Meta-epistemic logic: A minimal logic for reasoning about revealed beliefs

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October 8, 2010

Abstract

Reasoning about knowledge described in classical propositional logic is usually handled either in the meta-language as in belief revision, considering the dynamics of belief bases, or at the object level by means of modal logic. In the latter case, modalities express knowledge, belief, or absence thereof, about the truth of formulae. But the semantics is described in terms of accessibility relations, whose expressive power seems to be too powerful to account for mere epistemic states of an agent. This paper proposes a simpler logic whose atoms express beliefs about formulae expressed in another basic propositional language, and that allows for conjunctions, disjunctions and negations of beliefs. The idea is to model an agent reasoning about some beliefs of another agent as revealed by the latter. This logic, called Meta-Epistemic Logic (MEL), borrows its syntax and axioms from the modal logic KD. It can be be viewed as a fragment of KD, but it is an encapsulation of propositional logic rather than an extension thereof. Its semantics is given in terms of epistemic states understood as subsets of propositional interpretations. We prove soundness and completeness of this logic, and that any family of non-empty subsets of propositional interpretations can be expressed as a single formula in MEL. Inference rules and normal forms in MEL are discussed. We show that this logic is very similar to the consensus logic of Pauly. It is also simpler than many previous formalisms for reasoning about knowledge, and it avoids paradoxes of truth-functional accounts of incomplete information handling like partial logic. Our approach is in fact much closer to the logical rendering of uncertainty theories like possibilistic logic. MEL has indeed potential to be extended to deal with graded beliefs. For instance, we show that MEL can express a symbolic counterpart of the Möbius transform in the theory of belief functions.

1 Motivation

Reasoning about knowledge and beliefs requires more than the language of classical propositional logic. In classical propositional logic, it is only possible to express that certain propositions are believed. A set of logical formulae is then called a belief base [32], or a belief set (when it is deductively closed). This representation is used in belief revision for representing the dynamics of knowledge upon receiving new information [20]. It can be refined by introducing grades of beliefs as in possibilistic logic [12], or using kappa-rankings [39], let alone probabilities. However, stating that some propositions are acknowledged as being unknown to an agent requires the use of a more expressive language, since the language of classical propositional logic cannot really express the difference between statements like “not knowing α” and “knowing not α” (in fact it can only express the latter as $\neg\alpha$). In modal logic, the first case writes $\neg\Box\alpha$, and the second one is $\Box\neg\alpha$. This kind of syntax is used in epistemic logic [26, 25], but the usual semantics in terms of accessibility relations does not fit easily with uncertainty formalisms like probability or possibility theories, that rely on weights assigned to possible worlds. Kripke semantics are actually tailored for the multiple-agent setting.

Formal models of interaction between agents are the subject of current significant research effort. One important issue is to represent how an agent can reason about what is known about another agent's knowledge and beliefs. In this paper, we consider a much simpler problem: Consider two agents $\mathcal{E}$ (for emitter) and $\mathcal{R}$ (for receiver). Agent $\mathcal{E}$ supplies pieces of information to agent $\mathcal{R}$, explaining what (s)he believes and what (s)he thinks is only plausible or conceivable. For instance, $\mathcal{E}$ is a witness and $\mathcal{R}$ collects his or her testimony. How can agent $\mathcal{R}$ reason about what $\mathcal{E}$ accepts to tell the former, that is, $\mathcal{E}$'s revealed beliefs? On this basis, how can $\mathcal{R}$ decide that $\mathcal{E}$ believes or not a prescribed statement? It is supposed that $\mathcal{E}$ provides some pieces of information of the form *I believe* α*, I am not sure about* β, to $\mathcal{R}$. The question is: how can $\mathcal{R}$ reconstruct the epistemic state of $\mathcal{E}$ from this information?

The problem has two sides: a model-theoretic one and a syntactic one. Namely, what is a natural representation of $\mathcal{R}$'s epistemic state describing $\mathcal{E}$'s epistemic state? What is the proper language for representing information provided by $\mathcal{E}$, and reasoning about it? In the following, we do not care about whether $\mathcal{E}$'s beliefs are true or not. Also we do not assume $\mathcal{E}$ is lying. So we do not distinguish between knowledge and belief.

The aim of this paper is to define a minimal logic encoding the information provided by agent $\mathcal{E}$ and sufficient to let agent $\mathcal{R}$ reason about it. In this language, atomic propositions are expressed as $\Box\alpha$, where α is any formula from a propositional language used by $\mathcal{E}$ and $\Box$ is borrowed from modal logics. A set of formulae in this language is called a *meta-belief base*, because it represents what $\mathcal{R}$ knows about $\mathcal{E}$'s beliefs. In the sequel, if $\Box\alpha$ appears in $\mathcal{R}$'s meta-belief base, it means either that agent $\mathcal{E}$ has declared to believe that α is true to agent $\mathcal{R}$, or $\mathcal{R}$ can infer that agent $\mathcal{E}$ believes α, from what the latter previously said. The language is then completed by means of negation and conjunction, allowing for $\neg\Box\alpha$, $\Box\alpha \wedge \Box\beta$ and all combinations thereof. As a consequence, the formalism enables $\mathcal{E}$ to declare that (s)he does not know the truth status of a formula. However, the nesting of modalities is not allowed because we are not concerned with introspective reasoning of $\mathcal{R}$ about his or her own beliefs (e.g. whether $\mathcal{R}$ is aware of his or her beliefs about what $\mathcal{E}$ believes).

Some minimal axioms are proposed in such a way that the fragment of this logic restricted to propositions of the form $\Box\alpha$ is isomorphic to propositional logic, if the $\Box$ operator is dropped. This is because $\mathcal{R}$ assumes that $\mathcal{E}$ is a propositional logic reasoner. In particular, $\mathcal{R}$ assumes that $\mathcal{E}$ believes tautologies of the propositional calculus, and can reason using modus ponens. Moreover, $\mathcal{R}$ considers it equivalent for $\mathcal{E}$ to assert $\Box\alpha$ and $\Box\beta$ or to assert $\Box(\alpha \wedge \beta)$. In some sense, $\mathcal{E}$ is viewed by $\mathcal{R}$ as a source of information, or a witness, that communicates information on his or her epistemic state. We call the resulting logic *a Meta-Epistemic Logic* (MEL) so as to emphasize the fact that we deal with how an agent reasons about what (s)he knows about the beliefs of another agent.

At the semantic level, we search for the simplest basic representation of knowledge common to all uncertainty theories. Incomplete knowledge about the real world possessed by agent $\mathcal{E}$ has will be represented just by a subset E of interpretations of $\mathcal{E}$'s language, one and only one of which is true. All agent $\mathcal{R}$ knows about $\mathcal{E}$'s epistemic state stems from what $\mathcal{E}$ told him or her. So $\mathcal{R}$ has incomplete knowledge about $\mathcal{E}$'s epistemic state E. The epistemic state of an agent regarding another agent's beliefs is what we call a *meta-epistemic state*. Let $\mathcal{V}$ be the set of interpretations of a standard propositional language, and $[\alpha]$ the set of models of α. If $\mathcal{E}$ says $\Box\alpha$ then, only a subset E of $[\alpha]$ can stand for the epistemic state of $\mathcal{E}$ since the latter could not assert α otherwise. Similarly, if $\mathcal{E}$ says $\neg\Box\alpha$ then, no subset E of $[\alpha]$ can stand for the epistemic state of $\mathcal{E}$ while any other set of interpretations not contained in $[\alpha]$ can be a candidate representation of $\mathcal{E}$'s epistemic state. In other words, the meta-epistemic state of $\mathcal{R}$ (about $\mathcal{E}$'s beliefs) built from $\mathcal{E}$'s statements can be represented by a family $\mathcal{F}$ of non-empty subsets of $\mathcal{V}$, one and only one of which is the actual epistemic state of $\mathcal{E}$. Moreover, any such family $\mathcal{F}$ can stand for a meta-epistemic state. In order not to confuse models of propositional formulae with models of MEL formulae, we call the latter *meta-models* since they are subsets of interpretations. In the following, it is assumed that $\mathcal{F}$ does not contain the empty set, that is, agent $\mathcal{E}$ does not entertain contradictory beliefs [1].

In this paper, we describe such a minimal logic, the semantics of which exactly corresponds to meta-epistemic states modelled by families of non-empty subsets of propositional valuations (interpretations). In particular, we do not need Kripke-style semantics since we do not nest modalities. In this sense the proposed logic, even if formally a fragment of a known modal logic, is not really in the spirit of the modal logic trend for representing knowledge. In particular, the syntax of the logic is not tailored for reasoning about meta-epistemic states (i.e. about what $\mathcal{R}$ believes about what (s)he believes about $\mathcal{E}$), but only about the epistemic state of another agent. We view our logic as being of higher order because it encapsulates propositional logic inside, and there are thus two levels of syntax (as well as two levels of semantics), one on top of the other (respectively the one of standard propositional logic and the one of MEL). The language of MEL is not a flat extension of the one

[1] If we do not have this assumption then, whatever $\mathcal{E}$ declares, it is always possible to assume that this statement is due to the fact that $\mathcal{E}$ has a contradictory belief set which entails anything.

of propositional logic.

This kind of representation of higher order incomplete knowledge already exists in uncertainty theories. In Shafer's theory of evidence [34], an epistemic state is represented by a probability distribution m on $2^{\mathcal{V}} \setminus \{\emptyset\}$, the weight $m(E)$ being the probability that the actual epistemic state (resulting from collecting evidence) is E. The obtained formalism is also not new and it is a fragment of well-known modal systems. The originality of the paper lies in its specific perspective on the problem of reasoning about agent's revealed beliefs, the non-Kripke semantics of the system MEL, and the bridge between logic and uncertainty theories it suggests.

The paper [2] is organized as follows. In the next section, the syntax and the intended semantics of the logic are provided with stress on the kind of statements the emitter agent is allowed to use. An axiomatization of the logic is then supplied in Section 3. Soundness and completeness with respect to the intended semantics is established in Section 4. Section 5 focuses on how to encode any meta-epistemic state as a MEL formula. Similarly we exhibit the set of meta-models of any set of MEL formulae. We show that the set of all belief bases in MEL, quotiented by the semantic equivalence is isomorphic to the power set of the power set of the set of valuations (excluding the empty set from the latter). So, MEL can account for any meta-epistemic state of an agent about another agent. A bridge to Shafer's theory of evidence is also pointed out in Section 6. It is shown that there is a MEL-formula encoding a single epistemic state E that is the logical counterpart to the Möbius transform of a belief function. The latter is a probabilistic rendering of a meta-epistemic state. In the last sections, some related works are discussed further and perspectives are outlined. In particular, a comparative discussion of MEL with the consensus logic of Pauly [33] is included due to the identity of the syntax of both formalisms.

2 The logic MEL

Let us consider classical propositional logic PL, with (say) k propositional variables, $p_1, \dots, p_k$, and propositional constant $\top$. A propositional valua-

[2] A short version [2] of this paper was presented at the 10th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, Verona, July 2009, and at the Dagstuhl Seminar on "Information processing, rational belief change and social interaction", August 2009.

tion, as usual, is a map $w : PV \to \{0,1\}$, where $PV := \{p_1, \dots, p_k\}$. The set of all propositional valuations (interpretations) is denoted $\mathcal{V}$. For a PL-formula α, $w \models \alpha$ indicates that w satisfies α or w is a *model* of α, i.e. $w(\alpha) = 1$ (true). If $w \models \alpha$ for every α in a set $\mathcal{B}$ of PL-formulae, we write $w \models \mathcal{B}$. $[\alpha] := \{w : w \models \alpha\}$, is the set of models of α.

Let E denote the epistemic state of an agent $\mathcal{E}$. We assume that an epistemic state is represented by a subset of propositional valuations, understood as a disjunction thereof. Each valuation represents a 'possible world' consistent with the epistemic state of $\mathcal{E}$. So, $E \subseteq \mathcal{V}$, and it is further assumed that E is non-empty. Note that, for any E, $|E| \leq 2^k$.

2.1 The language for MEL

As suggested earlier, our idea is to encapsulate PL inside a belief modality denoted $\Box$. The base is PL, and $\alpha, \beta...$ denote PL-formulae. We add the unary connective $\Box$ to the PL-alphabet. Atomic formulae of MEL are of the form $\Box\alpha$, $\alpha \in PL$. $\Box\alpha$ is intended to be true for an agent $\mathcal{E}$, if α holds in *every* possible world compatible with $\mathcal{E}$'s epistemic state denoted by E.
The set of MEL-formulae, denoted $\phi, \psi...$, is then generated from the set At of atomic formulae, with the help of the Boolean connectives $\neg, \wedge$:

$$MEL := \ \Box\alpha \mid \neg\phi \mid \phi \wedge \psi.$$

One defines the connective $\vee$ and the modality $\Diamond$ in MEL in the usual way. Namely $\phi \vee \psi := \neg(\neg\phi \wedge \neg\psi)$ and $\Diamond\alpha := \neg\Box\neg\alpha$, where $\alpha \in PL$. Like $\Box$, modality $\Diamond$ applies only to PL-formulae. In the following, we denote by Γ a set of MEL-formulae, while $\mathcal{B}$ is used for sets of PL-formulae.

Remark 1

1. PL-formulae are not MEL-formulae, as they can only appear inside them.

2. Iteration of the modal operators $\Box, \Diamond$ is not allowed in MEL (as explained in Section 1).

The language is modal-like but the spirit of the approach is different: we aim at nesting a logic inside another one, so as to avoid mixing sentences referring to the real world with sentences referring to what an agent knows about it.

An agent $\mathcal{E}$ provides some information about his or her beliefs about the outside world to another agent $\mathcal{R}$ by means of the above language. Any set Γ of formulae in this language is interpreted as what an agent $\mathcal{E}$ declares to another agent $\mathcal{R}$. It forms the meta-belief base possessed by $\mathcal{R}$; on this basis, agent $\mathcal{R}$ tries to reconstruct the epistemic state of the other agent. Some of the basic statements that agent $\mathcal{E}$ can express in this language are as follows.

- For any propositional formula α, if $\Box\alpha \in \Gamma$, it means agent $\mathcal{E}$ declares that (s)he believes α is true.
- If $\Diamond\alpha \in \Gamma$, it means agent $\mathcal{E}$ declares that, to him or her, α is *possibly* true, that is (s)he has no argument as to the falsity of α. Note that this is equivalent to $\neg\Box\neg\alpha \in \Gamma$, that is, all that $\mathcal{R}$ can conclude is that either $\mathcal{E}$ believes α is true, or ignores whether α is true or not.
- If $\Diamond\alpha \wedge \Diamond\neg\alpha \in \Gamma$, it means agent $\mathcal{E}$ declares to ignore whether α is true or not.
- If $\Box\alpha \vee \Box\neg\alpha \in \Gamma$, it means agent $\mathcal{E}$ says (s)he knows whether α is true or not, but prefers not to reveal it.

Of course, the language enables agent $\mathcal{E}$ to declare more sophisticated (maybe unlikely) assertions like $(\Diamond\alpha \wedge \Diamond\beta) \vee \Box\gamma$, which means that either (s)he has no reason to disbelieve α nor to disbelieve β, or believes γ, or both. It reduces to the two assertions $\Diamond\alpha \vee \Box\gamma$ and $\Diamond\beta \vee \Box\gamma$. Note that the language allows to express that agent $\mathcal{R}$ believes that agent $\mathcal{E}$ ignores whether a proposition α is true or not, but it cannot express that $\mathcal{R}$ ignores if agent $\mathcal{E}$ believes α. To do it, we should expand the language of MEL to include additional modalities pertaining to agent $\mathcal{R}$. Indeed, in the language of MEL, the modalities refer solely to agent $\mathcal{E}$'s beliefs.

2.2 The semantics

For a given agent $\mathcal{E}$, we define satisfaction of MEL-formulae recursively, as follows. $\Box\alpha \in At$, ϕ, ψ are MEL-formulae, and E is the epistemic state of an agent $\mathcal{E}$. Note that $\emptyset \neq E \subseteq \mathcal{V}$, the set of all propositional valuations.

- $E \models \Box\alpha$, if and only if $E \subseteq [\alpha]$.
- $E \models \neg\phi$, if and only if $E \not\models \phi$.

- $E \models \phi \wedge \psi$, if and only if $E \models \phi$ and $E \models \psi$.

$E \models \Box\alpha$ means that in the epistemic state E, agent $\mathcal{E}$ believes α. Viewed from agent $\mathcal{R}$, if agent $\mathcal{E}$ declares (s)he believes α (i.e. $\Box\alpha \in \Gamma$), any E such that $E \models \Box\alpha$, is a possible epistemic state of $\mathcal{E}$.

It is then clear that

- $E \models \Diamond\alpha$, if and only if $E \cap [\alpha] \neq \emptyset$,

i.e. there is at least one possible world for agent $\mathcal{E}$, where α holds. If $\Diamond\alpha \in \Gamma$, it means that agent $\mathcal{E}$ declares that α is plausible (or conceivable) in the sense that there is no reason to disbelieve α. As a consequence, the epistemic state of $\mathcal{E}$ is known by agent $\mathcal{R}$ to be consistent with $[\alpha]$. Note that $\Diamond\alpha$ can be interpreted as an expression of partial ignorance. Especially, $\Diamond\alpha \wedge \Diamond\neg\alpha \in \Gamma$ corresponds to agent $\mathcal{E}$ explicitly declaring full ignorance about α so that its meta-models form the set $\{E, E \cap [\alpha] \neq \emptyset, E \cap [\alpha]^c \neq \emptyset\}$, i.e. it brings non-trivial information about $\mathcal{E}$'s epistemic state, even if it does not bring any information about the real world. Likewise, $\Box\alpha \vee \Box\neg\alpha \in \Gamma$ is not tautological. More generally, in the case of a disjunction $\Box\alpha \vee \Box\beta$, the only corresponding possible epistemic states form the set $\{E \subseteq [\alpha]\} \cup \{E \subseteq [\beta]\}$. It is clearly more informative than $\Box(\alpha \vee \beta)$, since the latter allows epistemic states where none of α or β can be asserted. Restricting the meta-models to singletons, so as to mimic the classical semantics, would make these two formulae equivalent.

Encoding a belief α by $\Box\alpha$ in MEL stands in contrast to, e.g. belief revision literature [20], where beliefs are represented by propositions of PL, keeping the modality implicit. But α and $\Box\alpha$ have models of a different nature, as shown above, and avoids confusion between $\neg\Box\alpha(\equiv \Diamond\neg\alpha)$ and $\Box\neg\alpha$. In MEL their sets of meta-models are again different.

As usual, we have the notion of semantic equivalence of formulae:

Definition 1 *ϕ is* semantically equivalent *to ψ, written $\phi \equiv \psi$, if for any epistemic state E, $E \models \phi$, if and only if $E \models \psi$.*

If Γ is a set of MEL-formulae, $E \models \Gamma$ means $E \models \phi$, for each $\phi \in \Gamma$. So the set of meta-models of Γ, which may be denoted $\mathcal{F}_\Gamma$, is precisely $\{E : E \models \Gamma\}$.

Now $\mathcal{R}$ can reason about what is known from agent $\mathcal{E}$'s assertions:

Definition 2 *For any set $\Gamma \cup \{\phi\}$ of MEL-formulae, ϕ is a* semantic consequence *of Γ, written $\Gamma \models_{MEL} \phi$, provided for every epistemic state E, $E \models \Gamma$ implies $E \models \phi$.*

For any family $\mathcal{F}$ of sets of propositional valuations, $\mathcal{F} \models \phi$ means that for each $E \in \mathcal{F}$, $E \models \phi$. A natural extension gives the notation $\mathcal{F} \models \Gamma$, for any set Γ of MEL-formulae. So, for instance, $\mathcal{F}_\Gamma \models \Gamma$.

3 Axiomatization

Observing valid formulae and rules in MEL, suggests immediately that the modal system KD may provide an axiomatization for it. We establish formally that this is indeed the case.

Let us denote as α, β wffs in PL, and ϕ, ψ, μ wffs in MEL. For any set $\mathcal{B} \cup \{\alpha\}$ of PL-formulae, $\mathcal{B} \vdash \alpha$ denotes that α is a syntactic PL-consequence of $\mathcal{B}$. In particular, $\vdash \alpha$ indicates that α is a PL-theorem.

3.1 The MEL axioms

We consider the following axioms and rule of inference:

Axioms:

$(PL):$ (i) $\phi \rightarrow (\psi \rightarrow \phi)$; (ii) $(\phi \rightarrow (\psi \rightarrow \mu)) \rightarrow ((\phi \rightarrow \psi) \rightarrow (\phi \rightarrow \mu))$;
(iii) $(\neg\phi \rightarrow \neg\psi) \rightarrow (\psi \rightarrow \phi)$.

$(RM):$ $\Box\alpha \rightarrow \Box\beta$, whenever $\vdash \alpha \rightarrow \beta$.

$(M):$ $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$.

$(C):$ $(\Box\alpha \wedge \Box\beta) \rightarrow \Box(\alpha \wedge \beta)$.

$(N):$ $\Box\top$.

$(D):$ $\Box\alpha \rightarrow \Diamond\alpha$.

Rule:

$(MP):$ If $\phi, \phi \rightarrow \psi$ then ψ.

The nomenclature of the axioms follows Chellas [9]. Axioms (M) and (C) (taken together) were justified in Section 1, and so was (N). They account for the logical sophistication of agent $\mathcal{E}$, in the classical sense. Namely if $\mathcal{E}$ claims to believe α and to believe β, this is equivalent to $\mathcal{E}$ believing their conjunction. As a consequence, $\mathcal{E}$ also follows (RM): if it is true that $\alpha \to \beta$ and $\mathcal{E}$ believes α, (s)he must believe β. This is the symbolic counterpart of the monotonicity of numerical belief measures for events, in the sense of set-inclusion. Namely, if $\vdash \alpha \to \beta$, the following inequality between probabilities hold: $P(\alpha) \leq P(\beta)$, and (RM) corresponds to when $\Box\alpha$ is understood as $P(\alpha) = 1$. Axiom (D) comes down to considering that asserting the certainty of α is stronger than asserting its plausibility. It is also a counterpart of numerical inequality between belief and plausibility functions [34], necessity and possibility measures [12] etc. in uncertainty theories. Finally, (PL) and (MP) enable agent $\mathcal{R}$ to infer from agent $\mathcal{E}$'s publicly declared beliefs, so as to reconstruct a picture of the latter agent's epistemic state.

Syntactically, MEL's axioms can be viewed as a Boolean version of those of the fuzzy logic of necessities briefly suggested by Hájek [22].
Taking any set Γ of MEL-formulae, one defines a compact syntactic consequence in MEL (written $\vdash_{MEL}$), in the standard way.

Definition 3 $\Gamma \vdash_{MEL} \phi$, *if and only if there is a finite sequence of MEL-formulae* $\phi_1, \ldots, \phi_n$ *with* $\phi_n := \phi$, *and each* ϕ_i *is either a MEL-axiom, or a member of* Γ, *or is derived from previous members of the sequence by the rule* (MP).

We have some immediate observations.

Observation 1

1. *Deduction theorem and its converse hold in MEL.*

2. *The axiom* (RM) *is equivalent to the rule:*
   If $\Box\alpha$ *then* $\Box\beta$, *whenever* $\vdash \alpha \to \beta$.

3. *Due to axioms* (M) *and* (C), (RM) *is also equivalent to*
   $(E) : \Box\alpha \leftrightarrow \Box\beta$, *whenever* $\vdash \alpha \leftrightarrow \beta$.

4. *MEL is the same as the normal modal system KD (=EMCND* [9]*) with a restricted language. In other words,* $\Gamma \vdash_{MEL} \phi$, *if and only if* $\Gamma \vdash_{KD} \phi$, *for any set* $\Gamma \cup \{\phi\}$ *of MEL-formulae.*

Soundness of MEL with respect to the semantics described in Section 2.2, is directly obtained.

Theorem 1 *(Soundness) If* $\Gamma \vdash_{MEL} \phi$ *then* $\Gamma \models_{MEL} \phi$.

In Section 4, we establish completeness of MEL.

3.2 The encapsulation of PL

Using soundness we get the following result, which demonstrates that deriving a $\Box$-formula, say $\Box\alpha$, in MEL from other $\Box$-formulae is equivalent to deriving α in PL. It may be noted that the result was proved in [11] for the modal system K having the standard Kripke semantics. As we shall see below, the proof immediately carries over to MEL. Axiom (D) is not used.

For any set $\mathcal{B}$ of PL-formulae, let $\Box\mathcal{B} := \{\Box\beta : \beta \in \mathcal{B}\}$, $\Diamond\mathcal{B} := \{\Diamond\beta : \beta \in \mathcal{B}\}$.

Theorem 2 *$\Box\mathcal{B} \vdash_{MEL} \Box\alpha$, if and only if $\mathcal{B} \vdash \alpha$.*

Proof: Let $\Box\mathcal{B} \vdash_{MEL} \Box\alpha$. Consider an agent with $E := \{w\}$, where w is any propositional valuation such that $w \models \mathcal{B}$. So $E \subseteq [\beta]$ for every $\beta \in \mathcal{B}$, and by definition of satisfaction, $E \models_{MEL} \Box\mathcal{B}$. Theorem 1 ensures that $\Box\mathcal{B} \models_{MEL} \Box\alpha$, giving $E \models_{MEL} \Box\alpha$. In other words, $w \models \alpha$. Thus by completeness of PL, $\mathcal{B} \vdash \alpha$.

For the other direction, using compactness of the PL-consequence and the deduction theorem for PL, we get a finite subset of $\mathcal{B}$, say $\mathcal{B}' := \{\alpha_1, \ldots, \alpha_n\}$, such that $\vdash \alpha_1 \to (\alpha_2 \to \ldots(\alpha_n \to \alpha)\ldots)$. By (RM) and axiom (K), $\vdash_{MEL} \Box\alpha_1 \to (\Box\alpha_2 \to \ldots(\Box\alpha_n \to \Box\alpha)\ldots)$. Thus by converse of deduction theorem and definition of $\vdash_{MEL}$, $\Box\mathcal{B} \vdash_{MEL} \Box\alpha$. ■

From the point of view of reasoning agents, this result means that agent $\mathcal{R}$ can reason about $\mathcal{E}$'s beliefs (leaving statements of ignorance aside) as if they were $\mathcal{R}$'s own beliefs. In case $\Box\mathcal{B} \vdash_{MEL} \Box\alpha$, if agent $\mathcal{R}$ were asked whether $\mathcal{E}$ believes α from what $\mathcal{E}$ previously declared to believe ($\Box\mathcal{B}$), the former's answer would be yes because $\mathcal{E}$ would reason likewise about α.

Thus, staying within the MEL-fragment of the language of system K, we obtain the above result. In fact,

Note 1 *Theorem 2 holds for the fragment of MEL given by the formula scheme $\Box\alpha \mid \phi \wedge \psi$, and the axioms (a) PL (i), (ii) and (b) $\Box\alpha$, whenever $\vdash \alpha$. The inference rules are: (a) (MP) and (b) if $\Box\alpha$ and $\Box(\alpha \to \beta)$ then $\Box\beta$.*

Remark 2 Theorem 2 confirms that propositional logic is *encapsulated* in MEL; MEL is not a usual modal extension of propositional logic: it is a two-tiered logic. In some sense, MEL is an encapsulation of PL within a special fragment of PL with atoms all of the form $\Box\alpha$, the modal axioms referring to the articulation between the languages of the bottom and the top levels.

In fact, we could totally do away with the connection to modal logic, by changing the syntax and proving MEL axioms from assuming two propositional languages, one of which is embedded in the other one. Namely, we can use the first propositional language L_1 with k propositional variables, $p_1, \ldots, p_k$, propositional constant $\top$, and formulae denoted by $\alpha, \beta, \ldots$. A second propositional language L_2 is then built, whose atoms are formulae in L_1. In the paper, we use a modal syntax to that effect, but an atomic formula in L_2 is intended to mean that the corresponding formula in L_1 *is known* to be true, which could be expressed by a pair of the form $(\alpha, \mathbf{T})$ instead of $\Box\alpha$, in the spirit of Labelled Deductive Systems (Gabbay [19]). Likewise, we could express in L_2 that a formula of L_1 is known to be false, or to be unknown, respectively as follows:

- Define $(\alpha, \mathbf{F})$ as meaning $(\neg\alpha, \mathbf{T})$ (instead of $\Box\neg\alpha$)
- Define $(\alpha, \mathbf{U})$ as short for $\neg(\alpha, \mathbf{T}) \wedge \neg(\alpha, \mathbf{F})$ (i.e. $\neg\Box\alpha \wedge \neg\Box\neg\alpha$).

Then MEL axioms could be retrieved easily in that setting, given that both languages L_1 and L_2 obey the rules of propositional logic. The following just rewrites MEL axioms in the new formalism:

(RM) : $\vdash_{L_2} (\alpha, \mathbf{T}) \to (\beta, \mathbf{T})$, whenever $\vdash_{L_1} \alpha \to \beta$.

(M) : $\vdash_{L_2} (\alpha \wedge \beta, \mathbf{T}) \to (\alpha, \mathbf{T}) \wedge (\beta, \mathbf{T})$.

(C) : $\vdash_{L_2} (\alpha, \mathbf{T}) \wedge (\beta, \mathbf{T}) \to (\alpha \wedge \beta, \mathbf{T})$.

(N) : $\vdash_{L_2} (\top, \mathbf{T})$.

(D) : $\vdash_{L_2} (\alpha, \mathbf{T}) \to \neg(\alpha, \mathbf{F})$.

We keep the same semantics as MEL (that is, the set of models of $(\alpha, \mathbf{T})$ is $\{E \subseteq \mathcal{V} : \emptyset \neq E \subseteq [\alpha]\}$. It is clear that $\Diamond\alpha$ in MEL equivalently writes $\neg(\alpha, \mathbf{F})$ or $(\alpha, \mathbf{T}) \vee (\alpha, \mathbf{U})$ in L_2. In fact, the law of excluded fourth holds in L_2, that is

$$\vdash_{L_2} (\alpha, \mathbf{T}) \vee (\alpha, \mathbf{U}) \vee (\alpha, \mathbf{F}),$$

each of the three formulae being mutually exclusive with the other. This formalism may solve some paradoxes of three-valued Kleene logic when, as often, it is viewed as a logic of incomplete information [10]. This is a topic for further research.

3.3 More inference rules in MEL

In the following we provide additional inference rules, valid in MEL, that emphasize the bridge between propositional logic and the higher order language.

Corollary 1 *The 'converse' of* (RM) *holds:* $\vdash \alpha \to \beta$*, if* $\vdash_{MEL} \Box\alpha \to \Box\beta$*. In other words, we have the equivalence:*

$\vdash_{MEL} \Box\alpha \to \Box\beta$*, if and only if* $\vdash \alpha \to \beta$*.*

In fact, this also yields the equivalence:

$\vdash_{MEL} \Box\alpha \to \Box\beta$ *if and only if* $\vdash_{MEL} \Box(\alpha \to \beta)$*.*

Theorem 2 holds for $\Diamond$-formulae, provided $\Diamond\mathcal{B}$ is a singleton (i.e. contains a single $\Diamond$-formula):

Observation 2 $\Diamond\alpha \vdash_{MEL} \Diamond\beta$*, if and only if* $\alpha \vdash \beta$*.*
Hence the equivalence : $\vdash_{MEL} \Diamond\alpha \to \Diamond\beta$*, if and only if* $\vdash \alpha \to \beta$*.*

Proof: Note that $\Diamond\alpha \vdash_{MEL} \Diamond\beta$ is equivalent to $\Box\neg\beta \vdash_{MEL} \Box\neg\alpha$, which, by Theorem 2, is clearly equivalent to $\alpha \vdash \beta$. ■

Observation 3 $\Box\mathcal{B} \vdash_{MEL} \Diamond\alpha$*, if and only if* $\mathcal{B} \vdash \alpha$*.*
Therefore, in particular,

$\vdash_{MEL} \Box\alpha \to \Diamond\beta$*, if and only if* $\vdash \alpha \to \beta$*.*

The 'if' part is a consequence of Theorem 2 and axiom (D). The proof of the converse part again follows the same lines as that of the 'only if' part of Theorem 2.

With the help of all the above, it is easy to derive some inference rules in MEL. The first collection exploits valid formulae in MEL.

Proposition 1

1. *If* $\vdash_{MEL} \Box\alpha \to \Box\beta$ *then* $\Diamond\alpha \vdash_{MEL} \Diamond\beta$*.*
2. *If* $\vdash_{MEL} \Diamond\alpha \to \Diamond\beta$ *then* $\Box\alpha \vdash_{MEL} \Box\beta$*.*
3. *If* $\vdash_{MEL} \Box(\alpha \to \beta)$ *then* $\Diamond\alpha \vdash_{MEL} \Diamond\beta$*.*
4. *If* $\vdash_{MEL} \Diamond(\alpha \to \beta)$ *then* $\Box\alpha \vdash_{MEL} \Box\beta$*.*
5. *If* $\vdash_{MEL} \Diamond(\neg\alpha \vee \beta)$ *then* $\Box(\alpha \vee \gamma) \vdash_{MEL} \Box(\beta \vee \gamma)$*.*

Proof: (1)-(3)follow from Corollary 1 and Observation 2.
Rule (4): Using $\vdash_{MEL} \Diamond(\alpha \to \beta) \leftrightarrow (\Box\alpha \to \Diamond\beta)$, one has $\Box\alpha \vdash_{MEL} \Diamond\beta$. By Observation 3, therefore, $\alpha \vdash \beta$. Then Corollary 1 gives the rule.
Rule (5): As for rule (4), $\vdash_{MEL} \Diamond(\neg\alpha \vee \beta)$ results in $\alpha \vdash \beta$. Hence $\alpha \vee \gamma \vdash \beta \vee \gamma$. Finally, we use Corollary 1. ■

The second collection consists of truth-preserving inference steps that apply to the encapsulated PL-formulae.

Proposition 2

1. $\{\Box\alpha, \Diamond(\alpha \to \beta)\} \vdash_{MEL} \Diamond\beta$.
2. $\{\Box(\neg\alpha \vee \beta),\ \Box(\alpha \vee \gamma)\} \vdash_{MEL} \Box(\beta \vee \gamma)$.
3. $\{\Box(\neg\alpha \vee \beta), \Diamond(\alpha \vee \gamma)\} \vdash_{MEL} \Diamond(\beta \vee \gamma)$.

Proof: These inference rules can be proved with theorems of the MEL-fragment of K, Theorem 2 and Observation 2. ■

Rules (1) and (3) in Proposition 1 ensure that if $\mathcal{E}$ says possibly α and belief in α always entails belief in β, then β is also possibly true for $\mathcal{E}$. Inference rules in Proposition 1 heavily rely on the equivalence between $\vdash_{MEL} \Box\alpha \to \Box\beta$, $\vdash_{MEL} \Box(\alpha \to \beta)$ and $\alpha \vdash \beta$. Inference rules in Proposition 2 are of a different nature. Rules (1)-(3) are "encapsulated" resolution rules, that is, inference rules pertaining to formulae of the inner PL language, operated in the outer language. Rule (1) is a weakened form of PL modus ponens, which from the point of view of encapsulated PL-formulae, preserves consistency, not certainty (hence not truth) of inner formulae. Rule (2) is related to the resolution rule in possibilistic logic [12] which preserves the weakest degree of certainty of premises. Rule (3) is the resolution counterpart of Rule (1) and was first proposed and semantically validated in the multivalued setting of possibility theory in [13].

Consistency-preserving inference rules (i.e. with premises and conclusion involving $\Diamond$-prefixed formulae) indicate that reasoning about the emitter's explicit partial ignorance is not completely trivial: via reasoning steps, $\mathcal{R}$ can better figure out what $\mathcal{E}$ is supposed to ignore.

4 Completeness

We prove two key propositions, that help establish completeness of MEL with respect to the semantics given in Section 2.2. The basic idea for getting completeness is to build a passage to and from the MEL and Kripke semantics. In Observation 1(4), it was pointed out that the MEL-language is a proper sub-language of the modal logic KD. We recall that a Kripke model [27] for the system KD, is a triple $M := (U, R, V)$, where the domain U is a non-empty set and R is a binary relation on U that is *serial/right unbounded*: for each $u \in U$, there is $u' \in U$ with uRu'. $V : PV \times U \to \{0, 1\}$ is the meaning function that may be extended to the set of all KD-formulae in a routine manner. $M, u \models \phi$ denotes that the KD-formula ϕ is satisfied at $u(\in U)$ by V, i.e. $V(\phi, u) = 1$. $M, u \models \Gamma$ for a set Γ of KD-formulae, indicates that $M, u \models \phi$ for each $\phi \in \Gamma$. A formula ϕ is a *local* semantic consequence of a set Γ in KD, written $\Gamma \models \phi$, if and only if for every KD-Kripke model $M := (U, R, V)$ and $u \in U$, if $M, u \models \Gamma$ then $M, u \models \phi$.

Proposition 3 *For every epistemic state E of an agent $\mathcal{E}$, there is a KD-Kripke model M_E with domain E such that for any MEL-formula ϕ,*

$$E \models_{MEL} \phi, \text{ if and only if for every } w \in E,\ M_E, w \models \phi.$$

Proof: We consider $M_E := (E, E \times E, V_E)$, where the meaning function V_E is defined for any $p \in PV,\ w \in E$ as: $V_E(p, w) := w(p)$. It is clear that M_E is a KD-Kripke model. It can be checked by induction on the complexity of any PL-formula α, that $V_E(\alpha, w) = w(\alpha)$. Proof of the proposition is by induction on the complexity of the MEL-formula ϕ.

We only consider the base case $\phi := \Box\alpha$, $\alpha \in PL$. Let $w \in E$. $V_E(\Box\alpha, w) = 1$, if and only if $V_E(\alpha, w') = 1$, for any w' with $w\ (E \times E)\ w'$. Therefore, for all $w' \in E$, $V_E(\alpha, w') = 1 = w'(\alpha)$, which happens if and only if $E \models_{MEL} \Box\alpha$. ■

In fact, what it means is that any meta-model E of MEL can be encoded as a relation $E \times E$ on $\mathcal{V}$, in the sense that $V_E(\Box\alpha, w) = 1$, which, in KD means $w' \models \alpha, \forall w' \in \mathcal{V}$ such that $(w, w') \in E \times E$, is equivalent to $E \subseteq [\alpha]$, the satisfiability relation in MEL.

Proposition 4 *For any KD-Kripke model $M := (U, R, V)$ and $u \in U$, there is an epistemic state E_u such that for any MEL-formula ϕ,*

$$M, u \models \phi, \text{ if and only if } E_u \models_{MEL} \phi.$$

Proof: We consider the 'R-neighbourhood' of u, viz. the set $[u]_R := \{u' \in U : uRu'\}$. For each $u' \in [u]_R$ and $p \in PV$, a propositional valuation $w_{u'}$ is defined as: $w_{u'}(p) := V(p, u')$.

Then $w_{u'}(\alpha) = V(\alpha, u')$, for any $\alpha \in PL$. This is proved by induction on the complexity of α.

The required epistemic state is the set $E_u := \{w_{u'} : u' \in [u]_R\}$. First note that $E_u \neq \emptyset$, as R is serial. The idea is to collect all the propositional valuations determined by V, in the R-neighbourhood of u. The proposition is now proved by induction on the complexity of ϕ. As before, we demonstrate the base case $\phi := \Box\alpha,\ \alpha \in PL$.
$E_u \models_{MEL} \Box\alpha$, if and only if $w_{u'}(\alpha) = 1$, for all u' with uRu'. The latter holds if and only if $V(\alpha, u') = 1$, which happens if and only if $V(\Box\alpha, u) = 1$, i.e. $M, u \models \Box\alpha$[3]. ■

The possibility of considering simplified versions of modal systems like $S5$ and $KD45$, omitting the Kripke relation in Kripke structures (assuming all possible worlds are related), thus reducing such structures to just the set of possible worlds, is pointed out in [25] p. 62. Here, we do not need axioms **4, 5** of positive and negative introspection, respectively, as we do not nest modalities. As a consequence, our non-Kripke semantics is valid for this specific fragment of KD and corresponds to our idea of reasoning about beliefs. Namely we do not refer to the "real state of the world" In fact, we need Kripke relations of the form $E \times E$, for any non-empty set $E \subseteq \mathcal{V}$, not just $\mathcal{V} \times \mathcal{V}$. As a by-product of Propositions 3 and 4, we obtain a result analogous to that for the systems $S5$ and $KD45$.

Proposition 5 *Let $M := (U, R, V)$ be a KD-Kripke model and $u \in U$. Then there is a structure $M_0 := (U_0, R_0, V_0)$ and a state $u_0 \in U_0$ such that, for any MEL-formula ϕ,*

$$M, u \models \phi, \text{ if and only if } M_0, u_0 \models \phi.$$

Proof: Consider any MEL-formula ϕ. By Proposition 4, $M, u \models \phi$, if and only if $E_u \models_{MEL} \phi$. Using Proposition 3, we get the Kripke model $M_{E_u} := (E_u, E_u \times E_u, V_{E_u})$ such that $E_u \models_{MEL} \phi$, if and only if for every

[3]Observe that, if the Kripke model M in the proposition above is, in particular, M_E of Proposition 3 for some epistemic state E, then $E_u = E$, for any $u \in E$.

$w \in E_u$, $M_{E_u}, w \models \phi$. Combining, $M, u \models \phi$, if and only if for every $w \in E_u$, $M_{E_u}, w \models \phi$.

As R is serial, $E_u \neq \emptyset$. Let $u_0 \in E_u$. So, if $M, u \models \phi$, $M_{E_u}, u_0 \models \phi$. Conversely, as we are considering the universal relation on E_u, it is not hard to show that if for one $u_0 \in E_u$, $M_{E_u}, u_0 \models \phi$, then for every $w \in E_u$, $M_{E_u}, w \models \phi$. Hence $M, u \models \phi$. ■

So we may omit the Kripke relation R_0 and obtain a simpler structure (U_0, V_0) that suffices for consideration of satisfiability of MEL-formulae in terms of Kripke models.

We now return to the completeness problem.

Theorem 3 *For any set* $\Gamma \cup \{\phi\}$ *of MEL-formulae,* $\Gamma \models \phi$*, if and only if* $\Gamma \models_{MEL} \phi$*.*

Proof: Let $\Gamma \models \phi$, and E be such that $E \models_{MEL} \Gamma$. By Proposition 3, the KD-Kripke model M_E satisfies $M_E, w \models \Gamma$, for every $w \in E$. So $M_E, w \models \phi$. By the same proposition, $E \models_{MEL} \phi$.

Conversely, let $\Gamma \models_{MEL} \phi$. Consider any KD-Kripke model $M := (U, R, V)$ and $u \in U$ with $M, u \models \Gamma$. Proposition 4 gives the set E_u such that $E_u \models_{MEL} \Gamma$. Hence $E_u \models_{MEL} \phi$. Again, by the same proposition, $M, u \models \phi$. ■

Corollary 2 *(Completeness) If* $\Gamma \models_{MEL} \phi$ *then* $\Gamma \vdash_{MEL} \phi$*.*

Proof: KD is strongly complete with respect to the class of serial frames. So by Observation 1(4) and Theorem 3, we get the result. ■

Therefore, in particular, we have the following soundness and completeness.

Theorem 4 $\vdash_{MEL} \phi$*, if and only if* $\models_{MEL} \phi$*, i.e.,* $E \models_{MEL} \phi$*, for all epistemic states of agent* $\mathcal{E}$*.*

5 The logical characterization of meta-epistemic states

Let $\mathcal{F}$ be any non-empty collection of non-empty sets of propositional valuations, representing the meta-epistemic state of an agent regarding another agent's beliefs. It is shown here that a MEL-formula $\delta_{\mathcal{F}}$ may be defined such that

- $\mathcal{F}$ satisfies $\delta_{\mathcal{F}}$;
- furthermore, if $\mathcal{F}$ satisfies any set Γ' of MEL-formulae, the syntactic consequences of Γ' must already be consequences of $\delta_{\mathcal{F}}$.

So the MEL-formula $\delta_{\mathcal{F}}$ completely characterizes the meta-epistemic state $\mathcal{F}$. To reach our goal, we follow the line of characterization of Kripke frames by Jankov-Fine formulae (cf. [4]). Here, a Jankov-Fine kind of formula for any non-empty epistemic state is considered, keeping in mind the correspondence with the simpler Kripke frame (with universal accessibility relation), used in the previous section. The formula is then extended naturally to a non-empty collection $\mathcal{F}$ of non-empty epistemic states.

5.1 Syntactic representation of meta-epistemic states

Let $E \subseteq \mathcal{V}, E \neq \emptyset$. Further, let $\alpha_E := \bigvee_{w \in E} \alpha_w$, where α_w is the PL-formula characterizing w, i.e. $\alpha_w := \bigwedge_{w(p)=1} p \quad \wedge \quad \bigwedge_{w(p)=0} \neg p$, where p ranges over PV. Observe that $E \models \Diamond\alpha_w$ if and only if $w \in E$, since $[\alpha_w] = \{w\}$. On the other hand, $E \models \Box\alpha_w$, if and only if $E = \{w\}$, since $E \neq \emptyset$.

Consider now a meta-epistemic state, say the non-empty collection $\mathcal{F} := \{E_1, \ldots, E_n\}$, where the E_i's are non-empty sets of propositional valuations. Note that $|\mathcal{F}| \leq 2^{2^k} - 1$. We take $\bigcup \mathcal{F} := E_1 \cup \ldots \cup E_n$, and let $\alpha_{\mathcal{F}} := \bigvee_{w \in \bigcup \mathcal{F}} \alpha_w$.

The following is then easy to observe.

Proposition 6 *Let $E_i \in \mathcal{F}$.*

1. *(i) For every $B \subseteq E_i$, $B \models_{MEL} \Box\alpha_{E_i}$. In fact, $B \models_{MEL} \Box\alpha_{\mathcal{F}}$.*

   (ii) If $E \not\subseteq \bigcup \mathcal{F}$, then $E \not\models_{MEL} \Box\alpha_{\mathcal{F}}$.

   (iii) Suppose $E \subseteq \bigcup \mathcal{F}$, but there is i ($\in \{1, \ldots, n\}$) such that $E \setminus E_i \neq \emptyset$ (e.g. when $E \supsetneq E_i$). In this case, $E \not\models_{MEL} \Box\alpha_{E_i}$.

2. *(i) $E_i \models_{MEL} \bigwedge_{w \in E_i} \Diamond\alpha_w$, since $w \in E_i \iff E_i \models_{MEL} \Diamond\alpha_w$.*

   (ii) For all E such that there is $w \in E_i \setminus E$, $E \not\models_{MEL} \bigwedge_{w \in E_i} \Diamond\alpha_w$. So, in particular, no proper subset of E_i can satisfy this formula.

In order to *exactly* describe the collection $\mathcal{F}$, we need a MEL-formula such that it is satisfied by all members of $\mathcal{F}$ *only*. In particular, it must *not* be satisfied by

(a) sets having elements from outside $\bigcup \mathcal{F}$,

(b) sets of valuations lying within $\bigcup \mathcal{F}$, but not equal to any of the E_i's,

(c) especially, subsets of members of $\mathcal{F}$.

Such a (non-unique) MEL-formula is denoted $\delta_{\mathcal{F}}$.

If $\mathcal{F} := \{E\}$, where $E := \{w_1, \ldots, w_m\}$, then $\delta_{\mathcal{F}}$ is denoted δ_E and can clearly be chosen as the conjunction of

1. $\Box(\alpha_{w_1} \vee \ldots \vee \alpha_{w_m})$
2. $\Diamond\alpha_{w_i},\ i = 1, \ldots, m,$

i.e., $\Box\alpha_E \ \wedge \ \bigwedge_{w \in E} \Diamond\alpha_w$. Because of Proposition 6 (1)(i) and (2)(i), we have $E \models_{MEL} \delta_E$, and it is simple to check that for any epistemic state E',

Observation 4 *$E' \models_{MEL} \delta_E$, if and only if $E' = E$.*

In the general case, $\mathcal{F} := \{E_1, \ldots, E_n\}$, and the following definition can be adopted:

Definition 4 $\delta_{\mathcal{F}} := \bigvee_{1 \leq i \leq n} \delta_{E_i} = \bigvee_{1 \leq i \leq n} (\Box\alpha_{E_i} \ \wedge \ \bigwedge_{w \in E_i} \Diamond\alpha_w)$.

Note that, strictly speaking, the Jankov-Fine kind of formula for E would be the conjunct of $\Box\alpha_E$ and $\Box(\alpha_{w_i} \to \Diamond\alpha_{w_j}),\ i \neq j$. But here, the latter components would not make sense, and we use the formulae $\Diamond\alpha_{w_j}$.

The following result shows that the set of meta-models of $\delta_{\mathcal{F}}$ is precisely $\mathcal{F}$, and any consequence of sets of formulae satisfied by all epistemic states of $\mathcal{F}$, is also a consequence of $\delta_{\mathcal{F}}$.

Theorem 5

1. *$\mathcal{F} \models_{MEL} \delta_{\mathcal{F}}$, i.e. for each $E_i \in \mathcal{F}$, $E_i \models_{MEL} \delta_{\mathcal{F}}$.*
2. *If $\mathcal{F}'$ is any other meta-epistemic state such that $\mathcal{F}' \models_{MEL} \delta_{\mathcal{F}}$, $\mathcal{F}' \subseteq \mathcal{F}$.*
3. *If Γ' is a set of MEL-formulae such that $\mathcal{F} \models_{MEL} \Gamma'$, $\Gamma' \vdash_{MEL} \phi$ would imply $\{\delta_{\mathcal{F}}\} \vdash_{MEL} \phi$, for any MEL-formula ϕ.*

Proof:

1. Observed earlier: follows from Proposition 6(1)(i), (2)(i).

2. Let $E \in \mathcal{F}'$. Suppose $E \notin \mathcal{F}$.
   If $E \not\subseteq \bigcup \mathcal{F}$, using Proposition 6(1)(ii), we get $E \not\models_{MEL} \delta_{\mathcal{F}}$, a contradiction to the assumption.
   So let $E \subseteq \bigcup \mathcal{F}$, and for each $i = 1, \ldots, n$, $E \neq E_i$.
   In case there is $w \in E_i \setminus E$, $E \not\models_{MEL} \bigwedge_{w \in E_i} \Diamond \alpha_w$, by Proposition 6(2)(ii).
   On the other hand, if there is $w \in E \setminus E_i$, $E \not\models_{MEL} \Box \alpha_{E_i}$, by Proposition 6(1)(iii).
   Thus in either case, $E \not\models_{MEL} \delta_{\mathcal{F}}$, a contradiction.

3. Suppose $\Gamma' \vdash_{MEL} \phi$, and let $E \models_{MEL} \delta_{\mathcal{F}}$. By part (2) of this theorem, $E \in \mathcal{F}$. Then $E \models_{MEL} \Gamma'$, by assumption. By soundness of MEL, $\Gamma' \models_{MEL} \phi$, and so $E \models_{MEL} \phi$. Thus $\{\delta_{\mathcal{F}}\} \models_{MEL} \phi$, and by completeness of MEL, we get the result. ■

To sum up, any meta-epistemic state, that is any non-empty family of non-empty subset of $\mathcal{V}$ can be expressed by a formula in MEL.

5.2 The meta-models of meta-belief bases

Conversely, let Γ be any consistent set of MEL-formulae representing a meta-belief base. We consider the family $\mathcal{F}_\Gamma$ of all meta-models (sets of propositional valuations) of Γ (cf. Section 2.2), viz.

Definition 5 $\mathcal{F}_\Gamma := \{E \subseteq \mathcal{V} : \emptyset \neq E \models \Gamma\}$.

When $\Gamma := \{\phi\}$, we write $\mathcal{F}_\phi$. So $\mathcal{F}_\Gamma = \bigcap_{\phi \in \Gamma} \mathcal{F}_\phi$.

As a simple consequence of the proposed semantics for MEL, we have

Observation 5

(a) *For* $\Box \alpha \in At$, $\mathcal{F}_{\Box \alpha} = \{E \subseteq \mathcal{V} : E \subseteq [\alpha]\}$.

(b) $\mathcal{F}_{\neg \psi} = (2^{\mathcal{V}} \setminus \{\emptyset\}) \setminus \mathcal{F}_\psi$; $\mathcal{F}_{\psi \wedge \psi'} = \mathcal{F}_\psi \cap \mathcal{F}_{\psi'}$.

(c) *For* $\alpha \in PL$, $\mathcal{F}_{\Diamond \alpha} = \{E \subseteq \mathcal{V} : E \cap [\alpha] \neq \emptyset\}$.

(d) *For any MEL-formulae* ϕ, ψ, $\phi \equiv \psi$ *(cf. Definition 1), if and only if* $\mathcal{F}_\phi = \mathcal{F}_\psi$.

It is clear that in the logic MEL, the meta-models, given by sets of valuations, play the same role as propositional valuations in classical logic. Moreover, it has been pointed out that the fragment of MEL restricted to classical formulae prefixed by a modality is isomorphic to propositional logic. So the encapsulation of PL in MEL at the syntactic level corresponds at the semantic level to the shift from the set of interpretations $\mathcal{V}$ to its power set (but for $\emptyset$).

The following theorem extends the classical properties of semantic entailment over meta-models. It is the companion of Theorem 5. We see that $\mathcal{F}_\Gamma$ is the maximal set of meta-models of Γ that satisfies *precisely* the consequences of Γ.

Theorem 6

1. *If Γ' is any set of MEL-formulae such that $\mathcal{F}_\Gamma \models_{MEL} \Gamma'$, $\Gamma' \vdash_{MEL} \phi$ would imply $\Gamma \vdash_{MEL} \phi$, for any MEL-formula ϕ.*

2. *Let $Con(\Gamma) := \{\phi : \Gamma \vdash_{MEL} \phi\}$ and $Th(\mathcal{F}_\Gamma) := \{\phi : \mathcal{F}_\Gamma \models \phi\}$. Then $Con(\Gamma) = Th(\mathcal{F}_\Gamma)$.*

Proof:

1. Let $E \models_{MEL} \Gamma$. By assumption, $E \models_{MEL} \Gamma'$. Therefore $E \models_{MEL} \phi$, using soundness of MEL, and so $\Gamma \models_{MEL} \phi$. Completeness of MEL gives the result.

2. Follows from part (1) of this theorem and definition of $\mathcal{F}_\Gamma$. ■

5.3 Main result

Definition 4 proposes an encoding of a meta-epistemic state into a MEL formula. Definition 5 gives the set of meta-models of any consistent meta-belief base. We can now establish the following connection between these definitions by iterating the construction. It shows the bijection between classes of semantically equivalent formulae in MEL and non-empty sets of non-empty subsets of valuations.

Theorem 7

1. *If $\Gamma \cup \{\phi\}$ is any set of MEL-formulae with Γ consistent, $\Gamma \vdash_{MEL} \phi$, if and only if $\{\delta_{\mathcal{F}_\Gamma}\} \vdash_{MEL} \phi$. In other words, the MEL-consequence sets of Γ and $\delta_{\mathcal{F}_\Gamma}$ are identical: $Con(\Gamma) = Con(\delta_{\mathcal{F}_\Gamma})$.*

2. *If $\mathcal{F}$ is any non-empty collection of non-empty sets of propositional valuations, $\mathcal{F} = \mathcal{F}_{\delta_\mathcal{F}}$.*

Proof:

1. As $\mathcal{F}_\Gamma \models_{MEL} \delta_{\mathcal{F}_\Gamma}$ (Theorem 5(a)), by Theorem 6(1), $\{\delta_{\mathcal{F}_\Gamma}\} \vdash_{MEL} \phi$ implies $\Gamma \vdash_{MEL} \phi$.
   Conversely, let $\Gamma \vdash_{MEL} \phi$. By soundness, $\Gamma \models_{MEL} \phi$. By Theorem 5(c), as $\mathcal{F}_\Gamma \models_{MEL} \Gamma$ (Theorem 6(2)), we have $\{\delta_{\mathcal{F}_\Gamma}\} \vdash_{MEL} \phi$.

2. Let $E \in \mathcal{F}$. Then by Theorem 5(a), $E \models_{MEL} \delta_\mathcal{F}$ and so $E \in \mathcal{F}_{\delta_\mathcal{F}}$.
   Conversely, let $E \in \mathcal{F}_{\delta_\mathcal{F}}$, i.e. $E \models_{MEL} \delta_\mathcal{F}$. Using Theorem 5(b), we get $E \in \mathcal{F}$. ■

This result shows that MEL can precisely account for non-empty families of subsets of valuations. Moreover, the following bijections can be established.

Corollary 3

(a) The Boolean algebra on the set of MEL-formulae quotiented by semantical equivalence $\equiv$, is isomorphic to the power set Boolean algebra with domain $2^{2^{\mathcal{V}} \setminus \{\emptyset\}}$. The correspondence, for any MEL-formula ϕ, is given by: $[\phi]_\equiv \mapsto \mathcal{F}_\phi$.

(b) There is a bijection between the set of all meta-epistemic states and the set of all (deductively closed) belief sets *of MEL, i.e. Γ such that $Con(\Gamma) = \Gamma$. For any family $\mathcal{F}$, the correspondence is given by: $\mathcal{F} \mapsto Con(\delta_\mathcal{F})$.*

Proof:

(a) We use Observation 5, and part (2) of Theorem 7.

(b) For this, we note that $\mathcal{F} = \mathcal{F}_{\delta_\mathcal{F}} = \mathcal{F}_{Con(\delta_\mathcal{F})}$, by soundness of MEL and part (2) of Theorem 7. That $Con(\Gamma) = Con(\delta_{\mathcal{F}_\Gamma})$ (part (1) of Theorem 7), suffices to show that the correspondence is surjective. ■

Item (b) of this corollary suggests that the situation of MEL is similar to the one of propositional logic at the semantic level, namely the one-to-one correspondence between sets of possible worlds and deductively closed sets of formulae. This result could be used for a proper development of MEL-theory revision in the style of Gärdenfors [20] (whereby one could revise what is explicitly known and what is explicitly unknown) since his treatment relies on deductively closed sets of formulae, in place of sets of possible worlds.

5.4 Normal forms in MEL

The next question would be the search for a *normal form* for well-formed formulae of MEL. Namely, since any subset Γ of MEL-formulae represents a family $\mathcal{F}_\Gamma$ of subsets of valuations and any meta-epistemic state $\mathcal{F} := \{E_1, \ldots, E_n\}$ can be exactly encoded as a formula of the form $\delta_\mathcal{F} := \bigvee_{1 \le i \le n} (\Box\alpha_{E_i} \wedge \bigwedge_{w \in E_i} \Diamond\alpha_w)$, it is interesting to see if the latter expression can lead or not to a normal form for the logic.

Using just one modality, the previous expression can also be expressed as $\bigvee_{1 \le i \le n}(\Box\alpha_{E_i} \wedge \bigwedge_{w \in E_i} \neg\Box\neg\alpha_w)$. In fact this is a disjunctive normal form for MEL-formulae which suggests the generic form

$$\bigvee_{1 \le i \le n} (\Box\alpha_i \wedge \bigwedge_{1 \le j \le n_i} \neg\Box\alpha_{ij}),$$

i.e. the propositional logic disjunctive normal form on atoms of the form $\Box\alpha$ for any propositional formula α. The corresponding conjunctive form is thus

$$\bigwedge_{1 \le i \le n} (\neg\Box\beta_i \vee \bigvee_{1 \le j \le n_i} \Box\beta_{ij}).$$

This gives an idea of the kind of information a belief source can provide in MEL, namely a disjunction of several beliefs[4] and of one plausible, but not ascertained, proposition. However, it is not clear that the canonical form obtained for representing families of epistemic states by a single MEL-formula is an attractive normal form for achieving efficient inference methods in this logic. It seems that it is not so obvious to find a useful normal form here. For instance, $\Diamond\alpha$ can be put in the form $\bigvee_{[\alpha] \cap E \neq \emptyset}(\Box\alpha_E \wedge \bigwedge_{w \in E} \neg\Box\neg\alpha_w)$, but this expression is not computationally appealing at all. The main problem is

[4] not to be confused with the belief in a disjunction of propositions

that if there are n propositional letters, then there are 2^{2^n} meta-models of the MEL language. So there is clearly an extra source of exponential complexity. The normal forms suggested above seem to be special cases of more general forms discussed by Moss [31]. That paper focuses on weak completeness and decidability results for several modal logics, especially those derived from K. At the proof-theoretic level, it is not clear at this stage how our logic MEL could benefit from normal forms proposed in more general modal logic systems, as this question seems to be open to a large extent to-date. (As Moss says in [31], "The topic is missing from most recent textbooks, and only a handful of papers discuss it.").

At the syntactic level, it may be useful to introduce an extra symbol similar to $\Box$, reversing the inclusion symbol at the semantic level. Namely $\Delta\alpha$, such that $E \models \Delta\alpha$ if and only if $[\alpha] \subseteq E$. This modality has been introduced in epistemic modal logic and uncertainty theories [11] in order to account for the idea of "guaranteed possibility" (as for instance explicit permission in a deontic acception). The counterpart of Δ in the setting of formal concept analysis is called "sufficiency operator" by some authors (see Düntsch and Orłowska [17]). It is clear that $\Delta\alpha$ is semantically equivalent to $\bigwedge_{w\in[\alpha]} \Diamond\alpha_w$, since obviously $\Delta(\alpha \vee \beta)$ is semantically equivalent to $\Delta\alpha \wedge \Delta\beta$ and $\Diamond\alpha_w$ to $\Delta\alpha_w$. Then if $E = [\alpha]$, $\mathcal{F}_{\Box\alpha\wedge\Delta\alpha} = \{E\}$, and also $\delta_{\mathcal{F}} \equiv \bigvee_{1\leq i\leq n}(\Box\alpha_{E_i} \wedge \Delta\alpha_{E_i})$. This could be a line to follow in search of a proper normal form.

6 From meta-epistemic states to belief functions

A connection between MEL and belief functions was pointed out in Section 1. A belief function [34] Bel is a non-additive monotonic set-function (a capacity) with domain $2^{\mathcal{V}}$ and range in the unit interval, that is super-additive at any order (also called ∞-monotone), that is, it verifies a weak version of the additivity axiom of probability measure. It generalizes probability measures. The degree of belief $Bel(A)$ in proposition A evaluates to what extent this proposition is logically implied by the available evidence. The plausibility function $Pl(A) := 1 - Bel(A^c)$ evaluates to what extent events are consistent with the available evidence. The pair (Bel, Pl) can be viewed as quantitative randomized versions of KD modalities $(\Box, \Diamond)$ [36], hence of

MEL. Interestingly, elementary forms of belief functions arose first, in the works of Bernoulli, for the modeling of unreliable testimonies [35], while MEL encodes the testimony of agent $\mathcal{E}$.

The function Bel can be mathematically defined from a (generally finite) random set on $\mathcal{V}$, that has a very specific interpretation. A so-called basic assignment $m(E)$ is assigned to each subset E of $\mathcal{V}$, and is such that $m(A) \geq 0$, for all $A \subseteq \mathcal{V}$; moreover:

$$\sum_{E \subseteq \mathcal{V}} m(E) = 1.$$

The degree $m(E)$ is understood as the weight given to the fact that all an agent knows is that the value of the variable of interest lies somewhere in set E, and nothing else. In other words, the probability allocation $m(E)$ could eventually be shared between elements of E, but remains suspended for lack of knowledge. For instance, agent $\mathcal{R}$ receives a testimony in the form of statements α such that $E = [\alpha]$; $m(E)$ reflects the probability that E correctly represents the available knowledge. A set E such that $m(E) > 0$ is called a focal set. In the absence of conflicting information it is generally assumed that $m(\emptyset) = 0$. It is then clear that a collection of focal sets is a meta-epistemic state in our terminology. Interestingly, a belief function Bel can be expressed in terms of the basic assignment m [34]:

$$Bel(A) = \sum_{E \subseteq A} m(E).$$

This formula is clearly related with the meta-model $\mathcal{F}_{\Box\alpha} = \{E \subseteq \mathcal{V} : \emptyset \neq E \subseteq [\alpha]\}$ (cf. Observation 5) of atomic belief $\Box\alpha$. The converse problem, namely, reconstructing the basic assignment from the belief function, has a unique solution via the so-called Möbius transform

$$m(E) = \sum_{A \subseteq E} (-1)^{|E \setminus A|} Bel(A).$$

It is clear that the assertion of a MEL formula $\Box\alpha$ is faithfully expressed by $Bel([\alpha]) = 1$. Especially, $Bel([\alpha])$ can be interpreted as the probability of $\Box\alpha$ [36]. Moreover, there is a similarity between the problem of reconstructing a mass assignment from the knowledge of a belief function and the problem of singling out an epistemic state in the language of MEL as in Section 5.1. Namely, consider the MEL-formula $\Box\alpha_E \ \wedge \ \neg \bigvee_{w \in E} \Box\neg\alpha_w \equiv \delta_E$, whose set of meta-models is $\{E\}$. We shall show that this expression can be written as an exact symbolic counterpart of the Möbius transform. To see it, in fact, rewrite the Möbius transform as

$$m(E) = \sum_{A \subseteq E : |E \setminus A| \text{ even}} Bel(A) \; - \sum_{A \subseteq E : |E \setminus A| \text{ odd}} Bel(A).$$

Now translate $\sum$ into $\bigvee$, $Bel(A)$ into $\Box\alpha$, "$-$" into $\wedge\neg$. We then can prove that the formula δ_E is the Boolean counterpart of the Möbius transform.

Proposition 7

$$\bigvee_{\alpha \models \alpha_E : |E \setminus [\alpha]| \text{ even}} \Box\alpha \wedge \neg \bigvee_{\alpha \models \alpha_E : |E \setminus [\alpha]| \text{ odd}} \Box\alpha \equiv \Box\alpha_E \;\wedge\; \neg \bigvee_{w \in E} \Box\neg\alpha_w$$

Proof: Indeed, note first that if $\beta \models \alpha$, $\Box\alpha \vee \Box\beta \equiv \Box\alpha$ in MEL, so, $\bigvee_{\alpha \models \alpha_E : |E \setminus [\alpha]| \text{ even}} \Box\alpha \equiv \Box\alpha_E$.

Now the set of meta-models of the formula $\Box\alpha_E \wedge \bigvee_{w \in E} \Box\neg\alpha_w$ is

$$\{A : A \subseteq E\} \cap \cup_{w \in E} \{A \subseteq \mathcal{V} : w \notin A\} = \cup_{w \in E} \{A \subseteq E : w \notin A\}.$$

It is not difficult to see that the above is also the set of meta-models of the formula $\bigvee_{w \in E} \Box\alpha_{E \setminus \{w\}}$, and equivalently of the more redundant formula $\bigvee_{\alpha \models \alpha_E : |E \setminus [\alpha]| \text{ odd}} \Box\alpha$. So the Möbius-like MEL-formula is semantically equivalent to $\Box\alpha_E \wedge \neg(\Box\alpha_E \wedge \bigvee_{w \in E} \Box\neg\alpha_w) \equiv \delta_E$. ■

So one may consider belief (resp: plausibility) functions as numerical generalisations of MEL boxed (diamonded) formulae, and formulae describing single epistemic states (totally informed meta-epistemic states) can be obtained via a symbolic counterpart to Möbius transform.

7 Related work

In this section, we review past works that either share similar technical tools or are closely related to our proposal. In the first group of works are modal logics having similar syntax such as consensus logics, or higher-order semantics, viz. neighborhood semantics. In the second group, there is a huge literature on reasoning about knowledge, that we can only briefly mention. Also, logics of incomplete information like partial logic and possibilistic logic consider incomplete epistemic states as models, and this is also true for more general uncertainty logics. Finally, we stress the similarity between the problem addressed by MEL of reconstructing an epistemic state from information made explicit by an agent, and the construction of a belief function from subjective probability assessments provided by an agent.

7.1 Consensus logic

Pauly [33] presents a logic for *consensus voting* that has a language and axiomatization identical to those of MEL. However, the semantics is set in a different context altogether. We take a brief look at it, in order to make a comparison with MEL. Assume, as before, that $|PV| = k$, and $\mathcal{V}$ is the set of all propositional valuations, so that $|\mathcal{V}| = 2^k$. At denotes the set of MEL atoms.

Let N be a finite set of n elements (n a positive integer), interpreted as voters. Let $\mathcal{W}^n$ be a collection of n propositional valuations $w_i,\ i = 1, \ldots, n$ expressing the votes for the various propositional formulae. A *collective* valuation expressing consensus over N is an assignment $W^n : At \to \{0, 1\}$, defined for any $\Box\alpha \in At$ by:

$$W^n(\Box\alpha) = 1, \text{ if and only if } w_i(\alpha) = 1,\ i = 1, \ldots, n.$$

W^n is called an *n-consensus model* or a consensus model for n individuals, and E_{W^n} denotes the corresponding *set* of PL-interpretations, that is $E_{W^n} := \{v \in \mathcal{V} : v = w_i, \exists w_i \in \mathcal{W}^n\}$. Then define $W^n \models_C \Box\alpha$ by $W^n(\Box\alpha) = 1$. $W^n : At \to \{0, 1\}$ is extended to the set of all MEL-formulae in a routine manner, to obtain $W^n \models_C \phi$. $\Gamma \models_C \phi$, for a set $\Gamma \cup \{\phi\}$ of MEL-formulae, is then defined as usual.

Remark 3 The n valuations w_i for an n-consensus model need not be distinct. In other words, the collection $\mathcal{W}^n$ formed by $w_i,\ i = 1, \ldots, n$, is a multiset on $\mathcal{V}$. So, though the motivation is entirely different, we may contrast $W^n \models_C \phi$ with $E_{W^n} \models \phi$ in MEL-semantics: E_{W^n} is a *set* of valuations, while W^n is associated with a *multiset* $\mathcal{W}^n$ of valuations.

W^n thus represents the result of a unanimous voting of a group N of n individuals, where each individual is equipped with some propositional valuation. "$\Box\alpha$ is true in consensus model W^n" indicates that the group unanimously accepts α, and it occurs provided (the valuations of) each individual makes α true.

Observation 6 *For any n-consensus model W^n, $W^n \models_C MEL$, i.e. $W^n \models_C \phi$, for all ϕ such that $\vdash_{MEL} \phi$.*

The following is proved in [33].

Proposition 8 *Let $n \geq 2^k$. For any assignment $W : At \to \{0,1\}$, $W \models_C MEL$, if and only if W is an n-consensus model.*

It is also observed that one can find an assignment $W : At \to \{0,1\}$ such that $W \models_C MEL$, but W is not an n-consensus model for any $n < 2^k$. Indeed, consider the MEL-formula $\phi_0 := \bigvee_{w \in \mathcal{V}} \Box(\bigvee_{w' \in \mathcal{V} \setminus \{w\}} \alpha_{w'})$[5]. Notice that for every n-consensus model W^n with $n < 2^k$, we have $W^n \models_C \phi_0$. However, take an m-consensus model W^m such that its associated collection $\mathcal{W}^m$ of propositional valuations contains all the 2^k valuations of $\mathcal{V}$, i.e. $E_{W^m} = \mathcal{V}$ so that $m \geq 2^k$. Then $W^m \not\models_C \phi_0$. So the assignment W^m cannot be an n-consensus model for any $n < 2^k$ (more precisely, there is no collection $\mathcal{W}^n$ of $n < 2^k$ valuations such that $W^n = W^m$). But $W^m \models_C MEL$, by Observation 6.

As indicated in Remark 3, an n-consensus model W^n is equivalent to an epistemic state E_{W^n} of an agent used in the MEL-semantics, if and only if it is associated with a collection of *distinct* propositional valuations. So '$W^n \models_C MEL$', per se, is not a meaningful statement in MEL, and we do not have a version of Proposition 8 for MEL.

The set of meta-epistemic states $\{E \subseteq \mathcal{V} : |E| = n\}$ may thus be identified with the collection Con_n^d of n-consensus models that are associated with n distinct propositional valuations. Let Con_n denote the set of all n-consensus models. We then have $Con_n^d \subsetneq Con_n$: for $n > 2^k$, $Con_n^d = \emptyset$. Let $n \leq 2^k$, and $W^n \in Con_n$, a consensus model where not all propositional valuations in the collection $\mathcal{W}^n$ are distinct. So $|E_{W^n}| < n$. Then one can show that W^n cannot become an n-consensus model in Con_n^d. Indeed, consider the MEL-formula $\phi_{W^n} := \Box \bigvee_{w_i \in \mathcal{W}^n} \alpha_{w_i}$. Clearly, $W^n \models_C \phi_{W^n}$. Moreover, for any $W_0^n \in Con_n^d$, $|E_{W_0^n}| = n$, so that $E_{W_0^n} \setminus E_{W^n} \neq \emptyset$. If $w'_j \in E_{W_0^n} \setminus E_{W^n}$, $w'_j \not\models \bigvee_{w_i \in \mathcal{W}^n} \alpha_{w_i}$. Therefore $W_0^n \not\models_C \phi_{W^n}$.

Rephrasing Theorem 4, we may say that

[5] ϕ_0 is logically equivalent to the formula $\neg \bigwedge_{w \in \mathcal{V}} \Diamond \alpha_w$. Interpreting $\Diamond \alpha_w$ as "interpretation w is possible", ϕ_0 means that at least one interpretation is impossible; namely for any epistemic state $E, E \models \phi_0$ iff E is a proper subset of $\mathcal{V}$.

Proposition 9 $\vdash_{MEL} \phi$, *if and only if for each* $n \leq 2^k$ *and* $W^n \in Con_n^d$, $W^n \models_C \phi$.

However, the general completeness result obtained for MEL (cf. Corollary 2) will not find an analogue in the setting of consensus logic.

7.2 Neighborhood semantics

At first glance this view seems to bring us closer to neighborhood semantics of modal logics proposed by D. Scott and R. Montague [9]. Indeed, in this approach, Kripke structures are replaced by neighborhood frames, that equip a set of possible worlds $\mathcal{V}$ with a neighborhood function N. The latter assigns to each element v of $\mathcal{V}$ a set $N(v)$ of subsets of $\mathcal{V}$ viewed as 'necessary propositions'. This kind of semantics is more general than relational semantics, hence can serve as semantics for modal logics weaker than the normal modal logic K. But again, despite the similarity between a meta-epistemic state and the set of 'necessary propositions', there does not appear to be a natural correlation between MEL and neighborhood semantics. Indeed, in MEL, the satisfiability condition is of the form $E \models \phi$, i.e. a model is a nonempty subset of $\mathcal{V}$, while in such modal logics, a model involves the whole collection of subsets of $\mathcal{V}$, and the satisfiability at world v of $\phi := \Box\alpha$, means that the set of models of α is part of $N(v)$. So the apparent connection between MEL and neighborhood semantics is fortuitous.

7.3 Modal logics of knowledge and belief

The modal logic approach to the representation of knowledge (and belief to some extent) is due to Hintikka [26]. Knowledge is then viewed as true belief. This approach, as well as subsequent work, relied on the $KD45$ modal logic. At the semantic level it uses Kripke semantics based on an accessibility relation R among possible worlds. A proposition α is necessarily true (i.e. $\Box\alpha$ is true) at world w if and only if it is true at all worlds w' such that wRw'. Indeed, modal logic accounts for relations having various properties. The possibility of nested modalities accounts for the composition of a relation with itself. As already said, our approach does not require axioms **4** and **5** (positive and negative introspection), since we are not concerned with an agent reasoning about his or her own beliefs. In the scope of belief representation, the meaning of the accessibility relation has been discussed in the literature. Basically, authors like Halpern and colleagues [25] consider wRw' to mean that w' is possible for the agent in state w, or that w and w' are not distinguishable [28].

In our approach, the fact that we rule out nested modalities and do not consider introspection does not make this kind of semantics very natural. It is not very clear what it means to say that an agent is in a possible world w. It may mean something like "given that the real world is w". This "actual" world always appear in semantic accounts of such epistemic modal logics. Recently, Aucher [1] indicates the possibility of different points of view on reasoning about agents' beliefs: the external point of view and the internal point of view. In the first case, there is an agent who describes what (s)he believes the real world is and also what are other agents' beliefs, or even mutual beliefs; this agent is not one of these agents and is not represented in the language. The internal point of view is the one of an agent who is one among other agents. In other words, the language describes what (s)he believes the other agents believe and what the other agents believe (s)he may believe. The point of view of usual epistemic logic discussed above is the external point of view with perfect knowledge. Aucher [1] deals with the internal point of view essentially. He also discusses the external point of view under incomplete knowledge and possibly erroneous beliefs. Our logic deals with a subcase of the latter point of view where an agent $\mathcal{R}$ models the beliefs of another agent $\mathcal{E}$, as expressed by the latter, but only $\mathcal{E}$ appears in the language (the $\Box$ symbol refers to $\mathcal{E}$). All we assume here is that what agent $\mathcal{E}$ believes about the world is summarized by a subset of valuations called 'epistemic state'; what agent $\mathcal{R}$ knows about the other agent's beliefs is thus a set of epistemic states, one of which is the correct one, hence it is a family of subsets of valuations (a 'meta-epistemic state'). The agent $\mathcal{R}$ is not concerned with the real world.

Nevertheless, our setting is clearly similar to the one proposed by Halpern and colleagues [25] reinterpreting knowledge bases as being fed by a "Teller" that makes statements supposed to be true in the real world. The knowledge base is what we call receiver and the teller what we call emitter. Important differences are that we are mainly concerned with beliefs held by the Teller (hence making no assumptions as to the truth of such beliefs), that these beliefs are incomplete, and that the Teller is allowed to explicitly declare partial ignorance about specific statements.

Finally, even if not concerned with nonmonotonic reasoning, MEL may be felt as akin to early nonmonotonic modal logics such as Moore's autoepistemic logic (AEL) [30], insofar as they share the ambition of reasoning about partial ignorance with the same interpretation for modal operators. Expansions of an AEL theory can be viewed as meta-models expressing epistemic states. However, there are several important differences. In autoepistemic logic an agent is reasoning about his or her own beliefs, or lack thereof, not about another agent's beliefs. So AEL naturally allows for the nesting of modalities, contrary to MEL. Moreover, sentences of the form $\Box\alpha \vee \neg\alpha$ (meaning that if α is not believed, then it is false) involving boxed and non-boxed formulae are allowed in AEL (and are the motivation for this logic), thus mixing propositional and modal formulae, which precisely MEL forbids, as the receiver agent is only allowed to store beliefs supplied by the emitter agent. These features of AEL, that are absent in MEL, are actually shared by many other nonmonotonic modal logics in the eighties, surveyed in [37] Chap. IV.

7.4 Partial logic

Partial logic Par [5], like MEL, uses sets of valuations in place of valuations, under the form of partial models. A partial model $[\sigma]$ assigns truth-values to a subset of propositional variables. The corresponding meta-model is formed of all completions of σ. Unfortunately, Par adopts a truth-functional view, and assumes the equivalence $\sigma \models \alpha \vee \beta$ if and only if $\sigma \models \alpha$ or $\sigma \models \beta$. So it loses classical tautologies, which sounds paradoxical when propositional variables are Boolean [10]. Actually, the basic Par keeps the syntax of classical logic, which forbids to make a difference between the fact of believing $\alpha \vee \beta$ and that of believing α or believing β.

7.5 Possibilistic logic and logics of uncertainty

Possibilistic logic has been essentially developed as a formalism for handling qualitative uncertainty with an inference mechanism that remains close to the one of classical logic [12, 14]. A standard possibilistic logic expression is a pair (α, a), where α is a propositional formula and $a \in (0, 1]$. Any discrete linearly ordered scale can be used in place of $[0, 1]$. The weight a is interpreted as a (positive) lower bound of the degree of certainty of α, i.e. $N(\alpha) \geq a$, where the function N is a necessity measure. N is a set-function that satisfies $N(\top) = 1; N(\bot) = 0; N(\alpha \wedge \beta) = \min(N(\alpha), N(\beta))$. A possibilistic belief base is a conjunction of such weighted formulae. Possibilistic knowledge bases have semantics in terms of a weak order over the set $\mathcal{V}$ of interpretations, encoded by means of a single possibility distribution $\pi : \mathcal{V} \to [0, 1]$. π can be viewed as a fuzzy set of models and derived as follows from a weighted formula (α, a): $\pi_{(\alpha,a)}(w) = 1$ if $w \models \alpha$, and $1 - a$ otherwise. In other words, an interpretation violating α is all the less tolerated as α is more certain. It can be checked that $N(\alpha) = \min_{w \not\models \alpha} 1 - \pi_{(\alpha,a)}(w) = a$. The possibility distribution induced by possibilistic belief base is obtained by the pointwise minimum of the possibility distributions induced by each possibilistic formula in it.

Disjunctions and negations of possibilistic formulae are not allowed in basic possibilistic logic, which is only a simple totally ordered extension of classical logic. One extended syntax of possibilistic logic also allows for propositions weighted by lower bounds of possibility degrees $\Pi(\alpha) = 1 - N(\neg\alpha)$, and express weak constraints of the form $\Pi(\alpha) \geq \beta$ [12].

Possibilistic logic is another so-called "encapsulated" two-tiered logic like MEL. It is propositional logic embedded within a multivalent logic, as the semantics of weighted formulae (α, a) is clearly many-valued. This multivalent logic is a fragment of Gödel logic (as coined by Hájek [22]), where the only allowed connective is conjunction expressed by the minimum. The characteristic axiom of necessity measures is a graded counterpart of axioms (M) and (C) of MEL, taken together. More generally, the encapsulation of classical propositions (then often called 'events') by means of degrees of belief is typical of uncertainty theories. As a consequence, reasoning under graded uncertainty in a logical format comes down to handling many-valued higher order propositions. For instance, a degree of probability $Prob(\alpha)$ can be modeled as the truth-value of the proposition "$Probable(\alpha)$" (which expresses the statement that α is probably true), where *Probable* is a many-valued predicate [23].

Assuming only maximal weights $a = 1$, possibilistic logic actually coincides with the fragment of MEL containing only conjunction of boxed formulae that was proved equivalent to propositional logic itself. At the semantic level, a possibility distribution is a graded extension of an epistemic state used as a meta-model in MEL. A possibilistic formula (α, a) can be viewed as satisfied by *any* possibility distribution such that[6]:

$$\pi \models (\alpha, a) \iff N(\alpha) = \min_{v \not\models \alpha} 1 - \pi(v) \geq \alpha.$$

So the semantics of possibilistic logics can be described in terms of generalized MEL meta-models. This kind of semantics was proposed by Boldrin and Sossai [6] under the name *forcing semantics.* It is thus patent that MEL and possibilistic logic are extensions of the same root logic in two complementary directions: syntactic handling of incomplete information on the one hand, and graded belief on the other hand, respectively. Moreover, formulae in MEL of the form $\Diamond\alpha$ correspond to constraints of the form $\Pi(\alpha) = 1$ in the extended possibilistic logic language. The introduction of connectives other than idempotent conjunctions between possibilistic formulae was studied by Boldrin and Sossai [6] in the scope of data fusion, and more recently by Dubois and Prade [15] in the scope of multiagent systems. In particular, disjunctions and negations of possibilistic formulae were interpreted similarly to disjunctions and negations of MEL-formulae, in terms of union and set-complement of families of generalized meta-models of weighted formulae.

As pointed out in section 2.2, MEL derived inference rules (2) and (3) in Proposition 2 are Boolean versions of the two resolution rules in possibilistic logic:

- $(\neg\alpha \vee \beta, a), (\alpha \vee \gamma, b)\} \vdash_{POSLOG} (\beta \vee \gamma, \min(a, b))$, where a and b are degrees of necessity [12]: Rule (2) in Proposition 2 is retrieved when $a = b = 1$.

- $\{\Pi(\neg\alpha \vee \beta) \geq a, N(\alpha \vee \gamma) \geq b\} \models_{POSLOG} \Pi(\beta \vee \gamma) \geq a$ [7], whenever $a + b > 1$ [13]. Rule (3) in Proposition 2 is retrieved when $a = b = 1$.

So a multivalued extension of MEL (involving multimodalities associated to degrees of possibility and necessity) is likely to provide a natural framework for generalizing possibilistic logic to a full-fledged uncertainty logic handling certainty and partial ignorance at a syntactic level.

[6]The previous semantics in terms of a single possibility distribution considers the least informative among them, which exists for basic possibilistic knowledge bases; however this semantics cannot extend to accommodate a richer language at the higher level.

[7]written in the semantic style, for clarity.

7.6 Interpreting revealed probabilistic beliefs

In this paper, agent $\mathcal{E}$ expresses beliefs and doubts in the langage MEL. However in the subjective probabilistic tradition [29], degrees of beliefs are elicited from an agent in the form of a betting odds yielding a unique probability distribution. While Bayesians take this probability distribution as representing actual beliefs of the agent, one may consider that agents often have incomplete knowledge and produce single subjective probability distributions only because they are forced into it by the elicitation procedure. This is one motivation for the development of imprecise probability theory [38]. Nevertheless, the obtained betting odds do inform about the agent beliefs. A probabilistic counterpart of the two-agent epistemic state reconstruction problem, here stated in the framework of MEL, is addressed by Dubois, Prade and Smets [16], where a Bayesian agent $\mathcal{E}$ provides a subjective probability distribution, and $\mathcal{R}$ reconstructs $\mathcal{E}$'s epistemic state as a least committed assignment function $m : 2^{\mathcal{V}} \setminus \{\emptyset\} \to [0, 1]$ defining a belief function. In this problem, the assumption is that at the credal level $\mathcal{E}$'s beliefs take the form of a Shafer belief function, while when engaged into a decision process $\mathcal{E}$ uses a subjective probability distribution in order to compute expected utilities. The transformation of a belief function into a subjective probability is called the pignistic transformation and comes down to uniformly distributing masses $m(A)$ over elements of set E. It is equivalent to Shapley value in game theory. The reconstruction method consists in choosing the least informed belief function in the set of belief functions with known pignistic probability. It turns out that the set of focal sets thus obtained is consonant, i.e. corresponds to a meta-epistemic state modelled by a nested family of subsets. Such kinds of meta-epistemic states express the idea that $\mathcal{R}$ has an incomplete but coherent idea of $\mathcal{E}$'s beliefs.

8 Conclusion and Perspectives

This paper lays the foundations for a belief logic that is in close agreement with more sophisticated uncertainty theories. It borrows from modal logic because it uses the standard modal symbols $\Box$ and $\Diamond$ for expressing ideas of certainty understood as validity in an epistemic state and possibility understood as consistency with an epistemic state. It differs from usual modal logics (even if borrowing much of their machinery) by a deliberate stand on not nesting modalities, and not mixing modal and non-modal formulae, thus yielding a two-tiered logic.

At the semantic level we have proved that the MEL language is capable of accounting for any meta-epistemic state, viewed as a family of non-empty subsets of classical valuations, just as propositional logic language is capable of accounting for any epistemic state, viewed as a family of classical valuations. In this sense, MEL is a higher-order logic with respect to classical logic.

In fact, MEL goes against the tradition of modern modal logics which, at the philosophical level are *de re* logics because even when the modalities have an epistemic flavor, they refer to the actual world via the Kripke relation. MEL is a *de dicto* logic, because formulae in MEL refer to an agent's epistemic state they try to account for, not to any objective reality. MEL underlies the assumption that only beliefs and doubts about the world can be expressed, forbidding direct access to the actual state of the world. In the belief environment of MEL, an agent is not allowed to claim that a proposition is true in the real world. We do not consider our modal formalism to be an extension of the classical logic language, but an encapsulation thereof, within an epistemic framework; hence combinations of objective and epistemic statements like $\alpha \wedge \Box\beta$ are considered meaningless in this perspective. This higher-order flavor is typical of uncertainty theories.

This subjectivist stand in MEL does not lead us to object to the study of languages where meta-statements relating belief and actual knowledge, observations and objective truths could be expressed. We only warn that epistemic statements expressing beliefs and doubts on the one hand and other pieces of information trying to bridge the gap between the real world and such beliefs (like deriving the latter from objective observations) should be handled separately.

One of the merits of MEL is to potentially offer a logical grounding to uncertainty theories of incomplete information. An obvious extension to be studied is towards possibilistic logics, using (graded) multimodalities and generalizing epistemic states to possibility distributions. In fact, modal logics capturing possibility and necessity measures have been around since the early nineties [18, 8, 24], but they were devised with a classical view of modal logic, (not so much under our two tiered view) and Kripke semantics. It would be of interest to see if MEL can help equip the extended possibilistic logic language with rigorous semantics and proof theory. One important contribution of the paper is to show that MEL is the Boolean version of

Shafer's theory of evidence, whereby a mass function is an extension of a meta-epistemic state. Interestingly, this mass function should be viewed as a *de dicto* probability assignment (to epistemic states), in opposition to usual probability assignments to states of the world. It suggests that beyond possibilistic logic, MEL could be extended to belief functions in a natural way, and it would be useful to compare MEL with the logic of belief functions devised by Godo and colleagues [21].

This study is a first step. Some technical aspects of MEL require more scrutiny, like devising proof methods and assessing their computational complexity. The framework of MEL also suggests other lines of further research. An interesting issue is to reconsider basic notions of belief change, like revision and contraction. In the classical propositional setting [20], a belief set viewed as a closed set of sentences, does not distinguish between sentences that are explicitly and implicitly ignored. This distinction is made possible by the syntax of MEL. Namely, explicitly ignoring α means deriving the formula $\Diamond\alpha \wedge \Diamond\neg\alpha$, while implicitly ignoring means not being capable of inferring any MEL-formula involving α and $\neg\alpha$ only. Booth and Nittka [7] make inferences about what an agent previously believed based on an observation of how the agent has responded to some sequence of previous belief revision inputs over time. This work uses propositional logic to encode epistemic states, and it would be interesting to figure out if their procedure for reconstructing initial beliefs can be accounted for in MEL.

Another direction is to handle conflicting beliefs in MEL. It is clear that an inconsistent propositional base $\mathcal{B}$ has no model. However, considering it as a belief base, one may try to restore a semantic view of it, by considering subsets of interpretations as meta-models, looking for those E such that $\forall\alpha \in \mathcal{B}$, $E \models \Diamond\alpha$ in MEL. The framework adopted here involving an emitter and a receiver is also somewhat reminding of Belnap's set-up based on information sources. His four-valued logic [3] extends partial logic to the handling of contradictions. In Belnap's logic, each source of information declares atoms to be true, false or ignored, but sources can be conflicting. It suggests the extension of MEL to the setting where several emitter agents provide information. More generally, assessing the role of a logic such as MEL and its possible extensions (to mutual or common beliefs) in the framework of multiagent systems is a topic for further research.

9 Acknowledgement

The first author acknowledges the support of IRIT, CNRS, during a visit to which the work was initiated. The second author wishes to thank colleagues for useful discussions on this paper, especially Guillaume Aucher, Andreas Herzig, and Vincenzo Marra. The authors also thank the anonymous reviewers for their valuable comments.

References

[1] G. Aucher. An internal version of epistemic logic. *Studia Logica*, 94(1):1–22, 2010.

[2] M. Banerjee and D. Dubois. A simple modal logic for reasoning about revealed beliefs. In C. Sossai and G. Chemello, editors, *Proc. ECSQARU 2009, Verona, Italy, LNAI 5590*, pages 805–816. Springer-Verlag, 2009.

[3] N. D. Belnap. A useful four-valued logic. In J. M. Dunn and G. Epstein, editors, *Modern Uses of Multiple-Valued Logic*, pages 8–37. D. Reidel Publishing Company, 1977.

[4] P. Blackburn, M. de Rijke, and Y. Venema. *Modal Logic*. Cambridge U. P., 2001.

[5] S. Blamey. Partial logic. In D. M. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 3, pages 1–70. D. Reidel Publishing Company, 1985.

[6] L. Boldrin and C. Sossai. Local possibilistic logic. *Journal of Applied Non-Classical Logics*, 7(3):309–333, 1997.

[7] R. Booth and A. Nittka. Reconstructing an agent's epistemic state from observations about its beliefs and non-beliefs. *J. Log. Comput.*, 18(5):755–782, 2008.

[8] C. Boutilier. Modal logics for qualitative possibility theory. *Int. J. Approximate Reasoning*, 10:173–201, 1994.

[9] B. F. Chellas. *Modal Logic: an Introduction*. Cambridge University Press, 1980.

[10] D. Dubois. On ignorance and contradiction considered as truth-values. *Logic Journal of the IGPL*, 16(2):195–216, 2008.

[11] D. Dubois, P. Hájek, and H. Prade. Knowledge-driven versus data-driven logics. *J. Logic, Language and Information*, 9:65–89, 2000.

[12] D. Dubois, J. Lang, and H. Prade. Possibilistic logic. In D. M. Gabbay, C. J. Hogger, J. A. Robinson, and D. Nute, editors, *Handbook of Logic in Artificial Intelligence and Logic Programming*, volume 3, pages 439–513. Oxford University Press, 1994.

[13] D. Dubois and H. Prade. Resolution principles in possibilistic logic. *Int. J. of Approximate Reasoning*, 4(1):1–21, 1990.

[14] D. Dubois and H. Prade. Possibilistic logic: a retrospective and prospective view. *Fuzzy Sets and Systems*, 144:3–23, 2004.

[15] D. Dubois and H. Prade. Toward multiple-agent extensions of possibilistic logic. In *Proc. IEEE International Conference on Fuzzy Systems (FUZZ-IEEE 2007)*, pages 187–192. IEEE, 2007.

[16] D. Dubois, H. Prade, and P. Smets. A definition of subjective possibility. *Badania Operacyjne i Decyzje (Operations Research and Decisions)*, 4:7–22, 2003 (revised version in *Int. J. Approximate Reasoning*, 48, 2008, 352-364).

[17] I. Düntsch and E. Orłowska. Boolean algebras arising from information systems. *Annals of Pure and Applied Logic*, 127:77–98, 2004.

[18] L. Farinas del Cerro and A. Herzig. A modal analysis of possibility theory. In Ph. Jorrand and J. Kelemen, editors, *Fundamentals of Artificial Intelligence Research (FAIR'91) LNCS 535*, pages 11–18. Springer-Verlag, 1991.

[19] D. Gabbay. *Labelled Deductive Systems.* Oxford University Press, Oxford, UK, 1996.

[20] P. Gärdenfors. *Knowledge in Flux.* MIT Press, 1988.

[21] L. Godo, P. Hájek, and F. Esteva. A fuzzy modal logic for belief functions. *Fundam. Inform.*, 57(2-4):127–146, 2003.

[22] P. Hájek. *The Metamathematics of Fuzzy Logics.* Kluwer Academic, 1998.

[23] P. Hájek, L. Godo, and F. Esteva. Fuzzy logic and probability. In P. Besnard and S. Hanks, editors, *UAI*, pages 237–244. Morgan Kaufmann, 1995.

[24] P. Hájek, D. Harmancova, F. Esteva, P. Garcia, and L. Godo. On modal logics for qualitative possibility in a fuzzy setting. In R. Lopez de Mantaras and D. Poole, editors, *UAI*, pages 278–285. Morgan Kaufmann, 1994.

[25] J. Y. Halpern, R. Fagin, Y. Moses, and M.Y. Vardi. *Reasoning About Knowledge.* MIT Press (Revised paperback edition), 2003.

[26] J. Hintikka. *Knowledge and Belief.* Cornell University Press, 1962.

[27] G. E. Hughes and M. J. Cresswell. *A New Introduction to Modal Logic.* Routledge, 1996.

[28] S. Kraus and D. J. Lehmann. Knowledge, belief and time. *Theor. Comput. Sci.*, 58:155–174, 1988.

[29] D. V. Lindley. Scoring rules and the inevitability of probability. *International Statistical Review / Revue Internationale de Statistique*, 50(1):1–11, 1982.

[30] R. C. Moore. Semantical considerations on nonmonotonic logic. *Artificial Intelligence*, 25:75–94, 1985.

[31] L. Moss. Finite models constructed from canonical formulas. *J. Philosophical Logic*, 36:605–640, 2007.

[32] B. Nebel. Belief revision and default reasoning: syntax-based approaches. In J. F. Allen, R. Fikes, and E. Sandewall, editors, *Proc. KR'91*, pages 417–428. Morgan Kaufmann, 1991.

[33] M. Pauly. Axiomatizing collective judgment sets in a minimal logical language. *Synthese*, 158(2):233–250, 2007.

[34] G. Shafer. *A Mathematical Theory of Evidence.* Princeton University Press, Princeton, N.J., 1976.

[35] G. Shafer. Nonadditive probabilities in the work of Bernoulli and Lambert. *Arch. Hist. Exact Sci.*, 19 (4):309–370, 1978/79.

[36] P. Smets. Comments on R. C. Moore's autoepistemic logic. In P. Smets, E. H. Mamdani, D. Dubois, and H. Prade, editors, *Non-standard Logics for Automated Reasoning*, pages 130–131. Academic Press, 1988.

[37] Léa Sombé. *Reasoning under Incomplete Information in Artificial Intelligence.* John Wiley & Sons, 1990.

[38] P. Walley. *Statistical Reasoning with Imprecise Probabilities.* Chapman and Hall, London, 1991.

[39] M.-A. Williams. Iterated theory base change: a computational model. In *Proc. IJCAI*, volume 2, pages 1541–1549. Morgan Kaufmann, 1995.