply. This view may be related to Weston (1987)'s idea of approximate truth as reflecting a distance between a statement and the ideal truth. This view is also related to the notion of truth-likeness of Niiniluoto (1987) and of similarity-based reasoning as developed in (Dubois et al., 1997).

Set-theoretic operations. Fuzzy set theory has developed an algebraic framework for defining truth-functional set-theoretic operations extending classical set operations. Intersection and unions are then point-wisely defined[2]. Depending on the operations used for defining the fuzzy set intersection, equalities (EM) and (NC) may fail or hold. However, the excluded-middle and contradiction laws, and idempotence ($A \cap A = A = A \cup A$) cannot be satisfied at the same time: idempotence holds only with min and max-based intersection and union, while (EM) and (NC) are preserved for non-idempotent operations such as the

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[2] That is, $\mu_{A \cap B}(u) = \mu_A(u) * \mu_B(u)$ and $\mu_{A \cup B}(u) = \mu_A(u) \perp \mu_B(u)$, where the two-place operations $*$ and $\perp$ on [0, 1] are commutative, associative, monotonically non-decreasing, with appropriate boundary conditions (for intersection $1 * x = x$; for union $0 \perp x = x$). They are named triangular norms and co-norms respectively (e.g., Alsina, Trillas and Valverde, 1983). Moreover $\mu_{´A}(u) = 1 - \mu_A(u)$ for complementation.

Lukasiewicz connectives[3] (although the supports of A and ¬A overlap!). This necessarily leads to structures weaker than Boolean algebras for fuzzy sets.

Clearly, truth-functionality of connectives is a nice property to have, when possible, for computation. This simplicity can be obtained when we assume a unique membership scale for all concepts on D_a. However, it is not easy, even if not impossible, to define (binary) connectives for concepts described only with two pre-orders (D_a, $\succcurlyeq_A$) and (D_a, $\succcurlyeq_B$) on the same domain. See (Lee, 2003) for an investigation of intersection and union connectives in this setting from the perspective of decomposability and ordinal conjoint structure in measurement theory. In any case, the most natural definitions are:

$u \succcurlyeq_{A \cap B} v$ *if and only if* $u \succcurlyeq_A v$ and $u \succcurlyeq_B v$ (Pareto-ordering)

$u \succcurlyeq_{A \cup B} v$ *if and only if* $u \succcurlyeq_A v$ or $u \succcurlyeq_B v$.

However the intersection will be very poorly discriminant as for only few pairs *(u, v)* will it be true that $u \succcurlyeq_{A \cap B} v$. Besides, $\succcurlyeq_{A \cup B}$ is generally not transitive (its strict part will neither be transitive nor acyclic) and its transitive closure may very well be trivial. This is clearly related to the impossibility theorem of Arrow (1963) in social sciences.

The truth-functionality assumption of membership functions is not without controversies. For instance, the above similarity-based model will be truth-functional for disjunction only. Moreover, this assumption may be found too

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[3] $x * y = \max(0, x + y - 1)$ and $x \perp y = \min(1, x + y)$.

simplistic for an accurate account of vagueness-originated phenomena; see Sanford (1975) for critical discussions.

In the discussion above, we have referred to fuzzy set operations on a unique attribute domain. Although properties whose definition involves several attribute domains make the discussion of vagueness more complicated, it does not bring any new important feature to the discussion of vagueness in relation with the idea of gradual properties.

Graduality is a useful form of vagueness. When caused by the use of gradual properties, vagueness should not be felt as a defect to remedy, but rather as a desirable capability of the language to capture the idea of typicality, and to interface linguistic categories with a continuum of attribute values (usually numerical), without introducing arbitrary discontinuities. This capability is accounted for by the fuzzy set representation. Mind that the other vagueness scenarios, in the next sections, assume classical (non-gradual) properties.

Some philosophers and logicians e. g. Haack, Parikh, Tyle (cited by Copeland (1997)) have pointed out the problem of Òinappropriate precisionÓ inherent to fuzzy set membership functions, which they find paradoxical when dealing with vagueness. However, the scenario considered in this section deals with the modeling of graduality, or partiality, which is mainly based on the idea of ordering, and which has nothing to do with imprecision. The notion of partial truth, as put forward by Lukasiewicz (1930), leads to changing the very notion of a proposition. The definition of a proposition is a matter of convention, as stressed by De Finetti (1936, our translation):

ÒPropositions are assigned two values, true or false, and no other, not because there "exists" an a priori truth called "excluded middle law", but because we call "propositions" logical entities built in such a way that only a yes/no answer is possibleÓ

Fuzzy sets deal with many-valuedness in a logical format, they are not primarily concerned with uncertainty or belief. Contrary to what the terminology (vague, fuzzy) may suggest, gradual predicates allow for a refined model of categories, more expressive than the Boolean setting, and reflecting the common usage of some words as underlying preferred meanings or default typicality orderings of situations they refer to. Membership functions are just convenient context-dependent numerical representations of this ordering. Gradual propositions contain more information than all-or-nothing ones. But the problem of the measurement of membership functions makes sense, and is discussed in the fuzzy set literature; see, e.g., (Turksen and Bilgic, 2000; Marchant 2004).

4. Precisely-defined vs. ill-defined properties

An agent may not be able to precisely delimit the extension of a clear-cut property *A*. By an imprecisely delimited extension, we mean the existence of a *borderline* region in D_a where there exist elements for which the agent cannot say whether they can be classified as A or to ´A. This is also called semantic ambiguity. Here, vagueness results from a lack of knowledge of the precise extension of property A, rather than from the lack of complete information regarding some attributes values of objects (for the latter scenario, see Section 7). Thus the most elementary representation of this situation, for properties which are not a matter of degree, is the partition of D_a into the three subsets

$$Y_A,\ N_A \text{ and } B_A = D_a - (Y_A \cup N_A)$$

where Y_A (resp. N_A) is the set of attribute values that the agent can classify in A (resp.´A) without any hesitation. B_A is the borderline (uncertainty) area containing the real boundary of A. It is the set of attribute values that the agent can neither classify in A, nor in ´A.

Some scholars, denying the existence of intrinsic graduality, model predicates like Ò*young*Ó in this way. Indeed the set Y_A of elements with sure membership sounds like the set A¡ of prototypical elements of a fuzzy set. However, we insist that in this section, we consider the hypothetical situation of a classical property whose precise meaning (i.e., the extension of the property) is not known by the agent who is unsure about the satisfaction of the property for some value or element.

The idea of sub-definite sets suggested by Narin'yani (1980) also acknowledges the fact that an agent may only have partial knowledge on the extensions of A and ´A. Then the non-membership of an element to a set does not determine its membership to the complement. Thus a sub-definite set S is a pair (A^+, A^-)of disjoint subsets of elements which definitely belong or definitely do not belong to S, together with some piece(s) of information on the cardinality of these subsets.

Classification ambiguity. When asked whether a certain value u in the domain D_a satisfies the property A or not, an agent may express his beliefs in the membership or nonmembership in A of values in D_a, by means of an uncertainty measure g^A_u:

$2^{\{y, n\}} \to [0, 1]$ for each u in D_a, where y and n stand for Òbelonging to AÓ and for Ònot belonging to AÓ respectively. In such a case, g^A_u induces two fuzzy sets on D_a with membership functions defined by $\mu_A(u) = g^A_u(y)$ and $\mu_{´A}(u) = g^A_u(n)$. A reasonable condition is that $g^A_u(n) = g^{´A}_u(y)$, and conversely. The sets Y_A and N_A correspond to cases when $g^A_u(y) = 1$ and $g^A_u(n) = 1$ respectively. Other elements, with uncertain membership, belong to the boundary B_A of A. Obviously, if the measures g^A_u are probabilities verifying the above condition then $\mu_A(u) = 1 — \mu_{´A}(u)$, and μ_A is similar to a likelihood function P(A|u). The interpretation of membership functions as conditional probabilities P(A|u) was stressed by Cheeseman (1986) and Hisdal (1988). Coletti and Scozzafava (2004) show that membership functions can be then cast in the theory of coherent conditional probability, that goes back to De Finetti. The work of Giles (1988) can be viewed as pertaining to the same subjective probability trend, whereby a membership grade is interpreted in terms of betting rates pertaining to (Boolean) membership.

Vagueness as limited perception. Parikh (1983)'s view can be related to the above representations. For him, the idea of vagueness stems from a perception problem, namely the difficulty of defining (crisp) predicates on "observationally connected spaces" (e.g., colors) having insufficiently separated elements. Then, rather than advocating a fuzzy set modeling in such a case (as, e.g. Kay and Mc Daniel (1975)), Parikh considers that the difficulty to assign borderline elements or values to A (or to ´A), is due to a lack of capability to discern or distinguish between them, since they are too close. So the boundary between the extensions of A and ´A is ill-known, even if there are elements that can be considered as clearly in A. More formally suppose two elements u and v in D_a are indiscernible as soon as d(u, v) † e, where d is a distance function on D_a and e is an indiscernibility threshold. Then, each element u in D_a is perceived as the subset [u] = {v, d(u, v) † e}. So Y_A = {u, [u] ⊆ A}, and N_A = {u, [u] ⊆ ´A}. Any elements u in A and v in ´A such that d(u, v) † e will be perceived as lying in the borderline area B_A .

Another case of the same kind, where a borderline area may occur, is when the attribute range D_a is replaced by clusters forming a partition of similar elements. This is the case when considering a coarsening (or granulation) of the attribute range (e.g., measuring heights in centimeters instead of millimeters). There is a classical equivalence relation on D_a and each element u in D_a is perceived as the equivalence class [u]. The partition (A, ´A) of D_a is again perceived as a trichotomy, as previously, here due to a coarse scale.

Supervaluations. In all the above settings, although the agent is not able to locate the boundary between A and ´A, he still assumes that the excluded-middle and contradiction laws (EM), (NC) hold. Indeed, K. Fine (1975), when advocating the idea of Òsuper-truth,Ó proposes that statements about a vague predicate be ÒsupertrueÒ if and only if they hold for all possible ways of making the predicate precise; see also (van Fraassen, 1969; Keefe, 2000). It enables all classical logical relationships between a vague predicate A and its negation not-A to be preserved. See Sanford (1976, 1979) for various points of view about the idea of super-truth. This view looks close to Williamson (1994)Õs view of vagueness. For him, Y_A corresponds to those elements which are Òclearly AÓ

Ill-known partial membership. Semantic ambiguity may also take place with gradual properties. The imprecision of μ_A can be captured by a type-2 fuzzy set (Mizumoto and Tanaka, 1976), where $\mu_A(u)$ is itself a fuzzy set of [0, 1]. A particular case called "interval-valued fuzzy set" is when $\mu_A(u)$ is an ordinary sub-interval of [0, 1] (Grattan-Guiness, 1975), also called *vague sets* by Gau and Buehrer (1993). Atanassov (1986; 1999) extends Narin'yani sub-definite sets by defining a so-called "intuitionistic fuzzy set (IFS)" *i*A as a pair of membership functions (μ_{A+}, μ_{A-}), where $\mu_{A+}(u)$ is the degree of membership of u in *i*A and $\mu_{A-}(u)$is its degree of non-membership. The two membership functions are supposed to verify the constraint $\mu_{A+}(u) + \mu_{A-}(u) \leq 1$. The name "intuitionistic" stems from this inequality that is supposed to express a rejection of the excluded middle law but, the negation being involutive in this theory (it amounts to swapping μ_{A+} and μ_{A-}),the name is misleading. In fact Atanassov's construct is isomorphic to interval-valued fuzzy sets. See Bustince and Burillo (1996), Deschrijver and Kerre (2003) for instance.

5. Refining precisely-defined properties using closeness relations

The situation considered in the previous section can be viewed as a case where the information for deciding between A and ¬A is poor or incomplete. Under rich information, precisely delimited extensions may also lead to a trichotomy of D_a if it is possible to measure how close any two attribute values are from each other. Here, we do not assume any perception deficiency: the agent can always distinguish between any two distinct attribute values u and v, no matter how close, so that *A* and *¬A* are *well-known* and form a partition of D_a. Consider there is a graded closeness or similarity relation S: $D_a \times D_a \to [0, 1]$ which is

- reflexive°: S(u, u) = 1
- symmetric°: S(u, v) = S(v, u)

- separating°: S(u, v) < 1 whenever u › v

S(u, v) is all the greater as u and v are close to each other. It is a monotonically decreasing function of a distance. The separating property is essential here to indicate perfect perception of the boundary of A by the agent. Then we can define the fuzzy set of *central elements* of A and ´A by means of the membership functions

$$\mu_{YA}(u) = 1 - \sup\{S(u, v) \mid v \in {}´A\}, \text{ and } \mu_{NA}(u) = 1 - \sup\{S(u, v) \mid v \in A\}.$$

Here, the Boolean representation is refined by making Y_A and N_A gradual. Elements not in A, but outside the core of N_A lie in the vicinity of A, and can be used for interpolation reasoning (Ruspini, 1991). If $\mu_{YA}(u) > 0$ then necessarily $u \in A$, so Y_A is indeed included in A. Moreover, u is a fully central element for A (i.e. $\mu_{YA}(u) = 1$) as soon as u is totally dissimilar from some element v of ´A (i.e. S(u, v) = 0).

In some sense this situation is opposite to the one in the previous section. There, assuming A is a binary property, we could explain the lack of knowledge about its boundary using an indiscernibility relation induced by a perception threshold, or some uncertainty measure. Making this indiscernibility gradual, we get formally the same expressions as above for computing valued (fuzzy) counterparts to Y_A and N_A, but the meaning is very different: In the previous section, the boundary region contained elements of uncertain membership. Here on the contrary A is well defined but we are interested in describing central elements of A, that lie far away from elements of ´A. The similarity relation enables a membership function for the fuzzy set of central elements of A to be derived.

For instance, consider marks, in the range [0, 20], of exams for pupils. It is perfectly known that those having mark 10 or more succeed, while the other fail. Yet, the really successful pupils are those whose marks are really higher than 10, while the really unsuccessful ones are those whose marks are really lower than 10. Here the graduality of Y_A and N_A makes the representation more expressive, and does not convey any idea of uncertainty.

6. Single agent vs. multiple agents

Another source of vagueness is when different extensions of a property A (and ´A) are provided by a set of agents, even if each agent perceives A as a classical property. Indeed, let Y_{Ai} and N_{Ai} = ´Y_{Ai} be the dichotomic (agent-dependent) representations of A for agent i (i = 1,Én). Assume for simplicity that they are classical extensions. This situation implicitly generates a partition of D_a in 3 regions:

$$Y_A = \cap_i Y_{Ai},\ N_A = \cap_i N_{Ai},\ B_A = D_a — (\cap_i Y_{Ai} \cup \cap_i N_{Ai}).$$

In extreme cases we may have $\cap_i Y_{Ai} = \emptyset$ or $\cap_i N_{Ai} = \emptyset$ if the agents are fully inconsistent altogether.

In the case of multiple agents, it is natural to try to summarize the different points of view. One way to do it is to attach to each (Y_{Ai}, N_{Ai}) the weight m(Y_{Ai}) given by the proportion of agents who consider that the correct extension of A is Y_{Ai}. Then $\sum_i m(Y_{Ai}) = 1$.

From the proportion of individuals thinking that A_i properly expresses A, we can define the grade of membership

$$\mu_A(u) = \Sigma_{Y_{Aj}:\, u \in Y_{Aj}}\, m(Y_{Ai}). \quad (1)$$

of u to the (agent-dependent) concept A. $\mu_A(u)$ estimates the extent to which the value u is globally compatible with the meaning of A. This is formally expressed under the form of a random set or equivalently of a body of evidence in the sense of Shafer. It becomes Zadeh's definition of a fuzzy set exactly, as soon as the family $\{Y_{Ai}, m(Y_{Ai}) > 0\}$ is a nested family so that the knowledge of the membership function μ_A is equivalent to that of the probabilities $m(Y_{Ai})$ (see Dubois and Prade, 1989, 1990). Of course this nested property is seldom observed in practice, since the Y_{Ai}'s come from different agents. However consonant (nested) approximations of dissonant bodies of evidence exist (Dubois and Prade, 1989) which are especially very good when $\cap_i Y_{Ai} \neq \emptyset$, a usually satisfied consistency requirement which expresses that they exists at least one value in D_a totally compatible with the concept for everybody in a given context. Hence a fuzzy set, with membership function μ_A: $D_a \rightarrow [0, 1]$, can always be used as an approximation of a random set. Such a construct can be used for measuring the membership function μ_A of a fuzzy set A (e.g., 'young') in a given context. Then A is a fuzzy set and Y_{Ai} is a crisp realization of the idea of fuzzy set A for an individual i.

A simpler, but related experiment consists in asking each agent i, for each value u, if u is or is not in the extension of A. Then $\mu_A(u)$ would just reflect the proportion of individuals who answer that u is in the extension of A; psychologists have used this for getting fuzzy set membership functions (e.g., Hersch and Caramazza, 1974). Then μ_A is obtained via a likelihood function P('A'|u). This view is also a translation of Cheeseman (1986)Õs definition of vagueness.

These two probability-oriented views (random sets and likelihood functions) of fuzzy sets are not antagonistic and can be reconciled. The random set view corresponds to an experiment whereby individuals are asked to point out a single crisp subset $Y_{Ai} \subseteq U$ that best represents some fuzzy concept A. The weight $m(Y_{Ai})$ represents the proportion of individuals for which A is best described by Y_{Ai}. It makes sense to relate this experiment with the likelihood Yes-No experiment provided that, when each individual chooses A_i as representing A, it means that, in the other experiment, he would answer Yes to the question "is u an 'A' ?" if and only if $u \in A_i$. Then, as pointed out in Dubois and Prade (1990), the likelihood function and the random set view are in agreement, i.e.,

$$P('A'|u) = \sum_{u \in Y_{Ai}} m(Y_{Ai}).$$

When performing logical operations, on the representations of A and B, there are then two possibilities: i) perform the operation for each agent i on Y_{Ai} and Y_{Bi} and then compute the resulting membership functions, or ii) perform the operation on the membership functions (the summarized views) of A and B. Clearly the first option is most respectful of the agents.

Another approach defines multiple-agent vagueness directly on the set of objects O without making the attribute scale explicit (Lawry, 2004). More precisely, it starts with a term set $\mathcal{T}$ of predicate symbols $\{A_k, k = 1,\ldots n\}$ which describe linguistic values of some attribute. For a given object $o \in O$, each agent points out a subset $S_i(o)$ of terms, each of which is considered compatible with the object. Let $m(S_i)$ be the proportion of agents who consider that the set $S_i(o)$ of terms is compatible with object o. The fuzzy extension of a predicate A on the set of objects is defined as

$$\mu_{Ext(A)}(o) = \sum_{Sj:\, A \in Sj} m(S_i(o)).$$

The advantage of this approach is the possibility to apply it to abstract predicates whose underlying measurement scales are not obvious to lay bare.

The idea of agent-dependent concept applied to a gradual property gives birth to the notion of probabilistic set (Hirota, 1981; Czogala and Hirota, 1986) where membership degrees are known only through probability distributions. Lastly, Halpern (2004) envisages an approach to vagueness combining both the idea of variability of the meaning across several agents and limited perception within each agent, modelled by non-transitive indiscernibility relations.

7. Ill-known attribute values and twofold sets

In most of the previous situations, vagueness stemmed from peculiarities of the way an agent perceived an attribute scale. In each case the vagueness of extension of A on the set of objects was a direct reflection of the vagueness of the representation of the property A on the attribute scale. Here we assume A is perceived as classical, so that Y_A and N_A partition the attribute domain D_a in the classical sense. However, the agent knowledge about the values of attribute *a* for objects is uncertain, or just incomplete. The set of objects which are A is then ill-known. For instance, if for each object *o*, $p_{a(o)}$ denotes the subjective probability distribution of the possible values for *a*(*o*) according to the agent, we note that

$$\text{Prob}(a(o) \in \text{A}) = \Sigma_{u \in \_ \text{A}}\, p_{a(o)}(u);$$

and then define the fuzzy set of objects satisfying A as having membership function

$$\mu_{\text{Ext(A)}}(o) = \text{Prob}(a(o) \in \text{A}).$$

Clearly, the boundary between objects that satisfy A and those that do not is blurred. Only for well-known objects do we have that $o \in$ Ext(A) or not. The excluded middle and contradiction laws on D_a imply $\mu_{\text{Ext(A)}}(o) + \mu_{\text{Ext(´A)}}(o) = 1$.

When the information is poorer, we can still define the extensions. Suppose all the agent knows about o is that $a(o) \in$ I(o), a subset of D_a. Then the agent knows for sure that ˙°o is A°˙˙ if and only if I(o) $\subseteq Y_A$ and that ˙°o is not A°˙˙ if and only if I(o) $\subseteq N_A$. If neither condition holds, o is borderline, it is neither in Ext(A) = {o, I(o) $\subseteq Y_A$ }, not in Ext(´A) = {o, I(o) $\subseteq N_A$}, which are disjoint and no longer form a partition of O.

More generally, for each object there is a possibility distribution $\pi_{a(o)}$ describing the more or less possible values of *a*(*o*) for each object. Then one may compute (Dubois and Prade, 1987):

- to what extent it is possible that object *o* has property A:

$$\forall\, o \in \text{O},\ \Pi(a(o) \in \text{A}) = \sup_{u \in \text{A}} \pi_{a(o)}(u) = 1 - \mu_{\text{Ext(´A)}}(o);$$

- to what extent it is certain that object *o* has property A:

$$\forall\, o \in \text{O},\ \text{N}(a(o) \in \text{A}) = 1 - \Pi(a(o) \in \text{´A}) = \mu_{\text{Ext(A)}}(o)$$

Since $\text{N}(a(o) \in \text{A}) > 0$ implies $\Pi(a(o) \in \text{A}) = 1$, the (fuzzy) set of objects Ext(A) which are more or less certainly A is disjoint from the (fuzzy) set of objects Ext(´A) which are more or less certainly ´A. In this situation, the set of objects having property A is ill-known because of imprecise descriptions of these objects. It gives birth to a trichotomic structure, not in the attribute domain D_a but on the set of objects O. Namely the set of objects A* which are certainly A (such that $\mu_{Ext(A)}(o) = 1$), the set (´A)* of those which are certainly ´A (such that $\mu_{Ext(´A)}(o) = 1$), and the boundary set O – {A* ∪ (´A)*}.

8. Approximately described sets

Yet another situation where a notion of vagueness appears in the set of objects O, rather than in the attribute scale, is when the language induced by the set of attributes $\mathcal{A}$ does not allow to accurately describe some subsets E of O. This is because there may be several objects sharing the same description. In such a case, we can only define a lower and an upper approximation of a given set E of objects, since objects having the same attribute values for an attribute *a* are indiscernible from the point of view of this attribute. This is actually the starting point of rough set theory (Pawlak, 1991).

Consider first an attribute *a* and define, for each $u \in D_a$, the equivalence classes

$$[O]_u = a^{-1}(u) = \{o \in O \mid a(o) = u\}.$$

Note that for $u \neq v$, $[O]_u \cap [O]_v = \emptyset$. This can be generalized to a subset $B = \{a_1, a_2, \ldots, a_r\}$ of attributes, taking $[O]_{\mathbf{u}} = \{o \in O \mid B(o) = \mathbf{u}\}$ for defining an equivalence class for each r-tuple $\mathbf{u} = (u_1, \ldots, u_r) \in D_{a_1} \times \ldots \times D_{a_r}$ where $B(o) = \mathbf{u}$ stands for $a_1(o) = u_1, \ldots, a_r(o) = u_r$. We can then define the lower and upper approximations of E in O respectively as:

$$E_* = \{o \in O \mid [O]_{B(o)} \subseteq E\}, \quad E^* = \{o \in O \mid [O]_{B(o)} \cap E \neq \emptyset\}.$$

And then clearly $E_* \subseteq E \subseteq E^*$. This again leads to a trichotomic structure: (E_*, $E^* - E_*$, ´(E^*)).

The rough set model can be enriched by dealing with fuzzy indiscernibility relations or fuzzy partitions instead of equivalence classes, or by approximating fuzzy subsets in O (rather than classical subsets only). See (Dubois and Prade, 1992). The latter is a combination between the rough set scenario and gradual predicates. Boixader et al. (2000) discuss the approximation of fuzzy sets using similarity relations.

9. Concluding remarks

Overall, there are basically three types of approaches to vagueness (see Table 1):

- Approaches admitting from the start that propositions or predicates may fail to be Boolean. It corresponds essentially to the first scenario underlying the fuzzy set paradigm and many-valued logics. The intended meaning of propositions is richer than what the use of Boolean variables to represent them might suggest. This situation is precisely what Zadeh calls fuzzy (and not vague). Scenario 3 (in table 1) can be viewed as belonging to the same family of thought, since after defining properties as essentially Boolean, some refinement separating central from peripheral elements is introduced by means of the underlying distance on the attribute scale. Again it is an enrichment of the Boolean representation. All other approaches preserve the Boolean representation convention, and consider vagueness (and graduality) as stemming from a deficiency of the information.

- Approaches which claim that the boundary between values satisfying a proposition exist but is ill-known due to the limited perception of an agent or the conflicting views of several agents. It corresponds to scenarios 2 and 4 respectively. Ill-defined properties induce a vague classification of objects. Moreover, since properties are still Boolean even if their boundaries are ill-known, the basic laws of classical logic are retained such as the laws of excluded middle and contradiction. This is a natural view of vagueness understood as semantic ambiguity.

Features / Scenarios	Boolean notion	Boundary on the attribute scale	Ill-known extensions	Cause of vagueness
1 Gradual predicates	No	no boundary	No	Typicality Continuity
2 Uncertain boundaries	Yes	Yes, but ill-known	Yes	Limited perception
3 Closeness	Yes, but refined	Yes	No	Metric space
4 Multiagent	Yes	Yes, but ill-known	Yes	Conflict
5 Ill-known objects	Yes	Yes	Yes	Incomplete information
6 Ill-described objects	Yes	Yes	Yes	Lack of attributes

Table 1: Overview of scenarios for vagueness

- Approaches where the information defect lies in the difficulty to describe objects by means of suitable attributes, either because attribute values of objects are ill-known (like in scenario 5) or because there are not enough attributes to ensure a bijection between the set of objects and the set of descriptions (scenario 6). In that case, vagueness is only reflected in a limited capability of classifying objects, but it does not affect the representation of properties on attribute ranges.

Clearly these basic scenarios can be combined into more complex ones where several key vagueness-generating features are present at the same time. A natural follow-up of the above investigation is the study of set-theoretic connectives that may be used to logically combine vague properties in the light of the above scenarios. This issue is the major cause for controversies in the philosophical literature and computer science journals (see Elkan, 1994). We believe that part of these controversies stem from misunderstandings due to a failure to acknowledge the existence of several types of vagueness phenomena and the temptation to comment proposals made within a certain scenario in the context of another scenario. A typical confusion is between degrees of truth (underlying scenario 1) and degrees of uncertainty (underlying scenarios 2 and 4).

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