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THEORY DiffEq
  IMPORT THEORY PROJECTS
    /SimpleDEq THEORIES /SimpleDEq/Piecewise.dtf|org.eventb.theory.core.deployedTheoryRoot#Piecewise
  TYPE PARAMETERS E,F,UF,STATES,F1,F2
  DATA TYPES
    DE(F)
    CONSTRUCTORS
      ode(fun:ℙ(RReal×F×F),initial:F,initialArg:RReal)
      aode(afun:ℙ(F×F),ainit:F,ainitArg:RReal)
    CDE(F,UF)
    CONSTRUCTORS
      code(cfun:ℙ((RReal×F×UF)×F),cinit:F,cinitArg:RReal)
      caode(cafun:ℙ((F×UF)×F),cainit:F,cainitArg:RReal)
  OPERATORS
    autonomousToODE <expression> (e: DE(F))
      well-definedness ∃ f,i,ia · f ∈ F ⇸ F ∧ i ∈ F ∧ ia ∈ RReal ∧ e = aode(f,i,ia)
      direct definition
        ode((λ t_↦eta_ · t_ ∈ RReal ∧ eta_ ∈ F ∧ eta_ ∈ dom(afun(e)) | afun(e)(eta_)),initial(e),
            initialArg(e))
    solutionOf <predicate> (DR: ℙ(RReal),eta: RReal ⇸ F,eq: DE(F))
      well-definedness DR ⊆ dom(eta)
    SolutionsOf <expression> (DR: ℙ(RReal),eq: DE(F))
      direct definition
        { eta_ | eta_ ∈ RReal ⇸ F ∧ DR ⊆ dom(eta_) ∧ solutionOf(DR,eta_,eq) }
    CauchyLipschitzCondition <predicate> (DR: ℙ(RReal),DF: ℙ(F),eq: DE(F))
    PiecewiseCauchyLipschitzCondition <predicate> (DRs: ℙ(ℙ(RReal)),DF: ℙ(F),eq: DE(F))
      well-definedness DRs ≠ ∅,∀ DR1,DR2 · DR1 ∈ DRs ∧ DR2 ∈ DRs ∧ DR1 ≠ DR2 ⇒ DR1 ∩ DR2 = ∅
    Solvable <predicate> (DR: ℙ(RReal),eq: DE(F))
      direct definition
        ∃ x · x ∈ RReal ⇸ F ∧ DR ⊆ dom(x) ∧ solutionOf(DR,x,eq)
    AppendSolutionBAP <predicate> (eq: DE(F),DR: ℙ(RReal),A: ℙ(DR),B: ℙ(DR),eta: RReal ⇸ F,etap: RReal
        ⇸ F)
      well-definedness A ∩ B = ∅,Solvable(B,eq),DR ⊆ dom(eta)
      direct definition
        DR ⊆ dom(etap) ∧
        (A ◁ etap = A ◁ eta) ∧
        solutionOf(B,B ◁ etap, eq)
    CBAP <predicate> (t: RRealPlus,tp: RRealPlus,eta: RReal ⇸ F,etap: RReal ⇸ F,Pred: ℙ((RRealPlus⇸F
        )×(RRealPlus⇸F)),Inv: ℙ(F))
      well-definedness Closed2Closed(Rzero,t) ⊆ dom(eta),Closed2Closed(Rzero,tp) ⊆ dom(etap)
      direct definition
        t ↦ tp ∈ lt ∧
        Closed2Open(Rzero,t) ◁ eta = Closed2Open(Rzero,t) ◁ etap ∧
        ((Closed2Closed(t,tp) ◁ eta) ↦ (Closed2Closed(t,tp) ◁ etap)) ∈ Pred ∧
        eta(t) = etap(t) ∧
        (∀ t_ · t_ ∈ Closed2Closed(t,tp) ⇒ etap(t_) ∈ Inv)
    CBAPsolutionOf <predicate> (t: RRealPlus,tp: RRealPlus,eta: RReal ⇸ F,etap: RReal ⇸ F,eq: DE(F),Inv
        : ℙ(F))
      well-definedness Closed2Closed(Rzero,t) ⊆ dom(eta),Closed2Closed(Rzero,tp) ⊆ dom(etap),Solvable(
          Closed2Closed(t,tp),eq),t ↦ tp ∈ lt
      direct definition
        CBAP(t,tp,eta,etap,(RReal⇸F)×SolutionsOf(Closed2Closed(t,tp),eq),Inv)
    CBAPParallelEq <predicate> (t: RRealPlus,tp: RRealPlus,eta1: RRealPlus ⇸ F1,eta1p: RRealPlus ⇸ F1,
        eq1: DE(F1),eta2: RRealPlus ⇸ F2,eta2p: RRealPlus ⇸ F2,eq2: DE(F2),Inv12: ℙ(F1×F2))
      well-definedness Closed2Closed(Rzero,t) ⊆ dom(eta1),Closed2Closed(Rzero,tp) ⊆ dom(eta1p),
          Closed2Closed(Rzero,t) ⊆ dom(eta2),Closed2Closed(Rzero,tp) ⊆ dom(eta2p),t ↦ tp ∈ lt
      direct definition
        t ↦ tp ∈ lt ∧
        Closed2Open(Rzero,t) ◁ eta1 = Closed2Open(Rzero,t) ◁ eta1p ∧
        solutionOf(Closed2Closed(t,tp),Closed2Closed(t,tp) ◁ eta1p,eq1) ∧
        eta1(t) = eta1p(t) ∧
        Closed2Open(Rzero,t) ◁ eta2 = Closed2Open(Rzero,t) ◁ eta2p ∧
        solutionOf(Closed2Closed(t,tp),Closed2Closed(t,tp) ◁ eta2p,eq2) ∧
        eta2(t) = eta2p(t) ∧
        (∀ t_ · t_ ∈ Closed2Closed(t,tp) ⇒ eta1p(t_)↦eta2p(t_) ∈ Inv12)
    VerifiesOn <predicate> (DR: ℙ(RReal),eta: RReal → F,Inv: ℙ(F))
      direct definition
        ∀ t_ · t_ ∈ DR ⇒ eta(t_) ∈ Inv
    withControl <expression> (DR: ℙ(RReal),ce: CDE(F,UF),u: RReal ⇸ UF)
      well-definedness DR ⊆ dom(u)
    Controllable <predicate> (DR: ℙ(RReal),ce: CDE(F,UF))
      direct definition
        ∃ u · u ∈ RReal ⇸ UF ∧ DR ⊆ dom(u) ∧ Solvable(DR,withControl(DR,ce,u))
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67    ControllableOn <predicate> (DR: ℙ(RReal),DU: ℙ(UF),ce: CDE(F,UF))
68      direct definition
69        ∃ u · u ∈ RReal ⇸ UF ∧ DR ⊆ dom(u) ∧ Solvable(DR,withControl(DR,ce,u)) ∧ (∀ t_ · t_ ∈ DR ⇒ u(
              t_) ∈ DU)
70    SolvableWith <predicate> (DR: ℙ(RReal),ce: CDE(F,UF),u: RReal ⇸ UF)
71      well-definedness DR ⊆ dom(u)
72      direct definition
73        Solvable(DR,withControl(DR,ce,u))
74    CBAPFIS <predicate> (t: RRealPlus,tp: RRealPlus,eta: RReal ⇸ F,Pred: ℙ((RRealPlus ⇸ F)×(RRealPlus
          ⇸ F)),Inv: ℙ(F))
75      well-definedness Closed2Closed(Rzero,t) ⊆ dom(eta),t ↦ tp ∈ lt
76      direct definition
77        ∃ etap ·
78          etap ∈ RRealPlus ⇸ F ∧ Closed2Closed(t,tp) ⊆ dom(etap) ∧
79          ((Closed2Closed(t,tp) ◁ eta) ↦ (Closed2Closed(t,tp) ◁ etap)) ∈ Pred ∧
80          eta(t) = etap(t) ∧
81          (∀ t_ · t_ ∈ Closed2Closed(t,tp) ⇒ etap(t_) ∈ Inv)
82    CBAPsolutionOfFIS <predicate> (t: RRealPlus,tp: RRealPlus,eta: RReal ⇸ F,eq: DE(F),Inv: ℙ(F))
83      well-definedness Closed2Closed(Rzero,t) ⊆ dom(eta),t ↦ tp ∈ lt,Solvable(Closed2Closed(t,tp),eq)
84      direct definition
85        ∃ etap ·
86          etap ∈ RRealPlus ⇸ F ∧ Closed2Closed(t,tp) ⊆ dom(etap) ∧
87          solutionOf(Closed2Closed(t,tp),etap,eq) ∧
88          eta(t) = etap(t) ∧
89          (∀ t_ · t_ ∈ Closed2Closed(t,tp) ⇒ etap(t_) ∈ Inv)
90    CBAPParallelEqFIS <predicate> (t: RRealPlus,tp: RRealPlus,eta1: RRealPlus ⇸ F1,eq1: DE(F1),eta2:
        RRealPlus ⇸ F2,eq2: DE(F2),Inv12: ℙ(F1×F2))
91      well-definedness t ↦ tp ∈ lt,Closed2Closed(Rzero,t) ⊆ dom(eta1),Closed2Closed(Rzero,t) ⊆ dom(
          eta2),Solvable(Closed2Closed(t,tp),eq1),Solvable(Closed2Closed(t,tp),eq2)
92      direct definition
93        ∃ eta1p,eta2p ·
94          eta1p ∈ RRealPlus ⇸ F1 ∧ Closed2Closed(t,tp) ⊆ dom(eta1p) ∧
95          eta2p ∈ RRealPlus ⇸ F2 ∧ Closed2Closed(t,tp) ⊆ dom(eta2p) ∧
96          solutionOf(Closed2Closed(t,tp),eta1p,eq1) ∧
97          solutionOf(Closed2Closed(t,tp),eta2p,eq2) ∧
98          eta1(t) = eta1p(t) ∧ eta2(t) = eta2p(t) ∧
99          (∀ t_ · t_ ∈ Closed2Closed(t,tp) ⇒ (eta1p(t_)↦eta2p(t_)) ∈ Inv12)
100 THEOREMS
101   solutionDer:
102     ∀ fu,DR,DF,init,initArg,eta,k ·
103       DR ⊆ RReal ∧ DF ⊆ F ∧
104       fu ∈ (DR×DF→F) ∧ k ∈ ℕ ∧ k > 0 ∧ (∀ x0 · x0 ∈ DF ⇒ partial1(fu,x0) ∈ Dn(k,DR,F)) ∧
105       init ∈ DF ∧ initArg ∈ DR ∧
106       eta ∈ RReal ⇸ F ∧ DR ⊆ dom(eta) ∧
107       solutionOf(DR,eta,ode(fu,init,initArg)) ⇒
108         eta ∈ Dn(k+1,DR,F)
109   SolutionsOf_solutionOf:
110     ∀ DR,eq,eta ·
111       DR ⊆ RReal ∧ eq ∈ DE(F) ∧ eta ∈ RReal ⇸ F ∧ DR ⊆ dom(eta) ⇒
112         (eta ∈ SolutionsOf(DR,eq) ⇔ solutionOf(DR,eta,eq))
113   CauchyLipschitz:
114     ∀ eq,DR,DF ·
115       DR ⊆ RReal ∧ DF ⊆ F ∧ eq ∈ DE(F) ∧
116       CauchyLipschitzCondition(DR,DF,eq)
117         ⇒
118           Solvable(DR,eq)
119   concatSolutions:
120     ∀ DR1,DR2,eta1,eta2,eq ·
121       DR1 ⊆ RReal ∧ DR2 ⊆ RReal ∧ DR1 ∩ DR2 = ∅ ∧
122       eta1 ∈ RReal ⇸ F ∧ DR1 ⊆ dom(eta1) ∧
123       eta2 ∈ RReal ⇸ F ∧ DR2 ⊆ dom(eta2) ∧
124       eq ∈ DE(F) ∧
125       solutionOf(DR1,eta1,eq) ∧ solutionOf(DR2,eta2,eq) ⇒
126         solutionOf(DR1 ∪ DR2,eta1 ∪ eta2, eq)
127   appendSolutionExistence:
128     ∀ DR, A, B, eq, eta ·
129       DR ⊆ RReal ∧
130       A ⊆ DR ∧ B ⊆ DR ∧ A ∩ B = ∅ ∧ A ∪ B = DR ∧
131       eq ∈ DE(F) ∧ Solvable(B,eq) ∧
132       eta ∈ RReal ⇸ F ∧ DR ⊆ dom(eta) ⇒
133         (∃ etap · etap ∈ DR → F ∧ AppendSolutionBAP(eq,DR,A,B,eta,etap))
134   concatSolvable:
135     ∀ DR1,DR2,eq ·
136       DR1 ⊆ RReal ∧ DR2 ⊆ RReal ∧ DR1 ∩ DR2 = ∅ ∧
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    eq ∈ DE(F) ∧ Solvable(DR1,eq) ∧ Solvable(DR2,eq) ⇒
      Solvable(DR1 ∪ DR2,eq)
  PiecewiseCauchyLipschitz:
    ∀ eq,DRs,DF ·
      DRs ⊆ ℙ(RReal) ∧ DRs ≠ ∅ ∧ (∀ DR1,DR2 · DR1 ∈ DRs ∧ DR2 ∈ DRs ∧ DR1 ≠ DR2 ⇒ DR1 ∩ DR2 = ∅) ∧
      DF ⊆ F ∧ eq ∈ DE(F) ∧
      PiecewiseCauchyLipschitzCondition(DRs,DF,eq) ⇒
        Solvable(union(DRs),eq)
  solution_restriction:
    ∀ DR1,DR2,eq,eta ·
      DR1 ⊆ RReal ∧ DR2 ⊆ DR1 ∧ eq ∈ DE(F) ∧
      eta ∈ RReal ⇸ F ∧ solutionOf(DR1,eta,eq)
        ⇒
          solutionOf(DR2,eta,eq)
  solvable_restriction:
    ∀ DR1,DR2,eq ·
      DR1 ⊆ RReal ∧ DR2 ⊆ DR1 ∧ eq ∈ DE(F) ∧
      Solvable(DR1,eq)
        ⇒
          Solvable(DR2,eq)
  CBAPsolutionOf_FIS:
    ∀ t,eta,eq,Inv ·
      t ∈ RRealPlus ∧
      eta ∈ RReal ⇸ F ∧ Closed2Closed(Rzero,t) ⊆ dom(eta) ∧
      eq ∈ DE(F) ∧
      Inv ∈ ℙ(F) ∧ IsOpen(Inv) ∧ eta(t) ∈ Inv ∧
      Solvable(Closed2Infinity(t),eq)
      ⇒ (
        ∃ tp,etap ·
          tp ∈ RRealPlus ∧ t ↦ tp ∈ lt ∧
          etap ∈ RReal ⇸ F ∧ Closed2Closed(Rzero,tp) ⊆ dom(etap) ∧
          Solvable(Closed2Closed(t,tp),eq) ∧
          CBAPsolutionOf(t,tp,eta,etap,eq,Inv)
      )
  CBAPsolutionOf_INV:
    ∀ t,tp,eta,etap,eq,LocalInv,GlobalInv ·
      t ∈ RRealPlus ∧ tp ∈ RRealPlus ∧ t ↦ tp ∈ lt ∧
      eta ∈ RReal ⇸ F ∧ Closed2Closed(Rzero,t) ⊆ dom(eta) ∧
      etap ∈ RReal ⇸ F ∧ Closed2Closed(Rzero,tp) ⊆ dom(etap) ∧
      eq ∈ DE(F) ∧ Solvable(Closed2Closed(t,tp),eq) ∧
      LocalInv ⊆ F ∧ eta(t) ∈ LocalInv ∧
      GlobalInv ⊆ F ∧ (∀ t_ · t_ ∈ Closed2Closed(Rzero,t) ⇒ eta(t_) ∈ GlobalInv) ∧
      CBAPsolutionOf(t,tp,eta,etap,eq,LocalInv ∩ GlobalInv)
        ⇒
          (∀ t_ · t_ ∈ Closed2Closed(Rzero,tp) ⇒ etap(t_) ∈ GlobalInv)
  CBAPFIS_act_FIS:
    ∀ t,tp,eta,Pred,Inv ·
      t ∈ RRealPlus ∧ tp ∈ RRealPlus ∧ eta ∈ RReal ⇸ F ∧
      Pred ∈ ℙ((RRealPlus ⇸ F)×(RRealPlus ⇸ F)) ∧ Inv ∈ ℙ(F) ∧
      Closed2Closed(Rzero,t) ⊆ dom(eta) ∧ t ↦ tp ∈ lt ∧
      CBAPFIS(t,tp,eta,Pred,Inv)
        ⇒ (
          ∃ etap · etap ∈ RReal ⇸ F ∧ Closed2Closed(Rzero,tp) ⊆ dom(etap) ∧
            CBAP(t,tp,eta,etap,Pred,Inv)
        )
  CBAPsolutionOfFIS_act_FIS:
    ∀ t,tp,eta,eq,Inv ·
      t ∈ RRealPlus ∧ tp ∈ RRealPlus ∧ eta ∈ RReal ⇸ F ∧
      eq ∈ DE(F) ∧ Inv ∈ ℙ(F) ∧
      Closed2Closed(Rzero,t) ⊆ dom(eta) ∧ t ↦ tp ∈ lt ∧
      CBAPsolutionOfFIS(t,tp,eta,eq,Inv)
        ⇒ (
          ∃ etap · etap ∈ RReal ⇸ F ∧ Closed2Closed(Rzero,tp) ⊆ dom(etap) ∧
            CBAPsolutionOf(t,tp,eta,etap,eq,Inv)
        )
  CBAPFIS_restriction:
    ∀ t,tp,eta,Pred,Inv ·
      t ∈ RRealPlus ∧ tp ∈ RRealPlus ∧ eta ∈ RReal ⇸ F ∧
      Pred ∈ ℙ((RRealPlus ⇸ F)×(RRealPlus ⇸ F)) ∧ Inv ∈ ℙ(F) ∧
      Closed2Closed(Rzero,t) ⊆ dom(eta) ∧ t ↦ tp ∈ lt ∧
      CBAPFIS(t,tp,eta,Pred,Inv) ⇒ (
        ∀ tc · tc ∈ RRealPlus ∧ t ↦ tc ∈ lt ∧ tc ↦ tp ∈ leq ⇒
            CBAPFIS(t,tc,eta,Pred,Inv)
      )
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    CBAPsolutionOfFIS_implies_CBAPFIS:
      ∀ t,tp,eta,eq,Inv ·
        t ∈ RRealPlus ∧ tp ∈ RRealPlus ∧ t ↦ tp ∈ lt ∧
        eta ∈ RReal ⇸ F ∧ Closed2Closed(Rzero,t) ⊆ dom(eta) ∧
        eq ∈ DE(F) ∧ Solvable(Closed2Closed(t,tp),eq) ∧
        Inv ∈ ℙ(F) ⇒ (
            CBAPsolutionOfFIS(t,tp,eta,eq,Inv)
          ⇔
            CBAPFIS(t,tp,eta,(RReal⇸F)×SolutionsOf(Closed2Closed(t,tp),eq),Inv)
        )
  PROOF RULES
    sof:
    Metavariables
      DR: ℙ(RReal)
      eq: DE(F)
      eta: ℙ(RReal×F)
    Rewrite Rules
      Solutions_to_solution: eta ∈ SolutionsOf(DR,eq)
        rhs1: ⊤ ⇒ solutionOf(DR,eta,eq)
      solution_to_Solutions: solutionOf(DR,eta,eq)
        rhs1: ⊤ ⇒ eta ∈ SolutionsOf(DR,eq)
    eq_typing:
    Metavariables
      DR: ℙ(RReal)
      ceq: CDE(F,UF)
      u: ℙ(RReal×UF)
    Rewrite Rules
      type_withControl: withControl(DR,ceq,u) ∈ DE(F)
        rhs1: ⊤ ⇒ ⊤
END
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