

THEORY Relation**IMPORT THEORY PROJECTS**/SimpleDEq **THEORIES** /SimpleDEq/Ring.dtf | org.eventb.theory.core.deployedTheoryRoot#Ring**TYPE PARAMETERS** S**OPERATORS****reflexive** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition** $\forall x \cdot x \in S \Rightarrow ((x \mapsto x) \in \text{rel})$ **antireflexive** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition** $\forall x \cdot x \in S \Rightarrow (x \mapsto x \notin \text{rel})$ **symmetrical** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition** $\forall x, y \cdot x \in S \wedge y \in S \Rightarrow ((x \mapsto y \in \text{rel}) \Rightarrow (y \mapsto x \in \text{rel}))$ **asymmetrical** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition** $\forall x, y \cdot x \in S \wedge y \in S \Rightarrow ((x \mapsto y \in \text{rel}) \Rightarrow (y \mapsto x \notin \text{rel}))$ **antisymmetrical** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition** $\forall x, y \cdot x \in S \wedge y \in S \Rightarrow ((x \mapsto y \in \text{rel}) \wedge (y \mapsto x \in \text{rel}) \Rightarrow x = y)$ **transitive** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition** $\forall x, y, z \cdot x \in S \wedge y \in S \wedge z \in S \Rightarrow ((x \mapsto y \in \text{rel}) \wedge (y \mapsto z \in \text{rel}) \Rightarrow (x \mapsto z \in \text{rel}))$ **total** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition** $\forall x, y \cdot x \in S \wedge y \in S \Rightarrow ((x \mapsto y \in \text{rel}) \vee (y \mapsto x \in \text{rel}))$ **equivalence** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition**reflexive(rel) \wedge symmetrical(rel) \wedge transitive(rel)**order** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition**reflexive(rel) \wedge transitive(rel) \wedge antisymmetrical(rel)**strict** <expression> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition**{ x \mapsto y | x \mapsto y \in rel \wedge x \neq y }**wellFounded** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition** $\forall X \cdot X \subseteq S \wedge X \neq \emptyset \Rightarrow (\exists m \cdot m \in X \wedge (\forall x \cdot x \in X \Rightarrow (x \mapsto m \notin \text{rel})))$ **wellPartialOrder** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition**order(rel) \wedge wellFounded(strict(rel))**wellOrder** <predicate> (rel: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$ **direct definition**order(rel) \wedge total(rel) \wedge wellFounded(strict(rel))**covers** <predicate> (rel: S \leftrightarrow S, a: S, b: S)**well-definedness** S $\neq \emptyset \wedge$ order(rel)**direct definition**(a \mapsto b \in rel) \wedge a \neq b \wedge ($\forall c \cdot c \in S \wedge (a \mapsto c \in \text{rel}) \wedge (c \mapsto b \in \text{rel}) \Rightarrow ((c = a) \vee (b = a)))$ **compose** <expression> (rel1: S \leftrightarrow S, rel2: S \leftrightarrow S)**well-definedness** S $\neq \emptyset$

74 **direct definition**

75 $\{ x, z \cdot x \in S \wedge z \in S \wedge (\exists y \cdot y \in S \Rightarrow ((x \mapsto y \in \text{rel1}) \wedge (y \mapsto z \in \text{rel2}))) \mid x \mapsto z \}$

76 **converse** <expression> (rel: S ↔ S)

77 **well-definedness** S ≠ ∅

78 **direct definition**

79 $\{ x, y \cdot x \in S \wedge y \in S \wedge y \mapsto x \in \text{rel} \mid x \mapsto y \}$

80 **complement** <expression> (rel: S ↔ S)

81 **well-definedness** S ≠ ∅

82 **direct definition**

83 $\{ x, y \cdot x \in S \wedge y \in S \wedge x \mapsto y \notin \text{rel} \mid x \mapsto y \}$

84 **equality** <expression> ()

85 **well-definedness** S ≠ ∅

86 **direct definition**

87 $\{ x \mapsto y \mid x \in S \wedge y \in S \wedge x = y \}$

88 **preorder** <predicate> (rel: S ↔ S)

89 **well-definedness** S ≠ ∅

90 **direct definition**

91 reflexive(rel) ∧ transitive(rel)

92 **equivalenceClass** <expression> (rel: S ↔ S, x: S)

93 **well-definedness** S ≠ ∅ ∧ equivalence(rel)

94 **direct definition**

95 $\{ y \cdot y \in S \wedge (x \mapsto y) \in \text{rel} \mid y \}$

96 **leftGeneralized** <expression> (rel: S ↔ S)

97 **well-definedness** S ≠ ∅

98 **direct definition**

99 $\{ x, P \cdot x \in S \wedge P \in \mathbb{P}(S) \wedge P \neq \emptyset \wedge (\forall y \cdot y \in P \Rightarrow (x \mapsto y \in \text{rel})) \mid x \mapsto P \}$

100 **rightGeneralized** <expression> (rel: S ↔ S)

101 **well-definedness** S ≠ ∅

102 **direct definition**

103 $\{ P, x \cdot P \in \mathbb{P}(S) \wedge P \neq \emptyset \wedge x \in S \wedge (\forall y \cdot y \in P \Rightarrow (y \mapsto x \in \text{rel})) \mid P \mapsto x \}$

104 **upperBound** <predicate> (ord: S ↔ S, T: ℙ(S), B: S)

105 **direct definition**

106 $\forall t \cdot t \in T \Rightarrow t \mapsto B \in \text{ord}$

107 **lowerBound** <predicate> (ord: S ↔ S, T: ℙ(S), B: S)

108 **direct definition**

109 $\forall t \cdot t \in T \Rightarrow B \mapsto t \in \text{ord}$

110 **bounds** <predicate> (ord: S ↔ S, T: ℙ(S), m: S, M: S)

111 **direct definition**

112 lowerBound(ord, T, m) ∧ upperBound(ord, T, M)

113 **upperBounded** <predicate> (ord: S ↔ S, T: ℙ(S))

114 **direct definition**

115 $\exists M \cdot M \in S \wedge \text{upperBound}(\text{ord}, T, M)$

116 **lowerBounded** <predicate> (ord: S ↔ S, T: ℙ(S))

117 **direct definition**

118 $\exists m \cdot m \in S \wedge \text{lowerBound}(\text{ord}, T, m)$

119 **bounded** <predicate> (ord: S ↔ S, T: ℙ(S))

120 **direct definition**

121 $\exists m, M \cdot m \in S \wedge M \in S \wedge \text{bounds}(\text{ord}, T, m, M)$

122 **supremum** <predicate> (ord: S ↔ S, T: ℙ(S), M: S)

123 **direct definition**

124 upperBound(ord, T, M) ∧
125 $(\forall m \cdot m \in S \wedge \text{upperBound}(\text{ord}, T, m) \Rightarrow M \mapsto m \in \text{ord})$

126 **infimum** <predicate> (ord: S ↔ S, T: ℙ(S), m: S)

127 **direct definition**

128 lowerBound(ord, T, m) ∧
129 $(\forall M \cdot M \in S \wedge \text{lowerBound}(\text{ord}, T, M) \Rightarrow M \mapsto m \in \text{ord})$

130 **maximal** <predicate> (ord: S ↔ S, T: ℙ(S), M: T)

131 **direct definition**

132 $\forall x \cdot x \in T \wedge M \mapsto x \in \text{ord} \Rightarrow x = M$

133 **minimal** <predicate> (ord: S ↔ S, T: ℙ(S), M: T)

134 **direct definition**

135 $\forall x \cdot x \in T \wedge x \mapsto M \in \text{ord} \Rightarrow x = M$

136 **maximum** <predicate> (ord: S ↔ S, T: ℙ(S), M: S)

137 **direct definition**

138 $(M \in T) \wedge (\forall x \cdot x \in T \Rightarrow x \mapsto M \in \text{ord})$

139 **minimum** <predicate> (ord: S ↔ S, T: ℙ(S), M: S)

140 **direct definition**

141 $(M \in T) \wedge (\forall x \cdot x \in T \Rightarrow M \mapsto x \in \text{ord})$

142 **hasMaximum** <predicate> (ord: S ↔ S, T: ℙ(S))

143 **direct definition**

144 $\exists M \cdot M \in T \wedge \text{maximum}(\text{ord}, T, M)$

145 **hasMinimum** <predicate> (ord: S ↔ S, T: ℙ(S))

146 **direct definition**

147 $\exists m \cdot m \in T \wedge \text{minimum}(\text{ord}, T, m)$

148 **monoidCompatible** <predicate> (op: (S × S) → S, e: S, rel: S ↔ S)

149 **well-definedness** S ≠ ∅ ∧ Monoid(op, e) ∧ order(rel)

150 **direct definition**

151 $\forall x, y, z \cdot x \in S \wedge y \in S \wedge z \in S \wedge (x \mapsto y \in \text{rel}) \Rightarrow ($

152 $((\text{op}(x \mapsto z) \mapsto \text{op}(y \mapsto z)) \in \text{rel}) \wedge$

153 $((\text{op}(z \mapsto x) \mapsto \text{op}(z \mapsto y)) \in \text{rel})$

154 $)$

155 **ringCompatible** <predicate> (oplus: (S × S) → S, otimes: (S × S) → S, azero: S, unit: S, rel: S ↔ S)

156 **well-definedness** S ≠ ∅ ∧ Ring(oplus, otimes, azero, unit) ∧ order(rel)

157 **direct definition**

158 monoidCompatible(oplus, azero, rel) ∧ (

159 $\forall x, y \cdot x \in S \wedge y \in S \wedge (\text{azero} \mapsto x \in \text{rel}) \wedge (\text{azero} \mapsto y \in \text{rel}) \Rightarrow$

160 $((\text{azero} \mapsto \text{otimes}(x \mapsto y)) \in \text{rel}) \wedge (\text{azero} \mapsto \text{otimes}(y \mapsto x)) \in \text{rel}))$

161 $)$

162 **AXIOMATIC DEFINITIONS** operations:

163 **OPERATORS**

164 **Gmax** <expression> (ord: S ↔ S, T: P(S)) : S

165 **Gmin** <expression> (ord: S ↔ S, T: P(S)) : S

166 **Gsup** <expression> (ord: S ↔ S, T: P(S)) : S

167 **Ginf** <expression> (ord: S ↔ S, T: P(S)) : S

168 **AXIOMS**

169 *GmaxDef:*

170 $\forall \text{ord}, T \cdot \text{ord} \in S \leftrightarrow S \wedge T \subseteq S \wedge \text{hasMaximum}(\text{ord}, T) \Rightarrow (\text{maximum}(\text{ord}, T, \text{Gmax}(\text{ord}, T)))$

171 *GminDef:*

172 $\forall \text{ord}, T \cdot \text{ord} \in S \leftrightarrow S \wedge T \subseteq S \wedge \text{hasMinimum}(\text{ord}, T) \Rightarrow (\text{minimum}(\text{ord}, T, \text{Gmin}(\text{ord}, T)))$

173 *GsupDef:*

174 $\forall \text{ord}, T \cdot \text{ord} \in S \leftrightarrow S \wedge T \subseteq S \wedge \text{upperBounded}(\text{ord}, T) \Rightarrow (\text{supremum}(\text{ord}, T, \text{Gsup}(\text{ord}, T)))$

175 *GinfDef:*

176 $\forall \text{ord}, T \cdot \text{ord} \in S \leftrightarrow S \wedge T \subseteq S \wedge \text{lowerBounded}(\text{ord}, T) \Rightarrow (\text{infimum}(\text{ord}, T, \text{Ginf}(\text{ord}, T)))$

177 **THEOREMS**

178 *converseDomain:*

179 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \Rightarrow (\text{dom}(\text{converse}(\text{rel})) = \text{ran}(\text{rel}))$

180 *converseRange:*

181 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \Rightarrow (\text{ran}(\text{converse}(\text{rel})) = \text{dom}(\text{rel}))$

182 *converseInvolutive:*

183 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \Rightarrow (\text{converse}(\text{converse}(\text{rel})) = \text{rel})$

184 *converseSymetry:*

185 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{symetrical}(\text{rel}) \Rightarrow \text{symetrical}(\text{converse}(\text{rel}))$

186 *converseReflexivity:*

187 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{reflexive}(\text{rel}) \Rightarrow \text{reflexive}(\text{converse}(\text{rel}))$

188 *converseAntireflexivity:*

189 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{antireflexive}(\text{rel}) \Rightarrow \text{antireflexive}(\text{converse}(\text{rel}))$

190 *complementInvolutive:*

191 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \Rightarrow (\text{complement}(\text{complement}(\text{rel})) = \text{rel})$

192 *complementConverse:*

193 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \Rightarrow (\text{complement}(\text{converse}(\text{rel})) = \text{converse}(\text{complement}(\text{rel})))$

194 *complementSymetry:*

195 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{symetrical}(\text{rel}) \Rightarrow \text{symetrical}(\text{complement}(\text{rel}))$

196 *complementReflexivity:*

197 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{reflexive}(\text{rel}) \Rightarrow \text{antireflexive}(\text{complement}(\text{rel}))$

198 *complementAntireflexivity:*

199 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{antireflexive}(\text{rel}) \Rightarrow \text{reflexive}(\text{complement}(\text{rel}))$

200 *totalRelationsReflexivity:*

201 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{total}(\text{rel}) \Rightarrow \text{reflexive}(\text{rel})$

202 *equivalenceClassEquity:*

203 $\forall \text{rel}, x, y \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{equivalence}(\text{rel}) \wedge x \in S \wedge y \in S \Rightarrow ($

204 $((x \mapsto y) \in \text{rel}) \Leftrightarrow (\text{equivalenceClass}(\text{rel}, x) = \text{equivalenceClass}(\text{rel}, y))$

205 $)$

206 *equivalenceClassNotEmpty:*

207 $\forall \text{rel}, x \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{equivalence}(\text{rel}) \wedge x \in S \Rightarrow$

208 $(\text{equivalenceClass}(\text{rel}, x) \neq \emptyset)$

209 *equivalenceClassCover:*

210 $\forall \text{rel} \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{equivalence}(\text{rel}) \Rightarrow ($

211 $(\bigcup \text{equivalenceClass}(\text{rel}, x) \mid x \in S) = S$

212 $)$

213 *equivalenceClassDisjoint:*

214 $\forall \text{rel}, x, y \cdot \text{rel} \in (S \leftrightarrow S) \wedge \text{equivalence}(\text{rel}) \wedge x \in S \wedge y \in S \wedge (x \mapsto y \notin \text{rel}) \Rightarrow ($

215 $(\text{equivalenceClass}(\text{rel}, x) \cap \text{equivalenceClass}(\text{rel}, y)) = \emptyset$

216 $)$

217 *supremumMaximal:*

218 $\forall \text{ord}, T, M \cdot \text{ord} \in (S \leftrightarrow S) \wedge \text{order}(\text{ord}) \wedge T \subseteq S \wedge M \in S \Rightarrow$

219 $(\text{supremum}(\text{ord}, T, M) \wedge M \in T \Rightarrow \text{maximal}(\text{ord}, T, M))$

220 *infimumMinimal:*

221 $\forall \text{ord}, T, M \cdot \text{ord} \in (S \leftrightarrow S) \wedge \text{order}(\text{ord}) \wedge T \subseteq S \wedge M \in S \Rightarrow$

222
223

$(\text{infimum}(\text{ord}, T, M) \wedge M \in T \Rightarrow \text{minimal}(\text{ord}, T, M))$

END