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THEORY
  IMPORT THEORY PROJECTS
    /SimpleDEq THEORIES /SimpleDEq/Approximation.dtf|org.eventb.theory.core.deployedTheoryRoot#Approximation
  OPERATORS
    PendulumRawFun <expression> (omega0: RReal)
      direct definition
        (λ t_↦(x1_↦x2_)↦u_ ·
          t_ ∈ RRealPlus ∧ x1_ ∈ RReal ∧ x2_ ∈ RReal ∧ u_ ∈ RReal
          | x2_ ↦ (plus(times(u_ ↦ cos(x1_)) ↦ times(times(omega0↦omega0)↦sin(x1_))))
        )
    PendulumRaw <expression> (omega0: RReal, x0: RReal×RReal, t0: RRealPlus)
      direct definition
        code(PendulumRawFun(omega0),x0,t0)
    PendulumLinFun <expression> (omega0: RReal)
      direct definition
        (λ t_↦(x1_↦x2_)↦u_ ·
          t_ ∈ RRealPlus ∧ x1_ ∈ RReal ∧ x2_ ∈ RReal ∧ u_ ∈ RReal
          | x2_ ↦ (plus(u_ ↦ times(times(omega0↦omega0)↦x1_)))
        )
    PendulumLin <expression> (omega0: RReal, x0: RReal×RReal, t0: RRealPlus)
      direct definition
        code(PendulumLinFun(omega0),x0,t0)
  AXIOMATIC DEFINITIONS  pendulum_solvability:
    OPERATORS
      theta_max <expression> (omega0: RReal) : RReal
    AXIOMS
      theta_max_bounds:
        ∀ omega0 · omega0 ∈ RReal ⇒ Rzero ↦ theta_max(omega0) ∈ leq ∧ theta_max(omega0) ↦ divide(pi↦Rtwo) ∈ leq
      pendulum_raw_controllability:
        ∀ omega0,theta0,thetap0,t0 ·
          omega0 ∈ RReal ∧
          theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
          thetap0 ∈ RReal ∧
          t0 ∈ RRealPlus
            ⇒ (
          ∃ t1 · t1 ∈ RRealPlus ∧ t0 ↦ t1 ∈ lt ∧
            Controllable(Closed2Closed(t0,t1),PendulumRaw(omega0,(theta0↦thetap0),t0))
        )
      pendulum_lin_controllability:
        ∀ omega0,theta0,thetap0,t0 ·
          omega0 ∈ RReal ∧
          theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
          thetap0 ∈ RReal ∧
          t0 ∈ RRealPlus
            ⇒ (
          ∃ t1 · t1 ∈ RRealPlus ∧ t0 ↦ t1 ∈ lt ∧
            Controllable(Closed2Closed(t0,t1),PendulumLin(omega0,(theta0↦thetap0),t0))
        )  pendulum_approx:
    OPERATORS
      PendulumApproxWD <predicate> (delta: RReal, omega0: RReal, theta_bound: RReal, ctrl_bound: RReal, ctrl_bound_lin: RReal, ctrl_delta: RReal, t0: RRealPlus, t1: RRealPlus) :
        well−definedness Rzero↦delta ∈ lt, Rzero↦theta_bound ∈ lt, theta_bound ↦ theta_max(omega0) ∈ lt, Rzero↦ctrl_bound ∈ lt, Rzero↦ctrl_bound_lin ∈ lt, Rzero↦ctrl_delta ∈ lt, t0 ↦ t1 ∈ lt
    AXIOMS
      PAWD_Approximation:
        ∀ delta,omega0,theta_bound,ctrl_bound,ctrl_bound_lin,ctrl_delta,t0,t1,x0_raw,x0_lin,u_raw,u_lin ·
          delta ∈ RReal ∧ Rzero ↦ delta ∈ lt ∧
          omega0 ∈ RReal ∧
          theta_bound ∈ RReal ∧ Rzero ↦ theta_bound ∈ lt ∧ theta_bound ↦ theta_max(omega0) ∈ lt ∧
          ctrl_bound ∈ RReal ∧ Rzero ↦ ctrl_bound ∈ lt ∧
          ctrl_bound_lin ∈ RReal ∧ Rzero ↦ ctrl_bound_lin ∈ lt ∧
          ctrl_delta ∈ RReal ∧ Rzero ↦ ctrl_delta ∈ lt ∧
          t0 ∈ RRealPlus ∧ t1 ∈ RRealPlus ∧ t0 ↦ t1 ∈ lt ∧
          PendulumApproxWD(delta,omega0,theta_bound,ctrl_bound,ctrl_bound_lin,ctrl_delta,t0,t1) ∧
          u_raw ∈ RReal ⇸ RReal ∧ Closed2Closed(t0,t1) ⊆ dom(u_raw) ∧ (∀ t_ · t_ ∈ Closed2Closed(t0,t1) ⇒ abs(u_raw(t_)) ↦ ctrl_bound ∈ lt) ∧
          u_lin ∈ RReal ⇸ RReal ∧ Closed2Closed(t0,t1) ⊆ dom(u_lin) ∧ (∀ t_ · t_ ∈ Closed2Closed(t0,t1) ⇒ abs(u_lin(t_)) ↦ ctrl_bound_lin ∈ lt) ∧
          DeltaApproximation(Closed2Closed(t0,t1),ctrl_delta,u_raw,u_lin) ∧
          x0_raw ∈ RReal×RReal ∧ x0_lin ∈ RReal×RReal ∧ DeltaNeighborhood(delta,x0_raw,x0_lin)
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        ⇒
          DeltaApproximationEq(Closed2Closed(t0,t1),delta,
            withControl(Closed2Closed(t0,t1),PendulumRaw(omega0,x0_raw,t0),u_raw),
            withControl(Closed2Closed(t0,t1),PendulumLin(omega0,x0_lin,t0),u_lin)
          )  pendulum_control:
OPERATORS
  PendulumRawControl <expression> (omega0: RReal,theta0: RReal,thetap0: RReal,t0: RRealPlus) : ℙ(
      RReal×RReal)
    well−definedness abs(theta0) ↦ theta_max(omega0) ∈ lt
  PendulumLinControl <expression> (omega0: RReal,theta0: RReal,thetap0: RReal,t0: RRealPlus) : ℙ(
      RReal×RReal)
    well−definedness abs(theta0) ↦ theta_max(omega0) ∈ lt
  PC_raw_bound <expression> () : RReal
  PC_lin_bound <expression> () : RReal
  PendulumControlDelta <expression> (omega0: RReal,delta: RReal) : RReal
    well−definedness Rzero ↦ delta ∈ lt
AXIOMS
  pendulum_raw_control_type:
    ∀ omega0,theta0,thetap0,t0 ·
      omega0 ∈ RReal ∧
      theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
      thetap0 ∈ RReal ∧
      t0 ∈ RRealPlus
        ⇒ (
          PendulumRawControl(omega0,theta0,thetap0,t0) ∈ RReal ⇸ RReal ∧
          Closed2Infinity(t0) ⊆ dom(PendulumRawControl(omega0,theta0,thetap0,t0))
    )
  pendulum_lin_control_type:
    ∀ omega0,theta0,thetap0,t0 ·
      omega0 ∈ RReal ∧
      theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
      thetap0 ∈ RReal ∧
      t0 ∈ RRealPlus
        ⇒ (
          PendulumLinControl(omega0,theta0,thetap0,t0) ∈ RReal ⇸ RReal ∧
          Closed2Infinity(t0) ⊆ dom(PendulumLinControl(omega0,theta0,thetap0,t0))
    )
  pendulum_raw_control_bound_def:
    Rzero ↦ PC_raw_bound ∈ lt
  pendulum_raw_control_bound:
    ∀ omega0,theta0,thetap0,t0 ·
      omega0 ∈ RReal ∧
      theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
      thetap0 ∈ RReal ∧
      t0 ∈ RRealPlus
        ⇒ (
          ∀ t_ · t_ ∈ RRealPlus ∧ t0 ↦ t_ ∈ leq ⇒
            abs(PendulumRawControl(omega0,theta0,thetap0,t0)(t_)) ↦ PC_raw_bound ∈ lt
    )
  pendulum_raw_control_acceptable:
    ∀ omega0,theta0,thetap0,t0 ·
      omega0 ∈ RReal ∧
      theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
      thetap0 ∈ RReal ∧
      t0 ∈ RRealPlus
        ⇒ (
        ∃ t1 · t1 ∈ RRealPlus ∧ t0 ↦ t1 ∈ lt ∧
          SolvableWith(
            Closed2Closed(t0,t1),
            PendulumRaw(omega0,(theta0↦thetap0),t0),
            PendulumRawControl(omega0,theta0,thetap0,t0)
          )
    )
  pendulum_raw_control_solution_bounded:
    ∀ omega0,theta0,thetap0,t0,t1 ·
      omega0 ∈ RReal ∧
      theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
      thetap0 ∈ RReal ∧
      t0 ∈ RRealPlus ∧ t1 ∈ RRealPlus ∧ t0 ↦ t1 ∈ lt ∧
      SolvableWith(Closed2Closed(t0,t1),PendulumRaw(omega0,(theta0↦thetap0),t0),PendulumRawControl
          (omega0,theta0,thetap0,t0))
      ⇒ (
        ∀ theta_,thetap_ ·
          theta_ ∈ RReal ⇸ RReal ∧ Closed2Closed(t0,t1) ⊆ dom(theta_) ∧
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            thetap_ ∈ RReal ⇸ RReal ∧ Closed2Closed(t0,t1) ⊆ dom(thetap_) ∧
            solutionOf(
              Closed2Closed(t0,t1),
              bind(theta_,thetap_),
              withControl(
                Closed2Closed(t0,t1),
                PendulumRaw(omega0,(theta0↦thetap0),t0),
                PendulumRawControl(omega0,theta0,thetap0,t0)
              )
            )
            ⇒ (
              ∀ t_ · t_ ∈ Closed2Closed(t0,t1) ⇒ abs(theta_(t_)) ↦ theta0 ∈ lt
        )
    )
pendulum_lin_control_bound_def:
  Rzero ↦ PC_lin_bound ∈ lt
pendulum_lin_control_bound:
  ∀ omega0,theta0,thetap0,t0 ·
    omega0 ∈ RReal ∧
    theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
    thetap0 ∈ RReal ∧
    t0 ∈ RRealPlus
      ⇒ (
        ∀ t_ · t_ ∈ RRealPlus ∧ t0 ↦ t_ ∈ leq ⇒
          abs(PendulumLinControl(omega0,theta0,thetap0,t0)(t_)) ↦ PC_lin_bound ∈ lt
  )
pendulum_lin_control_acceptable:
  ∀ omega0,theta0,thetap0,t0 ·
    omega0 ∈ RReal ∧
    theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
    thetap0 ∈ RReal ∧
    t0 ∈ RRealPlus
      ⇒ (
      ∃ t1 · t1 ∈ RRealPlus ∧ t0 ↦ t1 ∈ lt ∧
        SolvableWith(
          Closed2Closed(t0,t1),
          PendulumLin(omega0,(theta0↦thetap0),t0),
          PendulumLinControl(omega0,theta0,thetap0,t0)
        )
  )
pendulum_lin_control_solution_bounded:
  ∀ omega0,theta0,thetap0,t0,t1 ·
    omega0 ∈ RReal ∧
    theta0 ∈ RReal ∧ abs(theta0) ↦ theta_max(omega0) ∈ lt ∧
    thetap0 ∈ RReal ∧
    t0 ∈ RRealPlus ∧ t1 ∈ RRealPlus ∧ t0 ↦ t1 ∈ lt ∧
    SolvableWith(Closed2Closed(t0,t1),PendulumLin(omega0,(theta0↦thetap0),t0),PendulumLinControl(omega0,theta0,thetap0,t0))
    ⇒ (
      ∀ theta_,thetap_ ·
        theta_ ∈ RReal ⇸ RReal ∧ Closed2Closed(t0,t1) ⊆ dom(theta_) ∧
        thetap_ ∈ RReal ⇸ RReal ∧ Closed2Closed(t0,t1) ⊆ dom(thetap_) ∧
        solutionOf(
          Closed2Closed(t0,t1),
          bind(theta_,thetap_),
          withControl(
            Closed2Closed(t0,t1),
            PendulumLin(omega0,(theta0↦thetap0),t0),
            PendulumLinControl(omega0,theta0,thetap0,t0)
          )
        )
        ⇒ (
          ∀ t_ · t_ ∈ Closed2Closed(t0,t1) ⇒ abs(theta_(t_)) ↦ theta0 ∈ lt
      )
  )
pendulum_control_delta_def:
  ∀ omega0,delta ·
    omega0 ∈ RReal ∧
    delta ∈ RReal ∧ Rzero ↦ delta ∈ lt
      ⇒
        Rzero ↦ PendulumControlDelta(omega0,delta) ∈ lt
pendulum_control_approx:
  ∀ delta,omega0,theta0_raw,thetap0_raw,theta0_lin,thetap0_lin,t0 ·
    delta ∈ RReal ∧ Rzero ↦ delta ∈ lt ∧
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            omega0 ∈ RReal ∧
            theta0_raw ∈ RReal ∧ abs(theta0_raw) ↦ theta_max(omega0) ∈ lt ∧
            thetap0_raw ∈ RReal ∧
            theta0_lin ∈ RReal ∧ abs(theta0_lin) ↦ theta_max(omega0) ∈ lt ∧
            thetap0_lin ∈ RReal ∧
            t0 ∈ RRealPlus ∧
            DeltaNeighborhood(delta,(theta0_raw↦thetap0_raw),(theta0_lin↦thetap0_lin))
              ⇒
                DeltaApproximation(
                  Closed2Infinity(t0),
                  PendulumControlDelta(omega0, delta),
                  PendulumRawControl(omega0, theta0_raw, thetap0_raw, t0),
                  PendulumLinControl(omega0, theta0_lin, thetap0_lin, t0)
                )
END
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