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Erratum: A Continuous Exact ℓ_0 Penalty (CEL0) for Least Squares Regularized Problem*

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Abstract. Lemma 4.4 in [E. Soubies, L. Blanc-Féraud and G. Aubert, *SIAM J. Imaging Sci.*, 8 (2015), pp. 1607–1639] is wrong for local minimizers of the continuous exact ℓ_0 (CEL0) functional. The argument used to conclude the proof of this lemma is not sufficient in the case of local minimizers. In this note, we supply a revision of this lemma where new results are established for local minimizers. Theorem 4.8 in that paper remains unchanged but the proof has to be rewritten according to the new version of the lemma. Finally, some remarks of this paper are also rewritten using the corrected lemma.

Key words. inverse problems, ℓ_0 regularization, sparse modeling, underdetermined linear systems, global minimizers, local minimizers, minimizers equivalence, continuous exact ℓ_0 penalty, nonconvex nonsmooth penalty

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Let $\mathcal{C} \subset \mathbb{R}^N$ be a closed subset of $\mathbb{R}^N$ and $F : \mathbb{R}^N \to \mathbb{R}$. Assume that F is constant on $\mathcal{C}$ and that $\mathrm{int}(\mathcal{C}) = \emptyset$ (where int stands for interior). Then, the fact that $\mathcal{C}$ contains a local (not global) minimizer $\hat{\mathrm{x}}$ of F does *not* imply that any $\bar{\mathrm{x}} \in \mathcal{C}$ is a local minimizer of F, as it was argued in the proof of [2, Lemma 4.4] which is thus wrong for local (not global) minimizers.[1] In such a case, only points $\bar{\mathrm{x}} \in \mathcal{C} \cap \mathrm{int}(\mathcal{V})$, where $\mathcal{V} \ni \hat{\mathrm{x}}$ is the largest neighborhood such that $\forall \mathrm{x} \in \mathcal{V}\, F(\hat{\mathrm{x}}) \leq F(\mathrm{x})$, are local minimizers of F.

1. Corrections to the paper.

To correct this problem, we propose in this note a new version of [2, Lemma 4.4] stated and proved in section 2. Then, [2, Theorem 4.8] remains unchanged but its proof has to be rewritten which is done in section 3. Finally, some remarks of the original paper are also updated. In the following, we use the notation of [2].

2. Correction of Lemma 4.4.

First of all, in the proof of [2, Lemma 4.4], only the conclusion (for local minimizers) is wrong. The fact that

$$\forall i \in \sigma^+(\hat{\mathrm{x}}) \quad \forall t \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right], \; \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}^{(i)} - s_i e_i t) = C \tag{2.1}$$

(where C is a constant of $\mathbb{R}^+$), is true and will be used to prove the new version of [2, Lemma 4.4] which reads as follows.

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[1] The claim remains true for global minimizers.

Lemma 2.1 (Lemma 4.4 of [2]). *For* $d \in \mathbb{R}^M$ *and* $\lambda > 0$, *let* G_{CEL0} *have a minimum at* $\hat{\mathrm{x}} \in \mathbb{R}^N$. *Define* $\hat{\sigma}^+ := \sigma^+(\hat{\mathrm{x}})$ *and let* $s_i = \mathrm{sign}(\langle a_i, A\hat{\mathrm{x}}^{(i)} - d\rangle)$. *Then,*

(i) $\forall i \in \hat{\sigma}^+$, $\exists \mathcal{T}_i \subseteq \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right]$, *a nondegenerate interval of* $\mathbb{R}$, *s.t.* $\hat{\mathrm{x}}_i \in \mathcal{T}_i$ *and* $\forall t \in \mathcal{T}_i$,

$$\bar{\mathrm{x}} = \hat{\mathrm{x}}^{(i)} - s_i e_i t \quad \textit{is a minimizer of } \mathrm{G}_{\text{CEL0}}\,. \tag{2.2}$$

Furthermore $\forall i \in \hat{\sigma}^+$, $t \mapsto \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}^{(i)} - s_i e_i t)$ *is constant on* $\left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right]$;

(ii) *if* $\hat{\mathrm{x}}$ *is global,* $\forall i \in \hat{\sigma}^+$, $\mathcal{T}_i = \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right]$, *and* $\bar{\mathrm{x}}$ *is global.*

Proof.

(i). Since $\hat{\mathrm{x}}$ is a minimizer of G_{CEL0}, there exists $\varepsilon > 0$ such that

$$\forall \mathrm{x} \in \mathcal{B}_2(\hat{\mathrm{x}}, \varepsilon),\ \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}) \le \mathrm{G}_{\text{CEL0}}(\mathrm{x}), \tag{2.3}$$

where $\mathcal{B}_2(\hat{\mathrm{x}}, \varepsilon)$ denotes the open ℓ_2-ball with center $\hat{\mathrm{x}}$ and radius ε. Therefore, from (2.1) and (2.3), we get the result in (2.2). Indeed, $\forall i \in \sigma^+(\hat{\mathrm{x}})$ let

$$\mathcal{T}_i = \left\{ t \in \left[0, \frac{\sqrt{2\lambda}}{\|a_i\|}\right], \mathrm{x} = \hat{\mathrm{x}}^{(i)} - s_i e_i t \in \mathcal{B}_2(\hat{\mathrm{x}}, \varepsilon) \right\}. \tag{2.4}$$

Clearly, since $\varepsilon > 0$, $\mathcal{T}_i$ is a nondegenerate interval of $\mathbb{R}$. Then,

$$\forall t \in \mathcal{T}_i,\ \exists \varepsilon' \in (0, \varepsilon) \ \text{ s.t. } \ \mathcal{B}_2(\bar{\mathrm{x}}, \varepsilon') \subset \mathcal{B}_2(\hat{\mathrm{x}}, \varepsilon)\,, \tag{2.5}$$

where $\bar{\mathrm{x}} = \hat{\mathrm{x}}^{(i)} - s_i e_i t$, and we get

$$\forall \mathrm{x} \in \mathcal{B}_2(\bar{\mathrm{x}}, \varepsilon'),\ \mathrm{G}_{\text{CEL0}}(\bar{\mathrm{x}}) \underset{(2.1)}{=} \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}) \underset{(2.3)\&(2.5)}{\le} \mathrm{G}_{\text{CEL0}}(\mathrm{x})\,, \tag{2.6}$$

which proves (2.2).

(ii). Using the fact that $\hat{\mathrm{x}}$ is a global minimizer of G_{CEL0}, (2.1) completes the proof. ■

The comment after [2, Remark 4.2] about the strict minimizers of G_{CEL0} is still valid and its justification has to be rewritten according to the results in Lemma 2.1.

Lemma 2.1 ensures that all strict minimizers of G_{CEL0}*—i.e.,* $\hat{\mathrm{x}} \in \mathbb{R}^N$ *such that there exists a neighborhood* $\mathcal{V} \subset \mathbb{R}^N$ *containing* $\hat{\mathrm{x}}$ *for which* $\forall \mathrm{y} \in \mathcal{V}\backslash\{\hat{\mathrm{x}}\}$, $\mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}) < \mathrm{G}_{\text{CEL0}}(\mathrm{y})$*—verify* $\sigma^+(\hat{\mathrm{x}}) = \emptyset$. *Indeed, suppose that this claim is not verified by a strict minimizer* $\hat{\mathrm{x}} = \hat{\mathrm{x}}^{(i)} - s_i e_i \hat{t}$ *of* G_{CEL0}, *then Lemma 2.1 states that* $\forall i \in \sigma^+(\hat{\mathrm{x}})\, \exists \mathcal{T}_i \subseteq [0, \sqrt{2\lambda}/\|a_i\|]$, *a nondegenerate interval of* $\mathbb{R}$ *containing* $\hat{t}$, *s.t.* $\forall t \in \mathcal{T}_i\backslash\{\hat{t}\}$, $\bar{\mathrm{x}} = \hat{\mathrm{x}}^{(i)} - s_i e_i t$ is also a minimizer of G_{CEL0} and contradicts the fact that $\hat{\mathrm{x}}$ is strict.

Then, the comment before [2, Theorem 4.5] concerning the nonstrict minimizers should be read only for nonstrict *global* minimizers of G_{CEL0}.

3. Correction of the proof of Theorem 4.8. The claim of Theorem 4.8 is still valid but its proof has to be rewritten. In order to rewrite this proof, we need first to prove the following lemma.

Lemma 3.1. *Let* $d \in \mathbb{R}^M$, $\lambda > 0$, *and* G_{CEL0} *have a minimum at* $\hat{\mathrm{x}} \in \mathbb{R}^N$. *Then,*

$$\forall i \in \sigma^+(\hat{\mathrm{x}}) \quad \forall j \in \sigma(\hat{\mathrm{x}})\backslash\{i\},\ \langle a_i, a_j\rangle = 0\,. \tag{3.1}$$

Proof. Let $\hat{\sigma} = \sigma(\hat{\mathrm{x}})$, $\hat{\sigma}^- = \sigma^-(\hat{\mathrm{x}})$, and $\hat{\sigma}^+ = \sigma^+(\hat{\mathrm{x}})$. Let $i \in \hat{\sigma}^+$, $\mathcal{T}_i$ be defined by Lemma 2.1, $\hat{\mathrm{x}} = \hat{\mathrm{x}}^{(i)} - s_i e_i \hat{t}$, and $t \in \mathcal{T}_i \backslash \{\hat{t}\}$ then, from (2.2), $\bar{\mathrm{x}} = \hat{\mathrm{x}}^{(i)} - s_i e_i t$ is a minimizer of G_{CEL0}. Thus it verifies the conditions of [2, Lemma 4.1] (critical point). By construction,

$$\forall j \in \mathbb{I}_N \backslash \{i\},\ \hat{\mathrm{x}}_j = \bar{\mathrm{x}}_j\,. \tag{3.2}$$

Then, the conditions of [2, Lemma 4.1] lead to

$$\forall j \in \hat{\sigma}\backslash\{i\},\ \begin{cases} |\langle a_j, A\bar{\mathrm{x}}^{(j)} - d\rangle| = \sqrt{2\lambda}\|a_j\| & \text{if } j \in \hat{\sigma}^-, \\ \bar{\mathrm{x}}_j = -\dfrac{\langle a_j, A\bar{\mathrm{x}}^{(j)} - d\rangle}{\|a_j\|^2} & \text{if } j \in \hat{\sigma}\backslash\hat{\sigma}^-. \end{cases} \tag{3.3}$$

From (3.2), $\forall j \in \mathbb{I}_N\backslash\{i\}$, $\bar{\mathrm{x}}^{(j)} = \hat{\mathrm{x}}^{(j)} - \hat{\mathrm{x}}_i e_i + \bar{\mathrm{x}}_i e_i$ and (3.3) can be rewritten as follows:

$$\forall j \in \hat{\sigma}\backslash\{i\},\ \begin{cases} |\langle a_j, A\hat{\mathrm{x}}^{(j)} - d\rangle + \langle a_j, a_i\rangle(\bar{\mathrm{x}}_i - \hat{\mathrm{x}}_i)| = \sqrt{2\lambda}\|a_j\| & \text{if } j \in \hat{\sigma}^-, \\ \bar{\mathrm{x}}_j = -\dfrac{\langle a_j, A\hat{\mathrm{x}}^{(j)} - d\rangle}{\|a_j\|^2} + \dfrac{\langle a_j, a_i\rangle}{\|a_j\|^2}(\hat{\mathrm{x}}_i - \bar{\mathrm{x}}_i) & \text{if } j \in \hat{\sigma}\backslash\hat{\sigma}^-. \end{cases} \tag{3.4}$$

Since $\hat{\mathrm{x}}$ is a critical point of G_{CEL0} we get, $\forall j \in \hat{\sigma}\backslash\{i\}$

$$\begin{aligned} &\begin{cases} |s_j\sqrt{2\lambda}\|a_j\| + \langle a_j, a_i\rangle(\bar{\mathrm{x}}_i - \hat{\mathrm{x}}_i)| = \sqrt{2\lambda}\|a_j\| & \text{if } j \in \hat{\sigma}^-, \\ \bar{\mathrm{x}}_j = \hat{\mathrm{x}}_j + \dfrac{\langle a_j, a_i\rangle}{\|a_j\|^2}(\hat{\mathrm{x}}_i - \bar{\mathrm{x}}_i) & \text{if } j \in \hat{\sigma}\backslash\hat{\sigma}^-, \end{cases} \\ \underset{\hat{\mathrm{x}}_i \neq \bar{\mathrm{x}}_i\ \&\ (3.2)}{\Longrightarrow} &\begin{cases} \langle a_j, a_i\rangle = 0 & \text{if } j \in \hat{\sigma}^-, \\ \langle a_j, a_i\rangle = 0 & \text{if } j \in \hat{\sigma}\backslash\hat{\sigma}^-, \end{cases} \end{aligned} \tag{3.5}$$

which, together with the fact that $\hat{\sigma}^- \subseteq \hat{\sigma}$, completes the proof. ■

We are now able to rewrite the proof of [2, Theorem 4.8]. Let us recall this theorem.

Theorem 3.2 (Theorem 4.8 of [2]). *Let $d \in \mathbb{R}^M$, $\lambda > 0$, and G_{CEL0} have a local minimum (not global) at $\hat{\mathrm{x}} \in \mathbb{R}^N$. Then $\hat{\mathrm{x}}^0$ is a local minimizer (not global) of G_{ℓ_0} and $\mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}) = \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}^0) = \mathrm{G}_{\ell_0}(\hat{\mathrm{x}}^0)$.*

Proof. Let $\hat{\sigma} = \sigma(\hat{\mathrm{x}})$, $\hat{\sigma}^- = \sigma^-(\hat{\mathrm{x}})$, and $\hat{\sigma}^0 = \sigma(\hat{\mathrm{x}}^0)$. From Lemma 3.1, since $\hat{\mathrm{x}}$ is a minimizer of G_{CEL0}, we have (3.1). Moreover, $\hat{\mathrm{x}}$ is a critical point of G_{CEL0} and from [2, Lemma 4.1] we get

$$\begin{aligned} \forall i \in \hat{\sigma}^0 = \hat{\sigma}\backslash\hat{\sigma}^-,\ \hat{\mathrm{x}}_i = \hat{\mathrm{x}}_i^0 &= -\frac{\langle a_i, A\hat{\mathrm{x}}^{(i)} - d\rangle}{\|a_i\|^2} \\ &\Longleftrightarrow \hat{\mathrm{x}}_i^0 = -\frac{\langle a_i, A_{\hat{\sigma}^0}(\hat{\mathrm{x}}^0_{\hat{\sigma}^0})^{(i)} + \sum_{j\in\hat{\sigma}^-} a_j\hat{\mathrm{x}}_j - d\rangle}{\|a_i\|^2} \\ &\underset{(3.1)}{\Longleftrightarrow} \hat{\mathrm{x}}_i^0 = -\frac{\langle a_i, A_{\hat{\sigma}^0}(\hat{\mathrm{x}}^0_{\hat{\sigma}^0})^{(i)} - d\rangle}{\|a_i\|^2} - \frac{1}{\|a_i\|^2}\sum_{j\in\hat{\sigma}^-} \underbrace{\langle a_i, a_j\rangle}_{=0}\hat{\mathrm{x}}_j \\ &\Longleftrightarrow \langle a_i, A_{\hat{\sigma}^0}\hat{\mathrm{x}}^0_{\hat{\sigma}^0} - d\rangle = 0. \end{aligned}$$

Finally, we have $(A_{\hat{\sigma}^0})^T A_{\hat{\sigma}^0}\hat{\mathrm{x}}^0_{\hat{\sigma}^0} = (A_{\hat{\sigma}^0})^T d$ which, with [1, Corollary 2.5], ensures that $\hat{\mathrm{x}}^0$ is a local minimizer of G_{ℓ_0}. Then,

$$\begin{aligned}
\mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}) &= \frac{1}{2}\|A_{(\hat{\sigma}^-)^c}\hat{\mathrm{x}}_{(\hat{\sigma}^-)^c} + A_{\hat{\sigma}^-}\hat{\mathrm{x}}_{\hat{\sigma}^-} - d\|^2 + \sum_{i\in\mathbb{I}_N}\phi(\|a_i\|,\lambda;\hat{\mathrm{x}}_i)\\
&= \frac{1}{2}\|A_{(\hat{\sigma}^-)^c}\hat{\mathrm{x}}_{(\hat{\sigma}^-)^c} - d\|^2 + \langle A_{\hat{\sigma}^-}\hat{\mathrm{x}}_{\hat{\sigma}^-}, A_{(\hat{\sigma}^-)^c}\hat{\mathrm{x}}_{(\hat{\sigma}^-)^c} - d\rangle + \frac{1}{2}\|A_{\hat{\sigma}^-}\hat{\mathrm{x}}_{\hat{\sigma}^-}\|^2\\
&\quad + \sum_{i\in\mathbb{I}_N}\phi(\|a_i\|,\lambda;\hat{\mathrm{x}}_i)\\
&= \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}^0) + \sum_{i\in\hat{\sigma}^-}\left(\phi(\|a_i\|,\lambda;\hat{\mathrm{x}}_i) + \hat{\mathrm{x}}_i\langle a_i, A_{(\hat{\sigma}^-)^c}\hat{\mathrm{x}}_{(\hat{\sigma}^-)^c} - d\rangle + \frac{\hat{\mathrm{x}}_i}{2}\langle a_i, A_{\hat{\sigma}^-}\hat{\mathrm{x}}_{\hat{\sigma}^-}\rangle\right)\\
&\underset{(3.1)}{=} \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}^0) + \sum_{i\in\hat{\sigma}^-}\left(\phi(\|a_i\|,\lambda;\hat{\mathrm{x}}_i) + \hat{\mathrm{x}}_i\langle a_i, A\hat{\mathrm{x}}^{(i)} - d\rangle + \frac{\|a_i\|^2}{2}\hat{\mathrm{x}}_i^2\right),
\end{aligned}$$

and, since $\hat{\mathrm{x}}$ is a critical point of G_{CEL0}, we have

$$\mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}) = \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}^0) + \sum_{i\in\hat{\sigma}^-}\left(\phi(\|a_i\|,\lambda;\hat{\mathrm{x}}_i) - \mathrm{sign}(\hat{\mathrm{x}}_i)\sqrt{2\lambda}\|a_i\|\hat{\mathrm{x}}_i + \frac{\|a_i\|^2}{2}\hat{\mathrm{x}}_i^2\right) = \mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}^0). \tag{3.6}$$

Here we used the fact that from the definition of ϕ in [2, (2.9)] one has

$$\forall i \in \hat{\sigma}^-,\ \phi(\|a_i\|,\lambda;\hat{\mathrm{x}}_i) - \mathrm{sign}(\hat{\mathrm{x}}_i)\sqrt{2\lambda}\|a_i\|\hat{\mathrm{x}}_i + \frac{\|a_i\|^2}{2}\hat{\mathrm{x}}_i^2 = 0. \tag{3.7}$$

Finally, $\mathrm{G}_{\text{CEL0}}(\hat{\mathrm{x}}^0) = \mathrm{G}_{\ell_0}(\hat{\mathrm{x}}^0)$ comes from the same arguments as in the proof of [2, Theorem 4.5] and completes the proof. ■

In [2, Remark 4.4], the claim is based on the argument that when $\hat{\mathrm{x}}$ is a local (not global) minimizer of G_{CEL0}, $\hat{\mathrm{x}}^0$ is also a local (not global) minimizer of G_{CEL0} (and thus a critical point). However we have seen that this argument is wrong. From this erratum we see that $\hat{\mathrm{x}}^0$ is a local (not global) minimizer of G_{ℓ_0} (Theorem 3.2) but it may not be a critical point of G_{CEL0}. Therefore, an additional assumption is required in [2, Remark 4.4] that is "$\hat{\mathrm{x}}^0$ is a critical point of G_{CEL0}" which can be reduced to

$$\forall i \notin \sigma(\hat{\mathrm{x}}^0),\ |\langle a_i, A\hat{\mathrm{x}}^0 - d\rangle| \le \sqrt{2\lambda}\|a_i\|, \tag{3.8}$$

since other conditions of [2, Lemma 4.1] are verified, using Lemma 3.1, for such an $\hat{\mathrm{x}}^0$.

Finally, Remark 4.5 in [2] reads as follows.

Remark 3.1. For a (local) minimizer $\hat{\mathrm{x}}$ of G_{CEL0}, one can set $\forall i \in \hat{\sigma}^- = \sigma^-(\hat{\mathrm{x}})$ either $\hat{\mathrm{x}}_i = 0$ or $\hat{\mathrm{x}}_i = -s_i\frac{\sqrt{2\lambda}}{\|a_i\|}$ to obtain a (local) minimizer, denoted $\tilde{\mathrm{x}}$, of G_{ℓ_0}. Indeed, let $\{\omega_-, \omega_+\}$ be a partition of $\hat{\sigma}^-$ (*i.e.*, $\omega_- \subseteq \hat{\sigma}^-$, $\omega_+ \subseteq \hat{\sigma}^-$ such that $\omega_- \cup \omega_+ = \hat{\sigma}^-$ and $\omega_- \cap \omega_+ = \emptyset$) and

$$\forall i \in \mathbb{I}_N,\ \tilde{\mathrm{x}}_i = \begin{cases} \hat{\mathrm{x}}_i & \text{if } i \notin (\omega_- \cup \omega_+), \\ 0 & \text{if } i \in \omega_-, \\ -s_i\dfrac{\sqrt{2\lambda}}{\|a_i\|} & \text{if } i \in \omega_+, \end{cases} \tag{3.9}$$

where $s_i = \mathrm{sign}(\langle a_i, A\hat{\mathrm{x}}^{(i)} - d\rangle)$. Since $\hat{\mathrm{x}}$ is a critical point of $\mathrm{G}_{\mathrm{CEL0}}$, [2, Lemma 4.1] leads to

$$\forall i \in \sigma(\tilde{\mathrm{x}}),\ \tilde{\mathrm{x}}_i = -\frac{\langle a_i, A\hat{\mathrm{x}}^{(i)} - d\rangle}{\|a_i\|^2}$$

$$\underset{(3.1)\ \&\ (3.9)}{\iff} \tilde{\mathrm{x}}_i = -\frac{\langle a_i, A\tilde{\mathrm{x}}^{(i)} - d\rangle}{\|a_i\|^2} - \frac{1}{\|a_i\|^2}\sum_{j\in\omega-} \underbrace{\langle a_i, a_j\rangle}_{=0} \hat{\mathrm{x}}_j - \frac{1}{\|a_i\|^2}\sum_{\substack{j\in\omega+\\ j\neq i}} \underbrace{\langle a_i, a_j\rangle}_{=0} \left(\hat{\mathrm{x}}_j + s_j\frac{\sqrt{2\lambda}}{\|a_j\|}\right)$$

$$\iff \quad \langle a_i, A_{\sigma(\tilde{\mathrm{x}})}\tilde{\mathrm{x}}_{\sigma(\tilde{\mathrm{x}})} - d\rangle = 0$$

which, with [1, Corollary 2.5], ensures that $\tilde{\mathrm{x}}$ is a local minimizer of G_{ℓ_0}. There exist $2^{\sharp\hat{\sigma}^-}$ of such minimizers. Among them, $\hat{\mathrm{x}}^0$ is the sparsest. Note that this remark can be extended to points $\tilde{\mathrm{x}}$ defined by (3.9) with $\{\omega_-, \omega_+\}$ a partition of $\sigma^+(\hat{\mathrm{x}})$ such that $\forall (i,j) \in (\omega_+ \backslash \sigma(\hat{\mathrm{x}}))^2$, $\langle a_i, a_j\rangle = 0$.

Remark 3.2.As outlined in [2], some local (not global) minimizers of G_{ℓ_0} are not critical points of $\mathrm{G}_{\mathrm{CEL0}}$ and from each local (not global) minimizer of $\mathrm{G}_{\mathrm{CEL0}}$, we can easily extract a local (not global) minimizer, $\hat{\mathrm{x}}^0$, of G_{ℓ_0} (Theorem 3.2). However, we are not ensured that this extracted minimizer is a critical point of $\mathrm{G}_{\mathrm{CEL0}}$. Therefore, when using Theorem 3.2 in practice, it is important to verify if $\hat{\mathrm{x}}^0$ is a critical point of $\mathrm{G}_{\mathrm{CEL0}}$ which is a necessary condition to be global for G_{ℓ_0}. Note that when using the macro algorithm [2, Algorithm 1], the convergence point is ensured to be both a critical point of $\mathrm{G}_{\mathrm{CEL0}}$ and a (local) minimizer of G_{ℓ_0}.

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REFERENCES

[1] M. Nikolova, *Description of the minimizers of least squares regularized with ℓ_0-norm. Uniqueness of the global minimizer*, SIAM J. Imaging Sci., 6 (2013), pp. 904–937.

[2] E. Soubies, L. Blanc-Féraud, and G. Aubert, *A Continuous Exact ℓ_0 Penalty (CEL0) for Least Squares Regularized Problem*, SIAM J. Imaging Sci., 8 (2015), pp. 1607–1639.