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1 THEORY Functions
2 IMPORT THEORY PROJECTS
3 /SimpleDEq THEORIES /SimpleDEq/Intervals.dtf|org.eventb.theory.core.deployedTheoryRoot#Intervals
4 TYPE PARAMETERS E,F,G,H,I
5 OPERATORS
6 bind <expression> (fab: E → F,gac: E → G)
7 direct definition
8 (λ x · x ∈ dom(fab) ∩ dom(gac) | (fab(x) ↦ gac(x)))
9 fproj1 <expression> (fa_bc: E → F×G)
10 direct definition
11 (λ x↦y · x ∈ F ∧ y ∈ G | x) ∘ fa_bc
12 fproj2 <expression> (fa_bc: E → F×G)
13 direct definition
14 (λ x↦y · x ∈ F ∧ y ∈ G | y) ∘ fa_bc
15 Rfplus <expression> ()
16 direct definition
17 (λ rf ↦ rg · rf ∈ (RReal → RReal) ∧ rg ∈ (RReal → RReal) |
18 (λ x · x ∈ RReal | plus(rf(x) ↦ rg(x)))
19 )
20 Rftimes <expression> ()
21 direct definition
22 (λ rf ↦ rg · rf ∈ (RReal → RReal) ∧ rg ∈ (RReal → RReal) |
23 (λ x · x ∈ RReal | times(rf(x) ↦ rg(x)))
24 )
25 Rfscal <expression> ()
26 direct definition
27 (λ l ↦ rf · l ∈ RReal ∧ rf ∈ (RReal → RReal) |
28 (λ x · x ∈ RReal | times(l ↦ rf(x)))
29 )
30 Rfcste <expression> (l: RReal)
31 direct definition
32 (λ x · x ∈ RReal | l)
33 partial1 <expression> (fab_c: E×F→G,y: F)
34 direct definition
35 (λ x · (x ↦ y) ∈ dom(fab_c) | fab_c(x↦y))
36 partial2 <expression> (fab_c: E×F→G,x: E)
37 direct definition
38 (λ y · (x ↦ y) ∈ dom(fab_c) | fab_c(x↦y))
39 partialComp <expression> (fabb: E×F→G,gab: E→F)
40 direct definition
41 (λ t · t ∈ dom(gab) ∧ (t ↦ gab(t)) ∈ dom(fabb) | fabb(t ↦ gab(t)))
42 unpartialize1 <expression> (A: ℙ(E),fbc: F→G)
43 direct definition
44 (λ x↦y · x ∈ A ∧ y ∈ dom(fbc) | fbc(y))
45 unpartialize2 <expression> (B: ℙ(F),fac: E → G)
46 direct definition
47 (λ x↦y · x ∈ dom(fac) ∧ y ∈ B | fac(x))
48 increasing <predicate> (AR: ℙ(RReal),f: AR → RReal)
49 direct definition
50 ∀ x, y · x ∈ AR ∧ y ∈ AR ⇒ (
51 (x ↦ y ∈ leq) ⇔ (f(x) ↦ f(y) ∈ leq)
52 )
53 strictlyIncreasing <predicate> (AR: ℙ(RReal),f: AR → RReal)
54 direct definition
55 ∀ x, y · x ∈ AR ∧ y ∈ AR ⇒ (
56 (x ↦ y ∈ lt) ⇔ (f(x) ↦ f(y) ∈ lt)
57 )
58 decreasing <predicate> (AR: ℙ(RReal),f: AR → RReal)
59 direct definition
60 ∀ x, y · x ∈ AR ∧ y ∈ AR ⇒ (
61 (x ↦ y ∈ leq) ⇔ (f(x) ↦ f(y) ∈ geq)
62 )
63 strictlyDecreasing <predicate> (AR: ℙ(RReal),f: AR → RReal)
64 direct definition
65 ∀ x, y · x ∈ AR ∧ y ∈ AR ⇒ (
66 (x ↦ y ∈ lt) ⇔ (f(x) ↦ f(y) ∈ gt)
67 )
68 constant <predicate> (A: ℙ(E),fa: A → F)
69 direct definition
70 ∀ x · x ∈ A ⇒ (∀ y · y ∈ A ⇒ fa(x) = fa(y))
71 positive <predicate> (A: ℙ(E),far: A → RReal)
72 direct definition
73 ∀ x · x ∈ A ⇒ Rzero ↦ far(x) ∈ leq

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74 **strictlyPositive** <predicate> (A: $\mathbb{P}(E)$, far: $A \rightarrow \mathbb{RReal}$)
75 **direct definition**
76 $\forall x \cdot x \in A \Rightarrow \mathbb{Rzero} \mapsto \text{far}(x) \in \text{lt}$

77 **negative** <predicate> (A: $\mathbb{P}(E)$, far: $A \rightarrow \mathbb{RReal}$)
78 **direct definition**
79 $\forall x \cdot x \in A \Rightarrow \mathbb{Rzero} \mapsto \text{far}(x) \in \text{geq}$

80 **strictlyNegative** <predicate> (A: $\mathbb{P}(E)$, far: $A \rightarrow \mathbb{RReal}$)
81 **direct definition**
82 $\forall x \cdot x \in A \Rightarrow \mathbb{Rzero} \mapsto \text{far}(x) \in \text{gt}$

83 **application** <expression> ()
84 **direct definition**
85 $(\lambda f_ \mapsto e_ \cdot f_ \in E \Rightarrow F \wedge e_ \in \text{dom}(f_) \mid f_ (e_))$

86 **until** <expression> (start: \mathbb{RReal} , fre: $\mathbb{RReal} \Rightarrow E$, t0: \mathbb{RReal} , gre: $\mathbb{RReal} \Rightarrow E$)
87 **well-definedness** start \mapsto t0 \in leq, Closed2Open(start, t0) \subseteq dom(fre), Closed2Infinity(t0) \subseteq dom(gre)
88 **direct definition**
89 $(\text{Closed2Open}(\text{start}, \text{t0}) \triangleleft \text{fre}) \cup (\text{Closed2Infinity}(\text{t0}) \triangleleft \text{gre})$

90 **untilF** <expression> (start: \mathbb{RReal} , fref: $\mathbb{RReal} \times E \Rightarrow F$, t0: \mathbb{RReal} , gref: $\mathbb{RReal} \times E \Rightarrow F$)
91 **well-definedness** start \mapsto t0 \in leq, Closed2Open(start, t0) $\times \emptyset \subseteq$ dom(fref), Closed2Infinity(t0) $\times \emptyset \subseteq$ dom(gref)
92 **direct definition**
93 $((\text{Closed2Open}(\text{start}, \text{t0}) \times E) \triangleleft \text{fref}) \cup ((\text{Closed2Infinity}(\text{t0}) \times E) \triangleleft \text{gref})$

94 **fcste** <expression> (A: $\mathbb{P}(E)$, v: F)
95 **direct definition**
96 $(\lambda x \cdot x \in A \mid v)$

97 **boundedBy** <predicate> (A: $\mathbb{P}(E)$, far: $E \Rightarrow \mathbb{RReal}$, fmin: \mathbb{RReal} , fmax: \mathbb{RReal})
98 **well-definedness** A \subseteq dom(far)
99 **direct definition**
100 $\forall x_ \cdot x_ \in A \Rightarrow \text{fmin} \mapsto \text{far}(x_) \in \text{leq} \wedge \text{far}(x_) \mapsto \text{fmax} \in \text{leq}$

101 **AXIOMATIC DEFINITIONS** continuity:
102 **OPERATORS**

103 **C0** <expression> (A: $\mathbb{P}(E)$, B: $\mathbb{P}(F)$) : $\mathbb{P}(\mathbb{P}(E \times F))$
104 **well-definedness** A $\neq \emptyset$, B $\neq \emptyset$

105 **D1** <expression> (A2: $\mathbb{P}(\mathbb{RReal})$, B: $\mathbb{P}(F)$) : $\mathbb{P}(\mathbb{P}(\mathbb{RReal} \times F))$

106 **Cn** <expression> (n: \mathbb{N} , A: $\mathbb{P}(E)$, B: $\mathbb{P}(F)$) : $\mathbb{P}(\mathbb{P}(E \times F))$

107 **Dn** <expression> (n: \mathbb{N} , A2: $\mathbb{P}(\mathbb{RReal})$, B: $\mathbb{P}(F)$) : $\mathbb{P}(\mathbb{P}(\mathbb{RReal} \times F))$
108 **well-definedness** n > 0

109 **AXIOMS**
110 *cid*:
111 $\forall A, B \cdot A \subseteq E \wedge B \subseteq F \Rightarrow (\text{C0}(A, B) = \text{Cn}(0, A, B))$

112 *did*:
113 $\forall A, B \cdot A \subseteq \mathbb{RReal} \wedge B \subseteq F \Rightarrow (\text{D1}(A, B) = \text{Dn}(1, A, B))$

114 *c_in_c*:
115 $\forall A, B, k \cdot A \subseteq E \wedge B \subseteq F \wedge k \in \mathbb{N} \Rightarrow (\text{Cn}(k+1, A, B) \subseteq \text{Cn}(k, A, B))$

116 *d_in_d*:
117 $\forall A, B, k \cdot A \subseteq \mathbb{RReal} \wedge B \subseteq F \wedge k \in \mathbb{N} \wedge k > 0 \Rightarrow (\text{Dn}(k+1, A, B) \subseteq \text{Dn}(k, A, B))$

118 *c_in_d*:
119 $\forall A, B, k \cdot A \subseteq \mathbb{RReal} \wedge B \subseteq F \wedge k \in \mathbb{N} \wedge k > 0 \Rightarrow (\text{Cn}(k, A, B) \subseteq \text{Dn}(k, A, B))$

120 *d_in_c*:
121 $\forall A, B, k \cdot A \subseteq \mathbb{RReal} \wedge B \subseteq F \wedge k \in \mathbb{N} \wedge k > 0 \Rightarrow (\text{Dn}(k, A, B) \subseteq \text{Cn}(k-1, A, B))$

122 *c_inclusion_stable*:
123 $\forall A, C, B, k \cdot$
124 $A \subseteq E \wedge C \subseteq A \wedge B \subseteq F \wedge k \in \mathbb{N}$
125 \Rightarrow
126 $(\text{Cn}(k, A, B) \subseteq \text{Cn}(k, C, B))$

127 *d_inclusion_stable*:
128 $\forall A, C, B, k \cdot$
129 $A \subseteq \mathbb{RReal} \wedge C \subseteq A \wedge B \subseteq F \wedge k \in \mathbb{N} \wedge k > 0$
130 \Rightarrow
131 $(\text{Dn}(k, A, B) \subseteq \text{Dn}(k, C, B))$

132 *d_restriction*:
133 $\forall A, C, B, k, f \cdot$
134 $A \subseteq \mathbb{RReal} \wedge C \subseteq A \wedge B \subseteq F \wedge k \in \mathbb{N} \wedge k > 0 \wedge$
135 $f \in \mathbb{RReal} \Rightarrow B \wedge A \subseteq \text{dom}(f) \wedge f \in \text{Dn}(k, A, B) \Rightarrow$
136 $(C \triangleleft f) \in \text{Dn}(k, C, B)$ derivative:

137 **OPERATORS**

138 **Der** <expression> (A2: $\mathbb{P}(\mathbb{RReal})$, B: $\mathbb{P}(F)$, fa2b: $\mathbb{RReal} \Rightarrow B$) : $\mathbb{P}(\mathbb{RReal} \times F)$
139 **well-definedness** fa2b \in D1(A2, B), A2 \subseteq dom(fa2b)

140 **Der_n** <expression> (n: \mathbb{N} , A2: $\mathbb{P}(\mathbb{RReal})$, B: $\mathbb{P}(F)$, fa2b: $\mathbb{RReal} \Rightarrow B$) : $\mathbb{P}(\mathbb{RReal} \times F)$
141 **well-definedness** n > 0, A2 \subseteq dom(fa2b)

142 **AXIOMS**
143 *derType*:
144 $\forall A, B, f \cdot A \subseteq \mathbb{RReal} \wedge B \subseteq F \wedge f \in (\mathbb{RReal} \Rightarrow B) \wedge A \subseteq \text{dom}(f) \wedge f \in \text{D1}(A, B) \Rightarrow (\text{Der}(A, B, f) \in (A \rightarrow F))$

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145 derDef_1:
146    $\forall A, B, f \cdot A \subseteq \mathbb{R}\text{Real} \wedge B \subseteq F \wedge f \in (\mathbb{R}\text{Real} \rightarrow B) \wedge A \subseteq \text{dom}(f) \wedge f \in \text{Dn}(1, A, B) \Rightarrow ($ 
147      $\text{Dern}(1, A, B, f) = \text{Der}(A, B, f)$ 
148   )
149 dernDef_n:
150    $\forall A, B, f, k \cdot A \subseteq \mathbb{R}\text{Real} \wedge B \subseteq F \wedge f \in (\mathbb{R}\text{Real} \rightarrow B) \wedge A \subseteq \text{dom}(f) \wedge k \in \mathbb{N} \wedge k > 1 \wedge f \in \text{Dn}(k, A, B)$ 
151      $\Rightarrow ($ 
152        $\text{Dern}(k, A, B, f) = \text{Der}(A, B, \text{Dern}(k-1, A, B, f))$ 
153     )
154 c_def:
155    $\forall A, B, k, f \cdot A \subseteq \mathbb{R}\text{Real} \wedge B \subseteq F \wedge k \in \mathbb{N} \wedge k > 1 \wedge f \in (\mathbb{R}\text{Real} \rightarrow B) \wedge \text{dom}(f) \subseteq A \Rightarrow ($ 
156      $(f \in \text{Cn}(k, A, B) \Leftrightarrow (f \in \text{Dn}(k, A, B) \wedge \text{Dern}(k, A, B, f) \in \text{C0}(A, B)))$ 
157   ) cartesian:
158 AXIOMS
159 cartesian_continuity:
160    $\forall A, B, C, f, g, k \cdot$ 
161      $A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in (A \rightarrow B) \wedge g \in (A \rightarrow C) \wedge k \in \mathbb{N} \Rightarrow$ 
162      $(f \in \text{Cn}(k, A, B) \wedge g \in \text{Cn}(k, A, C) \Leftrightarrow (\text{bind}(f, g) \in \text{Cn}(k, A, B \times C)))$ 
163 cartesian_derivability:
164    $\forall A, B, C, f, g, k \cdot$ 
165      $A \subseteq \mathbb{R}\text{Real} \wedge B \subseteq F \wedge C \subseteq G \wedge f \in (A \rightarrow B) \wedge g \in (A \rightarrow C) \wedge k \in \mathbb{N} \wedge k > 0 \Rightarrow$ 
166      $(f \in \text{Dn}(k, A, B) \wedge g \in \text{Dn}(k, A, C) \Leftrightarrow (\text{bind}(f, g) \in \text{Dn}(k, A, B \times C)))$ 
167 cartesian_der:
168    $\forall A, B, C, f, g \cdot$ 
169      $A \subseteq \mathbb{R}\text{Real} \wedge B \subseteq F \wedge C \subseteq G \wedge f \in (A \rightarrow B) \wedge g \in (A \rightarrow C) \wedge$ 
170      $f \in \text{D1}(A, B) \wedge g \in \text{D1}(A, C) \Rightarrow$ 
171      $(\text{Der}(A, B \times C, \text{bind}(f, g)) = \text{bind}(\text{Der}(A, B, f), \text{Der}(A, C, g)))$ 
172 cartesian_dern:
173    $\forall A, B, C, f, g, k \cdot$ 
174      $A \subseteq \mathbb{R}\text{Real} \wedge B \subseteq F \wedge C \subseteq G \wedge f \in (A \rightarrow B) \wedge g \in (A \rightarrow C) \wedge k \in \mathbb{N} \wedge k > 0 \wedge$ 
175      $f \in \text{Dn}(k, A, B) \wedge g \in \text{Dn}(k, A, C) \Rightarrow$ 
176      $(\text{Dern}(k, A, B \times C, \text{bind}(f, g)) = \text{bind}(\text{Dern}(k, A, B, f), \text{Dern}(k, A, C, g)))$  partials:
177 AXIOMS
178 partial_cue:
179    $\forall A, B, C, f, k \cdot$ 
180      $A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in (A \times B \rightarrow C) \wedge k \in \mathbb{N} \Rightarrow ($ 
181      $f \in \text{Cn}(k, A \times B, C) \Leftrightarrow ((\forall y_0 \cdot y_0 \in B \Rightarrow \text{partial1}(f, y_0) \in \text{Cn}(k, A, C)) \wedge (\forall x_0 \cdot x_0 \in A \Rightarrow$ 
182      $\text{partial2}(f, x_0) \in \text{Cn}(k, B, C)))$ 
183   ) real_functions:
184 AXIOMS
185 cste_cue:
186    $\forall l \cdot l \in \mathbb{R}\text{Real} \Rightarrow (\forall k \cdot k \in \mathbb{N} \Rightarrow (\text{Rfcste}(l) \in \text{Cn}(k, \mathbb{R}\text{Real}, \mathbb{R}\text{Real})))$ 
187 cste_der:
188    $\forall l \cdot l \in \mathbb{R}\text{Real} \Rightarrow (\forall k \cdot k \in \mathbb{N} \wedge k > 0 \Rightarrow (\text{Rfcste}(l) \in \text{Dn}(k, \mathbb{R}\text{Real}, \mathbb{R}\text{Real})))$ 
189 cste_der_def:
190    $\forall l \cdot l \in \mathbb{R}\text{Real} \Rightarrow (\text{Der}(\mathbb{R}\text{Real}, \mathbb{R}\text{Real}, \text{Rfcste}(l)) = \text{Rfcste}(\text{Rzero}))$ 
191 plus_cue:
192    $\forall A, f, g, k \cdot$ 
193      $A \subseteq \mathbb{R}\text{Real} \wedge$ 
194      $f \in (A \rightarrow \mathbb{R}\text{Real}) \wedge g \in (A \rightarrow \mathbb{R}\text{Real}) \wedge$ 
195      $k \in \mathbb{N} \wedge$ 
196      $f \in \text{Cn}(k, A, \mathbb{R}\text{Real}) \wedge g \in \text{Cn}(k, A, \mathbb{R}\text{Real}) \Rightarrow$ 
197      $(\text{Rfplus}(f \mapsto g) \in \text{Cn}(k, A, \mathbb{R}\text{Real}))$ 
198 plus_der:
199    $\forall A, f, g, k \cdot$ 
200      $A \subseteq \mathbb{R}\text{Real} \wedge$ 
201      $f \in (A \rightarrow \mathbb{R}\text{Real}) \wedge g \in (A \rightarrow \mathbb{R}\text{Real}) \wedge$ 
202      $k \in \mathbb{N} \wedge k > 0 \wedge$ 
203      $f \in \text{Dn}(k, A, \mathbb{R}\text{Real}) \wedge g \in \text{Dn}(k, A, \mathbb{R}\text{Real}) \Rightarrow$ 
204      $(\text{Rfplus}(f \mapsto g) \in \text{Dn}(k, A, \mathbb{R}\text{Real}))$ 
205 plus_der_def:
206    $\forall A, f, g, k \cdot$ 
207      $A \subseteq \mathbb{R}\text{Real} \wedge$ 
208      $f \in (A \rightarrow \mathbb{R}\text{Real}) \wedge g \in (A \rightarrow \mathbb{R}\text{Real}) \wedge$ 
209      $k \in \mathbb{N} \wedge k > 0 \wedge$ 
210      $f \in \text{Dn}(k, A, \mathbb{R}\text{Real}) \wedge g \in \text{Dn}(k, A, \mathbb{R}\text{Real}) \Rightarrow$ 
211      $(\text{Dern}(k, A, \mathbb{R}\text{Real}, \text{Rfplus}(f \mapsto g)) = \text{Rfplus}(\text{Dern}(k, A, \mathbb{R}\text{Real}, f) \mapsto \text{Dern}(k, A, \mathbb{R}\text{Real}, g)))$ 
212 times_cue:
213    $\forall A, f, g, k \cdot$ 
214      $A \subseteq \mathbb{R}\text{Real} \wedge$ 
215      $f \in (A \rightarrow \mathbb{R}\text{Real}) \wedge g \in (A \rightarrow \mathbb{R}\text{Real}) \wedge$ 
216      $k \in \mathbb{N} \wedge$ 
217      $f \in \text{Cn}(k, A, \mathbb{R}\text{Real}) \wedge g \in \text{Cn}(k, A, \mathbb{R}\text{Real}) \Rightarrow$ 
218      $(\text{Rftimes}(f \mapsto g) \in \text{Cn}(k, A, \mathbb{R}\text{Real}))$ 

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217 *times_der*:
218 $\forall A, f, g, k \cdot$
219 $A \subseteq \mathbb{R}\text{Real} \wedge$
220 $f \in (A \rightarrow \mathbb{R}\text{Real}) \wedge g \in (A \rightarrow \mathbb{R}\text{Real}) \wedge$
221 $k \in \mathbb{N} \wedge k > 0 \wedge$
222 $f \in \text{Dn}(k, A, \mathbb{R}\text{Real}) \wedge g \in \text{Dn}(k, A, \mathbb{R}\text{Real}) \Rightarrow$
223 $(\text{Rftimes}(f \mapsto g) \in \text{Dn}(k, A, \mathbb{R}\text{Real}))$

224 *times_der_def*:
225 $\forall A, f, g \cdot$
226 $A \subseteq \mathbb{R}\text{Real} \wedge$
227 $f \in (A \rightarrow \mathbb{R}\text{Real}) \wedge g \in (A \rightarrow \mathbb{R}\text{Real}) \wedge$
228 $f \in \text{D1}(A, \mathbb{R}\text{Real}) \wedge g \in \text{D1}(A, \mathbb{R}\text{Real}) \Rightarrow$
229 $(\text{Der}(A, \mathbb{R}\text{Real}, \text{Rftimes}(f \mapsto g)) = \text{Rfplus}(\text{Rftimes}(\text{Der}(A, \mathbb{R}\text{Real}, f) \mapsto g) \mapsto \text{Rftimes}(f \mapsto \text{Der}(A, \mathbb{R}\text{Real}, g))))$

230 *scal_cue*:
231 $\forall A, l, f, k \cdot$
232 $A \subseteq \mathbb{R}\text{Real} \wedge$
233 $l \in \mathbb{R}\text{Real} \wedge f \in (A \rightarrow \mathbb{R}\text{Real}) \wedge$
234 $k \in \mathbb{N} \wedge$
235 $f \in \text{Cn}(k, A, \mathbb{R}\text{Real}) \Rightarrow$
236 $(\text{Rfscal}(l \mapsto f) \in \text{Cn}(k, A, \mathbb{R}\text{Real}))$

237 *scal_der*:
238 $\forall A, l, f, k \cdot$
239 $A \subseteq \mathbb{R}\text{Real} \wedge$
240 $l \in \mathbb{R}\text{Real} \wedge f \in (A \rightarrow \mathbb{R}\text{Real}) \wedge$
241 $k \in \mathbb{N} \wedge k > 0 \wedge$
242 $f \in \text{Dn}(k, A, \mathbb{R}\text{Real}) \Rightarrow$
243 $(\text{Rfscal}(l \mapsto f) \in \text{Dn}(k, A, \mathbb{R}\text{Real}))$

244 *scal_der_def*:
245 $\forall A, l, f, k \cdot$
246 $A \subseteq \mathbb{R}\text{Real} \wedge$
247 $l \in \mathbb{R}\text{Real} \wedge f \in (A \rightarrow \mathbb{R}\text{Real}) \wedge$
248 $k \in \mathbb{N} \wedge k > 0 \wedge$
249 $f \in \text{Dn}(k, A, \mathbb{R}\text{Real}) \Rightarrow$
250 $(\text{Dern}(k, A, \mathbb{R}\text{Real}, \text{Rfscal}(l \mapsto f)) = \text{Rfscal}(l \mapsto \text{Dern}(k, A, \mathbb{R}\text{Real}, f)))$

251 *comp_cue*:
252 $\forall A, B, f, g, k \cdot$
253 $A \subseteq \mathbb{R}\text{Real} \wedge B \subseteq \mathbb{R}\text{Real} \wedge$
254 $f \in A \rightarrow B \wedge g \in B \rightarrow \mathbb{R}\text{Real} \wedge$
255 $k \in \mathbb{N} \wedge$
256 $f \in \text{Cn}(k, A, B) \wedge g \in \text{Cn}(k, B, \mathbb{R}\text{Real}) \Rightarrow$
257 $(g \circ f \in \text{Cn}(k, A, \mathbb{R}\text{Real}))$

258 *comp_der*:
259 $\forall A, B, f, g, k \cdot$
260 $A \subseteq \mathbb{R}\text{Real} \wedge B \subseteq \mathbb{R}\text{Real} \wedge$
261 $f \in A \rightarrow B \wedge g \in B \rightarrow \mathbb{R}\text{Real} \wedge$
262 $k \in \mathbb{N} \wedge k > 0 \wedge$
263 $f \in \text{Dn}(k, A, B) \wedge g \in \text{Dn}(k, B, \mathbb{R}\text{Real}) \Rightarrow$
264 $(g \circ f \in \text{Dn}(k, A, \mathbb{R}\text{Real}))$

265 *comp_der_def*:
266 $\forall A, B, f, g \cdot$
267 $A \subseteq \mathbb{R}\text{Real} \wedge B \subseteq \mathbb{R}\text{Real} \wedge$
268 $f \in A \rightarrow B \wedge g \in B \rightarrow \mathbb{R}\text{Real} \wedge$
269 $f \in \text{D1}(A, B) \wedge g \in \text{D1}(B, \mathbb{R}\text{Real}) \Rightarrow$
270 $(\text{Der}(A, \mathbb{R}\text{Real}, g \circ f) = \text{Rftimes}((\text{Der}(B, \mathbb{R}\text{Real}, g) \circ f) \mapsto \text{Der}(A, B, f)))$ lipschitz :

271 **OPERATORS**

272 **lipschitzContinuous** <predicate> (A: $\mathbb{P}(E)$, B: $\mathbb{P}(F)$, fab: $A \rightarrow B$) :
273 **kLipschitzContinuous** <predicate> (A: $\mathbb{P}(E)$, B: $\mathbb{P}(F)$, fab: $A \rightarrow B$, r: $\mathbb{R}\text{Real}$) :

274 **AXIOMS**

275 *klip_lip*:
276 $\forall A, B, f \cdot A \subseteq E \wedge B \subseteq F \wedge f \in A \rightarrow B \Rightarrow ($
277 $\text{lipschitzContinuous}(A, B, f) \Leftrightarrow (\exists k \cdot k \in \mathbb{R}\text{Real} \Rightarrow \text{kLipschitzContinuous}(A, B, f, k))$
278 $)$

279 *lip_cue*:
280 $\forall A, B, f \cdot A \subseteq E \wedge B \subseteq F \wedge f \in A \rightarrow B \Rightarrow ($
281 $\text{lipschitzContinuous}(A, B, f) \Rightarrow (f \in \text{C0}(A, B))$
282 $)$

283 *lip_inclusion_stable*:
284 $\forall A, B, C, f \cdot$
285 $A \subseteq E \wedge B \subseteq F \wedge C \subseteq A \wedge f \in A \rightarrow B \Rightarrow ($
286 $\text{lipschitzContinuous}(A, B, f) \Rightarrow \text{lipschitzContinuous}(C, B, f)$
287 $)$

288 *klip_inclusion_stable*:
289 $\forall A, B, C, f, k \cdot$

$A \subseteq E \wedge B \subseteq F \wedge C \subseteq A \wedge f \in A \rightarrow B \wedge k \in \mathbb{RReal} \Rightarrow ($
 $k\text{LipschitzContinuous}(A,B,f,k) \Rightarrow k\text{LipschitzContinuous}(C,B,f,k)$
 $) \text{ trigo} :$

OPERATORS

pi <expression> () : \mathbb{RReal}
sin <expression> () : $\mathbb{P}(\mathbb{RReal} \times \mathbb{RReal})$
cos <expression> () : $\mathbb{P}(\mathbb{RReal} \times \mathbb{RReal})$

AXIOMS

pi_pos:
 $\text{Rzero} \mapsto \text{pi} \in \text{lt}$
sin_dom:
 $\text{sin} \in \mathbb{RReal} \rightarrow \mathbb{RReal}$
cos_dom:
 $\text{cos} \in \mathbb{RReal} \rightarrow \mathbb{RReal}$
sin_bounded:
 $\text{boundedBy}(\mathbb{RReal}, \text{sin}, \text{uminus}(\text{Rone}), \text{Rone})$
sin_ran:
 $\text{ran}(\text{sin}) = \text{Closed2Closed}(\text{uminus}(\text{Rone}), \text{Rone})$
cos_bounded:
 $\text{boundedBy}(\mathbb{RReal}, \text{cos}, \text{uminus}(\text{Rone}), \text{Rone})$
cos_ran:
 $\text{ran}(\text{cos}) = \text{Closed2Closed}(\text{uminus}(\text{Rone}), \text{Rone})$
cos_cinf:
 $\forall n \cdot n \in \mathbb{N} \Rightarrow \text{cos} \in \text{Cn}(n, \mathbb{RReal}, \mathbb{RReal})$
sin_cinf:
 $\forall n \cdot n \in \mathbb{N} \Rightarrow \text{sin} \in \text{Cn}(n, \mathbb{RReal}, \mathbb{RReal})$
cos0:
 $\text{cos}(\text{Rzero}) = \text{Rone}$
cospi:
 $\text{cos}(\text{pi}) = \text{uminus}(\text{Rone})$
cospi2:
 $\text{cos}(\text{divide}(\text{pi} \mapsto \text{Rtwo})) = \text{Rzero}$
cosmpi2:
 $\text{cos}(\text{uminus}(\text{divide}(\text{pi} \mapsto \text{Rtwo}))) = \text{Rzero}$
sin0:
 $\text{sin}(\text{Rzero}) = \text{Rzero}$
sinpi:
 $\text{sin}(\text{pi}) = \text{uminus}(\text{Rzero})$
sinpi2:
 $\text{sin}(\text{divide}(\text{pi} \mapsto \text{Rtwo})) = \text{Rone}$
sinmpi2:
 $\text{sin}(\text{uminus}(\text{divide}(\text{pi} \mapsto \text{Rtwo}))) = \text{uminus}(\text{Rone})$
sin_odd:
 $\forall x \cdot x \in \mathbb{RReal} \Rightarrow \text{sin}(\text{uminus}(x)) = \text{uminus}(\text{sin}(x))$
cos_even:
 $\forall x \cdot x \in \mathbb{RReal} \Rightarrow \text{cos}(\text{uminus}(x)) = \text{cos}(x)$
sin_refl_pi4:
 $\forall x \cdot x \in \mathbb{RReal} \Rightarrow \text{sin}(\text{minus}(\text{divide}(\text{pi} \mapsto \text{Rtwo}) \mapsto x)) = \text{cos}(x)$
cos_refl_pi4:
 $\forall x \cdot x \in \mathbb{RReal} \Rightarrow \text{cos}(\text{minus}(\text{divide}(\text{pi} \mapsto \text{Rtwo}) \mapsto x)) = \text{sin}(x)$
sin_refl_pi2:
 $\forall x \cdot x \in \mathbb{RReal} \Rightarrow \text{sin}(\text{minus}(\text{pi} \mapsto x)) = \text{sin}(x)$
cos_refl_pi2:
 $\forall x \cdot x \in \mathbb{RReal} \Rightarrow \text{cos}(\text{minus}(\text{pi} \mapsto x)) = \text{uminus}(\text{cos}(x))$
sin_2pi:
 $\forall x \cdot x \in \mathbb{RReal} \Rightarrow \text{sin}(\text{plus}(x \mapsto \text{times}(\text{Rtwo} \mapsto \text{pi}))) = \text{sin}(x)$
cos_2pi:
 $\forall x \cdot x \in \mathbb{RReal} \Rightarrow \text{cos}(\text{plus}(x \mapsto \text{times}(\text{Rtwo} \mapsto \text{pi}))) = \text{cos}(x)$

OPERATORS

IsOpen <predicate> (A: $\mathbb{P}(E)$) :

AXIOMS

open_continuity:
 $\forall x, f, A, B \cdot$
 $x \in A \wedge A \subseteq E \wedge B \subseteq F \wedge$
 $f \in E \mapsto F \wedge A \subseteq \text{dom}(f) \wedge f \in \text{C0}(A, F) \wedge$
 $\text{IsOpen}(B) \wedge f(x) \in B \Rightarrow$
 $(\exists y \cdot y \in A \wedge f(y) \in B)$
open_continuity_R:
 $\forall x, f, A, B \cdot$
 $A \subseteq \mathbb{RReal} \wedge B \subseteq F \wedge$
 $x \in \mathbb{RReal} \wedge$
 $f \in \mathbb{RReal} \mapsto F \wedge f \in \text{C0}(A, F) \wedge$
 $\text{IsOpen}(B) \wedge f(x) \in B \Rightarrow$
 $(\exists y \cdot y \in A \wedge x \mapsto y \in \text{lt} \wedge f(y) \in B)$

364 **THEOREMS**

365 *functionEquality:*

366 $\forall f, g \cdot f \in E \rightarrow F \wedge g \in E \rightarrow F \Rightarrow ($
 367 $(f = g) \Leftrightarrow (\text{dom}(f) = \text{dom}(g) \wedge (\forall x \cdot x \in \text{dom}(f) \Rightarrow (f(x) = g(x))))$
 368 $)$

369 *bind_type:*

370 $\forall A, B, C, f, g \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \rightarrow B \wedge g \in A \rightarrow C \Rightarrow$
 371 $\text{bind}(f, g) \in A \rightarrow B \times C$

372 *proj1_type:*

373 $\forall A, B, C, f \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \rightarrow B \times C \Rightarrow ($
 374 $\text{fproj1}(f) \in A \rightarrow B$
 375 $)$

376 *proj2_type:*

377 $\forall A, B, C, f \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \rightarrow B \times C \Rightarrow ($
 378 $\text{fproj2}(f) \in A \rightarrow C$
 379 $)$

380 *partial1_type:*

381 $\forall A, B, C, f, y \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \times B \rightarrow C \wedge y \in B \Rightarrow ($
 382 $\text{partial1}(f, y) \in A \rightarrow C$
 383 $)$

384 *partial2_type:*

385 $\forall A, B, C, f, x \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \times B \rightarrow C \wedge x \in A \Rightarrow ($
 386 $\text{partial2}(f, x) \in B \rightarrow C$
 387 $)$

388 *partialComp_type:*

389 $\forall A, B, B2, f, g \cdot$
 390 $A \subseteq E \wedge B \subseteq F \wedge B2 \subseteq G \wedge$
 391 $f \in A \times B \rightarrow B2 \wedge g \in A \rightarrow B \Rightarrow$
 392 $\text{partialComp}(f, g) \in A \rightarrow B2$

393 *proj1_bind:*

394 $\forall A, B, C, f, g \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \rightarrow B \wedge g \in A \rightarrow C \Rightarrow ($
 395 $\text{fproj1}(\text{bind}(f, g)) = f$
 396 $)$

397 *proj2_bind:*

398 $\forall A, B, C, f, g \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \rightarrow B \wedge g \in A \rightarrow C \Rightarrow ($
 399 $\text{fproj2}(\text{bind}(f, g)) = g$
 400 $)$

401 *bind_proj:*

402 $\forall A, B, C, f \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \rightarrow B \times C \Rightarrow ($
 403 $\text{bind}(\text{fproj1}(f), \text{fproj2}(f)) = f$
 404 $)$

405 *proj_cue:*

406 $\forall A, B, C, f, k \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \rightarrow B \times C \wedge k \in \mathbb{N} \Rightarrow ($
 407 $f \in \text{Cn}(k, A, B \times C) \Leftrightarrow (\text{fproj1}(f) \in \text{Cn}(k, A, B) \wedge \text{fproj2}(f) \in \text{Cn}(k, A, C))$
 408 $)$

409 *unpartialized1_cue:*

410 $\forall A, B, C, f, k \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in B \rightarrow C \wedge k \in \mathbb{N} \wedge f \in \text{Cn}(k, B, C) \Rightarrow$
 411 $(\text{unpartialize1}(A, f) \in \text{Cn}(k, A \times B, C))$

412 *unpartialized2_cue:*

413 $\forall A, B, C, f, k \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \rightarrow C \wedge k \in \mathbb{N} \wedge f \in \text{Cn}(k, A, C) \Rightarrow$
 414 $(\text{unpartialize2}(B, f) \in \text{Cn}(k, A \times B, C))$

415 *unpartialize2_partial1:*

416 $\forall A, B, C, f \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in A \rightarrow C$
 417 \Rightarrow
 418 $(\forall y \cdot y \in B \Rightarrow \text{partial1}(\text{unpartialize2}(B, f), y) = f)$

419 *unpartialize1_partial2:*

420 $\forall A, B, C, f \cdot A \subseteq E \wedge B \subseteq F \wedge C \subseteq G \wedge f \in B \rightarrow C$
 421 \Rightarrow
 422 $(\forall x \cdot x \in A \Rightarrow \text{partial2}(\text{unpartialize1}(A, f), x) = f)$

423 *MeanValue:*

424 $\forall a, b, f, M \cdot$
 425 $a \in \text{RReal} \wedge b \in \text{RReal} \wedge a \mapsto b \in \text{It} \wedge$
 426 $f \in \text{RReal} \rightarrow \text{RReal} \wedge \text{Closed2Closed}(a, b) \subseteq \text{dom}(f) \wedge f \in \text{D1}(\text{Closed2Closed}(a, b), \text{RReal}) \wedge$
 427 $M \in \text{RReal} \wedge (\forall x \cdot x \in \text{Closed2Closed}(a, b) \Rightarrow (\text{abs}(\text{Der}(\text{Closed2Closed}(a, b), \text{RReal}, f)(x)) \mapsto M \in$
 428 $\text{leq})) \Rightarrow$
 429 $(\text{abs}(\text{divide}(\text{minus}(f(b) \mapsto f(a)) \mapsto \text{minus}(b \mapsto a)))) \mapsto M \in \text{leq})$

429 *functionConcat:*

430 $\forall A, B, f, g \cdot$
 431 $A \subseteq E \wedge B \subseteq E \wedge f \in A \rightarrow F \wedge g \in B \rightarrow F \wedge$
 432 $(\forall x \cdot x \in (A \cap B) \Rightarrow f(x) = g(x)) \Rightarrow$
 433 $(f \cup g) \in (A \cup B) \rightarrow F$

434 *functionConcatTyping:*

435 $\forall f, g \cdot$
 436 $f \in E \rightarrow F \wedge g \in E \rightarrow F \wedge$

437 $(\forall x \cdot x \in (\text{dom}(f) \cap \text{dom}(g)) \Rightarrow f(x) = g(x)) \Rightarrow$
438 $(f \cup g) \in (\text{dom}(f) \cup \text{dom}(g)) \rightarrow F$
439 *restrictionTyping:*
440 $\forall A, B, f \cdot A \subseteq E \wedge B \subseteq A \wedge f \in A \rightarrow F \Rightarrow ((B \triangleleft f) \in B \rightarrow F)$
441 *restrictionDomain:*
442 $\forall f, A \cdot f \in E \rightarrow F \wedge A \subseteq E \Rightarrow \text{dom}(A \triangleleft f) = A$
443 *partialFunction:*
444 $\forall A, B, f1, f2 \cdot$
445 $A \subseteq E \wedge B \subseteq E \wedge A \cap B = \emptyset \wedge$
446 $f1 \in A \rightarrow F \wedge f2 \in B \rightarrow F \Rightarrow ($
447 $(A \triangleleft (f1 \cup f2)) = f1 \wedge$
448 $(B \triangleleft (f1 \cup f2)) = f2$
449 $)$
450 *strictlyPositiveWeakening:*
451 $\forall A, f \cdot$
452 $A \subseteq \text{RReal} \wedge f \in \text{RReal} \rightarrow \text{RReal} \wedge A \subseteq \text{dom}(f) \wedge$
453 $\text{strictlyPositive}(A, f) \Rightarrow$
454 $\text{positive}(A, f)$
455 *strictlyNegativeWeakening:*
456 $\forall A, f \cdot$
457 $A \subseteq \text{RReal} \wedge f \in \text{RReal} \rightarrow \text{RReal} \wedge A \subseteq \text{dom}(f) \wedge$
458 $\text{strictlyNegative}(A, f) \Rightarrow$
459 $\text{negative}(A, f)$
460 *positiveDer2Increasing:*
461 $\forall A, f \cdot$
462 $A \subseteq \text{RReal} \wedge$
463 $f \in A \rightarrow \text{RReal} \wedge f \in \text{D1}(A, \text{RReal}) \Rightarrow ($
464 $\text{positive}(A, \text{Der}(A, \text{RReal}, f))$
465 \Leftrightarrow
466 $\text{increasing}(A, f)$
467 $)$
468 *strictlyPositiveDer2StrictlyIncreasing:*
469 $\forall A, f \cdot$
470 $A \subseteq \text{RReal} \wedge$
471 $f \in \text{RReal} \rightarrow \text{RReal} \wedge A \subseteq \text{dom}(f) \wedge f \in \text{D1}(A, \text{RReal}) \Rightarrow ($
472 $\text{strictlyPositive}(A, \text{Der}(A, \text{RReal}, f))$
473 \Leftrightarrow
474 $\text{strictlyIncreasing}(A, f)$
475 $)$
476 *negativeDer2Decreasing:*
477 $\forall A, f \cdot$
478 $A \subseteq \text{RReal} \wedge$
479 $f \in A \rightarrow \text{RReal} \wedge f \in \text{D1}(A, \text{RReal}) \Rightarrow ($
480 $\text{negative}(A, \text{Der}(A, \text{RReal}, f))$
481 \Leftrightarrow
482 $\text{decreasing}(A, f)$
483 $)$
484 *strictlyNegativeDer2StrictlyDecreasing:*
485 $\forall A, f \cdot$
486 $A \subseteq \text{RReal} \wedge$
487 $f \in \text{RReal} \rightarrow \text{RReal} \wedge A \subseteq \text{dom}(f) \wedge f \in \text{D1}(A, \text{RReal}) \Rightarrow ($
488 $\text{strictlyNegative}(A, \text{Der}(A, \text{RReal}, f))$
489 \Leftrightarrow
490 $\text{strictlyDecreasing}(A, f)$
491 $)$
492 *zeroDer2Constant:*
493 $\forall A, f \cdot$
494 $A \subseteq \text{RReal} \wedge$
495 $f \in \text{RReal} \rightarrow \text{RReal} \wedge A \subseteq \text{dom}(f) \wedge f \in \text{D1}(A, \text{RReal}) \Rightarrow ($
496 $(\forall x \cdot x \in A \Rightarrow \text{Der}(A, \text{RReal}, f)(x) = \text{Rzero})$
497 \Leftrightarrow
498 $\text{constant}(A, f)$
499 $)$
500 *meanValue_geq:*
501 $\forall a, b, m, f \cdot$
502 $a \in \text{RReal} \wedge b \in \text{RReal} \wedge a \mapsto b \in \text{It} \wedge m \in \text{RReal} \wedge$
503 $f \in \text{RReal} \rightarrow \text{RReal} \wedge \text{Closed2Closed}(a, b) \subseteq \text{dom}(f) \wedge f \in \text{D1}(\text{Closed2Closed}(a, b), \text{RReal}) \wedge$
504 $f(a) \mapsto m \in \text{geq} \wedge \text{positive}(\text{Closed2Closed}(a, b), \text{Der}(\text{Closed2Closed}(a, b), \text{RReal}, f)) \Rightarrow$
505 $(\forall t \cdot t \in \text{Closed2Closed}(a, b) \Rightarrow f(t) \mapsto m \in \text{geq})$
506 *meanValue_geq_strict:*
507 $\forall a, b, m, f \cdot$
508 $a \in \text{RReal} \wedge b \in \text{RReal} \wedge a \mapsto b \in \text{It} \wedge m \in \text{RReal} \wedge$
509 $f \in \text{RReal} \rightarrow \text{RReal} \wedge \text{Closed2Closed}(a, b) \subseteq \text{dom}(f) \wedge f \in \text{D1}(\text{Closed2Closed}(a, b), \text{RReal}) \wedge$
510 $f(a) \mapsto m \in \text{geq} \wedge \text{strictlyPositive}(\text{Closed2Closed}(a, b), \text{Der}(\text{Closed2Closed}(a, b), \text{RReal}, f)) \Rightarrow$

511 $(\forall t \cdot t \in \text{Open2Closed}(a,b) \Rightarrow f(t) \mapsto m \in \text{gt})$

512 *meanValue_leq*:

513 $\forall a, b, m, f \cdot$

514 $a \in \text{RReal} \wedge b \in \text{RReal} \wedge a \mapsto b \in \text{lt} \wedge m \in \text{RReal} \wedge$

515 $f \in \text{RReal} \mapsto \text{RReal} \wedge \text{Closed2Closed}(a,b) \subseteq \text{dom}(f) \wedge f \in \text{D1}(\text{Closed2Closed}(a,b), \text{RReal}) \wedge$

516 $f(a) \mapsto m \in \text{leq} \wedge \text{negative}(\text{Closed2Closed}(a,b), \text{Der}(\text{Closed2Closed}(a,b), \text{RReal}, f)) \Rightarrow$

517 $(\forall t \cdot t \in \text{Closed2Closed}(a,b) \Rightarrow f(t) \mapsto m \in \text{leq})$

518 *meanValue_leq_strict*:

519 $\forall a, b, m, f \cdot$

520 $a \in \text{RReal} \wedge b \in \text{RReal} \wedge a \mapsto b \in \text{lt} \wedge m \in \text{RReal} \wedge$

521 $f \in \text{RReal} \mapsto \text{RReal} \wedge \text{Closed2Closed}(a,b) \subseteq \text{dom}(f) \wedge f \in \text{D1}(\text{Closed2Closed}(a,b), \text{RReal}) \wedge$

522 $f(a) \mapsto m \in \text{leq} \wedge \text{strictlyNegative}(\text{Closed2Closed}(a,b), \text{Der}(\text{Closed2Closed}(a,b), \text{RReal}, f)) \Rightarrow$

523 $(\forall t \cdot t \in \text{Open2Closed}(a,b) \Rightarrow f(t) \mapsto m \in \text{lt})$

524 *meanValue_positivity*:

525 $\forall a, b, f \cdot$

526 $a \in \text{RReal} \wedge b \in \text{RReal} \wedge a \mapsto b \in \text{lt} \wedge$

527 $f \in \text{RReal} \mapsto \text{RReal} \wedge \text{Closed2Closed}(a,b) \subseteq \text{dom}(f) \wedge f \in \text{D1}(\text{Closed2Closed}(a,b), \text{RReal}) \wedge$

528 $f(a) \mapsto \text{Rzero} \in \text{geq} \wedge \text{positive}(\text{Closed2Closed}(a,b), \text{Der}(\text{Closed2Closed}(a,b), \text{RReal}, f)) \Rightarrow$

529 $\text{positive}(\text{Closed2Closed}(a,b), f)$

530 *meanValue_negativity*:

531 $\forall a, b, f \cdot$

532 $a \in \text{RReal} \wedge b \in \text{RReal} \wedge a \mapsto b \in \text{lt} \wedge$

533 $f \in \text{RReal} \mapsto \text{RReal} \wedge \text{Closed2Closed}(a,b) \subseteq \text{dom}(f) \wedge f \in \text{D1}(\text{Closed2Closed}(a,b), \text{RReal}) \wedge$

534 $f(a) \mapsto \text{Rzero} \in \text{leq} \wedge \text{negative}(\text{Closed2Closed}(a,b), \text{Der}(\text{Closed2Closed}(a,b), \text{RReal}, f)) \Rightarrow$

535 $\text{negative}(\text{Closed2Closed}(a,b), f)$

536 *bind_bind_partialComp*:

537 $\forall f1, f2, x1, x2, Af, Bf, Cf, Ax, Bx, Cx \cdot$

538 $Af \subseteq E \wedge Bf \subseteq F \wedge Cf \subseteq G \wedge$

539 $Ax \subseteq Af \wedge Bx \subseteq Bf \wedge Cx \subseteq Cf \wedge$

540 $f1 \in Af \times (Bf \times Cf) \rightarrow Bf \wedge f2 \in Af \times (Bf \times Cf) \rightarrow Cf \wedge$

541 $x1 \in Ax \rightarrow Bx \wedge x2 \in Ax \rightarrow Cx \Rightarrow$

542 $\text{partialComp}(\text{bind}(f1, f2),$

543 $\text{bind}(x1, x2))$

544 $) = \text{bind}(\text{partialComp}(f1, \text{bind}(x1, x2)),$

545 $\text{partialComp}(f2, \text{bind}(x1, x2)))$

546 $)$

547 $)$

548 $)$

549 *until_type*:

550 $\forall s, t0, f, g \cdot$

551 $s \in \text{RReal} \wedge t0 \in \text{RReal} \wedge s \mapsto t0 \in \text{leq} \wedge$

552 $f \in \text{RReal} \mapsto E \wedge g \in \text{RReal} \mapsto E \wedge$

553 $\text{Closed2Open}(s, t0) \subseteq \text{dom}(f) \wedge \text{Closed2Infinity}(t0) \subseteq \text{dom}(g) \Rightarrow$

554 $\text{until}(s, f, t0, g) \in \text{Closed2Infinity}(s) \rightarrow E$

555 *until_restrict1*:

556 $\forall s, t0, f, g \cdot$

557 $s \in \text{RReal} \wedge t0 \in \text{RReal} \wedge s \mapsto t0 \in \text{leq} \wedge$

558 $f \in \text{RReal} \mapsto E \wedge g \in \text{RReal} \mapsto E \wedge$

559 $\text{Closed2Open}(s, t0) \subseteq \text{dom}(f) \wedge \text{Closed2Infinity}(t0) \subseteq \text{dom}(g) \Rightarrow$

560 $(\text{Closed2Open}(s, t0) \triangleleft \text{until}(s, f, t0, g)) = f$

561 *until_restrict2*:

562 $\forall s, t0, f, g \cdot$

563 $s \in \text{RReal} \wedge t0 \in \text{RReal} \wedge s \mapsto t0 \in \text{leq} \wedge$

564 $f \in \text{RReal} \mapsto E \wedge g \in \text{RReal} \mapsto E \wedge$

565 $\text{Closed2Open}(s, t0) \subseteq \text{dom}(f) \wedge \text{Closed2Infinity}(t0) \subseteq \text{dom}(g) \Rightarrow$

566 $(\text{Closed2Infinity}(t0) \triangleleft \text{until}(s, f, t0, g)) = g$

567 *until_bind1*:

568 $\forall s, t0, f1, f2, g \cdot$

569 $s \in \text{RReal} \wedge t0 \in \text{RReal} \wedge s \mapsto t0 \in \text{leq} \wedge$

570 $f1 \in \text{RReal} \mapsto E \wedge f2 \in \text{RReal} \mapsto E \wedge g \in \text{RReal} \mapsto F \wedge$

571 $\text{Closed2Open}(s, t0) \subseteq \text{dom}(f1) \wedge \text{Closed2Infinity}(t0) \subseteq \text{dom}(f2) \Rightarrow ($

572 $\text{bind}(\text{until}(s, f1, t0, f2), g) = \text{until}(s, \text{bind}(f1, g), t0, \text{bind}(f2, g))$

573 $)$

574 *until_bind2*:

575 $\forall s, t0, f, g1, g2 \cdot$

576 $s \in \text{RReal} \wedge t0 \in \text{RReal} \wedge s \mapsto t0 \in \text{leq} \wedge$

577 $f \in \text{RReal} \mapsto E \wedge g1 \in \text{RReal} \mapsto F \wedge g2 \in \text{RReal} \mapsto F \wedge$

578 $\text{Closed2Open}(s, t0) \subseteq \text{dom}(g1) \wedge \text{Closed2Infinity}(t0) \subseteq \text{dom}(g2) \Rightarrow ($

579 $\text{bind}(f, \text{until}(s, g1, t0, g2)) = \text{until}(s, \text{bind}(f, g1), t0, \text{bind}(f, g2))$

580 $)$

581 *untilF_type*:

582 $\forall s, t0, f, g \cdot$

583 $s \in \text{RReal} \wedge t0 \in \text{RReal} \wedge s \mapsto t0 \in \text{leq} \wedge$

584 $f \in \text{RReal} \times E \mapsto F \wedge g \in \text{RReal} \times E \mapsto F \wedge$

585 $\text{Closed2Open}(s, t0) \times \emptyset \subseteq \text{dom}(f) \wedge \text{Closed2Infinity}(t0) \times \emptyset \subseteq \text{dom}(g) \Rightarrow$
586 $\text{untilF}(s, f, t0, g) \in \mathbb{R}\text{Real} \times E \Rightarrow F$
587 *untilF_type_strong:*
588 $\forall s, t0, f, g, A .$
589 $A \subseteq E \wedge$
590 $s \in \mathbb{R}\text{Real} \wedge t0 \in \mathbb{R}\text{Real} \wedge s \mapsto t0 \in \text{leq} \wedge$
591 $f \in \mathbb{R}\text{Real} \times A \rightarrow F \wedge g \in \mathbb{R}\text{Real} \times A \rightarrow F \Rightarrow$
592 $\text{untilF}(s, f, t0, g) \in \text{Closed2Infinity}(s) \times A \rightarrow F$
593 *untilF_bind1:*
594 $\forall s, t0, f1, f2, g .$
595 $s \in \mathbb{R}\text{Real} \wedge t0 \in \mathbb{R}\text{Real} \wedge s \mapsto t0 \in \text{leq} \wedge$
596 $f1 \in \mathbb{R}\text{Real} \times E \rightarrow F \wedge f2 \in \mathbb{R}\text{Real} \times E \rightarrow F \wedge g \in \mathbb{R}\text{Real} \times E \rightarrow G \wedge$
597 $\text{Closed2Open}(s, t0) \times \emptyset \subseteq \text{dom}(f1) \wedge \text{Closed2Infinity}(t0) \times \emptyset \subseteq \text{dom}(f2) \Rightarrow ($
598 $\text{bind}(\text{untilF}(s, f1, t0, f2), g) = \text{untilF}(s, \text{bind}(f1, g), t0, \text{bind}(f2, g))$
599 $)$
600 *untilF_bind2:*
601 $\forall s, t0, f, g1, g2 .$
602 $s \in \mathbb{R}\text{Real} \wedge t0 \in \mathbb{R}\text{Real} \wedge s \mapsto t0 \in \text{leq} \wedge$
603 $f \in \mathbb{R}\text{Real} \times E \rightarrow F \wedge g1 \in \mathbb{R}\text{Real} \times E \rightarrow G \wedge g2 \in \mathbb{R}\text{Real} \times E \rightarrow G \wedge$
604 $\text{Closed2Open}(s, t0) \times \emptyset \subseteq \text{dom}(g1) \wedge \text{Closed2Infinity}(t0) \times \emptyset \subseteq \text{dom}(g2) \Rightarrow ($
605 $\text{bind}(f, \text{untilF}(s, g1, t0, g2)) = \text{untilF}(s, \text{bind}(f, g1), t0, \text{bind}(f, g2))$
606 $)$
607 *fcste_cue:*
608 $\forall A, l . A \subseteq E \wedge l \in F \Rightarrow (\text{fcste}(A, l) \in C0(A, F))$
609 *fcste_type:*
610 $\forall A, l . A \subseteq E \wedge l \in F \Rightarrow \text{fcste}(A, l) \in A \rightarrow F$

PROOF RULES

612 *typing:*

Metavariables

614 $A: \mathbb{P}(E)$
615 $B: \mathbb{P}(F)$
616 $C: \mathbb{P}(G)$
617 $\text{fab}: \mathbb{P}(E \times F)$
618 $\text{gac}: \mathbb{P}(E \times G)$
619 $\text{fa_bc}: \mathbb{P}(E \times (F \times G))$
620 $\text{fab_c}: \mathbb{P}(E \times F \times G)$
621 $x: E$
622 $y: F$
623 $A2: \mathbb{P}(E)$

Rewrite Rules

625 *domainRestrictionInvolutive:* $A \triangleleft (A \triangleleft \text{fab})$
626 $\text{rhs1}: T \Rightarrow A \triangleleft \text{fab}$
627 *domainRestrictionComplete:* $A \triangleleft \text{fab}$
628 $\text{rhs1}: \text{dom}(\text{fab}) = A \Rightarrow \text{fab}$
629 *basicDomainRestrictionTyping:* $(A \triangleleft \text{fab}) \in E \Rightarrow F$
630 $\text{rhs1}: T \Rightarrow \text{fab} \in E \Rightarrow F$
631 *basicBindTyping:* $\text{bind}(\text{fab}, \text{gac}) \in E \Rightarrow F \times G$
632 $\text{rhs1}: T \Rightarrow \text{fab} \in E \Rightarrow F \wedge \text{gac} \in E \Rightarrow G$
633 *bindDomain:* $\text{dom}(\text{bind}(\text{fab}, \text{gac}))$
634 $\text{rhs1}: T \Rightarrow \text{dom}(\text{fab}) \cap \text{dom}(\text{gac})$
635 *fcsteTyping:* $\text{fcste}(A, y) \in A \rightarrow F$
636 $\text{rhs1}: T \Rightarrow T$
637 *bindDomSubset:* $A \subseteq \text{dom}(\text{bind}(\text{fab}, \text{gac}))$
638 $\text{rhs1}: T \Rightarrow A \subseteq \text{dom}(\text{fab}) \wedge A \subseteq \text{dom}(\text{gac})$
639 *fproj1Typing:* $\text{fproj1}(\text{fa_bc}) \in E \Rightarrow F$
640 $\text{rhs1}: T \Rightarrow \text{fa_bc} \in E \Rightarrow F \times G$
641 *fproj2Typing:* $\text{fproj2}(\text{fa_bc}) \in E \Rightarrow G$
642 $\text{rhs1}: T \Rightarrow \text{fa_bc} \in E \Rightarrow F \times G$
643 *fproj1Domain:* $\text{dom}(\text{fproj1}(\text{fa_bc}))$
644 $\text{rhs1}: T \Rightarrow \text{dom}(\text{fa_bc})$
645 *fproj2Domain:* $\text{dom}(\text{fproj2}(\text{fa_bc}))$
646 $\text{rhs1}: T \Rightarrow \text{dom}(\text{fa_bc})$

Inference Rules

648 *bindType:* $\text{fab} \in A \rightarrow B, \text{gac} \in A \rightarrow C \vdash \text{bind}(\text{fab}, \text{gac}) \in A \rightarrow B \times C$
649 *proj1Type:* $\text{fa_bc} \in A \rightarrow B \times C \vdash \text{fproj1}(\text{fa_bc}) \in A \rightarrow B$
650 *proj2Type:* $\text{fa_bc} \in A \rightarrow B \times C \vdash \text{fproj2}(\text{fa_bc}) \in A \rightarrow C$
651 *partial1Type:* $\text{fab_c} \in A \times B \rightarrow C, y \in B \vdash \text{partial1}(\text{fab_c}, y) \in A \rightarrow C$
652 *partial2Type:* $\text{fab_c} \in A \times B \rightarrow C, x \in A \vdash \text{partial2}(\text{fab_c}, x) \in B \rightarrow C$
653 *domainRestrictionType:* $A2 \subseteq A, \text{fab} \in A \rightarrow B \vdash (A2 \triangleleft \text{fab}) \in A2 \rightarrow B$

654 *binding:*

Metavariables

656 $A: \mathbb{P}(E)$
657 $B: \mathbb{P}(F)$
658 $C: \mathbb{P}(G)$

659 AR: $\mathbb{P}(\mathbb{R}\text{Real})$
660 fab: $\mathbb{P}(\mathbb{E}\times\mathbb{F})$
661 gac: $\mathbb{P}(\mathbb{E}\times\mathbb{G})$
662 fab2: $\mathbb{P}(\mathbb{E}\times\mathbb{F})$
663 gac2: $\mathbb{P}(\mathbb{E}\times\mathbb{G})$
664 fa_bc: $\mathbb{P}(\mathbb{E}\times(\mathbb{F}\times\mathbb{G}))$
665 farb: $\mathbb{P}(\mathbb{R}\text{Real}\times\mathbb{F})$
666 garc: $\mathbb{P}(\mathbb{R}\text{Real}\times\mathbb{G})$
667 far_bc: $\mathbb{P}(\mathbb{R}\text{Real}\times(\mathbb{F}\times\mathbb{G}))$
668 n: \mathbb{Z}
669 x: E

Rewrite Rules

670 *bindProjRew*: $\text{bind}(\text{fproj1}(\text{fa_bc}), \text{fproj2}(\text{fa_bc}))$
671 rhs1: $T \Rightarrow \text{fa_bc}$
672 *bindDerRew*: $\text{Der}(\text{AR}, \text{B}\times\text{C}, \text{bind}(\text{farb}, \text{garc}))$
673 rhs1: $T \Rightarrow \text{bind}(\text{Der}(\text{AR}, \text{B}, \text{farb}), \text{Der}(\text{AR}, \text{C}, \text{garc}))$
674 *bindEquality*: $\text{bind}(\text{fab}, \text{gac}) = \text{bind}(\text{fab2}, \text{gac2})$
675 rhs1: $T \Rightarrow \text{fab} = \text{fab2} \wedge \text{gac} = \text{gac2}$
676 *bindC0Rew*: $\text{bind}(\text{fab}, \text{gac}) \in \text{C0}(\text{A}, \text{B}\times\text{C})$
677 rhs1: $T \Rightarrow \text{fab} \in \text{C0}(\text{A}, \text{B}) \wedge \text{gac} \in \text{C0}(\text{A}, \text{C})$
678 *bindD1Rew*: $\text{bind}(\text{farb}, \text{garc}) \in \text{D1}(\text{AR}, \text{B}\times\text{C})$
679 rhs1: $T \Rightarrow \text{farb} \in \text{D1}(\text{AR}, \text{B}) \wedge \text{garc} \in \text{D1}(\text{AR}, \text{C})$
680 *bindCnRew*: $\text{bind}(\text{farb}, \text{garc}) \in \text{Cn}(n, \text{AR}, \text{B}\times\text{C})$
681 rhs1: $n \geq 0 \Rightarrow \text{farb} \in \text{Cn}(n, \text{AR}, \text{B}) \wedge \text{garc} \in \text{Cn}(n, \text{AR}, \text{C})$
682 *bindDnRew*: $\text{bind}(\text{farb}, \text{garc}) \in \text{Dn}(n, \text{AR}, \text{B}\times\text{C})$
683 rhs1: $n > 0 \Rightarrow \text{farb} \in \text{Dn}(n, \text{AR}, \text{B}) \wedge \text{garc} \in \text{Dn}(n, \text{AR}, \text{C})$
684 *bindRestrict*: $A \triangleleft \text{bind}(\text{fab}, \text{gac})$
685 rhs1: $T \Rightarrow \text{bind}(A \triangleleft \text{fab}, A \triangleleft \text{gac})$
686 *bindEvaulate*: $\text{bind}(\text{fab}, \text{gac})(x)$
687 rhs1: $T \Rightarrow \text{fab}(x) \mapsto \text{gac}(x)$

continuity:

Metavariables

690 A: $\mathbb{P}(\mathbb{E})$
691 B: $\mathbb{P}(\mathbb{F})$
692 AR: $\mathbb{P}(\mathbb{R}\text{Real})$
693 n: \mathbb{Z}

Rewrite Rules

694 *C0_to_Cn0*: $\text{C0}(\text{A}, \text{B})$
695 rhs1: $T \Rightarrow \text{Cn}(0, \text{A}, \text{B})$
696 *D1_toDn1*: $\text{D1}(\text{AR}, \text{B})$
697 rhs1: $T \Rightarrow \text{Dn}(1, \text{AR}, \text{B})$
698 *Cn0_to_C0*: $\text{Cn}(0, \text{A}, \text{B})$
699 rhs1: $T \Rightarrow \text{C0}(\text{A}, \text{B})$
700 *Dn1_to_D1*: $\text{Dn}(1, \text{AR}, \text{B})$
701 rhs1: $T \Rightarrow \text{D1}(\text{AR}, \text{B})$

misc:

Metavariables

702 f: $\mathbb{P}(\mathbb{E}\times\mathbb{F})$
703 g: $\mathbb{P}(\mathbb{E}\times\mathbb{F})$

Rewrite Rules

704 *fun_equality*: $f = g$
705 rhs1: $T \Rightarrow \text{dom}(f) = \text{dom}(g) \wedge (\forall x \cdot x \in \text{dom}(f) \Rightarrow f(x) = g(x))$

direct_product:

Metavariables

706 fab: $\mathbb{P}(\mathbb{E}\times\mathbb{F})$
707 gac: $\mathbb{P}(\mathbb{E}\times\mathbb{G})$

Rewrite Rules

708 *dirprodType*: $\text{fab} \otimes \text{gac} \in \mathbb{E} \leftrightarrow \mathbb{F}\times\mathbb{G}$
709 rhs1: $T \Rightarrow \text{fab} \in \mathbb{E} \leftrightarrow \mathbb{F} \wedge \text{gac} \in \mathbb{E} \leftrightarrow \mathbb{G}$
710 *dirprodDomain*: $\text{dom}(\text{fab} \otimes \text{gac})$
711 rhs1: $T \Rightarrow \text{dom}(\text{fab}) \cap \text{dom}(\text{gac})$

END