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THEORY Intervals
  IMPORT THEORY PROJECTS
    /SimpleDEq THEORIES /SimpleDEq/Reals.dtf|org.eventb.theory.core.deployedTheoryRoot#Reals
  OPERATORS
    Infinity2Open <expression> (b: RReal)
      direct definition
        { t | t ↦ b ∈ lt }
    Infinity2Closed <expression> (b: RReal)
      direct definition
        { t | t ↦ b ∈ leq }
    Open2Infinity <expression> (a: RReal)
      direct definition
        { t | a ↦ t ∈ lt }
    Closed2Infinity <expression> (a: RReal)
      direct definition
        { t | a ↦ t ∈ leq }
    Open2Open <expression> (a: RReal,b: RReal)
      direct definition
        { t | a ↦ t ∈ lt ∧ t ↦ b ∈ lt }
    Open2Closed <expression> (a: RReal,b: RReal)
      direct definition
        { t | a ↦ t ∈ lt ∧ t ↦ b ∈ leq }
    Closed2Open <expression> (a: RReal,b: RReal)
      direct definition
        { t | a ↦ t ∈ leq ∧ t ↦ b ∈ lt }
    Closed2Closed <expression> (a: RReal,b: RReal)
      direct definition
        { t | a ↦ t ∈ leq ∧ t ↦ b ∈ leq }
  THEOREMS
    realPlusIsZero2Infinity:
      RRealPlus = Closed2Infinity(Rzero)
    realMinusIsInfinity2Zero:
      RRealMinus = Infinity2Closed(Rzero)
    c2c_existence:
      ∀ a,b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ leq ⇒ (∃ x · x ∈ Closed2Closed(a,b))
    c2o_existence:
      ∀ a,b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒ (∃ x · x ∈ Closed2Open(a,b))
    o2c_existence:
      ∀ a,b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒ (∃ x · x ∈ Open2Closed(a,b))
    o2o_existence:
      ∀ a,b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒ (∃ x · x ∈ Open2Open(a,b))
    boundaryInClosed2Closed:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ⇒ (a ∈ Closed2Closed(a,b) ∧ b ∈ Closed2Closed(a,b))
    boundaryInClosed2Open:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ⇒ (a ∈ Closed2Open(a,b))
    boundaryInOpen2Closed:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ⇒ (b ∈ Open2Closed(a,b))
    boundaryInInfinity2Closed:
      ∀ b · b ∈ RReal ⇒ (b ∈ Infinity2Closed(b))
    boundaryInClosed2Infinity:
      ∀ a · a ∈ RReal ⇒ (a ∈ Closed2Infinity(a))
    closed2ClosedLowerBound:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ leq ⇒
        lowerBound(leq, Closed2Closed(a,b), a)
    closed2OpenLowerBound:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        lowerBound(leq, Closed2Open(a,b), a)
    open2ClosedLowerBound:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        lowerBound(leq, Open2Closed(a,b), a)
    open2OpenLowerBound:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        lowerBound(leq, Open2Open(a,b), a)
    closed2InfinityLowerBound:
      ∀ a · a ∈ RReal ⇒
        lowerBound(leq, Closed2Infinity(a), a)
    open2InfinityLowerBound:
      ∀ a · a ∈ RReal ⇒
        lowerBound(leq, Open2Infinity(a), a)
    infinity2ClosedLowerBound:
      ∀ b · b ∈ RReal ⇒
        ¬ lowerBounded(leq, Infinity2Closed(b))
    infinity2OpenLowerBound:
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      ∀ b · b ∈ RReal ⇒
        ¬ lowerBounded(leq, Infinity2Open(b))
    closed2ClosedInfimum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ leq ⇒
        infimum(leq, Closed2Closed(a,b), a)
    closed2OpenInfimum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        infimum(leq, Closed2Open(a,b), a)
    open2ClosedInfimum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        infimum(leq, Open2Closed(a,b), a)
    open2OpenInfimum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        infimum(leq, Open2Open(a,b), a)
    closed2InfinityInfimum:
      ∀ a · a ∈ RReal ⇒
        infimum(leq, Closed2Infinity(a), a)
    open2InfinityInfimum:
      ∀ a · a ∈ RReal ⇒
        infimum(leq, Open2Infinity(a), a)
    closed2ClosedMinimum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ leq ⇒
        minimum(leq, Closed2Closed(a,b), a)
    closed2OpenMinimum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        minimum(leq, Closed2Open(a,b), a)
    open2ClosedMinimum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        ¬ hasMinimum(leq, Open2Closed(a,b))
    open2OpenMinimum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        ¬ hasMinimum(leq, Open2Open(a,b))
    closed2InfinityMinimum:
      ∀ a · a ∈ RReal ⇒
        minimum(leq, Closed2Infinity(a), a)
    open2InfinityMinimum:
      ∀ a · a ∈ RReal ⇒
        ¬ hasMinimum(leq, Open2Infinity(a))
    infinity2ClosedMinimum:
      ∀ b · b ∈ RReal ⇒
        ¬ hasMinimum(leq, Infinity2Closed(b))
    infinity2OpenMinimum:
      ∀ b · b ∈ RReal ⇒
        ¬ hasMinimum(leq, Infinity2Open(b))
    closed2ClosedUpperBound:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ leq ⇒
        upperBound(leq, Closed2Closed(a,b), b)
    closed2OpenUpperBound:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        upperBound(leq, Closed2Open(a,b), b)
    open2ClosedUpperBound:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        upperBound(leq, Open2Closed(a,b), b)
    open2OpenUpperBound:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        upperBound(leq, Open2Open(a,b), b)
    closed2InfinityUpperBound:
      ∀ a · a ∈ RReal ⇒
        ¬ upperBounded(leq, Closed2Infinity(a))
    open2InfinityUpperBound:
      ∀ a · a ∈ RReal ⇒
        ¬ upperBounded(leq, Open2Infinity(a))
    infinity2ClosedUpperBound:
      ∀ b · b ∈ RReal ⇒
        upperBound(leq, Infinity2Closed(b), b)
    infinity2OpenUpperBound:
      ∀ b · b ∈ RReal ⇒
        upperBound(leq, Infinity2Open(b), b)
    closed2ClosedSupremum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ leq ⇒
        supremum(leq, Closed2Closed(a,b), b)
    closed2OpenSupremum:
      ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
        supremum(leq, Closed2Open(a,b), b)
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open2ClosedSupremum:
  ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
    supremum(leq, Open2Closed(a,b), b)
open2OpenSupremum:
  ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
    supremum(leq, Open2Open(a,b), b)
infinity2ClosedSupremum:
  ∀ b · b ∈ RReal ⇒
    supremum(leq, Infinity2Closed(b), b)
infinity2OpenSupremum:
  ∀ b · b ∈ RReal ⇒
    supremum(leq, Open2Infinity(b), b)
closed2ClosedMaximum:
  ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ leq ⇒
    maximum(leq, Closed2Closed(a,b), b)
closed2OpenMaximum:
  ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
    ¬ hasMaximum(leq, Closed2Open(a,b))
open2ClosedMaximum:
  ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
    maximum(leq, Open2Closed(a,b), b)
open2OpenMaximum:
  ∀ a, b · a ∈ RReal ∧ b ∈ RReal ∧ a ↦ b ∈ lt ⇒
    ¬ hasMaximum(leq, Open2Open(a,b))
closed2InfinityMaximum:
  ∀ a · a ∈ RReal ⇒
    ¬ hasMaximum(leq, Closed2Infinity(a))
open2InfinityMaximum:
  ∀ a · a ∈ RReal ⇒
    ¬ hasMaximum(leq, Open2Infinity(a))
infinity2ClosedMaximum:
  ∀ b · b ∈ RReal ⇒
    maximum(leq, Infinity2Closed(b), b)
infinity2OpenMaximum:
  ∀ b · b ∈ RReal ⇒
    ¬ hasMaximum(leq, Infinity2Open(b))
c2cUc2c:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ leq ∧ b ↦ c ∈ leq ⇒
      Closed2Closed(a,b) ∪ Closed2Closed(b,c) = Closed2Closed(a,c)
c2cUc2o:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ leq ∧ b ↦ c ∈ lt ⇒
      Closed2Closed(a,b) ∪ Closed2Open(b,c) = Closed2Open(a,c)
o2cUc2c:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ leq ⇒
      Open2Closed(a,b) ∪ Closed2Closed(b,c) = Open2Closed(a,c)
o2cUc2o:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
      Open2Closed(a,b) ∪ Closed2Open(b,c) = Open2Open(a,c)
inf2cUc2c:
  ∀ b, c ·
    b ∈ RReal ∧ c ∈ RReal ∧
    b ↦ c ∈ leq ⇒
      Infinity2Closed(b) ∪ Closed2Closed(b,c) = Infinity2Closed(c)
inf2cUc2o:
  ∀ b, c ·
    b ∈ RReal ∧ c ∈ RReal ∧
    b ↦ c ∈ lt ⇒
      Infinity2Closed(b) ∪ Closed2Open(b,c) = Infinity2Open(c)
c2cUc2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
    a ↦ b ∈ leq ⇒
      Closed2Closed(a,b) ∪ Closed2Infinity(b) = Closed2Infinity(a)
o2cUc2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
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        a ↦ b ∈ lt ⇒
          Open2Closed(a,b) ∪ Closed2Infinity(b) = Open2Infinity(a)
    inf2cUc2inf:
      ∀ a ·
        a ∈ RReal ⇒
          Infinity2Closed(a) ∪ Closed2Infinity(a) = RReal
    c2cUo2c:
      ∀ a, b, c ·
        a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
        a ↦ b ∈ leq ∧ b ↦ c ∈ lt ⇒
          Closed2Closed(a,b) ∪ Open2Closed(b,c) = Closed2Closed(a,c)
    c2cUo2o:
      ∀ a, b, c ·
        a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
        a ↦ b ∈ leq ∧ b ↦ c ∈ lt ⇒
          Closed2Closed(a,b) ∪ Open2Open(b,c) = Closed2Open(a,c)
    o2cUo2c:
      ∀ a, b, c ·
        a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
        a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
          Open2Closed(a,b) ∪ Open2Closed(b,c) = Open2Closed(a,c)
    o2cUo2o:
      ∀ a, b, c ·
        a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
        a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
          Open2Closed(a,b) ∪ Open2Open(b,c) = Open2Open(a,c)
    inf2cUo2c:
      ∀ b, c ·
        b ∈ RReal ∧ c ∈ RReal ∧
        b ↦ c ∈ lt ⇒
          Infinity2Closed(b) ∪ Open2Closed(b,c) = Infinity2Closed(c)
    inf2cUo2o:
      ∀ b, c ·
        b ∈ RReal ∧ c ∈ RReal ∧
        b ↦ c ∈ lt ⇒
          Infinity2Closed(b) ∪ Open2Open(b,c) = Infinity2Open(c)
    c2cUo2inf:
      ∀ a, b ·
        a ∈ RReal ∧ b ∈ RReal ∧
        a ↦ b ∈ leq ⇒
          Closed2Closed(a,b) ∪ Open2Infinity(b) = Closed2Infinity(a)
    o2cUo2inf:
      ∀ a, b ·
        a ∈ RReal ∧ b ∈ RReal ∧
        a ↦ b ∈ lt ⇒
          Open2Closed(a,b) ∪ Open2Infinity(b) = Open2Infinity(a)
    inf2cUo2inf:
      ∀ a ·
        a ∈ RReal ⇒
          Infinity2Closed(a) ∪ Open2Infinity(a) = RReal
    c2oUc2c:
      ∀ a, b, c ·
        a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
        a ↦ b ∈ lt ∧ b ↦ c ∈ leq ⇒
          Closed2Open(a,b) ∪ Closed2Closed(b,c) = Closed2Closed(a,c)
    c2oUc2o:
      ∀ a, b, c ·
        a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
        a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
          Closed2Open(a,b) ∪ Closed2Open(b,c) = Closed2Open(a,c)
    o2oUc2c:
      ∀ a, b, c ·
        a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
        a ↦ b ∈ lt ∧ b ↦ c ∈ leq ⇒
          Open2Open(a,b) ∪ Closed2Closed(b,c) = Open2Closed(a,c)
    o2oUc2o:
      ∀ a, b, c ·
        a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
        a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
          Open2Open(a,b) ∪ Closed2Open(b,c) = Open2Open(a,c)
    inf2oUc2c:
      ∀ b, c ·
        b ∈ RReal ∧ c ∈ RReal ∧
        b ↦ c ∈ leq ⇒
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      Infinity2Open(b) ∪ Closed2Closed(b,c) = Infinity2Closed(c)
inf2oUc2o:
  ∀ b, c ·
    b ∈ RReal ∧ c ∈ RReal ∧
    b ↦ c ∈ lt ⇒
      Infinity2Open(b) ∪ Closed2Open(b,c) = Infinity2Open(c)
c2oUc2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
    a ↦ b ∈ lt ⇒
      Closed2Open(a,b) ∪ Closed2Infinity(b) = Closed2Infinity(a)
o2oUc2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
    a ↦ b ∈ lt ⇒
      Open2Open(a,b) ∪ Closed2Infinity(b) = Open2Infinity(a)
inf2oUc2inf:
  ∀ a ·
    a ∈ RReal ⇒
      Infinity2Open(a) ∪ Closed2Infinity(a) = RReal
c2cCc2c:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ leq ∧ b ↦ c ∈ leq ⇒
      Closed2Closed(a,b) ∩ Closed2Closed(b,c) = {b}
o2cCc2c:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ leq ⇒
      Open2Closed(a,b) ∩ Closed2Closed(b,c) = {b}
c2cCc2o:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ leq ∧ b ↦ c ∈ lt ⇒
      Closed2Closed(a,b) ∩ Closed2Open(b,c) = {b}
o2cCc2o:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
      Open2Closed(a,b) ∩ Closed2Open(b,c) = {b}
inf2cCc2c:
  ∀ b, c ·
    b ∈ RReal ∧ c ∈ RReal ∧
    b ↦ c ∈ leq ⇒
      Infinity2Closed(b) ∩ Closed2Closed(b,c) = {b}
inf2cCc2o:
  ∀ b, c ·
    b ∈ RReal ∧ c ∈ RReal ∧
    b ↦ c ∈ lt ⇒
      Infinity2Closed(b) ∩ Closed2Open(b,c) = {b}
c2cCc2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
    a ↦ b ∈ leq ⇒
      Closed2Closed(a,b) ∩ Closed2Infinity(b) = {b}
o2cCc2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
    a ↦ b ∈ lt ⇒
      Open2Closed(a,b) ∩ Closed2Infinity(b) = {b}
inf2cCc2inf:
  ∀ a ·
    a ∈ RReal ⇒
      Infinity2Closed(a) ∩ Closed2Infinity(a) = {a}
c2oCc2c:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ leq ⇒
      Closed2Open(a,b) ∩ Closed2Closed(b,c) = ∅
o2oCc2c:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ leq ⇒
      Open2Open(a,b) ∩ Closed2Closed(b,c) = ∅
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c2oCc2o:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
      Closed2Open(a,b) ∩ Closed2Open(b,c) = ∅
o2oCc2o:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
      Open2Open(a,b) ∩ Closed2Open(b,c) = ∅
inf2oCc2c:
  ∀ b, c ·
    b ∈ RReal ∧ c ∈ RReal ∧
    b ↦ c ∈ leq ⇒
      Infinity2Open(b) ∩ Closed2Closed(b,c) = ∅
inf2oCc2o:
  ∀ b, c ·
    b ∈ RReal ∧ c ∈ RReal ∧
    b ↦ c ∈ lt ⇒
      Infinity2Open(b) ∩ Closed2Open(b,c) = ∅
c2oCc2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
    a ↦ b ∈ lt ⇒
      Closed2Open(a,b) ∩ Closed2Infinity(b) = ∅
o2oCc2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
    a ↦ b ∈ lt ⇒
      Open2Open(a,b) ∩ Closed2Infinity(b) = ∅
inf2oCc2inf:
  ∀ a ·
    a ∈ RReal ⇒
      Infinity2Closed(a) ∩ Closed2Infinity(a) = ∅
c2cCo2c:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ leq ∧ b ↦ c ∈ lt ⇒
      Closed2Closed(a,b) ∩ Open2Closed(b,c) = ∅
o2cCo2c:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
      Open2Closed(a,b) ∩ Open2Closed(b,c) = ∅
c2cCo2o:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ leq ∧ b ↦ c ∈ lt ⇒
      Closed2Closed(a,b) ∩ Open2Open(b,c) = ∅
o2cCo2o:
  ∀ a, b, c ·
    a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
    a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
      Open2Closed(a,b) ∩ Open2Open(b,c) = ∅
inf2cCo2c:
  ∀ b, c ·
    b ∈ RReal ∧ c ∈ RReal ∧
    b ↦ c ∈ lt ⇒
      Infinity2Closed(b) ∩ Open2Closed(b,c) = ∅
inf2cCo2o:
  ∀ b, c ·
    b ∈ RReal ∧ c ∈ RReal ∧
    b ↦ c ∈ lt ⇒
      Infinity2Closed(b) ∩ Open2Open(b,c) = ∅
c2cCo2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
    a ↦ b ∈ leq ⇒
      Closed2Closed(a,b) ∩ Open2Infinity(b) = ∅
o2cCo2inf:
  ∀ a, b ·
    a ∈ RReal ∧ b ∈ RReal ∧
    a ↦ b ∈ lt ⇒
      Open2Closed(a,b) ∩ Open2Infinity(b) = ∅
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  inf2cCo2inf:
    ∀ a ·
      a ∈ RReal ⇒
        Infinity2Closed(a) ∩ Open2Infinity(a) = ∅
  c2oCo2c:
    ∀ a, b, c ·
      a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
      a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
        Closed2Open(a,b) ∩ Open2Closed(b,c) = ∅
  o2oCo2c:
    ∀ a, b, c ·
      a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
      a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
        Open2Open(a,b) ∩ Open2Closed(b,c) = ∅
  c2oCo2o:
    ∀ a, b, c ·
      a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
      a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
        Closed2Open(a,b) ∩ Open2Open(b,c) = ∅
  o2oCo2o:
    ∀ a, b, c ·
      a ∈ RReal ∧ b ∈ RReal ∧ c ∈ RReal ∧
      a ↦ b ∈ lt ∧ b ↦ c ∈ lt ⇒
        Open2Open(a,b) ∩ Open2Open(b,c) = ∅
  inf2oCo2c:
    ∀ b, c ·
      b ∈ RReal ∧ c ∈ RReal ∧
      b ↦ c ∈ lt ⇒
        Infinity2Open(b) ∩ Open2Closed(b,c) = ∅
  inf2oCo2o:
    ∀ b, c ·
      b ∈ RReal ∧ c ∈ RReal ∧
      b ↦ c ∈ lt ⇒
        Infinity2Open(b) ∩ Open2Open(b,c) = ∅
  c2oCo2inf:
    ∀ a, b ·
      a ∈ RReal ∧ b ∈ RReal ∧
      a ↦ b ∈ lt ⇒
        Closed2Open(a,b) ∩ Open2Infinity(b) = ∅
  o2oCo2inf:
    ∀ a, b ·
      a ∈ RReal ∧ b ∈ RReal ∧
      a ↦ b ∈ lt ⇒
        Open2Open(a,b) ∩ Open2Infinity(b) = ∅
  inf2oCo2inf:
    ∀ a ·
      a ∈ RReal ⇒
        Infinity2Open(a) ∩ Open2Infinity(a) = ∅
PROOF RULES
  minimumRew:
  Metavariables
    a: RReal
    b: RReal
  Rewrite Rules
    minClosed2Closed: Rmin(Closed2Closed(a,b))
      rhs1: a ↦ b ∈ leq ⇒ a
    minClosed2Open: Rmin(Closed2Open(a,b))
      rhs1: a ↦ b ∈ lt ⇒ a
    minClosed2Infinity: Rmin(Closed2Infinity(a))
      rhs1: ⊤ ⇒ a
  maximumRew:
  Metavariables
    a: RReal
    b: RReal
  Rewrite Rules
    maxClosed2Closed: Rmax(Closed2Closed(a,b))
      rhs1: a ↦ b ∈ leq ⇒ b
    maxOpen2Closed: Rmax(Open2Closed(a,b))
      rhs1: a ↦ b ∈ lt ⇒ b
    maxInfinity2Closed: Rmax(Infinity2Closed(b))
      rhs1: ⊤ ⇒ b
  infimumRew:
  Metavariables
    a: RReal
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      b: RReal
    Rewrite Rules
      infClosed2Closed: Rinf(Closed2Closed(a,b))
        rhs1: a ↦ b ∈ leq ⇒ a
      infClosed2Open: Rinf(Closed2Open(a,b))
        rhs1: a ↦ b ∈ lt ⇒ a
      infOpen2Closed: Rinf(Open2Closed(a,b))
        rhs1: a ↦ b ∈ lt ⇒ a
      infOpen2Open: Rinf(Open2Open(a,b))
        rhs1: a ↦ b ∈ lt ⇒ a
      infClosed2Infinity: Rinf(Closed2Infinity(a))
        rhs1: ⊤ ⇒ a
      infOpen2Infinity: Rinf(Open2Infinity(a))
        rhs1: ⊤ ⇒ a
    supremumRew:
    Metavariables
      a: RReal
      b: RReal
    Rewrite Rules
      supClosed2Closed: Rsup(Closed2Closed(a,b))
        rhs1: a ↦ b ∈ leq ⇒ b
      supClosed2Open: Rsup(Closed2Open(a,b))
        rhs1: a ↦ b ∈ lt ⇒ b
      supOpen2Closed: Rsup(Open2Closed(a,b))
        rhs1: a ↦ b ∈ lt ⇒ b
      supOpen2Open: Rsup(Open2Open(a,b))
        rhs1: a ↦ b ∈ lt ⇒ b
      supInfinity2Closed: Rsup(Infinity2Closed(b))
        rhs1: ⊤ ⇒ b
      supInfinity2Open: Rsup(Infinity2Open(b))
        rhs1: ⊤ ⇒ b
    intervalURew:
    Metavariables
      a: RReal
      b: RReal
      c: RReal
    Rewrite Rules
      c2cUc2c_rew: Closed2Closed(a,b) ∪ Closed2Closed(b,c)
        rhs1: ⊤ ⇒ Closed2Closed(a,c)
      o2cUc2c_rew: Open2Closed(a,b) ∪ Closed2Closed(b,c)
        rhs1: ⊤ ⇒ Open2Closed(a,c)
      c2cUc2o_rew: Closed2Closed(a,b) ∪ Closed2Open(b,c)
        rhs1: ⊤ ⇒ Closed2Open(a,c)
      o2cUc2o_rew: Open2Closed(a,b) ∪ Closed2Open(b,c)
        rhs1: ⊤ ⇒ Open2Open(a,c)
      inf2cUc2c_rew: Infinity2Closed(b) ∪ Closed2Closed(b,c)
        rhs1: ⊤ ⇒ Infinity2Closed(c)
      inf2cUc2o_rew: Infinity2Closed(b) ∪ Closed2Open(b,c)
        rhs1: ⊤ ⇒ Infinity2Open(c)
      c2cUc2inf_rew: Closed2Closed(a,b) ∪ Closed2Infinity(b)
        rhs1: ⊤ ⇒ Closed2Infinity(a)
      o2cUc2inf_rew: Open2Closed(a,b) ∪ Closed2Infinity(b)
        rhs1: ⊤ ⇒ Open2Infinity(a)
      c2oUc2c_rew: Closed2Open(a,b) ∪ Closed2Closed(b,c)
        rhs1: ⊤ ⇒ Closed2Closed(a,c)
      o2oUc2c_rew: Open2Open(a,b) ∪ Closed2Closed(b,c)
        rhs1: ⊤ ⇒ Open2Closed(a,c)
      c2oUc2o_rew: Closed2Open(a,b) ∪ Closed2Open(b,c)
        rhs1: ⊤ ⇒ Closed2Open(a,c)
      o2oUc2o_rew: Open2Open(a,b) ∪ Closed2Open(b,c)
        rhs1: ⊤ ⇒ Open2Open(a,c)
      inf2oUc2c_rew: Infinity2Open(b) ∪ Closed2Closed(b,c)
        rhs1: ⊤ ⇒ Infinity2Closed(c)
      inf2oUc2o_rew: Infinity2Open(b) ∪ Closed2Open(b,c)
        rhs1: ⊤ ⇒ Infinity2Open(c)
      c2oUc2inf_rew: Closed2Open(a,b) ∪ Closed2Infinity(b)
        rhs1: ⊤ ⇒ Closed2Infinity(a)
      o2oUc2inf_rew: Open2Open(a,b) ∪ Closed2Infinity(b)
        rhs1: ⊤ ⇒ Open2Infinity(a)
      c2cUo2c_rew: Closed2Closed(a,b) ∪ Open2Closed(b,c)
        rhs1: ⊤ ⇒ Closed2Closed(a,c)
      o2cUo2c_rew: Open2Closed(a,b) ∪ Open2Closed(b,c)
        rhs1: ⊤ ⇒ Open2Closed(a,c)
      c2cUo2o_rew: Closed2Closed(a,b) ∪ Open2Open(b,c)
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      rhs1: ⊤ ⇒ Closed2Open(a,c)
    o2cUo2o_rew: Open2Closed(a,b) ∪ Open2Open(b,c)
      rhs1: ⊤ ⇒ Open2Open(a,c)
    inf2cUo2c_rew: Infinity2Closed(b) ∪ Open2Closed(b,c)
      rhs1: ⊤ ⇒ Infinity2Closed(c)
    inf2cUo2o_rew: Infinity2Closed(b) ∪ Open2Open(b,c)
      rhs1: ⊤ ⇒ Infinity2Open(c)
    c2cUo2inf_rew: Closed2Closed(a,b) ∪ Open2Infinity(b)
      rhs1: ⊤ ⇒ Closed2Infinity(a)
    o2cUo2inf_rew: Open2Closed(a,b) ∪ Open2Infinity(b)
      rhs1: ⊤ ⇒ Open2Infinity(a)
  intervalCRew:
  Metavariables
    a: RReal
    b: RReal
    c: RReal
  Rewrite Rules
    c2cCc2c_rew: Closed2Closed(a,b) ∩ Closed2Closed(b,c)
      rhs1: ⊤ ⇒ {b}
    o2cCc2c_rew: Open2Closed(a,b) ∩ Closed2Closed(b,c)
      rhs1: ⊤ ⇒ {b}
    c2cCc2o_rew: Closed2Closed(a,b) ∩ Closed2Open(b,c)
      rhs1: ⊤ ⇒ {b}
    o2cCc2o_rew: Open2Closed(a,b) ∩ Closed2Open(b,c)
      rhs1: ⊤ ⇒ {b}
    inf2cCc2c_rew: Infinity2Closed(b) ∩ Closed2Closed(b,c)
      rhs1: ⊤ ⇒ {b}
    inf2cCc2o_rew: Infinity2Closed(b) ∩ Closed2Open(b,c)
      rhs1: ⊤ ⇒ {b}
    c2cCc2inf_rew: Closed2Closed(a,b) ∩ Closed2Infinity(b)
      rhs1: ⊤ ⇒ {b}
    o2cCc2inf_rew: Open2Closed(a,b) ∩ Closed2Infinity(b)
      rhs1: ⊤ ⇒ {b}
  intervalRealParts:
  Rewrite Rules
    RPlus2Int: RRealPlus
      rhs1: ⊤ ⇒ Closed2Infinity(Rzero)
    Int2RPlus: Closed2Infinity(Rzero)
      rhs1: ⊤ ⇒ RRealPlus
    RMinus2Int: RRealMinus
      rhs1: ⊤ ⇒ Infinity2Closed(Rzero)
    Int2RMinus: Infinity2Closed(Rzero)
      rhs1: ⊤ ⇒ RRealMinus
    RPlusStar2Int: RRealPlusStar
      rhs1: ⊤ ⇒ Open2Infinity(Rzero)
    Int2RPlusStar: Open2Infinity(Rzero)
      rhs1: ⊤ ⇒ RRealPlusStar
    RMinusStar2Int: RRealMinusStar
      rhs1: ⊤ ⇒ Infinity2Open(Rzero)
    Int2RMinusStar: Infinity2Open(Rzero)
      rhs1: ⊤ ⇒ RRealMinusStar
  intervalInclusion:
  Metavariables
    a1: RReal
    b1: RReal
    a2: RReal
    b2: RReal
  Rewrite Rules
    c2InfIncc2Inf: Closed2Infinity(a1) ⊆ Closed2Infinity(a2)
      rhs1: ⊤ ⇒ a2 ↦ a1 ∈ leq
    o2InfIncc2Inf: Open2Infinity(a1) ⊆ Closed2Infinity(a2)
      rhs1: ⊤ ⇒ a2 ↦ a1 ∈ leq
    c2InfInco2Inf: Closed2Infinity(a1) ⊆ Open2Infinity(a2)
      rhs1: ⊤ ⇒ a2 ↦ a1 ∈ lt
    o2InfInco2Inf: Open2Infinity(a1) ⊆ Open2Infinity(a2)
      rhs1: ⊤ ⇒ a2 ↦ a1 ∈ leq
    inf2cIncinf2c: Infinity2Closed(b1) ⊆ Infinity2Closed(b2)
      rhs1: ⊤ ⇒ b1 ↦ b2 ∈ leq
    inf2oIncinf2c: Infinity2Open(b1) ⊆ Infinity2Closed(b2)
      rhs1: ⊤ ⇒ b1 ↦ b2 ∈ leq
    inf2cIncinf2o: Infinity2Closed(b1) ⊆ Infinity2Open(b2)
      rhs1: ⊤ ⇒ b1 ↦ b2 ∈ lt
    inf2oIncinf2o: Infinity2Open(b1) ⊆ Infinity2Open(b2)
      rhs1: ⊤ ⇒ b1 ↦ b2 ∈ leq
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    o2oIno2o: Open2Open(a1,b1) ⊆ Open2Open(a2,b2)
      rhs1: ⊤ ⇒ a1 ↦ a2 ∈ lt ∧ b1 ↦ b2 ∈ gt
    c2oInc2c: Closed2Open(a1,b1) ⊆ Closed2Closed(a2,b2)
      rhs1: ⊤ ⇒ a1 ↦ a2 ∈ leq ∧ b1 ↦ b2 ∈ geq
  intervalInclusion2:
  Metavariables
    a: RReal
    b: RReal
  Rewrite Rules
    s_o2oIno2Inf: Open2Open(a,b) ⊆ Open2Infinity(a)
      rhs1: ⊤ ⇒ ⊤
    s_o2cIno2Inf: Open2Closed(a,b) ⊆ Open2Infinity(a)
      rhs1: ⊤ ⇒ ⊤
    s_o2oInc2Inf: Open2Open(a,b) ⊆ Closed2Infinity(a)
      rhs1: ⊤ ⇒ ⊤
    s_o2cInc2Inf: Open2Closed(a,b) ⊆ Closed2Infinity(a)
      rhs1: ⊤ ⇒ ⊤
    s_c2oInc2Inf: Closed2Open(a,b) ⊆ Closed2Infinity(a)
      rhs1: ⊤ ⇒ ⊤
    s_c2cInc2Inf: Closed2Closed(a,b) ⊆ Closed2Infinity(a)
      rhs1: ⊤ ⇒ ⊤
    s_o2oIno2c: Open2Open(a,b) ⊆ Open2Closed(a,b)
      rhs1: ⊤ ⇒ ⊤
    s_o2oInc2o: Open2Open(a,b) ⊆ Closed2Open(a,b)
      rhs1: ⊤ ⇒ ⊤
    s_o2oInc2c: Open2Open(a,b) ⊆ Closed2Closed(a,b)
      rhs1: ⊤ ⇒ ⊤
    s_c2oInc2c: Closed2Open(a,b) ⊆ Closed2Closed(a,b)
      rhs1: ⊤ ⇒ ⊤
    s_o2cInc2c: Open2Closed(a,b) ⊆ Closed2Closed(a,b)
      rhs1: ⊤ ⇒ ⊤
  degenerated_intervals:
  Metavariables
    a: RReal
  Rewrite Rules
    c2c_single: Closed2Closed(a,a)
      rhs1: ⊤ ⇒ {a}
    o2c_empty: Open2Closed(a,a)
      rhs1: ⊤ ⇒ ∅ : ℙ(RReal)
    c2o_empty: Closed2Open(a,a)
      rhs1: ⊤ ⇒ ∅ : ℙ(RReal)
    o2o_empty: Open2Open(a,a)
      rhs1: ⊤ ⇒ ∅ : ℙ(RReal)
  intervalElements:
  Metavariables
    a: RReal
    b: RReal
  Rewrite Rules
    a_in_c2c: a ∈ Closed2Closed(a,b)
      rhs1: ⊤ ⇒ a ↦ b ∈ leq
    a_in_c2o: a ∈ Closed2Open(a,b)
      rhs1: ⊤ ⇒ a ↦ b ∈ lt
    b_in_c2c: b ∈ Closed2Closed(a,b)
      rhs1: ⊤ ⇒ a ↦ b ∈ leq
    b_in_o2c: b ∈ Open2Closed(a,b)
      rhs1: ⊤ ⇒ a ↦ b ∈ lt
    a_in_c2inf: a ∈ Closed2Infinity(a)
      rhs1: ⊤ ⇒ ⊤
    b_in_inf2c: b ∈ Infinity2Closed(b)
      rhs1: ⊤ ⇒ ⊤
END
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