Unifying practical uncertainty representations: II. Clouds

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Abstract

There exist many tools for capturing imprecision in probabilistic representations. Among them are random sets, possibility distributions, probability intervals, and the more recents Ferson's p-boxes and Neumaier's clouds. Both for theoretical and practical considerations, it is very useful to know whether one representation can be turned into or approximated by other ones. In the companion paper, we have thoroughly studied a generalized form of p-box, relating it with other models. This paper focuses on so-called clouds and their links with other representations. In particular, it is shown that they are more general than generalized p-boxes and generally cannot be represented by convex capacities, hence by random sets.

Key words: imprecise probability representations, p-boxes, possibility theory, random sets, clouds, probability intervals

1 Introduction

There exist many different representations of imprecise probabilities. Usually, the more general, the more difficult they are to handle. Simpler representations, although less expressive, usually have the advantage of being more tractable. Over the years, several such practical representations have been proposed. Among them are possibility distributions [20], probability intervals [2], and more recently p-boxes [12] and clouds [15, 16]. With such a diversity of

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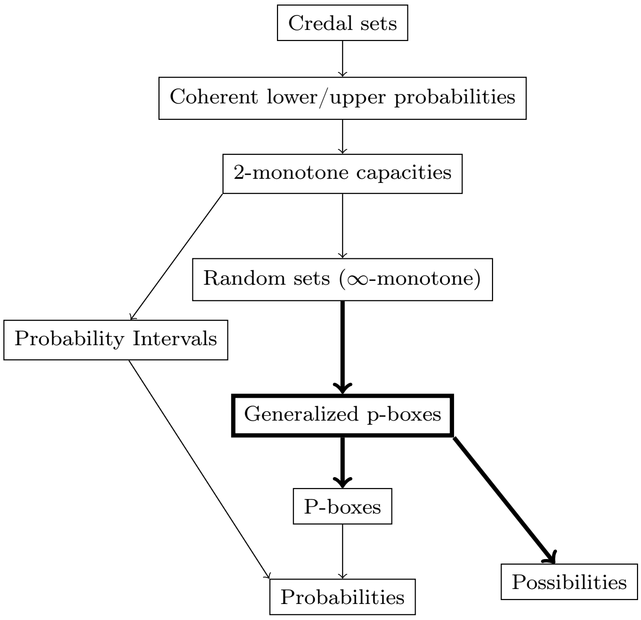

Figure 1. Relationships among representations. $A \longrightarrow B$ means B is a special case of A

simplified representations, it is then natural to compare their respective expressive power. Finding formal relations between such representations also facilitates a unified handling of uncertainty.

In the companion paper, a generalized notion of p-boxes is studied and related to representations mentioned above. It is shown that uncertainty modelled by any generalized p-box can be represented by an equivalent pair of possibility distributions, or by a particular random set. Generalised p-boxes are thus more general than single possibility distributions, and a special case of random sets. Moreover, their interpretation in term of lower and upper confidence bounds on collection of nested subsets makes them intuitive simple representations. Figure 1 recalls the situation reached at the end of the companion paper. It shows the known relation between the studied representations, going from the most (top) to the least (bottom) general.

The study of the present paper will allow us to complete Figure 1 by adding clouds to it, making one step further towards the unification of uncertainty models. As we shall see, generalised p-boxes are a bridge between clouds, possibility distributions and usual p-boxes.

The paper is divided into two main sections:

- Section 2 briefly recalls the uncertainty frameworks as well as the main results of the companion paper that will be needed here.
- Section 3 studies the formalism of clouds and relates them to pairs of possibility distributions and to generalized p-boxes. It is shown that general-

ized p-boxes are equivalent to a particular subfamily of clouds, named here comonotonic clouds.
- Section 4 studies non-comonotonic clouds, that are not equivalent to generalized p-boxes. Since the lower probability they induce are not 2-monotone, simpler outer and inner approximations are proposed.
- Section 5 then studies relations between clouds and probability intervals. As neither of them is a special case of the other, some transformation of probability intervals into covering clouds are proposed.
- Section 6 extends some of our results to the case of continuous models defined on the real line, since such models are often encountered in applications. The particular case of thin clouds is emphasized, as they have non-empty credal sets in the continuous setting.

To make the paper easier to read, longer proofs have been moved to the appendix.

2 Preliminaries

In this section, we briefly recall the notions introduced in the companion paper, as well as the main results useful in the present study. Unless explicitly mentioned otherwise, the paper sticks to a finite space X of n elements denoted x. More details can be found in the companion paper.

2.1 Basic notions

Capacities

Definition 2.1 *Given a finite space X, a capacity is a function μ defined on subsets of X such that:*

- $\mu(\emptyset) = 0, \mu(X) = 1$
- $A \subset B \Rightarrow \mu(A) \leq \mu(B)$

A capacity is said to be super-additive *if* $\forall A, B \subset X, A \cap B = \emptyset$, $\mu(A \cup B) \geq \mu(A) + \mu(B)$. *The dual notion, called* sub-additivity, *is obtained by reversing the inequality. A capacity is said* additive *if the inequality is turned into an equality.*

Capacities are often characterized by their n-monotonicity, defined as:

Definition 2.2 *A super-additive capacity μ is $n-monotone$, where $n > 0$ and $n \in \mathbb{N}$, if and only if for any set $\mathcal{A} = \{A_i | i \in \mathbb{N}, 0 < i \leq n\}$ of events A_i,* *it holds that*

$$\mu(\bigcup_{A_i \in \mathcal{A}} A_i) \geq \sum_{I \subseteq \mathcal{A}} (-1)^{|I|+1} \mu(\bigcap_{A_i \in I} A_i)$$

And a capacity that is n-monotone for any n is said ∞-monotone. An n-monotone capacity is also $n-1$-monotone, but not forcefully $n+1$-monotone. The dual capacity μ^c of a capacity μ is such that $\mu^c(A) = \mu(X) - \mu(A) = 1 - \mu(A)$ for any event $A \subseteq X$.

Credal Sets and coherent lower/upper probabilities

A credal set $\mathcal{P}$ is a closed convex set of finitely additive probability distributions P. Walley [19] systematized their use as uncertainty models. Here, we restrict ourselves to credal sets induced by coherent lower probability measures. A lower probability measure is a *super-additive* capacity, and it is coherent if it coincides with the lower envelope of the credal sets it induces.

The credal set $\mathcal{P}_{\underline{P},\overline{P}}$ induced by a coherent lower probability $\underline{P}$ is the set of probability distributions P dominating this lower probability:

$$\mathcal{P}_{\underline{P},\overline{P}} = \{P | \forall A \subset X,\ P(A) \geq \underline{P}(A)\}.$$

The coherent upper probability $\overline{P}$ such that for any event A, $\overline{P}(A) = 1 - \underline{P}(A^c)$, with A^c the complement of A is the conjugate measure of the coherent lower probability $\underline{P}$. A credal set $\mathcal{P}_{\underline{P},\overline{P}}$ can also be described by a set of constraints on probability masses:

$$\underline{P}(A) \leq \sum_{x \in A} p(x) \leq \overline{P}(A).$$

Probability intervals

Probability intervals are lower and upper confidence bounds on a probability distribution. They are defined by a set of intervals $L = \{[l(x), u(x)] | x \in X\}$ inducing the credal set

$$\mathcal{P}_L = \{P | l(x) \leq p(x) \leq u(x),\ x \in X\}$$

where $p(x)$ is the probability mass of x. In terms of constraints, $\mathcal{P}_L$ is only described by bounds on probability masses of elements ($l(x) \leq p(x) \leq u(x)$). Lower and upper probabilities $\underline{P}(A), \overline{P}(A)$ on all events $A \subset X$ of $\mathcal{P}_L$ are calculated by the following expressions

$$\underline{P}(A) = \max(\sum_{x \in A} l(x), 1 - \sum_{x \in A^c} u(x)), \overline{P}(A) = \min(\sum_{x \in A} u(x), 1 - \sum_{x \in A^c} l(x)). \quad (1)$$

Probability intervals are extensively studied by De Campos *et al.* [2]. They show that the induced lower and upper probabilities are Choquet capacities of order 2.

Random sets

When X is finite, a random set can be represented as a distribution of positive masses m over the power set $\wp(X)$ and such that $\sum_{E\subseteq X} m(E) = 1$ and $m(\emptyset) = 0$. A set E that receives strict positive mass is called a focal set. From this distribution of masses, Shafer [17] defines two set functions, the belief and plausibility functions:

$$Bel(A) = \sum_{E, E\subseteq A} m(E); \qquad Pl(A) = 1 - Bel(A^c) = \sum_{E, E\cap A\neq\emptyset} m(E).$$

It can be shown that a belief function induced by a random set is an ∞-monotone capacity, and that to any ∞-monotone capacity corresponds one and only one random set. As ∞-monotone capacities, belief functions can be identified as special cases of lower probabilities. In this case, a random set induces the credal set $\mathcal{P}_{Bel} = \{P | \forall A \subseteq X,\ Bel(A) \leq P(A) \leq Pl(A)\}$.

Possibility distributions

A possibility distribution [7] is a mapping $\pi : X \to [0, 1]$ from a space X to the unit interval such that $\pi(x) = 1$ for at least one element x in X. From this distribution, two dual measures on events $A \subseteq X$ are defined, the possibility and necessity measures:

$$\Pi(A) = \sup_{x\in A} \pi(x); \qquad N(A) = 1 - \Pi(A^c) \tag{2}$$

and another that will be useful in the paper, called the sufficiency measure, such that:

$$\Delta(A) = \inf_{x\in A} \pi(x) \tag{3}$$

Given a possibility distribution π and a degree $\alpha \in [0, 1]$, strong and regular α-cuts are respectively defined as the sets $A_{\overline{\alpha}} = \{x \in X | \pi(x) > \alpha\}$ and $A_\alpha = \{x \in X | \pi(x) \geq \alpha\}$. These α-cuts are nested, since if $\alpha > \beta$, then $A_\alpha \subseteq A_\beta$. On finite spaces, a possibility distribution can only take a finite set of distinct values on elements x in X. Let us note $\alpha_0 = 0 < \alpha_1 < \ldots < \alpha_m = 1$ these distinct values. Then, there will only be m distinct α-cuts.

As necessity measures are ∞-monotone capacities, they is also a particular instance of belief functions. A possibility distribution is equivalent to a random set whose focal elements are nested. Given a possibility distribution π, the corresponding random set has nested focal elements E_i with masses $m(E_i)$,

$i = 1, \ldots, m$:

$$\begin{cases} E_i = \{x \in X | \pi(x) \geq \alpha_i\} = A_{\alpha_i} \\ m(E_i) = \alpha_i - \alpha_{i-1} \end{cases} \tag{4}$$

A ncessity measure is also a particular instance of a coherent lower probability $\underline{P}$, and it induces the credal set $\mathcal{P}_\pi = \{P | \forall A \subseteq X,\ N(A) \leq P(A) \leq \Pi(A)\}$.

We recall here a result, proved by Dubois *et al.* [5], which relates probabilities P that are in $\mathcal{P}_\pi$ with constraints on α-cuts, and that will be useful in the sequel:

Proposition 2.3 *Given a possibility distribution* π *and the induced convex set* $\mathcal{P}_\pi, P \in \mathcal{P}_\pi$ *if and only if* $1 - \alpha \leq P(\{x \in X | \pi(x) > \alpha\}), \forall \alpha \in (0, 1]$

This result means that the probabilities P in the credal set $\mathcal{P}_\pi$ can also be described in terms of constraints on strong α-cuts of π (i.e. $1 - \alpha \leq P(A_{\overline{\alpha}})$).

2.2 Generalized p-boxes

Generalized p-boxes extend the usual notion of p-boxes [12] to arbitrary finite spaces X. They are fully studied in the companion paper, and we simply recall here some of their features as well as the main results needed in the sequel.

Two mappings from X to $[0,1]$, $\underline{F} : X \to [0,1]$ and $\overline{F} : X \to [0,1]$, are said to be comonotonic if there exists a total ranking $\{x_1, \ldots, x_n\}$ of X such that $\underline{F}(x_1) \geq \ldots, \underline{F}(x_n)$ and $\overline{F}(x_1) \geq \ldots, \overline{F}(x_n)$. A generalized p-box is defined as follows:

Definition 2.4 *A generalized p-box* $[\underline{F}, \overline{F}]$ *over a finite space* X *is a pair of comonotonic mappings,* $\underline{F} : X \to [0,1]$ *and* $\overline{F} : X \to [0,1]$ *such that* $\underline{F}$ *is point-wise lower than* $\overline{F}$ *(i.e.* $\underline{F} \leq \overline{F}$*) and there is at least one element* x *in* X *for which* $\overline{F}(x) = \underline{F}(x) = 1$.

Any generalized p-box thus induces a complete (pre-)order $\leq_{[\underline{F},\overline{F}]}$ on elements x of X, such that $x \leq_{[\underline{F},\overline{F}]} y$ if $\underline{F}(x) \leq \underline{F}(y)$ and $\overline{F}(x) \leq \overline{F}(y)$. To simplify notations in the sequel, elements x of X are indexed so as to ensure $x_i \leq_{[\underline{F},\overline{F}]} x_j$ as soon as $i < j$.

The credal set $\mathcal{P}_{[\underline{F},\overline{F}]}$ induced by a generalized p-box is such that

$$\mathcal{P}_{[\underline{F},\overline{F}]} = \{P | i = 1, \ldots, n,\ \underline{F}(x_i) \leq P(A_i) \leq \overline{F}(x_i)\}.$$

with A_i the set such that $A_i = \{x \in X | x \leq_{[\underline{F},\overline{F}]} x_i\}$. These sets are nested ($\emptyset \subset A_1 \subseteq \ldots \subseteq A_n = X$).

Let $\underline{F}(x_i) = \alpha_i$ and $\overline{F}(x_i) = \beta_i$ for all $i = 1, \ldots, n$. Then, the credal set $\mathcal{P}_{[\underline{F},\overline{F}]}$ can also be described by the following constraints bearing on probabilities of nested sets A_i:

$$i = 1, \ldots, n \qquad \alpha_i \leq P(A_i) \leq \beta_i \tag{5}$$

with $0 = \alpha_0 \leq \alpha_1 \leq \ldots \leq \alpha_n = 1$, $0 = \beta_0 < \beta_1 \leq \beta_2 \leq \ldots \leq \beta_n = 1$ and $\alpha_i \leq \beta_i$.

Generalized p-boxes and possibility distributions are linked by the following proposition showing that any generalized p-box can be represented by a pair of possibility distributions:

Proposition 2.5 *Uncertainty modeled on X by a generalized p-box $[\underline{F}, \overline{F}]$ can also be encoded by a pair of possibility distributions $\pi_{\overline{F}}, \pi_{\underline{F}}$ such that $\mathcal{P}_{[\underline{F},\overline{F}]} = \mathcal{P}_{\pi_{\overline{F}}} \cap \mathcal{P}_{\pi_{\underline{F}}}$ with, for $i = 1, \ldots, n$,*

$$\pi_{\overline{F}}(x_i) = \beta_i \text{ and } \pi_{\underline{F}}(x_i) = 1 - \max\{\alpha_j | j = 0, \ldots, i\ \alpha_j < \alpha_i\}$$

with $\alpha_0 = 0$.

Conversely, any possibility distribution can be seen as a generalized p-box where one of $\underline{F}$ or $\overline{F}$ has extreme values 0 or 1.

The uncertainty modeled by a generalized p-box $[\underline{F}, \overline{F}]$ can also be mapped into an equivalent random set m, in the sense that $\mathcal{P}_{[\underline{F},\overline{F}]} = \mathcal{P}_{Bel}$. The focal sets of the random set equivalent to the generalized p-box are of the form $A_{i+1} \setminus A_j$, with mass

$$m(A_{i+1} \setminus A_j) = \min(\alpha_{i+1}, \beta_{j+1}) - \max(\alpha_i, \beta_j). \tag{6}$$

where $\alpha_{i+1} > \theta \geq \alpha_i$ and $\beta_{j+1} > \theta \geq \beta_j$, for all thresholds $\theta \in (0, 1]$.

As emphasized in the companion paper, there is no direct relationships between sets of probability intervals and generalized p-boxes. Nevertheless, it is possible to transform one representation into the other, as well as to relate the two representations. Here, we briefly restate results that will be useful to study clouds.

Consider a set L of probability intervals defined on an indexed space $X = \{x_1, \ldots, x_2\}$. For all i from 1 to n, we note $l(x_i) = l_i$ and $u(x_i) = u_i$. An approximated generalized p-box $[\underline{F}', \overline{F}']$ covering the set L of probability intervals can be computed by using Equations (1) of Section 2.1 in the following way:

$$\underline{F}'(x_i) = \underline{P}(A_i) = \alpha'_i = \max(\sum_{x_i \in A_i} l_i, 1 - \sum_{x_i \notin A_i} u_i) \tag{7}$$
$$\overline{F}'(x_i) = \overline{P}(A_i) = \beta'_i = \min(\sum_{x_i \in A_i} u_i, 1 - \sum_{x_i \notin A_i} l_i)$$

where $\underline{P}, \overline{P}$ are respectively the lower and upper probabilities of $\mathcal{P}_L$ on events A_i, given by Equations (1). Note that the obtained p-box depends on the chosen permutation of elements of X. Let Σ_σ denotes the set of permutations σ of elements in X and $[\underline{F}', \overline{F}']_\sigma$ the corresponding p-box obtained by equation (7). It holds that $\mathcal{P}_L = \bigcap_{\sigma \in \Sigma_\sigma} \mathcal{P}_{[\underline{F}', \overline{F}']_\sigma}$, showing that the informative content of a given set L of probability intervals can also be modeled by a set of generalized p-boxes.

3 Clouds

A cloud is defined as a pair of mappings $\delta : X \to [0,1]$ and $\pi : X \to [0,1]$ from the space X to $[0,1]$, such that δ is point-wise less than π (i.e. $\delta \leq \pi$. Moreover, $\pi(x) = 1$ for at least one element x in X, and $\delta(y) = 0$ for at least one element y in X. δ and π are respectively the lower and upper distributions of a cloud.

Mappings δ, π forming the cloud $[\delta, \pi]$ are formally equivalent to fuzzy membership functions. A cloud $[\delta, \pi]$ is formally equivalent to an interval-valued fuzzy set (IVF for short). More precisely, since $\delta \leq \pi$, a cloud $[\delta, \pi]$ is formally equivalent to an interval-valued membership function whereby the membership value of each element x of X is $[\delta(x), \pi(x)]$. Since a cloud is equivalent to a pair of fuzzy membership functions, at most $2|X| - 2$ values (notwithstanding boundary constraints on δ and π) are needed to fully determine a cloud on a finite set. Two subcases of clouds considered by Neumaier [15] are the thin and fuzzy clouds. A *thin cloud* is defined as a cloud for which $\delta = \pi$, while a *fuzzy cloud* is a cloud for which $\delta = 0$.

The credal set $\mathcal{P}_{[\delta,\pi]}$ induced by a cloud is defined by Neumaier [15] :

$$\mathcal{P}_{[\delta,\pi]} = \{P, P(\{x \in X, \delta(x) \geq \alpha\}) \leq 1 - \alpha \leq P(\{x \in X, \pi(x) > \alpha\})\} \tag{8}$$

where P is a probability measure. In the finite setting, let $0 = \gamma_0 < \gamma_1 < \ldots < \gamma_m = 1$ be the ordered distinct values taken by both δ and π on elements of X, then denote the strong and regular cuts as

$$B_{\overline{\gamma}_i} = \{x \in X | \pi(x) > \gamma_i\} \text{ and } B_{\gamma_i} = \{x \in X | \pi(x) \geq \gamma_i\} \tag{9}$$

for the upper distribution π and

$$C_{\overline{\gamma_i}} = \{x \in X | \delta(x) > \gamma_i\} \text{ and } C_{\gamma_i} = \{x \in X | \delta(x) \geq \gamma_i\} \quad (10)$$

for the lower distribution δ. Note that in the finite case, $B_{\overline{\gamma_i}} = B_{\gamma_{i+1}}$ and $C_{\overline{\gamma_i}} = C_{\gamma_{i+1}}$, with $\gamma_{m+1} = 1$, and also

$$\emptyset = B_{\overline{\gamma_m}} \subset B_{\overline{\gamma_{m-1}}} \subseteq \ldots \subseteq B_{\overline{\gamma_0}} = X;$$

$$\emptyset = C_{\gamma_m} \subseteq C_{\gamma_{m-1}} \subseteq \ldots \subseteq C_{\gamma_0} = X$$

and since $\delta \leq \pi$, this implies that $C_{\gamma_i} \subseteq B_{\gamma_i}$, hence $C_{\gamma_i} \subseteq B_{\overline{\gamma_{i-1}}}, \forall i = 1, \ldots, m$. In such a finite case, a cloud is said to be discrete. In terms of constraints bearing on probabilities, the credal set $\mathcal{P}_{[\delta,\pi]}$ of a finite cloud is described by the finite set of inequalities:

$$i = 0, \ldots, m, \quad P(C_{\gamma_i}) \leq 1 - \gamma_i \leq P(B_{\overline{\gamma_i}}) \quad (11)$$

under the above inclusion constraints.

Note that some conditions must hold for $\mathcal{P}_{[\delta,\pi]}$ to be non-empty in the finite case. In particular, distribution δ must not be equal to π (i.e. $\delta < \pi$). Otherwise, consider the case where $C_{\gamma_i} = B_{\overline{\gamma_{i-1}}} (= B_{\gamma_i})$, that is π and δ have the same γ_i-cut. Clearly, there is no probability distribution satisfying the constraint $1 - \gamma_{i-1} \leq P(C_{\gamma_i}) \leq 1 - \gamma_i$ since $\gamma_{i-1} < \gamma_i$. So, finite clouds cannot be thin.

Example 3.1 illustrates the notion of cloud and will be used in the next sections to illustrate various results.

Example 3.1 *Let us consider a space $X = \{u, v, w, x, y, z\}$ and the following cloud $[\delta, \pi]$, pictured in Figure 2, defined on this space:*

	u	v	w	x	y	z
π	0.75	1	1	0.75	0.75	0.5
δ	0.5	0.5	0.75	0.5	0	0

The values γ_i corresponding to this cloud are

$$\begin{array}{c} 0 \leq 0.5 \leq 0.75 \leq 1 \\ \gamma_0 \leq \gamma_1 \leq \gamma_2 \leq \gamma_3 \end{array}$$

and the constraints associated to this cloud and corresponding to Equation (11)

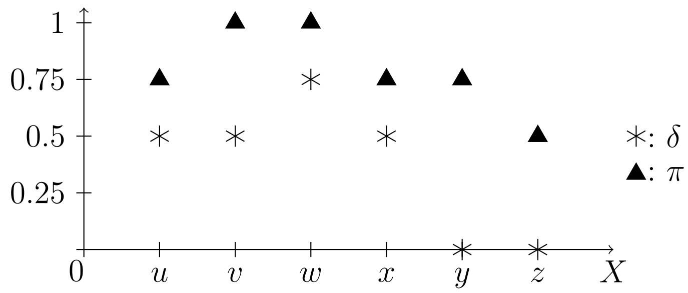

Figure 2. Cloud $[\delta, \pi]$ of Example 3.1

are

$$\begin{array}{c} P(C_{\gamma_3} = \emptyset) \leq 1 - 1 \leq P(B_{\overline{\gamma}_3} = \emptyset) \\ P(C_{\gamma_2} = \{w\}) \leq 1 - 0.75 \leq P(B_{\overline{\gamma}_2} = \{v, w\}) \\ P(C_{\gamma_1} = \{u, v, w, x\}) \leq 1 - 0.5 \leq P(B_{\overline{\gamma}_1} = \{u, v, w, x, y\}) \\ P(C_{\gamma_0} = X) \leq 1 - 0 \leq P(B_{\overline{\gamma}_0} = X) \end{array}$$

3.1 Clouds in the setting of possibility theory

To relate clouds with possibility distributions, first consider the case of *fuzzy clouds* $[\delta, \pi]$. In this case, $\delta = 0$ and, $C_{\gamma_i} = \emptyset$ for $i = 1, \ldots, m$, which means that constraints given by Equations (11) reduce to

$$i = 0, \ldots, m \quad 1 - \gamma_i \leq P(B_{\overline{\gamma}_i})$$

which, by using Proposition 2.3, induces a credal set equivalent to $\mathcal{P}_\pi$. This shows that *fuzzy clouds* are equivalent to possibility distributions. The following proposition is a direct consequence of this observation:

Proposition 3.2 *Uncertainty modeled by a cloud* $[\delta, \pi]$ *is equivalent to the uncertainty modeled by the pair of possibility distributions* $1 - \delta$ *and* π*, and the following relation holds:*

$$\mathcal{P}_{\delta,\pi} = \mathcal{P}_\pi \cap \mathcal{P}_{1-\delta}$$

Proof of Proposition 3.2 Consider a cloud $[\delta, \pi]$ and the constraints inducing the credal set $\mathcal{P}_{[\delta,\pi]}$. As for generalized p-boxes, these constraints can be split into two sets of constraints, namely, for $i = 0, \ldots, m$, $P(C_{\gamma_i}) \leq 1 - \gamma_i$ and $1 - \gamma_i \leq P(B_{\overline{\gamma}_i})$. Since $B_{\overline{\gamma}_i}$ are strong cuts of π, then by Proposition 2.3 we know that these constraints define a credal set equivalent to $\mathcal{P}_\pi$.

Note then that $P(C_{\gamma_i}) \leq 1 - \gamma_i$ is equivalent to $P(C^c_{\gamma_i}) \geq \gamma_i$ (where $C^c_{\gamma_i} = \{x \in X, 1 - \delta(x) > 1 - \gamma_i\}$). By construction, $1 - \delta$ is a normalized possibility distribution. Interpreting these inequalities in the light of Proposition 2.3, it is clear that they define the credal set $\mathcal{P}_{1-\delta}$. By merging the two set of constraints, we get $\mathcal{P}_{\delta,\pi} = \mathcal{P}_\pi \cap \mathcal{P}_{1-\delta}$. $\square$

This proposition shows that, as for generalized p-boxes, the credal set induced by a cloud is equivalent to the conjunction of two credal sets dominated by possibility distributions [10]. This analogy between generalized p-boxes and clouds is fully studied in section 3.3. This result also makes it cleat that a cloud $[\delta, \pi]$ is equivalent to its mirror cloud $[1-\pi, 1-\delta]$.

Example 3.3 shows the two possibility distributions induced from the cloud $[\delta, \pi]$ of Example 3.1

Example 3.3 *We consider the same space X and the same cloud as in Example 3.1. Then, possibility distributions $\pi, 1-\delta$ are:*

	u	v	w	x	y	z
π	0.75	1	1	0.75	0.75	0.5
$1-\delta$	0.5	0.5	0.25	0.5	1	1

3.2 Clouds with non-empty credal sets

A natural question is, given two possibility distributions π_1, π_2, whether we can try to find conditions under which $\mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2}$ is ensured to be empty or non-empty. This would shed light on clouds with non-empty credal sets.

As the minimum is a usual conjunction operator in possibility theory, one could be tempted to jump to the conclusion that $\mathcal{P}_\pi \cap \mathcal{P}_{1-\delta} = \mathcal{P}_{\min(\pi,1-\delta)}$. In fact given any two possibility distributions π_1, π_2, we do have $\mathcal{P}_{\min(\pi_1,\pi_2)} \subseteq \mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2}$, but not the converse inclusion [8].From this remark, it is clear that

- $\mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2} \neq \emptyset$ as soon as $\min(\pi_1, \pi_2)$ is a normalized possibility distribution.
- Not all pairs of possibility distributions such that $\mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2} \neq \emptyset$ derive from a cloud $[1-\pi_2, \pi_1]$. Indeed the normalization of $\min(\pi_1, \pi_2)$ does not imply that $1-\pi_2 \leq \pi_1$.

Chateauneuf [3] has found a characteristic condition under which the credal sets associated to two belief functions intersect. We can thus apply this result to a pair of possibility distributions and get the following necessary and sufficient condition for a cloud $[\delta, \pi]$ to have an non-empty credal set:

Proposition 3.4 *A cloud $[\delta, \pi]$ has a non-empty credal set if and only if*

$$\forall A \subseteq X, \max_{x \in A} \pi(x) \geq \min_{y \notin A} \delta(y)$$

Proof Chateauneuf's condition applied to possibility distributions π_1 and π_2 reads $\forall A \subseteq X, \Pi_1(A) + \Pi_2(A^c) \geq 1$. Choose $\pi_1 = \pi$ and $\pi_2 = 1 - \delta$. In particular $\Pi_2(A^c) = 1 - \min_{y \notin A} \delta(y)$. □

However this characterization has exponential complexity. It can be simplified as follows. Suppose $\pi(x_1) \leq \pi(x_2) \cdots \leq \pi(x_n)$ and $\max_{x \in A} \pi(x) = \pi(x_i)$. The tightest constraint of the form $\max_{x \in A} \pi(x) = \pi(x_i) \geq \min_{y \notin A} \delta(y)$ is when choosing $A = \{x_1, \ldots x_i\}$. Hence, Chateauneuf condition comes down to the following set of $n - 1$ inequalities to be checked:

$$\pi(x_i) \geq \min_{j > i} \delta(x_j), j = 1, n - 1.$$

There are simpler sufficient conditions for $\mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2} = \emptyset$. For instance, if we consider a finite partition $\{G_1, \ldots, G_m\}$ of X, any probability distribution P defined on X satisfies $\sum_{i=1}^{m} P(G_i) = 1$. As the possibility measures Π_1, Π_2 induced from π_1, π_2 are upper probabilities of the credal sets $\mathcal{P}_{\pi_1}, \mathcal{P}_{\pi_2}$, a sufficient condition for $\mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2}$ to be empty is

$$\sum_{i=1}^{m} \min(\Pi_{\pi_1}(G_i), \Pi_{\pi_2}(G_i)) < 1 \tag{12}$$

and in particular, if the two distributions $\pi, 1 - \delta$ derive from a cloud $[\delta, \pi]$ defined on X, at least the inequality $\sum_{x \in X} \min(\pi(x), 1 - \delta(x)) \geq 1$ must hold for $\mathcal{P}_{[\delta, \pi]}$ to be non-empty.

Another simple sufficient condition for the emptiness of $\mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2}$ can be established:

Proposition 3.5 *If* $\sup_{x \in X} \min(\pi_1(x), \pi_2(x)) < 0.5$ *then* $\mathcal{P}_{\pi_1} \cap \mathcal{P}_{\pi_2} = \emptyset$.

Proof of Proposition 3.5 If the premise holds then there is $\alpha < 0.5$ and a subset A of X such that

$$A_{\pi_1} = \{x \in X | \pi_1(x) > \alpha\} \subseteq A \text{ and } A_{\pi_2} = \{x \in X | \pi_2(x) > \alpha\} \subseteq A^c.$$

Then $\Pi_1(A^c) + \Pi_2(A) \leq \Pi_1(A^c_{\pi_1}) + \Pi_2(A^c_{\pi_2}) \leq 2\alpha < 1$, thus violating Equation (12). □

In particular, this proposition applies when for all x in X, $\pi_1(x) + \pi_2(x) < 1$. However, this is a case when $[1 - \pi_2, \pi_1]$ does not correspond to the definition of a cloud.

We already know that, a thin finite cloud has an empty credal set. Now consider the extreme case of clouds for which $C_{\gamma_i} = B_{\overline{\gamma}_i}, \forall i$ in equation (11). Rank-ordering X in increasing values of $\pi(x)$ ($\pi(x_i) \geq \pi(x_{i-1}), \forall i$) enforces $\delta(x_i) = \pi(x_{i-1})$, with $\delta(x_1) = 0$. Let δ_π be this lower distribution. As then $P(B_{\overline{\gamma}_i}) = 1 - \gamma_i, \forall i$, it follows that this (almost thin) cloud $[\delta_\pi, \pi]$ contains the single probability measure P with distribution $p_i = \pi(x_i) - \pi(x_{i-1}), \forall x_i \in X$. So if a finite cloud $[\delta, \pi]$ is such that if $\delta > \delta_\pi$, it has empty credal set; and if $\delta \leq \delta_\pi$, then the credal set is not empty.

Note that there may exist clouds $[\delta, \pi]$ with non-empty credal set while $\delta(x_i) = \pi(x_i)$ for some i. For instance, if $\delta(x_i) = \pi(x_i) < 1$ and $\delta(x_j) = 0$ if $j \neq i$, it defines a non-empty credal set since $\sup_{x \in X} \min(\pi(x), 1 - \delta(x)) = 1$.

3.3 Generalized p-boxes as a special kind of clouds

The previous subsections show that, similarly to generalized p-boxes, clouds correspond to pairs of possibility distributions. Moreover, the constraints defining a finite cloud are quite close to the ones defining a generalized p-box on a finite set, as per equations (5). The lemma below lays bare the nature of the relationship between the two representations:

Lemma 3.6 *A finite cloud $[\delta, \pi]$ can be encoded as as a generalized p-box if and only if sets $\{B_{\overline{\gamma}_i}, C_{\gamma_j} | i, j = 0, \ldots, m\}$ defined from Equations (9) and (10) form a nested sequence (i.e. these sets are completely (pre-)ordered with respect to inclusion).*

Proof of Lemma 3.6 Assume the sets $B_{\overline{\gamma}_i}$ and C_{γ_j} form a globally nested sequence whose current element is A_k. Then the set of constraints defining a cloud can be rewritten in the form $\alpha_k \leq P(A_k) \leq \beta_k$, where $\alpha_k = 1 - \gamma_i$ and $\beta_k = \min\{1 - \gamma_j : B_{\overline{\gamma}_i} \subseteq C_{\gamma_j}\}$ if $A_k = B_{\overline{\gamma}_i}$; $\beta_k = 1 - \gamma_i$ and $\alpha_k = \max\{1 - \gamma_j : B_{\overline{\gamma_j}} \subseteq C_{\gamma_i}\}$ if $A_k = C_{\gamma_i}$.

Since $0 = \gamma_0 < \alpha_1 < \ldots < \alpha_M = 1$, these constraints are equivalent to those describing a generalized p-box (Equations (5)). But if $\exists\ C_{\gamma_j}, B_{\overline{\gamma}_i}$ with $j < i$ s.t. $C_{\gamma_j} \not\subset B_{\overline{\gamma}_i}$ and $B_{\overline{\gamma}_i} \not\subset C_{\gamma_j}$, then uncertainty modeled by the corresponding cloud cannot be exactly modeled by a generalized p-box, since confidence sets $\{B_{\overline{\gamma}_i}, C_{\gamma_j} | i, j = 0, \ldots, m\}$ would not form a complete preordering with respect to inclusion anymore. □

If a cloud $[\delta, \pi]$ satisfies Lemma 3.6, then sets $\{B_{\overline{\gamma}_i}, C_{\gamma_j} | i, j = 0, \ldots, m\}$ can be interpreted as nested downsets $(x]$. In other words, there does not exist $x, y \in X$ such that $\pi(x) > \pi(y)$ and $\delta(y) > \delta(x)$; otherwise the regular $\delta(y)$-

cut of δ (i.e. $C_{\delta(y)}$) contains y and not x, and the $\pi(x)$-cut of π (i.e. $B_{\pi(x)}$) contains x and not y, and these two sets wouldn't be nested. Such distributions δ and π are comonotonic, and from now on, if a cloud satisfies Lemma 3.6, the cloud $[\delta, \pi]$ is said to be comonotonic.

To completely relate comonotonic clouds and generalized p-boxes, it remains to express a given comonotonic cloud $[\delta, \pi]$ as a generalized p-box $[\underline{F}, \overline{F}]$. As both clouds and generalized p-boxes correspond to pairs of possibility distribution, we can define $\pi = \pi_{\overline{F}}$ and $\delta = 1 - \pi_{\underline{F}}$, where δ, π are the distributions of the cloud and $\pi_{\overline{F}}, 1 - \pi_{\underline{F}}$ are the possibility distributions describing the generalized p-box equivalent to the cloud $[\delta, \pi]$. By using Proposition 2.5, $\underline{F}, \overline{F}$ can then be computed for all x in X:

$$\overline{F}(x) = \pi(x) \text{ and } \underline{F}(x) = \min\{\delta(y)|y \in X, \delta(y) > \delta(x)\} \tag{13}$$

Let us also note that a comonotonic cloud $[\delta, \pi]$ and the corresponding generalized p-box $[\underline{F}, \overline{F}]$ induce the same complete pre-orders on elements of X, that we will note $\leq_{[\underline{F}, \overline{F}]}$ to remain coherent with the notations of the companion paper. We will consider that elements x of X are indexed accordingly, as already specified.

In practice, this relation between comonotonic clouds and generalized p-boxes means that all the results that hold for generalized p-boxes also hold for comonotonic clouds, and conversely. In particular, a comonotonic cloud $[\delta, \pi]$ can be encoded as an equivalent random set, and if we adapt Equations (6) to the case of the comonotonic cloud $[\delta, \pi]$, we get the random set such that for $j = 1, \ldots, M$

$$\begin{cases} E_j = \{x \in X | (\pi(x) \geq \gamma_j) \wedge (\delta(x) < \gamma_j)\} \\ \qquad m(E_j) = \gamma_j - \gamma_{j-1} \end{cases} \tag{14}$$

Note that in the formalism of clouds this random set can be expressed in terms of the sets $\{B_{\overline{\gamma_i}}, C_{\gamma_i} | i = 0, \ldots, m\}$. Namely, for $j = 1, \ldots, M$:

$$\begin{cases} E_j = B_{\overline{\gamma_{j-1}}} \setminus C_{\gamma_j} = B_{\gamma_j} \setminus C_{\gamma_j} \\ \qquad m(E_j) = \gamma_j - \gamma_{j-1} \end{cases} \tag{15}$$

Example 3.7 illustrates the above relations on the cloud $[\delta, \pi]$ used in Example 3.1, which is comonotonic.

Example 3.7 *From the cloud of Example 3.1,* $C_{\gamma_3} \subset C_{\gamma_2} \subset B_{\overline{\gamma_2}} \subset C_{\gamma_1} \subset$

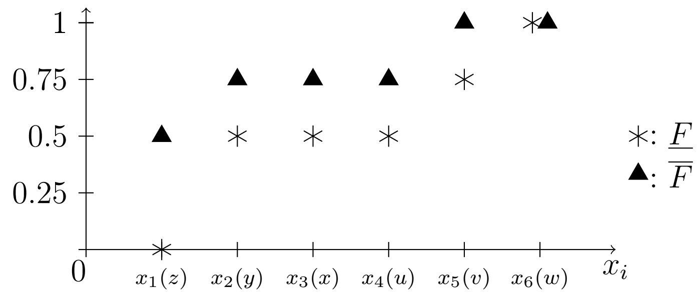

Figure 3. Generalized p-box $[\underline{F}, \overline{F}]$ corresponding to cloud of Example 3.1

.

$B_{\overline{\gamma_1}} \subset B_{\overline{\gamma_0}}$, and the constraints defining $\mathcal{P}_{[\delta,\pi]}$ can be transformed into

$$\begin{aligned} 0 \leq & C_{\gamma_2} = \{w\} \leq 0.25 \\ 0.25 \leq & B_{\overline{\gamma_2}} = \{v, w\} \leq 0.5 \\ 0.25 \leq & C_{\gamma_1} = \{u, v, w, x\} \leq 0.5 \\ 0.5 \leq & B_{\overline{\gamma_1}} = \{u, v, w, x, y\} \leq 1. \end{aligned}$$

They are equivalent to the generalized p-box $[\underline{F}, \overline{F}]$ pictured on Figure 3:

	u	v	w	x	y	z
$\overline{F}$	0.75	1	1	0.75	0.75	0.5
$\underline{F}$	0.5	0.75	1	0.5	0.5	0

The following ranking of elements of X is compatible with the two distributions (see Figure 3):

$$z <_{[\underline{F},\overline{F}]} y <_{[\underline{F},\overline{F}]} x =_{[\underline{F},\overline{F}]} u <_{[\underline{F},\overline{F}]} v <_{[\underline{F},\overline{F}]} w$$

The corresponding random set, given by Equations (15) or (14), is:

$$\begin{aligned} & m(\{x_5, x_6\}) = 0.25 \\ & m(\{x_2, x_3, x_4, x_5\}) = 0.25 \\ & m(\{x_1, x_2\}) = 0.5 \end{aligned}$$

These results provide insight in uncertainty representations based on pairs of comonotonic possibility distributions. They emphasize different views of the same tool. Comonotonic clouds being special cases of clouds, it is then natural to wonder if some of the results presented in this section extend to clouds that are not comonotonic (and called non-comonotonic). In particular, can uncertainty modeled by a non-comonotonic cloud be exactly modeled by an equivalent random set?

4 The Nature of Non-comonotonic Clouds

We will now study the case of non-comonotonic clouds. For this kind of clouds, Proposition 3.2 linking clouds and possibility distributions still holds, but non-comonotonic clouds are no longer equivalent to generalized p-boxes, thus results valid for comonotonic clouds cannot be used anymore. As we shall see, non-comonotonic clouds appear to be less interesting, at least from a practical point of view, than comonotonic ones.

4.1 Characterization

One way of characterizing an uncertainty model is to find the maximal natural number n such that the lower measure induced by this uncertainty model is always n-monotone. This is how we will proceed with non-comonotonic clouds.

Let $[\delta, \pi]$ be a non-comonotonic cloud, and $\mathcal{P}_{[\delta,\pi]}$ the derived credal set. The question is: what is the (maximal) n-monotonicity of the associated lower probability $\underline{P}$ of $\mathcal{P}_{[\delta,\pi]}$? The next lemma will be useful to address this question:

Lemma 4.1 *Let $(F_1, F_2), (G_1, G_2)$ be two pairs of sets such that $F_1 \subset F_2$, $G_1 \subset G_2$, $G_1 \not\subseteq F_2$ and $G_1 \cap F_1 \neq \emptyset$. Let also π_F, π_G be two possibility distributions s.t. the corresponding belief functions are defined by mass assignments $m_F(F_1) = m_G(G_2) = \lambda$, $m_F(F_2) = m_G(G_1) = 1 - \lambda$. Then, the lower probability of the non-empty credal set $\mathcal{P} = \mathcal{P}_{\pi_F} \cap \mathcal{P}_{\pi_G}$ is not $2-$monotone.*

Note that in the above lemma, $[1-\pi_G, \pi_F]$ is a not a cloud, since the inequality $\pi_G + \pi_F \geq 1$ does not hold, even if by construction, $\mathcal{P} = \mathcal{P}_{\pi_F} \cap \mathcal{P}_{\pi_G}$ is not empty. Non-emptiness of $\mathcal{P}_{\pi_F} \cap \mathcal{P}_{\pi_G}$ comes from $\pi_F(x) = \pi_G(x) = 1$ for an element $x \in G_1 \cap F_1$. Although $[1 - \pi_G, \pi_F]$ does not satisfy the definition of a cloud, Example 4.2 shows that the situation described in Lemma 4.1 also occurs in non-comonotonic clouds.

Example 4.2 *Consider a space X of five elements $\{v, w, x, y, z\}$ and the following non-comonotonic cloud $[\delta, \pi]$ pictured on Figure 4:*

	v	w	x	y	z
π	1	1	0.5	0.5	0.25
δ	0	0.5	0.25	0	0

This cloud is non-comonotonic, since $\pi(v) > \pi(x)$ and $\delta(v) < \delta(x)$. The credal

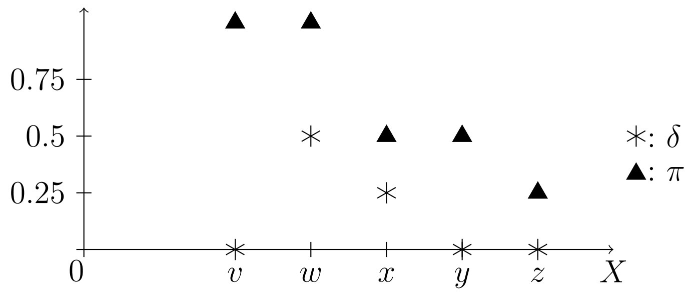

Figure 4. Cloud $[\delta, \pi]$ of Example 4.2

set $\mathcal{P}_{[\delta,\pi]}$ can also be defined by the following constraints:

$$P(C_{\gamma_2} = \{w\}) \leq 1 - 0.5 \leq P(B_{\overline{\gamma}_2} = \{v, w\})$$
$$P(C_{\gamma_1} = \{w, x\}) \leq 1 - 0.25 \leq P(B_{\overline{\gamma}_1} = \{v, w, x, y\})$$

with $\gamma_2 = 0.5$ and $\gamma_1 = 0.25$. Now, consider the events $B_{\overline{\gamma}_2}, C^c_{\gamma_1}, B_{\overline{\gamma}_2} \cap C^c_{\gamma_1}$, $B_{\overline{\gamma}_2} \cup C^c_{\gamma_1}$. We can check that

$$\underline{P}(B_{\overline{\gamma}_2}) = 0.5 \qquad \underline{P}(C^c_{\gamma_1}) = 0.25$$
$$\underline{P}(B_{\overline{\gamma}_2} \cap C^c_{\gamma_1} = \{v\}) = 0 \qquad \underline{P}(B_{\overline{\gamma}_2} \cup C^c_{\gamma_1} = \{v, w, y, z\}) = 0.5$$

since at most a 0.5 probability mass can be assigned to x. Then the inequality $\underline{P}(B_{\overline{\gamma}_2} \cap C^c_{\gamma_1}) + \underline{P}(B_{\overline{\gamma}_2} \cup C^c_{\gamma_1}) < \underline{P}(B_{\overline{\gamma}_2}) + \underline{P}(C^c_{\gamma_1})$ holds, which shows that the lower probability induced by the cloud is not 2-monotone.

This example is sufficient to show that at least some non-comonotonic clouds induce lower probability measures that are not 2-monotone. It suggests that such non-comonotonic clouds are likely to be less tractable and thus of a limited practical interest. The following proposition gives a general characterization of such non-comonotonic clouds:

Proposition 4.3 *Let $[\delta, \pi]$ be a non-comonotonic cloud and assume there is a pair of events $B_{\overline{\gamma}_i}, C_{\gamma_j}$ in the cloud s.t. $B_{\overline{\gamma}_i} \cap C_{\gamma_j} \notin \{B_{\overline{\gamma}_i}, C_{\gamma_j}, \emptyset\}$ (i.e. $B_{\overline{\gamma}_i}, C_{\gamma_j}$ are just overlapping) . Then, the lower probability measure of the credal set $\mathcal{P}_{\delta,\pi}$ is not $2-$monotone.*

The proof of Proposition 4.3 can be found in the appendix. It comes down to showing that for any non-comonotonic cloud with a pair $B_{\overline{\gamma}_i}, C_{\gamma_j}$ of events such that $B_{\overline{\gamma}_i} \cap C_{\gamma_j} \neq \{B_{\overline{\gamma}_i}, C_{\gamma_j}, \emptyset\}$, the situation exhibited in Lemma 4.1 and the above example always occurs, namely the existence of two subsets F and G (respectively of the form $B_{\overline{\gamma}_i}$ and $C^c_{\gamma_j}$) for which 2-monotonicity fails. Proposition 4.3 also shows that non-comonotonic clouds satisfying this proposition cannot be viewed as random sets. Note that, although comonotonic clouds and clouds described by Proposition 4.3 cover a large number of possible discrete clouds, there remains a "small" subfamily of non- comonotonic clouds not covered by Proposition 4.3. The following subsectionsheds some light on them.

4.2 Disjoint-nested finite clouds

Non-monotonic clouds violating conditions of Lemma 3.6 correspond to the case where, for any pair of events $\{B_{\overline{\gamma}_i}, C_{\gamma_j}\}$ $i, j = 1, \ldots, n$, we have $B_{\overline{\gamma}_i} \cap C_{\gamma_j} \in \{B_{\overline{\gamma}_i}, C_{\gamma_j}, \emptyset\}$, with at least one pair $B_{\overline{\gamma}_i}, C_{\gamma_j}$ of non-empty sets such that $B_{\overline{\gamma}_i} \cap C_{\gamma_j} = \emptyset$. In other words, $\forall i, j, B_{\overline{\gamma}_i}$ and C_{γ_j} are either nested or disjoint. They can be called *nested-disjoint* clouds and they are fully described by the existence of three indices $j > k \geq l$ s.t.

$$\text{the sets } \{B_{\overline{\gamma}_i}, i < j\} \cup \{C_{\gamma_i}, i \leq k\} \text{ form a nested sequence} \tag{16}$$

$$C_{\gamma_i} \cap B_{\overline{\gamma}_j} = \emptyset \quad \forall i \quad k \geq i \geq l \tag{17}$$

$$C_{\gamma_i} = \emptyset \quad \forall i > k \text{ and } C_{\gamma_i} \neq \emptyset \quad \forall i \leq k \tag{18}$$

$$B_{\overline{\gamma}_i} \cap C_{\gamma_f} = B_{\overline{\gamma}_i} \text{ or } C_{\gamma_f} \quad i > k, f < l. \tag{19}$$

and these four statements induce the fact that $(C_{\gamma_l} \cup B_{\overline{\gamma}_j}) \subset B_{\overline{\gamma}_{j-1}}$, since we know that $B_{\overline{\gamma}_j} \subset B_{\overline{\gamma}_{j-1}}$,$C_{\gamma_l} \cap B_{\overline{\gamma}_j} = \emptyset$ (Statement (17)) and $C_{\gamma_l} \cap B_{\overline{\gamma}_{j-1}} \in \{C_{\gamma_l}, B_{\overline{\gamma}_{j-1}}\}$ (Statement (16)). Given these facts, $C_{\gamma_l} \cap B_{\overline{\gamma}_{j-1}} = C_{\gamma_l}$, otherwise we end up with a contradiction. The structure of this particular case is summarized by Figure 5 (where only the most important sets are represented).

We have strong reasons to think that these particular clouds, although not generalized p-boxes, can still be represented by random sets. A first reason is that we can associate to the sets $B_{\overline{\gamma}_1}, \ldots, B_{\overline{\gamma}_j}$ a possibility distribution (i.e. they are nested and are associated to lower probability bounds), while the sets $\{B_{\overline{\gamma}_i}, i < j\} \cup \{C_{\gamma_i}, i \leq k\}$ form a nested sequence and can thus be associated to a generalized p-box. The nested-disjoint clouds could then be seen as a convex mixture of two random sets, thus giving again a random set. Secondly, this conjecture is reinforced by the following simple example: Let us consider a cloud whose cuts are such that $B_{\overline{\gamma}_2}, B_{\overline{\gamma}_1}, C_{\gamma_2}$ with $B_{\overline{\gamma}_2} \subset B_{\overline{\gamma}_1}$, $C_{\gamma_2} \subset B_{\overline{\gamma}_1}$ and $C_{\gamma_1} \cap B_{\overline{\gamma}_2} = \emptyset$, together with the two weights $\gamma_2 > \gamma_1$. This cloud is nested-disjoint, and the belief function such that $m(B_{\overline{\gamma}_2}) = 1 - \gamma_2$, $m(B_{\overline{\gamma}_1}) = \gamma_2 - \gamma_1$ and $m(C^c_{\gamma_1}) = \gamma_1$ clearly models the same credal set as this cloud (since the presence of C_{γ_1} only induces a bound over its upper probability).

Finally, it should be noted that, in the case of continuous clouds, this subclass doesn't exist, since when distribution δ is such that $\sup_{x \in X} \delta(x) > 0$ and δ, π are not comonotonic, we can always exhibit two cuts that are overlapping.

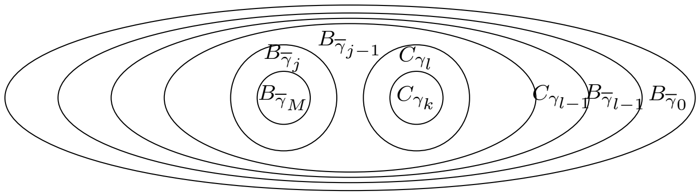

Figure 5. Structure of a nested-disjoint cloud

4.3 Outer approximation of a non-monotonic cloud

Since non-comonotonic clouds satisfying property 4.3 are likely to be hard to handle in practice, we provide, in this section and the next one, some practical means to compute guaranteed outer and inner approximations of the exact probability bounds induced by a non-comonotonic cloud. To this aim, we rely on previous results.

Given a cloud $[\delta, \pi]$, we have proven that $\mathcal{P}_{[\delta,\pi]} = \mathcal{P}_\pi \cap \mathcal{P}_{1-\delta}$, where π and $1 - \delta$ are possibility distributions. As a consequence, the upper and lower probabilities of $\mathcal{P}_{[\delta,\pi]}$ on any event can be bounded from above (resp. from below), using the possibility measures and the necessity measures induced by π and $\overline{\pi} = 1-\delta$. The following bounds, originally considered by Neumaier [15], provide, for all event A of X, an outer approximation of the range of $P(A)$:

$$\max(N_\pi(A), N_{1-\delta}(A)) \leq \underline{P}(A) \leq P(A) \leq \overline{P}(A) \leq \min(\Pi_\pi(A), \Pi_{1-\delta}(A)), \tag{20}$$

where $\underline{P}(A), \overline{P}(A)$ are the lower and upper probabilities induced by $\mathcal{P}_{[\delta,\pi]}$. Remember that probability bounds generated by possibility distributions alone are of the form $[0, \beta]$ or $[\alpha, 1]$. Using a cloud and applying Equation (20) lead to tighter bounds of the form $[\alpha, \beta] \subseteq [0, 1]$, and thus to more precise information, while remaining simple to compute. Nevertheless, these bounds are not, in general, the infinimum and the supremum of $P(A)$ over $\mathcal{P}_{[\delta,\pi]}$. To see this, consider the following example:

Example 4.4 *Let $[\delta, \pi]$ be a cloud defined on a space X, such that distributions δ and π takes up to four different values on elements x of X (including 0 and 1). These values are such that $0 = \gamma_0 < \gamma_1 < \gamma_2 < \gamma_3 = 1$, and the distributions δ, π are such that*

$$\begin{aligned}
\pi(x) &= 1 \textit{ if } x \in B_{\overline{\gamma}_2}; \\
&= \gamma_2 \textit{ if } x \in B_{\overline{\gamma}_1} \setminus B_{\overline{\gamma_2}}; \\
&= \gamma_1 \textit{ if } x \notin B_{\overline{\gamma_1}}. \\
\delta(x) &= \gamma_2 \textit{ if } x \in C_{\gamma_2}; \\
&= \gamma_1 \textit{ if } x \in C_{\gamma_1} \setminus C_{\gamma_2}; \\
&= 0 \textit{ if } x \notin C_{\gamma_1}.
\end{aligned}$$

Since $P(B_{\overline{\gamma}_1}) \geq 1 - \gamma_1$ and $P(C_{\gamma_2}) \leq 1 - \gamma_2$, from Equations (11), we can check that $\underline{P}(B_{\overline{\gamma}_1} \setminus C_{\gamma_2}) = \underline{P}(B_{\overline{\gamma}_1} \cap C^c_{\gamma_2}) = \gamma_2 - \gamma_1$. Now, by definition of a necessity measure, $N_\pi(B_{\overline{\gamma_1}} \cap C^c_{\gamma_2}) = \min(N_\pi(B_{\overline{\gamma}_1}), N_\pi(C^c_{\gamma_2})) = 0$ since $\Pi_\pi(C_{\gamma_2}) = 1$ because $C_{\gamma_2} \subseteq B_{\overline{\gamma}_1}$ and $\Pi_\pi(B_{\overline{\gamma}_1}) = 1$. Considering distribution δ, we can have $N_{1-\delta}(B_{\overline{\gamma}_1} \cap C^c_{\gamma_2}) = \min(N_{1-\delta}(B_{\overline{\gamma}_1}), N_{1-\delta}(C^c_{\gamma_2})) = 0$ since $N_{1-\delta}(B_{\overline{\gamma}_1}) = \Delta_\delta(B^c_{\overline{\gamma}_1}) = 0$ and $C_{\gamma_1} \subseteq B_{\overline{\gamma}_1}$ (which means that the elements x of X that are in $B^c_{\overline{\gamma}_1}$ are such that $\delta(x) = 0$). Equation (20) can thus result in a trivial lower bound (i.e. equal to 0), different from $\underline{P}(B_{\overline{\gamma}_1} \cap C^c_{\gamma_2})$.

Bounds given by Equation (20), are the main motivation for clouds, after Neumaier [15]. Since these bounds are, in general, not the infinimum and supremum of $P(A)$ on $\mathcal{P}_{[\delta,\pi]}$, Neumaier's claim that they're only vaguely related to Walley's previsions or to random sets is not surprising. If a cloud is comonotonic, Equation (20) becomes less useful. Indeed, since comonotonic clouds are equivalent to generalized p-boxes, we can easily compute exact values of lower and upper probabilities of $\mathcal{P}_{[\delta,\pi]}$, eg. via the random set representation.

4.4 Inner approximation of a non-comonotonic cloud

The previous outer approximation is easy to compute and allows to make some of Neumaier's claims more clear. Nevertheless, it is still unclear how to practically use these outer bounds in subsequent treatments (e.g., propagation, fusion). The inner approximation of a cloud $[\delta, \pi]$ proposed now is a random set, which is easy to exploit in practice. This inner approximation is given by the following proposition:

Proposition 4.5 *Let $[\delta, \pi]$ be a non-comonotonic cloud defined on a space X. Let us then define, for $j = 1, \ldots, M$, the following random set:*

$$\begin{cases} E_j = \{x \in X | (\pi(x) \geq \gamma_j) \wedge (\delta(x) < \gamma_j)\} \\ \qquad m(E_j) = \gamma_j - \gamma_{j-1} \end{cases}$$

where $0 = \gamma_0 < \ldots \gamma_j < \ldots < \gamma_M = 1$ are the distinct values taken by δ, π on elements of X, E_j are the focal elements with masses $m(E_j)$ of the random set. This random set is an inner approximation of $[\delta, \pi]$, in the sense that the credal set $\mathcal{P}_{Bel}$ induced by this random set is such that $\mathcal{P}_{Bel} \subseteq \mathcal{P}_{[\delta,\pi]}$.

In the case of non-comonotonic clouds satisfying Proposition 4.3, the inclusion is strict. This inner approximation appears to be a natural candidate, since on events of the type $\{B_{\overline{\gamma_i}}, C_{\gamma_i}, B_{\overline{\gamma_i}} \setminus C_{\gamma_j}, i = 0, \ldots, M; j = 0, \ldots, M; i \leq j\}$ it gives exact bounds, and is exact when the cloud $[\delta, \pi]$ is comonotonic.

5 Clouds and probability intervals

Since in many cases a cloud is either equivalent to a random set or does not lead to 2-monotone capacites, there is no direct relationship between clouds and probability intervals. Nevertheless, we can study how to transform a set of probability intervals into a cloud. Such a transformation can be useful when one wishes to work with clouds but information is obtained in terms of sets of probability intervals.

There are mainly two paths that can be followed to do this transformation:

- the first one is to use the fact that clouds are equivalent to pairs of possibility distributions, and to extend existing transformations that transform a set of probability intervals into a single possibility distribution.
- The second uses the correspondence between generalized p-boxes and comonotonic clouds, and simply apply the results obtained for generalized p-boxes.

Section 5.1 proposes a transformation following the first path, while Section 5.2 explores the second one. Some elements of comparison between the two methods are then given in Section **??**.

5.1 Exploiting probability-possibility transformations

The problem of transforming a probability distribution into a quantitative possibility distribution has been addressed by many authors (see [9] for an extended discussion). A consistency principle between (precise) probabilities and possibility distributions was first informally stated by Zadeh [20]: what is probable should be possible. It was later translated by Dubois and Prade [6,11] as a mathematical constraint. Given a possibility distribution π obtained by the transformation of a probability measure P, one should have, for all events A of X:

$$P(A) \leq \Pi(A)$$

In this case, the possibility measure Π is said to dominate P, and the transformation from probability to possibility then consists of choosing a possibility distribution amongst the ones inducing a possibility measure dominating P. Dubois and Prade [5, 11] proposed to add the following ordinal equivalence constraint, such that for two elements x, y in X

$$p(x) \leq p(y) \iff \pi(x) \leq \pi(y)$$

and to choose the least specific possibility distribution (π' is more specific than π if $\pi' \leq \pi$) respecting these two constraints.

Dubois and Prade [6] showed that the solution exists and is unique. Let us consider probability masses such that the order on probability masses is such that $p_1 \leq \ldots \leq p_n$ with $p_j = p(x_j)$. When all probability masses are different, Dubois and Prade probability-possibility transformation can be formulated as

$$\pi_i = \sum_{j=1}^{i} p_j$$

with $\pi_i = \pi(x_i)$. When some elements have equal probability, the above equation must be used on the ordered partition induced by the probability weights, using uniform probabilities inside each element of the partition.

Reversing the ordering of the p_i's in the above formula yields another possibility distribution $\overline{\pi}_i = \sum_{j=i}^{n} p_j$, with $\overline{\pi}_i = \overline{\pi}(x_i)$. Letting $\delta = 1 - \overline{\pi}$, distribution δ is of the form δ_π introduced in section 3.2, that is, $[\delta, \pi]$ is a cloud such that $\delta_i = \pi_{i-1}$ for all $i > 1$, with $\delta_1 = 0$ and $\delta_i = \delta(x_i)$. It is the tightest cloud containing P, in the sense that $\mathcal{P}(\pi) \cap \mathcal{P}(\overline{\pi}) = \{P\}$. This shows that, at least when probability masses are precise, transformation into possibility distributions can be extended to get a second possibility distribution such that this pair of distributions is equivalent to a cloud. Moreover, the fact that $\mathcal{P}(\pi) \cap \mathcal{P}(\overline{\pi}) = \{P\}$ shows that the cloud models exactly the same information as the (precise) probability distribution.

When working with imprecise probability assignments, i.e. with a set L of probability intervals, the order induced by probability weights on X is a partial order $\leq_L$ (actually, an interval order) defined by:

$$x \leq_L y \iff u(x) \leq l(y)$$

and two elements x, y are incomparable if intervals $[l(x), u(x)], [l(y), u(y)]$ intersect. The problem of transforming a set L of probability intervals into a covering possibility distribution by extending Dubois and Prade information is studied in detail by Masson and Denoeux [14]. We first recall their method, before proposing its extension to clouds.

Let $\mathcal{C}_L$ be set of linear extensions of the partial order $\leq_L$: a linear extension $<_l \in \mathcal{C}_L$ is a ranking of X compatible with the partial order $\leq_L$. Let σ_l be the permutation such that $\sigma_l(x)$ is the rank of element x in the linear extension $<_l$. Given this partial order, Masson and Denoeux [14] propose the following procedure transforming the set of probability intervals into a possibility distribution:

(1) For each linear order $<_l \in \mathcal{C}_L$ and each element x, solve

$$\pi^l(x) = \max_{\{p(y) | y \in X\}} \sum_{\sigma_l(y) \leq \sigma_l(x)} p(y) \qquad (21)$$

under the constraints

$$\begin{cases} \sum_{x \in X} p(x) = 1 \\ \forall x \in X,\ l(x) \leq p(x) \leq u(x) \\ p(\sigma_l^{-1}(1)) \leq p(\sigma_l^{-1}(2)) \leq \ldots \leq p(\sigma_l^{-1}(n)) \end{cases}$$

(2) The most informative distribution π dominating all distributions π^l is:

$$\pi(x) = \max_{<_l \in \mathcal{C}} \pi^l(x). \tag{22}$$

This procedure ensures that the resulting possibility distribution π dominates every probability distribution contained in $\mathcal{P}_L$. In other words, the convex set $\mathcal{P}_\pi$ is such that $\mathcal{P}_L \subseteq \mathcal{P}_\pi$.

To extend this transformation to a pair of possibility distributions equivalent to a cloud, we consider that the possibility distribution π given by Equation (22) is the upper distribution of a cloud $[\delta, \pi]$. To build the lower distribution δ of a cloud containing $\mathcal{P}_L$, we need to build a second possibility distribution π_δ such that $\mathcal{P}_L \subseteq \mathcal{P}_{\pi_\delta}$ and such that the pair $[1 - \pi_\delta, \pi]$ defines a cloud (with $1 - \pi_\delta = \delta$). To achieve this, we propose to use the same method as Masson and Denoeux [14], simply reversing the inequality under the summation sign in Equation (21). The procedure to build π_δ then becomes

(1) For each order $<_l \in \mathcal{C}_L$ and each element x, solve

$$\pi_\delta^l(x) = \max_{\{p(y) | y \in X\}} \sum_{\sigma_l(x) \leq \sigma_l(y)} p(y) \tag{23}$$

$$= 1 - \min_{\{p(y) | y \in X\}} \sum_{\sigma_l(y) < \sigma_l(x)} p(y) = 1 - \delta^l(x) \tag{24}$$

with the same constraints as in the first transformation.

(2) The most informative distribution dominating all distributions $\pi_\delta^l(x)$ is:

$$\pi_\delta(x) = 1 - \delta(x) = \max_{<_l \in \mathcal{C}} \pi_\delta^l(x) \tag{25}$$

Example 5.1 illustrates this procedure.

Example 5.1 *Let us take the same four probability intervals as in the example given by Masson and Denoeux [14], on the space $X = \{w, x, y, z\}$, and summarized in the following table*

	w	x	y	z
l	*0.10*	*0.34*	*0.25*	*0*
u	*0.28*	*0.56*	*0.46*	*0.08*

The partial order is given by $L_y < L_x; L_z < \{L_x, L_w, L_y\}$. There are three possible linear extensions $<_l \in \mathcal{C}_L$

$$\begin{aligned} <_l^1 &= (L_z, L_w, L_y, L_x) \\ <_l^2 &= (L_z, L_w, L_x, L_y) \\ <_l^3 &= (L_z, L_y, L_w, L_x) \end{aligned}$$

corresponding to the following π_δ's:

$<_l^i$	$\pi_\delta(w)$	$\pi_\delta(x)$	$\pi_\delta(y)$	$\pi_\delta(z)$
1	*1*	*0.16*	*0.63*	*1*
2	*1*	*0.9*	*0.46*	*1*
3	*0.75*	*0.5*	*1*	*1*
max	*1*	*0.9*	*1*	*1*

and, finally, the obtained cloud is:

	w	x	y	z
π	*0.64*	*1*	*1*	*0.08*
δ	*0*	*0.1*	*0*	*0*

where π is the possibility distribution obtained by Masson and Denoeux [14] for the same example by applying the first transformation. Note that the cloud is only a little more informative than the upper distribution taken alone (indeed, the only added constraint is that $p(x) \leq 0.9$).

We can verify the following property:

Proposition 5.2 *Given a set L of probability intervals, the cloud $[1 - \pi_\delta, \pi]$ built from the two possibility distributions π_δ, π obtained via the above procedures is such that the induced credal set $\mathcal{P}_{[1-\pi_\delta,\pi]}$ contains $\mathcal{P}_L$. In the degenerate case of a precise probability distribution, this cloud contains this distribution only.*

The proof can be found in the appendix. So, this method allows to get a cloud encompassing the information contained in any set of probability intervals. It directly extends known methods used in possibility theory.

5.2 *Using generalized p-boxes*

We have previously shown that generalized p-boxes and comonotonic clouds were equivalent representations. Thus, we can directly use transformations from probability intervals to generalized p-boxes and get an approximation as a comonotonic cloud .

Consider the following example:

Example 5.3 *Let us consider the same probability intervals as in example 5.1 and the following order relationship R on the elements: $z <_R w <_R y <_R x$. We can then build the generalized p-box associated to this order (using Equations (7)), and then take the comonotonic cloud associated to this p-box (by using transformations in Proposition 2.5)*

	w	x	y	z
$\overline{F} = \pi$	0.36	1	0.66	0.08
$\underline{F}$	0.1	1	0.44	0
δ	0	0.44	0.1	0

And, by Proposition 3.8 and related results in the companion paper, we know that the credal set $\mathcal{P}_{[\delta,\pi]}$ induced by this cloud is such that $\mathcal{P}_L \subseteq \mathcal{P}_{[\delta,\pi]}$ and that we can recover the information modeled by a set L of probability intervals by means of at least $|X|/2$ clouds built by this method.

Both methods transform a set L of probability intervals into a cloud $[\delta, \pi]$ such that $\mathcal{P}_L \subset \mathcal{P}_{[\delta,\pi]}$, thus guaranteeing that no extra information is added in the transformation.

In general, since finite clouds can model precise probability distributions without any loss of information, the cloud resulting from any transformation of a discrete probability distribution should contain this probability distribution only. Both methods proposed here satisfy these requirements in the finite case.

However, if we compare the clouds resulting from Examples 5.1 and 5.3, it is clear that the cloud resulting from the second method (Example 5.3) is more precise than the one resulting from the first one (Example 5.1). Moreover, using the first method, it is in general impossible to recover the information provided by the original set L of probability intervals. This shows that the first method can be very conservative. This is mainly due to the fact that even if it considers every possible ranking of elements, it is only based on the partial order induced by probability intervals.

If a ranking of elements is naturally present in a considered problem, then the first method seems to be the best solution. If no natural order is present, it is hard to justify the fact of considering one particular order rather than another one, and the first method should be applied. In this case, one has to be aware that a lot of information can be lost in the process. One may also use the ranking inducing one of the most precise comonotonic clouds, but this question remains open.

6 Continuous clouds on the real line

In many applications, the available information concerning some parameters is defined on the (continuous) real line. It is thus important to know if results obtained so far can be extended to this particular setting. We consider clouds defined on a bounded interval $[\underline{r}, \overline{r}]$.

First, let us recall that, as in the discrete case, a cloud $[\delta, \pi]$ defined on the real line is a pair of distributions such that, for any element $r \in \mathbb{R}$, $[\delta(r), \pi(r)]$ is an interval and there is an element r for which $\delta(r) = 0$, and another r' for which $\pi(r') = 1$. *Thin clouds* ($\pi = \delta$) and *Fuzzy clouds* ($\delta = 0$) have the same definition as in the case of finite space. The credal set $\mathcal{P}_{[\delta,\pi]}$ induced by a cloud on the real line is such that:

$$\mathcal{P}_{[\delta,\pi]} = \{P|P(\{r \in \mathbb{R}, \delta(r) \geq \alpha\}) \leq 1 - \alpha \leq P(\{r \in \mathbb{R}, \pi(r) > \alpha\})\} \quad (26)$$

where P is a σ-measurable probability distribution [1]

As Proposition 2.3 has been originally proven for the case of continuous possibility distributions π, results whose proof is based on this proposition directly extend to continuous models on the real line. In particular, the following statements still hold:

- if $[\delta, \pi]$ is a cloud, $1 - \delta, \pi$ are possibility distributions, and $\mathcal{P}_{[\delta,\pi]} = \mathcal{P}_\delta \cap \mathcal{P}_\pi$,
- if $[\underline{F}, \overline{F}]$ is a generalized p-box defined on the reals, then $\mathcal{P}_{[\underline{F},\overline{F}]} = \mathcal{P}_{\pi_{\underline{F}}} \cap \mathcal{P}_{\pi_{\overline{F}}}$ with, for all $r \in \mathbb{R}$:

$$\pi_{\overline{F}}(r) = \overline{F}(r)$$

[1] To avoid mathematical subtleties that would require a study of their own, we restrict ourselves to σ-measurable probability distributions rather than considering finitely additive probabilities

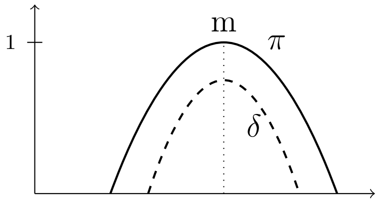

Fig. 6.A: Comonotonic cloud

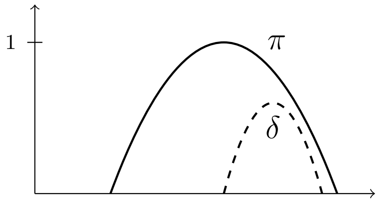

Fig. 6.B: Non-comonotonic cloud

Figure 6. Illustration of comonotonic and non-comonotonic clouds on the real line.

and

$$\pi_{\underline{F}}(r) = 1 - \sup\{\underline{F}(r')|r' \in \mathbb{R}; \underline{F}(r') < \underline{F}(r)\}$$

with $\pi_{\underline{F}}(\underline{r}) = 0$.

- a generalized p-box $[\underline{F}, \overline{F}]$ represents the same information as the comonotonic cloud $[1 - \pi_{\underline{F}}, \pi_{\overline{F}}]$ (and, similarly, any comonotonic cloud can be mapped into a generalized p-box).

Note that $\pi = \pi_{\overline{F}}$ and $\delta = 1 - \pi_{\underline{F}}$ satisfy the nestedness property of $\{r \in \mathbb{R}, \delta(r) \geq \alpha\}$ and $\{r \in \mathbb{R}, \pi(r) > \alpha\}, \forall \alpha \in (0, 1)$. To preserve it, we need more than comonotonicity here. One way of preserving this property is to require that π and δ be *strongly* comonotonic. Namely for any two numbers r, r', $\pi(r) = \pi(r')$ if and only if $\delta(r) = \delta(r')$. In other words, there exists a mapping f from $[0, 1)$ to $[0, 1]$ that is non-decreasing and for which $\{f(r) \leq r|r \in [0, 1]\}$ and such that for any $r \in \mathbb{R}$, $\delta(r) = f(\pi(r))$. The non-decreasingness condition ensures the strong comonotonicity of δ, π, while the condition $\{f(r) \leq r|r \in [0, 1]\}$ ensures that $\delta \leq \pi$. On the values r for which $\pi(r) = 1$, δ can take any value in $[f(1), 1]$. Figures 6.A and 6.B respectively illustrate the notion of strongly comonotonic and non-comonotonic clouds on the reals. Figure 6.A illustrates a comonotonic cloud (and, consequently, a generalized p-box) for which elements are ordered according to their distance to the mode m (i.e., for this particular cloud, two values x, y in $\mathbb{R}$ are such that $x <_{[\delta,\pi]} y$ if and only if $|m - x| > |m - y|$). MONTRER UN COMON CLOUD NON STRONGLY COMON.

We can now extend the propositions linking clouds and generalized p-boxes with random sets. In particular, the following result extends Proposition 4.3 to the continuous case:

Proposition 6.1 *Let the distributions* $[\delta, \pi]$ *describe a continuous cloud on the reals and* $\mathcal{P}_{[\delta,\pi]}$ *be the induced credal set. Then, the random set defined by the Lebesgue measure on the unit interval* $\alpha \in [0, 1]$ *and the multimapping* $\alpha \longrightarrow E_\alpha$ *such that*

$$E_\alpha = \{x \in X|(\pi(x) \geq \alpha) \wedge (\delta(x) < \alpha)\}$$

describe a credal set $\mathcal{P}_{Bel}$ which is an inner approximation of $\mathcal{P}_{\pi,\delta}$ ($\mathcal{P}_{Bel} \subset \mathcal{P}_{\pi,\delta}$).

The proof can be found in the appendix. This proposition has two corollaries:

Corollary 6.2 *Let $[\delta, \pi]$ be a strongly comonotonic cloud with continuous distributions on the real line. Then the credal set is $\mathcal{P}_{[\delta,\pi]}$ also the credal set of a continuous random set with uniform mass density, whose focal sets are of the form, for $\alpha \in [0,1]$:*

$$E_\alpha = \{x \in X | (\pi(x) \geq \alpha) \wedge (\delta(x) < \alpha)\}$$

Another interesting particular case is the one of uniformly continuous p-boxes.

Corollary 6.3 *The credal set $\mathcal{P}_{[\underline{F},\overline{F}]}$ described by two continuous cumulative distributions $\underline{F}, \overline{F}$ on the reals is equivalent to the credal set described by the continuous random set with uniform mass density, whose focal sets are sets of the form $[x(\alpha), y(\alpha)]$ where $x(\alpha) = \overline{F}^{-1}(\alpha)$ and $y(\alpha) = \underline{F}^{-1}(\alpha)$.*

This is because strictly increasing continuous p-boxes are special cases of strongly comonotonic clouds (or, equivalently, of generalized p-boxes). The strict increasingness property can be relaxed to intervals where the cumulative functions are constant (the cloud will not be strongly monotonic, but the nestedness property of all cuts is then preserved).

From a practical and computational perspective, these results are appealing. For example, they can facilitate the computation of lower and upper expectations over continuous generalized p-boxes. Another interesting point is that all the framework developed by Smets [18] concerning belief functions on reals can be applied to comonotonic clouds (generalized p-boxes). Let us also note that the results given in this section extend and give alternative proofs to some results given by Alvarez [1] concerning continuous p-boxes.

6.1 Thin continuous clouds

Thin clouds are defined by Neumaier [15] as clouds $[\delta, \pi]$ for which $\pi = \delta$. Hence, constraints defining the credal set, given by Equation (11), reduce to $P(\pi(x) \geq \alpha) = P(\pi(x) > \alpha) = 1 - \alpha$ for all $\alpha \in (0,1)$. On a finite space X, these constraints are generally conflicting as shown earlier, because for some α, $P(\{x \in X | \pi(x) \geq \alpha\}) > P(\{x \in X | \pi(x) > \alpha\})$ will hold.

When the *thin* cloud is a uniformly continuous distribution defined on the real line, this is no longer a difficulty, and the following proposition holds:

Proposition 6.4 *If π is a continuous possibility distribution, then its credal set $\mathcal{P}_\pi \cap \mathcal{P}_{1-\pi}$ is not empty.*

Proof of Proposition 6.4 Let $F(x) = \Pi((-\infty, x])$, with $x \in \mathbb{R}$. F is the distribution function of a probability measure P_π such that $\forall \alpha \in [0,1], P_\pi(\pi(x) > \alpha) = 1 - \alpha$, where the sets $\{x \in \mathbb{R} | \pi(x) > \alpha\}$ form a continuous nested sequence (see [5] p. 285). Such a probability lies in $\mathcal{P}_\pi$. Moreover,

$$P_\pi(\{x \in \mathbb{R}, \pi(x) > \alpha\}) = P_\pi(\{x \in \mathbb{R} | \pi(x) > \alpha\})$$

due to uniform continuity of π. We also have
$P_\pi(\{x \in \mathbb{R} | \pi(x) > \alpha\}) = 1 - \Pi(\{x \in \mathbb{R} | \pi(x) \geq \alpha\}^c) = 1 - \Delta(\{x \in \mathbb{R} | \pi(x) \geq \alpha\})$
again due to uniform continuity. Since
$1 - \Delta(\{x \in \mathbb{R} | \pi(x) \geq \alpha\}) = \sup_{x|\pi(x)\geq\alpha} 1 - \pi(x)$, this means $P_\pi \in \mathcal{P}(1-\pi)$. □

A continuous *thin* cloud is obviously a particular case of strongly comonotonic cloud. It induces a complete pre-ordering on the reals. If there are no ties, meaning that this pre-order is linear, it means that for any $\alpha \in [0,1]$, there is only one value $r \in \mathbb{R}$ for which $\pi(r) = \alpha$, and that $\mathcal{P}_\pi \cap \mathcal{P}_{1-\pi}$ contains only one probability measure. In particular, if the order is the natural order of real numbers, this *thin* cloud reduces to a usual continuous cumulative distribution.

When the pre-order has ties, it means that for some $\alpha \in [0,1]$, there are several values in $x \in \mathbb{R}$ such that $\pi(x) = \alpha$. Using Corollary 6.2, we can model the credal set $P_\pi \in \mathcal{P}(1-\pi)$ by the random set with uniform mass density, whose focal sets are of the form

$$E_\alpha = \{r \in \mathbb{R} | \pi(r) = \alpha\}$$

In this case, we can check that $Bel(\{r \in \mathbb{R} | \pi(r) \geq \alpha\}) = 1 - \alpha$, in accordance with Equation (11).

Finally, consider the specific case of a continuous thin cloud modeled by an unimodal distribution π (formally, a fuzzy interval). In this case, each focal set associated to a value α is a doubleton $\{x(\alpha), y(\alpha)\}$ where $\{x | \pi(x) \geq \alpha\} = [x(\alpha), y(\alpha)]$. Noticeable probability distributions that are inside the credal set modeled by such a *thin* cloud are the cumulative distributions F_+ and F_- such that for all α in $[0,1]$ $F_+^{-1}(\alpha) = x(\alpha)$ and $1 - F_-^{-1}(\alpha) = y(\alpha)$ (they respectively correspond to the case where the mass density of the random set is concentrated on values $x(\alpha)$ and $y(\alpha)$). All probability measures with cumulative finctions of the form $\lambda \cdot F_+ + (1-\lambda) \cdot F_-$ also belong to the credal set (for $\lambda = \frac{1}{2}$, this distribution correspond to the case where mass density is evenly divided between elements $x(\alpha)$ and $y(\alpha)$). Other distributions inside this set are considered by Dubois *et al.* [5].

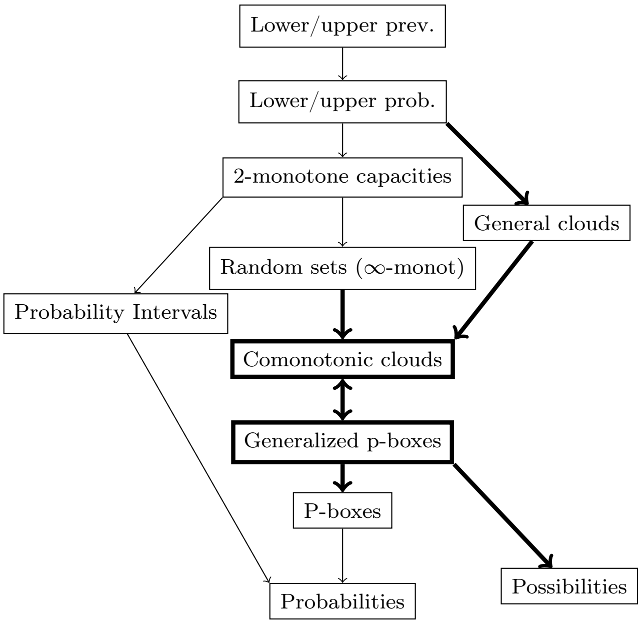

Figure 7. Representation relationships: completed summary with clouds. $A \longrightarrow B$: B is a special case of A

Representation	Capacity of order	Fully determined by .. values
Random set	∞	$2^{\|X\|} - 2$
Possibility	∞	$\|X\| - 1$
Probability Interval	2	$2 \cdot \|X\|$
Generalized p-box	∞	$2 \cdot (\|X\| - 1)$
Comonotonic cloud	∞	$2 \cdot (\|X\| - 1)$
Non-comonotonic cloud	1	$2 \cdot (\|X\| - 1)$

Table 1
Characteristics of representations

7 Conclusion

In this paper clouds are compared to other practical representations of uncertainty, including generalized p-boxes introduced in the companion paper. Properties of the cloud formalism are explained in the light of other representations. The fact that lower probabilities induced by non-comonotonic clouds are not 2-monotone capacities tends to indicate that, from a computational standpoint, they look less attractive than the other formalisms. Nevertheless, as far as we know, clouds are the only simple numerical model generating capacities that are not 2-monotone.

We are now ready to complete Figure 1 with clouds. This completed picture is given by Figure 7. New relationships and representations coming from this paper and its companion are in bold lines. Table 1 recalls the complexity of each representation. Such practical representations are easier to handle than more general models: they often require less evaluations to be fully specified and they allow many mathematical simplifications, which increase computational efficiency (except, perhaps, for non-comonotonic clouds).

The next step is to explore computational aspects of each formalism as done by De Campos *et al.* [2] for probability intervals. In particular, we need to answer the following questions: how do we define operations of fusion, marginalization, conditioning or propagation for each of these models? Are the representations preserved after such operations, and under which assumptions? What is the computational complexity of these operations? Can the models presented here be easily elicited or integrated? If many results already exist for random sets, possibility distributions and probability intervals, few have been derived for generalized p-boxes or clouds, due to their novelty. Note that all the results presented in this paper and its companion can be helpful to perform such a study. In particular, we have shown that generalized p-boxes and comonotonic clouds are equivalent formalisms that can be represented by random sets. It implies that sampling methods and interval analysis can be easily applied to them, just like for p-boxes and possibility distributions, without fear of a loss of information.

Nevertheless it is not clear that random set calculation methods would preserve the simplified representations described here. So their merits are at the elicitation level and also to provide simple accounts of results in the forms of credal sets, so as to explain them to a user. In connection with the latter issue, it would be useful to evaluate the cognitive relevance of these representations, particularly from a psychological standpoint (even if some results already exist [**?**, 13] for possibility distributions, for instance).

Another issue is to extend presented results to more general spaces, to general lower/upper previsions or to cases not considered here (e.g. continuous clouds with some discontinuities), possibly by using existing results [4, 18].

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Appendix

Proof of Lemma 4.1 To prove Lemma 4.1, we first recall a useful result by Chateauneuf [3] concerning the intersection of credal sets induced by random sets. This result is then applied to the possibility distributions defined in Lemma 4.1 to prove that the associated lower probability is not 2-monotone. The main idea is to exhibit two events such that 2-monotonicity is not satisfied for them. Consider the set $\mathcal{M}$ of matrices M of the form

	G_1	G_2
F_1	m_{11}	m_{12}
F_2	m_{21}	m_{22}

where

$$m_{11} + m_{12} = m_{22} + m_{12} = \lambda$$
$$m_{21} + m_{22} = m_{21} + m_{11} = 1 - \lambda$$
$$\sum m_{ij} = 1$$

Each such matrix is a normalized (i.e. such that $m(\emptyset) = 0$) joint mass distribution for the random sets induced from possibility distributions π_F, π_G, viewed as marginal belief functions. Following Chateauneuf [3], the lower probability $\underline{P}$ induced by the credal set $\mathcal{P} = \mathcal{P}_{\pi_F} \cap \mathcal{P}_{\pi_G}$ has, for any event $E \subseteq X$, value

$$\underline{P}(E) = \min_{M \in \mathcal{M}} \sum_{(F_i \cap G_j) \subset E} m_{ij} \quad (27)$$

Now consider the four events $F_1, G_1, F_1 \cap G_1, F_1 \cup G_1$. Studying the relations between sets and the constraints on the values m_{ij}, we can see that

$$\begin{aligned}
\underline{P}(F_1) &= \lambda \\
\underline{P}(G_1) &= 1 - \lambda \\
\underline{P}(F_1 \cap G_1) &= 0.
\end{aligned}$$

For $F_1 \cap G_1$, just consider the matrix $m_{12} = \lambda, m_{21} = 1 - \lambda$. To show that the lower probability is not even $2-$monotone, it is enough to show that $\underline{P}(F_1 \cup G_1) < 1$. To achieve this, consider the following mass distribution

$$\begin{aligned}
m_{11} &= \min(\lambda, 1 - \lambda) \\
m_{12} &= \lambda - m_{11} \\
m_{21} &= 1 - \lambda - m_{11} \\
m_{22} &= \min(\lambda, 1 - \lambda)
\end{aligned}$$

it can be checked that the matrix corresponding to this distribution is in the set $\mathcal{M}$, and yields

$$\begin{aligned}
P(F_1 \cup G_1) &= m_{12} + m_{11} + m_{21} \\
&= m_{11} + \lambda - m_{11} + 1 - \lambda - m_{11} \\
&= 1 - m_{11} = 1 - \min(\lambda, 1 - \lambda) \\
&= \max(1 - \lambda, \lambda) < 1
\end{aligned}$$

since $(F_2 \cap G_2) \not\subseteq (F_1 \cup G_1)$ (due to the fact that $G_1 \not\subseteq F_2$). Then the inequality

$$\underline{P}(F_1 \cup G_1) + \underline{P}(F_1 \cap G_1) < \underline{P}(F_1) + \underline{P}(G_1)$$

holds, which ends the proof. $\square$

Proof of Proposition 4.3 To prove Proposition 4.3, we again use the result by Chateauneuf [3] as in the proof of Lemma 4.1, and given . These results are clearly applicable to clouds, since possibility distributions are equivalent to nested random sets. Consider a finite cloud described by the general Equation

(11) and the following matrix Q of weights q_{ij}

	$C^c_{\gamma_1}$	$\cdots$	$C^c_{\gamma_j}$	$\cdot$	$C^c_{\gamma_{i+1}}$	$\cdots$	$C^c_{\gamma_m}$
$B_{\overline{\gamma}_0}$	q_{11}	$\cdots$	q_{1j}	$\cdot$	$q_{1(i+1)}$	$\cdots$	q_{1m}
$\vdots$	$\vdots$	$\ddots$			$\vdots$		$\vdots$
$B_{\overline{\gamma}_{j-1}}$	q_{j1}	$\cdots$	$\mathbf{q_{jj}}$	$\cdot$	$\mathbf{q_{j(i+1)}}$	$\cdots$	q_{jm}
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
$B_{\overline{\gamma}_i}$	$q_{(i+1)1}$	$\cdots$	$\mathbf{q_{(i+1)j}}$	$\cdot$	$\mathbf{q_{(i+1)(i+1)}}$	$\cdots$	$q_{(i+1)m}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$
$B_{\overline{\gamma}_{m-1}}$	q_{m1}	$\cdots$	q_{mj}	$\cdot$	$q_{m(i+1)}$	$\cdots$	$q_{mm)}$

Respectively call Bel_1 and Bel_2 the belief functions equivalent to the possibility distributions respectively generated by the collections of sets $\{B_{\overline{\gamma}_i} | i = 0, \dots, m-1\}$ and $\{C^c_{\gamma_i} | i = 1, \dots, m\}$. From Equation (4), $m_1(B_{\overline{\gamma}_i}) = \gamma_{i+1} - \gamma_i$ for $i = 0, \dots, m-1$, and $m_2(C^c_{\gamma_j}) = \gamma_j - \gamma_{j-1}$ for $j = 1, \dots, m$. As in the proof of Lemma 4.1, we consider every possible joint random set such that $m(\emptyset) = 0$ built from the two marginal belief functions Bel_1, Bel_2.

Following Chateauneuf, let $\mathcal{Q}$ be the set of matrices Q s.t.

$$q_{i\cdot} = \sum_{j=1}^{m} q_{ij} = \gamma_i - \gamma_{i-1}$$
$$q_{\cdot j} = \sum_{i=1}^{m} q_{ij} = \gamma_j - \gamma_{j-1}$$
$$\text{If } i, j \text{ s.t. } B_{\overline{\gamma}_i} \cap C^c_{\gamma_j} = \emptyset \text{ then } q_{ij} = 0$$

and the lower probability of the credal set $\mathcal{P}_{[\delta,\pi]}$ on event E is such that

$$\underline{P}(E) = \min_{Q \in \mathcal{Q}} \sum_{(B_{\overline{\gamma}_i} \cap C^c_{\gamma_j}) \subset E} q_{ij}. \tag{28}$$

Now, by hypothesis, there are at least two overlapping sets $B_{\overline{\gamma}_i}, C_{\gamma_j}$ $i > j$ that are not included in each other (i.e. $B_{\overline{\gamma}_i} \cap C_{\gamma_j} \notin \{B_{\overline{\gamma}_i}, C_{\gamma_j}, \emptyset\}$). Let us consider the four events $B_{\overline{\gamma}_i}, C^c_{\gamma_j}, B_{\overline{\gamma}_i} \cap C^c_{\gamma_j}, B_{\overline{\gamma}_i} \cup C^c_{\gamma_j}$. Considering Equation (28), the matrix Q and the relations between sets, inclusions $B_{\overline{\gamma}_m} \subset \ldots \subset B_{\overline{\gamma}_0}$,

$C^c_{\gamma_0} \subset \ldots \subset C^c_{\gamma_m}$ and, for $i = 0, \ldots, m$, $C_{\gamma_i} \subset B_{\overline{\gamma}_i}$ imply:

$$\begin{aligned}
&\underline{P}(B_{\overline{\gamma}_i}) = 1 - \gamma_i \\
&\underline{P}(C^c_{\gamma_j}) = \gamma_j \\
&\underline{P}(B_{\overline{\gamma}_i} \cap C^c_{\gamma_j}) = 0
\end{aligned}$$

for the last result, just consider the mass distribution $q_{kk} = \gamma_{k-1} - \gamma_k$ for $k = 1, \ldots, m$.

Next, consider event $B_{\overline{\gamma}_i} \cup C^c_{\gamma_j}$ (which is different from X by hypothesis). Suppose all masses are such that $q_{kk} = \gamma_{k-1} - \gamma_k$, except for masses (in boldface in the matrix) $q_{jj}, q_{(i+1)(i+1)}$. Then, $C^c_{\gamma_j} \subset C^c_{\gamma_{i+1}}$, $B_{\overline{\gamma}_i} \subset B_{\overline{\gamma}_{j-1}}$, $C^c_{\gamma_j} \not\subseteq B_{\overline{\gamma}_{j-1}}$ by definition of a cloud and $B_{\overline{\gamma}_i} \cap C^c_{\gamma_j} \neq \emptyset$ by hypothesis. Finally, using Lemma 4.1, consider the mass distribution

$$\begin{aligned}
&q_{(i+1)j} = \min(\gamma_{i+1} - \gamma_i, \gamma_j - \gamma_{j-1}) \\
&q_{(i+1)(i+1)} = \gamma_{i+1} - \gamma_i - q_{(i+1)j} \\
&q_{jj} = \gamma_j - \gamma_{j-1} - q_{(i+1)j} \\
&q_{j(i+1)} = \min(\gamma_{i+1} - \gamma_i, \gamma_j - \gamma_{j-1}.)
\end{aligned}$$

It always gives a matrix in the set $\mathcal{Q}$. By considering every subset of $B_{\overline{\gamma}_i} \cup C^c_{\gamma_j}$, we thus get the following inequality

$$\underline{P}(B_{\overline{\gamma}_i} \cup C^c_{\gamma_j}) \leq \gamma_{j-1} + 1 - \gamma_{i+1} + \max(\gamma_{i+1} - \gamma_i, \gamma_j - \gamma_{j-1}).$$

And, similarly to what was found in Lemma 4.1, we get

$$\underline{P}(B_{\overline{\gamma}_i} \cup C^c_{\gamma_j}) + \underline{P}(B_{\overline{\gamma}_i} \cap C^c_{\gamma_j}) < \underline{P}(B_{\overline{\gamma}_i}) + \underline{P}(C^c_{\gamma_j}),$$

which shows that the lower probability is not 2−monotone. □

Proof of Proposition 4.5 First, we know that the random set given in Proposition 4.5 is equivalent to

$$\begin{cases} E_j = B_{\overline{\gamma}_{j-1}} \setminus C_{\gamma_j} = B_{\gamma_j} \setminus C_{\gamma_j} \\ m(E_j) = \gamma_j - \gamma_{j-1} \end{cases}$$

Now, if we consider the matrix given in the proof of Proposition 4.3, this random set comes down, for $i = 1, \ldots, M$ to assign masses $q_{ii} = \gamma_i - \gamma_{i-1}$. Since this is a legal assignment, we are sure that for all events $E \subseteq X$, the belief function of this random set is such that $Bel(E) \geq \underline{P}(E)$, where $\underline{P}$ is the lower probability induced by the cloud. The proof of Proposition 4.3 shows that this inclusion is strict for clouds satisfying the latter proposition (since the lower probability induced by such clouds is not 2-monotone). □

Proof of Proposition 5.2 The two possibility distributions π, π_δ are such that $\mathcal{P}_L \subset \mathcal{P}_\pi$ and $\mathcal{P}_L \subset \mathcal{P}_{\pi_\delta}$ by construction, so $\mathcal{P}_L \subset (\mathcal{P}_\pi \cap \mathcal{P}_{\pi_\delta})$. The final result is thus more precise than a single possibility distribution dominating $\mathcal{P}_L$. When L reduces to a precise masses $\{p\}$, the transformations give the following possibility distributions (elements of X are ordered in accordance with the order of probability masses):

$$\pi(x_i) = \sum_{j \leq i} p_j$$

and

$$\pi_\delta(x_i) = \sum_{j \geq i} p_j = 1 - \sum_{j < i} p_j = 1 - \delta(x_i) = 1 - \pi(x_{i-1}).$$

Hence, the only probability distribution in the cloud $[\delta, \pi]$ is given by $p_i = \pi(x_i) - \pi(x_{i-1})$. □

Proof of Proposition 6.1 We build outer and inner approximations of the continuous random set that converge to the belief measure of the continuous random set, while the corresponding clouds of which they are inner approximations themselves converge to the uniformly continuous cloud.

First, consider a finite collection of equidistant levels α_i s.t. $0 = \alpha_0 < \alpha_1 < \ldots < \alpha_n = 1$ ($\alpha_{i-1} - \alpha_i = 1/n \forall i = 1, \ldots, n$). Then, consider the following discrete non-comonotonic clouds $[\underline{\delta}_n, \underline{\pi}_n]$, $[\overline{\delta}_n, \overline{\pi}_n]$ that are respectively outer and inner approximations of the cloud $[\delta, \pi]$: for every value r in $\mathbb{R}$, do the following transformation

$$\pi(r) = \alpha \text{ with } \alpha \in [\alpha_{i-1}, \alpha_i] \quad \underline{\pi}_n(r) = \alpha_i \quad \overline{\pi}_n(r) = \alpha_{i-1}$$
$$\delta(r) = \alpha' \text{ with } \alpha' \in [\alpha_{j-1}, \alpha_j] \; \underline{\delta}_n(r) = \alpha_{j-1} \quad \overline{\delta}_n(r) = \alpha_j$$

This construction is illustrated in Figure 8 for the particular case when both π and δ are unimodal. In this particular case, for $i = 1, \ldots, n$

$$\{x \in \mathbb{R} | \underline{\pi}(x) \geq \alpha\} = [x(\alpha_{i-1}), y(\alpha_{i-1})] \alpha \in [\alpha_{i-1}, \alpha_i]$$
$$\{x \in \mathbb{R} | \underline{\delta}(x) > \alpha\} = [u(\alpha_i), v(\alpha_i)] \alpha \in [\alpha_{i-1}, \alpha_i]$$

$$\{x \in \mathbb{R} | \overline{\pi}(x) \geq \alpha\} = [x(\alpha_i), y(\alpha_i)] \alpha \in [\alpha_{i-1}, \alpha_i]$$
$$\{x \in \mathbb{R} | \overline{\delta}(x) > \alpha\} = [u(\alpha_{i-1}), v(\alpha_{i-1})] \alpha \in [\alpha_{i-1}, \alpha_i]$$

Given the above transformations, $\mathcal{P}(\underline{\pi}_n) \subset \mathcal{P}(\pi) \subset \mathcal{P}(\overline{\pi}_n)$, and $\lim_{n \to \infty} \mathcal{P}(\underline{\pi}_n) = \mathcal{P}(\pi)$ and also $\lim_{n \to \infty} \mathcal{P}(\overline{\pi}_n) = \mathcal{P}(\pi)$. Similarly, $\mathcal{P}(1 - \underline{\delta}_n) \subset \mathcal{P}(1 - \delta) \subset \mathcal{P}(1 - \overline{\delta}_n)$, $\lim_{n \to \infty} \mathcal{P}(1 - \overline{\delta}_n) = \mathcal{P}(1 - \delta)$ and $\lim_{n \to \infty} \mathcal{P}(1 - \underline{\delta}_n) = \mathcal{P}(1 - \delta)$. Since the set of probabilities induced by the cloud $[\delta, \pi]$ is $\mathcal{P}(\pi) \cap \mathcal{P}(1 - \delta)$, it is clear that the two credal sets $\mathcal{P}(\underline{\pi}_n) \cap \mathcal{P}(1 - \underline{\delta}_n)$ and $\mathcal{P}(\overline{\pi}_n) \cap \mathcal{P}(1 - \overline{\delta}_n)$, are

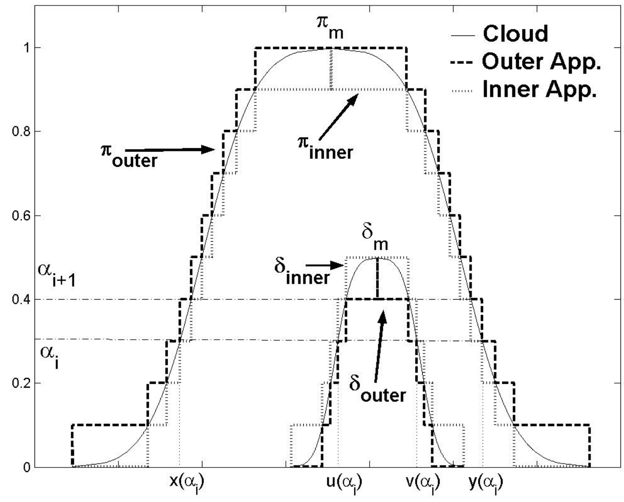

Figure 8. Inner and outer approximations of a non-comonotonic clouds

respectively inner and outer approximations of $\mathcal{P}(\pi) \cap \mathcal{P}(1-\delta)$. Moreover:

$$\lim_{n \to \infty} \mathcal{P}(\underline{\pi}_n) \cap \mathcal{P}(1-\underline{\delta}_n) = \mathcal{P}(\pi) \cap \mathcal{P}(1-\delta)$$

and

$$\lim_{n \to \infty} \mathcal{P}(\overline{\pi}_n) \cap \mathcal{P}(1-\overline{\delta}_n) = \mathcal{P}(\pi) \cap \mathcal{P}(1-\delta).$$

The random sets that are inner approximations (by proposition 4.5) of the finite clouds $[\underline{\delta}_n, \underline{\pi}_n]$ and $[\underline{\delta}_n, \underline{\pi}_n]$ converge to the continuous random set defined by the Lebesgue measure on the unit interval $\alpha \in [0, 1]$ and the multimapping $\alpha \longrightarrow E_\alpha$ such that

$$E_\alpha = \{x \in X | (\pi(x) \geq \alpha) \wedge (\delta(x) < \alpha)\}.$$

In the limit, it follows that this continuous random set is an inner approximation of the continuous cloud. □