Strategies and Interactive Beliefs in Dynamic Games

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Introduction

- **Topic:** formal analysis of strategic thinking in dynamic games, backward and forward induction reasoning.
- **Tools:** epistemic (type) structures where states determine truth value of conditional statements.
- **Distinctive feature:** we use only epistemic conditionals of the form: "if I learned h he would believe E with probability p"; state=(actual actions, epistemic conditionals).
- **Motivation:** strategies cannot be (irreversibly) chosen, they are just beliefs on own contingent choices obtained from planning (objective vs subjective strategies).
- **Focus:** generic perfect information games.
Background literature

- **Hierarchies of probabilistic beliefs**: Mertens & Zamir (IJGT 1985), Brandenburger & Dekel (JET 1993), Heifetz & Samet (JET 1998)
- **Rationalizability in games**: Bernheim (Ecma 1984), Pearce (Ecma 1984)
- **Dynamic interactive epistemology**: Ben Porath (REStud 1997), Battigalli & Siniscalchi (JET 1999,2002)
Road map

1. Preview on conditionals and Backward Induction (BI): semi-formal analysis of an example
2. Setup: perfect information (PI) games and epistemic structures
3. Strategies as epistemic constructs: some results
4. Conclusions
Variants of Battigalli & Siniscalchi JET99: types are implicit representations of hierarchies of conditional beliefs. But, what are first-order beliefs about?

- $\mathcal{H}=\text{histories, } \mathcal{Z}=\text{paths (terminal nodes), } \mathcal{S}_i, \mathcal{S}_{-i}, \mathcal{S}=\text{strategies}$

- "Traditional" $\mathcal{S}$-based structures: $(\bar{T}_i, \bar{\beta}_i)_{i\in I}$, states in $\Omega = \mathcal{S} \times \mathcal{T}$, $(s_i, t_i)=\text{state of } i$ ($s_i$ is "objective"), $\mathcal{H}_i=\text{conditioning events}$,

$$\mathcal{H}_i = \{ \mathcal{S}_{-i}(h) \times \bar{T}_{-i} : h \in \mathcal{H} \}$$

$$\bar{\beta}_i : \bar{T}_i \rightarrow \Delta^{\mathcal{H}_i}(\mathcal{S}_{-i} \times \bar{T}_{-i}) \subset [\Delta(\mathcal{S}_{-i} \times \bar{T}_{-i})]^{\mathcal{H}_i}$$

- Our $\mathcal{Z}$-based structures: $(T_i, \beta_i)_{i\in I}$, states in $\Omega = \mathcal{Z} \times \mathcal{T}$, $t_i=\text{epist.state of } i$ (beliefs about others+plan),

$$\mathcal{H} \cong \{ \mathcal{Z}(h) \times T_{-i} : h \in \mathcal{H} \}$$

$$\beta_i : T_i \rightarrow \Delta^{\mathcal{H}}(\mathcal{Z} \times T_{-i}) \subset [\Delta(\mathcal{Z} \times T_{-i})]^\mathcal{H}$$
Preview on conditionals and BI: an example

Stackelberg Minigame

<table>
<thead>
<tr>
<th>outputs:</th>
<th>low (left)</th>
<th>high (right)</th>
</tr>
</thead>
<tbody>
<tr>
<td>high (up)</td>
<td>3,1</td>
<td>0,0</td>
</tr>
<tr>
<td>low (down)</td>
<td>2,2</td>
<td>1,3</td>
</tr>
</tbody>
</table>
Ann believes "Bob would go Right given Up"
= belief about a behavioral conditional

Given Up, Ann would believe "Bob goes Right"
= conditional belief about behavior

In traditional analysis: behavioral conditionals and beliefs about conditionals and opponents’ beliefs

In our analysis: actual actions (paths) and conditional beliefs about actions and beliefs
States $\omega$ specify strategies $s_i = \sigma_i(\omega)$ objectively

‘Bob would go Right given Up’ false at $\omega$ iff $\sigma_{Bob}(\omega) \in \{L.r, L.l\}$

$R_{Bob} \subset [L.r] = \{\omega : \sigma_i(\omega) = L.r\}$ \quad \Rightarrow \quad (3, 1)

$B_{Ann}(R_{Bob}) \subset B_{Ann}(L.r)$ (uncond. belief) \quad \Leftarrow \quad L \quad \Rightarrow \quad R$

States also specify conditional beliefs

By built in independence:

$B_{Ann}(L.r) \subset B_{Ann}(L|U) \cap B_{Ann}(r|D) = B_{Ann}(\text{util} = 3|U) \cap B_{Ann}(\text{util} = 1|D)$

$\Rightarrow R_{Ann} \cap R_{Bob} \cap B_{Ann}(R_{Bob}) \subset \subset [U, L.r] \subset [U, L]$ \quad (2, 2)

$\Rightarrow BI$ strategies and path obtain

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“strategy” is an epistemic concept:
i’s cond. beliefs $\Rightarrow$ (beliefs about others and) contingent plan of i

Rat. Planning: $\text{RP}_{\text{Bob}} = B_{\text{Bob}}(L|U) \cap B_{\text{Bob}}(r|D)$

Material Consistency = path cons. with plan

Rationality = Material cons. + Rat. plan.

$R_{\text{Bob}} \subseteq [U, L] \cup [D, r]$

But no event $[L, r]$. $B_{\text{Ann}}(L, r)$ not expressible!

$B_{\text{Ann}}(R_{\text{Bob}}) \subseteq B_{\text{Ann}}([U, L] \cup [D, r]) \not\subseteq B_{\text{Ann}}(L|U) \cap B_{\text{Ann}}(r|D)$ e.g. if $B_{\text{Ann}}(D)$

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Recall: $R_{Bob} \subseteq [U, L] \cup [D, r]$, but $B_{Ann}(R_{Bob}) \not\subseteq B_{Ann}(L | U) \cap B_{Ann}(r | D)$ e.g. if $B_{Ann}(D)$ (Ann plans $D$)

Since there is no event $[L, r]$, $B_{Ann}(L, r)$ not expressible

If $B_{Ann}(D)$ (Ann plans $D$) the following is possible

$B_{Ann}([U, L] \cup [D, r]) \cap \neg B_{Ann}(L | U)$

$\exists \omega \in [D, r] \cap R_{Ann} \cap R_{Bob} \cap B_{Ann}(R_{Bob})$

$\Rightarrow$ Imperfect Nash equilibrium may obtain
Ann strongly believes $E$ if she believes $E$ given each $C$ with $C \cap E \neq \emptyset$

$[U] \cap R_{Bob} \neq \emptyset, [D] \cap R_{Bob} \neq \emptyset$

hence $SB_{Ann}(R_{Bob}) \subset B_{Ann}(R_{Bob}|U) \cap B_{Ann}(R_{Bob}|D)$

but $R_{Bob} \subset [U, L] \cup [D, r]$

hence $SB_{Ann}(R_{Bob}) \subset B_{Ann}(L|U) \cap B_{Ann}(r|D)$ (BI cond. beliefs)

thus $R_{Ann} \cap R_{Bob} \cap SB_{Ann}(R_{Bob}) \subset [U, L]$ (BI path)
Suppose
(i) Ann’s conditional beliefs about Bob’s beliefs (hence his plan) are \textit{independent} of her actions (\textit{Ind}),
(ii) Ann strongly believes that Bob is materially consistent:
\[ \text{Ind} \cap B_{Ann}(RP_{Bob}) \cap SB_{Ann}(MC_{Bob}) \subseteq B_{Ann}(L|U) \cap B_{Ann}(r|D) \]
Thus
\[ R_{Ann} \cap R_{Bob} \cap \text{Ind} \cap B_{Ann}(RP_{Bob}) \cap SB_{Ann}(MC_{Bob}) \subseteq [U, L] \]
(Note: such independence and belief in consistency are implicit in traditional analysis)
Route 1 (SB, with higher-level epistemic assumptions) leads to the BI-path in generic PI games

Route 2 (OAI) leads to the BI-path in generic two-stage PI games, not in longer games such as the Centipede (even with higher-level epistemic assumptions)
"Centipede" example: strong belief analysis
Forward induction reasoning yields the BI outcome

\[
\begin{align*}
\text{Ann} & \rightarrow C \quad \text{Bob} \rightarrow c \quad \text{Ann} \rightarrow C' \\
\downarrow D & \quad \downarrow d & \downarrow D' \\
(1,0) & \quad (0,2) & \quad (3,0)
\end{align*}
\]

- \( R_{\text{Ann}} \subseteq \neg [C, c, C'] \)
- \( R_{\text{Bob}} \cap SB_{\text{Bob}}(R_{\text{Ann}}) \subseteq [D] \cup [C, d] \)
- \( R_{\text{Ann}} \cap SB_{\text{Ann}}(R_{\text{Bob}} \cap SB_{\text{Bob}}(R_{\text{Ann}})) \subseteq [D] \)
"Centipede" example: initial common belief analysis
Not enough to get the BI outcome

\[
\begin{align*}
&\text{Ann} \quad \rightarrow \quad \text{Bob} \\
&\downarrow \quad D \quad \rightarrow \quad \downarrow \quad d \\
&(1,0) \quad \rightarrow \quad (0,2)
\end{align*}
\]

\[
\begin{align*}
&\text{Ann} \quad \rightarrow \quad \text{Ann} \\
&\downarrow \quad D' \quad \rightarrow \quad (0,3)
\end{align*}
\]

- \(R_{Ann} \subset \neg [C, c, C']\)
- But, without \(SB_{Bob}(R_{Ann})\), \(R_{Bob}\) has no behavioral implication!
- Thus

\[
R_{Ann} \cap R_{Bob} \cap Ind \cap B_{Ann}(RP_{Bob}) \cap SB_{Ann}(MC_{Bob}) \subset \neg [C, c, C']
\]

- \(\Rightarrow\) initial common belief in \(R \cap Ind \cap SB(MC)\) only buys \(\neg [C, c, C']\)

(We can show this by example and as corollary of a general theorem)
Setup
Perfect Information (PI) games

- $i \in I$, players
- $h \in H$, histories/nodes ($H_i$, owned by $i$), $H$ finite
- $z \in Z \subset H$, terminal histories/paths; $Z(h) = \{z : h \preceq z\}$
- $a \in A(h)$, actions at $h \in H \setminus Z$
- $u_i : Z \rightarrow \mathbb{R}$, utility/payoff s.t. no relevant ties
Setup
Epistemic structures for PI games: general

- $\omega \in \Omega = X \times T = X \times \prod_{i \in I} T_i$, *states of the world*
- $x \in X$, *external* (non-epistemic) *states*, finite (e.g. $X = S$ or $X = Z$)
- (implicitly understood *path function* $\pi : X \to Z$)
- for $Y \subset X$, $[Y] = Y \times T$ eXternal events
- $\beta_i : T_i \to B_i$, $\beta_i(t_i)$ epistemic state of $i$ at $\omega$ ($B_i = \text{Beliefs}_i$ to be specified)
Setup
Conditional Probability Systems

- Conditioning events (hypotheses) correspond to histories:
  \[
  [h] = \{(x, t) \in X \times T_{-i} : \pi(x) \in Z(h)\}
  \]

- Probability measures concentrated on conditioning events \(C = [h], D = [h']\) ... related by *chain rule*:
  \[
  E \subset C \subset D \Rightarrow \mu_i(E|D) = \mu_i(E|C)\mu_i(C|D) \quad \text{(ch.r)}
  \]

- Write
  \[
  \mu_{i,h}(\cdot) = \mu_i(\cdot|[h]), \mu_i = (\mu_{i,h}(\cdot))_{h \in H} \in [\Delta(X \times T)]^H
  \]
  Conditional probability systems (*CPS*s) on \((X \times T, H)\):
  \[
  \Delta^H(X \times T_{-i}) = \{\mu_i : \mu_i,h([h]) = 1 \text{ and (ch.r) holds}\}
We focus on epistemic type structures where external states are complete sequences of actions: $X = Z$

$$\Omega = Z \times \prod_{i \in I} T_i, \ \beta_i : T_i \rightarrow \Delta^H(Z \times T_{-i}), \ (i \in I)$$

meaning:

- **no knowledge** about $z$ at the outset
- **information** about moves **cannot** directly **disclose** anything about types/beliefs
Type structures

\[ \Omega = \mathbb{Z} \times \prod_{i} T_i, \quad \beta_i : T_i \to \Delta^H(\mathbb{Z} \times T_{-i}), \]

Technical assumptions (in a sense, w.l.o.g.): for each \( i \in I \)

- \( T_i \) compact metrizable, hence \( \Delta^H(\mathbb{Z} \times T_{-i}) \) compact metrizable (B&S JET99)
- \( \beta_i \) continuous

**Definition**

A type structure is **complete** if \( \beta_i \) is **onto** for every \( i \in I \).

Main example: the canonical structure of hierarchies of beliefs.
Canonical structure: types as hierarchies of beliefs

- First-order beliefs: $T^1_i = \Delta^H(Z)$
- Second-order beliefs:

$$T^2_i = \left\{ (\mu^1_i, \mu^2_i) \in T^1_i \times \Delta^H(Z \times T^1_{-i}) : \forall h \in H, \text{marg}_Z \mu^2_{i,h}(\cdot) = \mu^1_{i,h}(\cdot) \right\}$$

... recursive def. of $T^n_i \subset T^1_i \times \Delta^H(Z \times T^{n-1}_{-i})$...

$T_i$ = projective limit of $(T^n_i)_{n \in \mathbb{N}}$, set of infinite hierarchies of beliefs satisfying common full belief in coherence

homeomorphism $\beta_i : T_i \to \Delta^H(Z \times T_{-i})$ by a generalization of Kolmogorov’s extension theorem
Setup
Belief operators

Standard (monotonic) belief operators:
[for $E \subseteq Z \times T$, let $E_{t_i} = \{(z, t_{-i}) : (z, t_i, t_{-i}) \in E\}$ section at $t_i$]

- conditional belief $B_i(E|h) = \{(z, t_i, t_{-i}) : \beta_i(t_i)(E_{t_i}|[h]) = 1\}$,
- initial belief $B_i(E) = B_i(E|h^0) (h^0=$empty hist., root$)$

Nonmonotonic belief operator

- strong belief $SB_i(E) = \bigcap_{h:E \cap [h] \neq \emptyset} B_i(E|h)$
Given $\mu_i \in \Delta^H(Z \times T_{-i})$ derive

$$\mu_i(a|h) = \mu_i([h, a]|[h])$$

for all $h \in H \setminus Z$, $a \in A(h)$.

Consider only the conditional prob. of $i$'s opponents' actions:

$$\mu_i(a|h), h \in H_{-i}, a \in A(h)$$

$\Rightarrow$ subjective decision tree $\Gamma_i(\mu_i)$ for $i$,

$\mu_i$ is consistent with dynamic programming on $\Gamma_i(\mu_i)$ iff (OSD)

$$\forall h \in H_i, \mu_i \left( \underset{a \in A(h)}{\text{arg max}} V_i((h, a), \mu_i)|h \right) = 1,$$

with $V_i((h, a), \mu_i) = \sum_{z \in Z(h,a)} u_i(z)\mu_i(z|h, a)$
The plan of \( i \) at \((z, t_i, t_{-i})\) (plan of \( t_i \)) is derived from \( \beta_i(t_i) \)

**Definition**

Pl. \( i \) plans rationally at \((z, t_i, t_{-i})\) if

\[
\forall h \in H_i, \beta_i(t_i) \left( \arg \max_{a \in A(h)} V_i((h, a), \mu_i|h) \right) = 1.
\]

Event: \( RP_i \)

- \( RP_i \) is just a property of \( i \)'s beliefs/types: \( i \) expects to take locally maximizing actions conditional on each \( h \in H_i \).
- **Interpretation:** \( i \) has beliefs about others and computes his plan (beliefs about himself) by dynamic programming ("folding back") on the corresponding subjective decision tree.
Connect beliefs to behavior:

**Definition**

Pl. \( i \) is *materially consistent at* \((z, t_i, t_{-i})\) if he does not violate his plan on path \( z \)

\[
\forall h \in H_i, \forall a \in A(h), (h, a) \leq z \Rightarrow \beta_i(t_i)(a|h) > 0.
\]

Event: \( MC_i \)

**Definition**

Pl. \( i \) is *materially rational at* \((z, t_i, t_{-i})\) if he plans rationally and does not violate his plan at \((z, t_i, t_{-i})\). Event: \( R_i = MC_i \cap RP_i \).
Recall: results apply to finite PI games with NRT, in such games the BI strategy is unique and (mixed) Nash equilibria yield a unique path with prob. 1.

Similar to traditional analysis: initial common belief in material rationality does not yield BI or Nash paths in games of depth $d > 2$ (e.g. Centipede).
Unlike traditional analysis: in games of depth 2, correct belief in rationality does not yield BI, only a Nash path (cf. initial example).

**Proposition**

*In a game $\Gamma$ of depth 2, $\forall z \in Z$, $z$ is a Nash path if and only if*

$$(z, t) \in \bigcap_i R_i \cap B_i(R_{-i})$$

*for some $t$, in some type structure $(T_i, \beta_i)_{i \in I}$ for $\Gamma$.***
One way to obtain elementary BI (games of depth 2) is to assume that the first mover strongly believes in the material rationality of the co-player.

We can go further and replicate "traditional" results on common strong belief in rationality and Nash eq. (Battigalli-Friedenberg, 2010), or EFR and BI in complete structures (Battigalli-Siniscalchi, 2002).
**Strategies as beliefs**

**Common Strong Belief in Material Rationality**

- $R_i^0 = R_i$, $R_i^{k+1} = R_i^k \cap SB_i(R_{-i}^k)$
- $CSBR = \bigcap_{k,i} R_i^k$, correct Common Strong Belief in MR
- $\pi : S \rightarrow Z$ strategy-path function

**Proposition**

*(i) In any type structure*

\[ proj_Z CSBR \subset \{ \text{Nash-paths} \} \]

*(ii) In a complete (or otherwise "sufficiently rich") type structure*

\[ proj_Z CSBR = \pi(EFR) = \{ \text{BI-path} \}. \]
Reason for non-BI result: Ann may initially believe in $R_{Bob}$ but she need not believe in $R_{Bob}$ conditional on taking an unplanned action.

This cannot happen if Ann’s beliefs about Bob’s type *conditional on her own actions* do not depend on the conditioning action, and she strongly (hence always) believes in $MC_{Bob}$.

*Own-action independence* = i’s conditional beliefs about $t_{-i}$ do not depend on i’s actions (event $Ind_i$)
Proposition

In every "rich" (e.g., complete) type structure for a game of depth 2

\[ \text{proj}_Z \left( \bigcap_i R_i \cap \text{Ind}_i \cap B_i(R_{-i}) \cap SB_i(MC_{-i}) \right) = \{ \text{BI-path} \} \]

What about longer games (e.g. Centipede)?
Strategies as beliefs

Independence

- \( R\text{Ind}_i^0 = R_i \cap \text{Ind}_i \cap SB_i(MC_{-i}) \),
- \( R\text{Ind}_i^{k+1} = R\text{Ind}_i^k \cap B_i(R\text{Ind}_{-i}^k) \)
- \( \text{CBRInd} = \bigcap_{i,k} R\text{Ind}_i^k \)

All the paths consistent with the "Dekel-Fudenberg procedure", \( \pi(S^\infty W) \), are also consistent with \( \text{CBRInd} \):

Proposition

*There are type structures (including the canonical one) such that \( \forall s \in S^\infty W, \exists (z, t) \in \text{CBRInd} \) such that \( z = \pi(s) \).*

Corollary

*There are non-BI paths consistent with \( \text{CBRInd} \) in Centipede (of depth \( d > 2 \)).*
Conclusions

- Many results of the “traditional analysis” with behavioral conditionals in the state of the world make a lot of sense. They are built on often implicit assumptions of consistency of plans with actual behavior, strong belief in consistency (perceived intentionality) and self/opponents independence.

- We rule out behavioral conditionals, but allow for epistemic ones. This forces an interpretation of strategies as epistemic constructs and imposes discipline: in this setup strategies cannot be chosen, they can only be planned.

- Consistency and own-action independence have to be assumed explicitly. This seems fitting for a formal analysis of strategic reasoning.
Conclusions

Imperfect information (with perfect recall)

- $h_i \in H_i$ information sets (personal histories)
- Conditions/hypotheses for $i$: $[h_i]$ and $[h_i, a_i]$ ($a_i \in A_i(h_i)$)
- Value of action $a_i$ at $h_i$: $E_{\beta_i(t_i)}[u_{i|h_i, a_i}]$
- Potential conflict between our analysis and standard decision theory.
- *Own-action independence* and *strong belief in material consistency* allow to reconcile our analysis with traditional theory.


References


References


