

## **A framework for iterated belief revision using possibilistic counterparts to Jeffrey's rule**

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**Abstract.** Intelligent agents require methods to revise their epistemic state as they acquire new information. Jeffrey's rule, which extends conditioning to probabilistic inputs, is appropriate for revising probabilistic epistemic states when new information comes in the form of a partition of events with new probabilities and has priority over prior beliefs. This paper analyses the expressive power of two possibilistic counterparts to Jeffrey's rule for modeling belief revision in intelligent agents. We show that this rule can be used to recover several existing approaches proposed in knowledge base revision, such as adjustment, natural belief revision, drastic belief revision, and the revision of an epistemic state by another epistemic state. In addition, we also show that some recent forms of revision, called improvement operators, can also be recovered in our framework.

**Keywords:** Belief revision, Jeffrey's rule, possibility theory.

### **1. Introduction**

The information available to intelligent agents is often uncertain, inconsistent and incomplete. It is then crucially important to define tools to manage it in response to the acquisition of new, possibly conflicting,

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information. The term ‘information’ covers a broad range of entities such as knowledge, perceptions, beliefs, expectations, preferences, or causal relations. It can describe the agent’s view of the world, its actions and its understanding of changes. During the past twenty years, many approaches have been proposed to address the problem of belief change from the axiomatic point of view (e.g., [17], [9]), from the semantic point of view (e.g., [32], [6], [31]) and from the computational point of view [24], [2] (see also [11] for a thorough discussion on different forms of revision applied to different kinds of epistemic situations).

This paper focuses on semantic aspects of belief revision in the framework of possibility theory. The basic object in possibility theory is a possibility distribution, which is a mapping from the set of possible worlds to a totally ordered structure, usually the interval  $[0, 1]$ . In logical settings, a possible world is an interpretation of a propositional language. A possibility degree expresses notions of normality, plausibility, lack of surprise and the like.

The revision of a possibility distribution can be viewed as a so-called “transmutation” [32] that modifies the ranking of possible worlds so as to give priority to the input information. Jeffrey-like possibilistic revision consists in modifying a possibility distribution representing the epistemic state of an agent, based on the knowledge of updated possibility values over a partition of the set of possible worlds. This new information acts as a partial epistemic state enforcing constraints on the possible resulting posterior epistemic states. This mode of revision is a counterpart to Jeffrey’s so-called updating rule in probability theory [19], where epistemic states are represented by probability distributions and the input information consists in enforcing new probability values on a partition of the set of possible worlds. Two forms of possibilistic revision (based on minimum and product respectively) are investigated as counterparts to Jeffrey’s rule of revision in probability theory. The choice between these two methods crucially depends on the type of scale used to evaluate degrees of possibility: ordinal, qualitative or quantitative. In each case, possibilistic revision comes down to modifying the prior possibility distribution such that the proposition pertaining to each element of the partition must be plausible to a degree prescribed by the input information. Input possibility degrees may either be determined for example by an expert, or via some function of the original possibility degrees associated with each partition element. For instance, in the so-called improvement revision [22], the new weight associated is simply the initial weight augmented with a plausibility “unit”, simply expressing that the corresponding proposition is slightly more believed (see section 4.4).

This paper<sup>1</sup> first extends natural properties described in [2] in order to take into account the new form of the input, namely a partial epistemic state. Then we recall two definitions of possibilistic revision operators [15] that are similar to the probabilistic revision rule due to Jeffrey. They naturally extend, to inputs with possibility degrees, two forms of conditioning that have been defined in the possibility theory framework. We check whether these possibilistic counterparts to Jeffrey’s generalized conditioning verify the proposed natural properties of epistemic state revision. In its second half, the paper shows that most of existing belief revision operators can be recovered by one of the two forms of possibilistic Jeffrey-like revision. But first, in order to establish the new results, we need to restate the necessary background on possibility theory.

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## 2. Possibilistic representations of epistemic states

Let  $\Omega$  be set of possible worlds. Subsets of possible worlds will be denoted by Greek letters such as  $\phi$ , or  $\psi$ . For simplicity, we assume that  $\Omega$  is a finite set, for instance the set of interpretations of a finite propositional language  $L$ . We do not distinguish here between subsets of possible worlds (i.e. *events*) and propositions expressed in this language;  $\omega \models \phi$  denotes the membership of  $\omega$  in set  $\phi$ , modeling the satisfiability of proposition  $\phi$  in possible world  $\omega$ . Then  $\omega$  is said to be a *model* of  $\phi$ . The negation of  $\phi$  is  $\neg\phi$ . We also use set-theoretic union and intersection symbols  $\cup, \cap$ .

An epistemic state on possible worlds is often encoded by means of a total preorder  $\succeq$  telling which worlds are more normal, less surprising than other ones. Throughout this paper, we also use a general representation of a total preorder of possible worlds, namely a possibility distribution. A possibility distribution is a mapping  $\pi$  from  $\Omega$  to a totally ordered bounded scale  $L$  (with top 1 and bottom 0). Given an possible world  $\omega \in \Omega$ ,  $\pi(\omega)$  represents the degree of compatibility of  $\omega$  with the available information (or beliefs) about the real world.  $\pi(\omega) = 0$  means that  $\omega$  is impossible, and  $\pi(\omega) = 1$  means that nothing prevents  $\omega$  from being the real world. When  $\pi(\omega_1) > \pi(\omega_2)$ ,  $\omega_1$  is preferred to  $\omega_2$  as a candidate for being the real state of the world. The less  $\pi(\omega)$ , the less plausible  $\omega$ , or the less likely it is the real world. A possibility distribution  $\pi$  is said to be *normalized* if  $\exists \omega \in \Omega$ , such that  $\pi(\omega) = 1$ , in other words if at least one possible world is a fully plausible candidate for being the actual world. Interpretations  $\omega$  where  $\pi(\omega) = 1$  are considered to be normal (they are not at all surprising). Note that in this setting the endpoints of the scale have a strong meaning. A subnormalized possibility distribution is considered self-conflicting to some extent (and is ruled out in the first stance). The case where  $\forall \omega, \pi(\omega) = 0$  encodes a full contradiction. A consistent epistemic state is thus always encoded by a normalized possibility distribution.

The purely ordinal representation (a plausibility relation  $\succeq$ ) is less expressive than the qualitative encoding of a possibility distribution on a totally ordered scale as the former cannot express impossibility. A possibility distribution can be used for representing any total preorder of possible worlds. The possibility distribution  $\pi$  is said to *represent* the total preorder  $\succeq$  when it is such that  $\omega_1 \succeq \omega_2$  if and only if  $\pi(\omega_1) \geq \pi(\omega_2)$ . As an ordinal representation  $\succeq$  of an epistemic state is understood as being consistent, and formally excluding no possible world, we represent  $\succeq$  by a normalized and *positive* possibility distribution  $\pi$ , i.e. such that  $\forall \omega, \pi(\omega) > 0$ .

Given a possibility distribution  $\pi$ , the possibility degree of proposition  $\phi$  is defined as:

$$\Pi(\phi) = \max\{\pi(\omega) : \omega \models \phi\}.$$

It evaluates the extent to which  $\phi$  is consistent with the available information expressed by  $\pi$ . Note that  $\Pi(\phi)$  is evaluated under the assumption that the situation where  $\phi$  is true is as normal as it can be (since  $\Pi(\phi)$  reflects the maximal plausibility of a model of  $\phi$ ).

Given a possibility distribution  $\pi$ , we define a belief set [17], denoted by  $Bel(\pi)$ , which is the set of accepted beliefs [12], [1] in the epistemic state  $\pi$ , obtained by considering all propositions that are more plausible than their negation, namely:

$$Bel(\pi) = \{\phi : \Pi(\phi) > \Pi(\neg\phi)\}.$$

It is a deductively closed set of propositions whose set of models  $M^*(\pi)$  contains the best possible worlds in terms of  $\pi$ . When  $\pi$  is normalized,  $M^*(\pi)$  contains the completely possible worlds, namely

$M^*(\pi) = \{\omega : \pi(\omega) = 1\}$ . Moreover,  $Bel(\pi) = \{\phi, M^*(\pi) \subseteq \phi\}$ . The proposition  $\phi$  belongs to  $Bel(\pi)$  when  $\phi$  holds in all the most normal or plausible situations (hence  $\phi$  is expected, or accepted as being true).

There are several representations of epistemic states in agreement with the above setting such as: well-ordered partitions of  $\Omega$  [29], Lewis' systems of spheres [18], Spohn's Ordinal Conditional Functions (OCF) [29, 30], etc. But all these representations of epistemic states do not have the same expressive power. In fact we can distinguish several representation settings according to the expressiveness of the scale used:

1. The pure ordinal finite setting (ORDFI for short): Only a plausibility relation  $\succeq$  on possible worlds is used. The quotient set  $\Omega / \sim$ , built from the equivalence relation  $\sim$  extracted from  $\succeq$ , forms a well-ordered partition (WOP)  $\psi_0, \dots, \psi_k$  such that  $\Pi(\psi_0) = 1 > \dots > \Pi(\psi_k) > 0$ , where  $\Pi$  is the possibility measure induced by any possibility distribution  $\pi$  (with values on any totally ordered scale) that represents  $\succeq$ . This is the setting used by Grove [18], and Gärdenfors [17]<sup>2</sup>.
2. The qualitative finite setting (QUALFI for short) with possibility degrees in a finite totally ordered scale :  $L = \{\alpha_0 = 1 > \alpha_1 > \dots > \alpha_{m-1} > 0\}$ . This setting is used in possibilistic logic [14].
3. The denumerable setting (DENUM for short), using a scale  $L = \{\alpha^0 = 1 > \alpha^1 > \dots > \alpha^i > \dots > 0\}$ , for some  $\alpha \in (0, 1)$ <sup>3</sup>. This is isomorphic to the use of integers in so-called  $\kappa$ -functions by Spohn [29]. Note that this scale is quite expressive as it is equipped with semi-group operations min, max, product, and also division.
4. The dense ordinal setting (DORD for short) using  $L = [0, 1]$ , seen as an ordinal scale. In this case, the possibility distribution  $\pi$  is defined up to any monotone increasing transformation  $f : [0, 1] \rightarrow [0, 1]$ ,  $f(0) = 0$ ,  $f(1) = 1$ . This setting is also used in possibilistic logic [14].
5. The dense absolute setting (DABS for short) where  $L = [0, 1]$ , seen as a genuine numerical scale equipped with product. In this case, a possibility measure can be viewed as special case of a Shafer [28] plausibility function, actually a consonant plausibility function, and  $1 - \pi$  a potential surprise function in the sense of Shackle [27].

**Example 2.1.** *Let us consider the following example where we are interested to know if a given researcher, named JM, is attending or not a given Artificial Intelligence conference. We assume that JM works in a computer science laboratory in the North of France, while the conference is held in Newark, N.J. We are also interested to know whether JM is accommodated in the hotel recommended by the conference. Lastly, we are interested to know whether JM has a biometric passport and if he applied for electronic visa. For the sake of simplicity we will only use the following four propositional variables to encode available information:*

- *a*: to express that JM is attending the conference in Newark area;
- *h*: to express that JM booked a room in the conference hotel;

<sup>2</sup>In the AI literature, preference is often denoted  $\preceq$ , rather than  $\succeq$ , but we do not follow this convention in the possibility theory setting.

<sup>3</sup>As usual  $\alpha^i$  stands for the  $i$ th power of  $\alpha$ .

- $b$ : to express that JM has a biometric passport;
- $v$ : to express that JM applied for an electronic visa.

For the sake of simplicity, we use the *DENUM* setting to assess the possibility degree of each possible world. We assume that there is no excluded worlds. This setting is instrumental for the running example of this paper and for easy computations when illustrating different changes mechanisms.

The possibility distribution in Table 1, provides an encoding of our initial beliefs. The most plausible event is the one where JM is attending the conference (explained for instance by the fact that JM has a paper accepted at the conference), is in the conference hotel (since in general, researchers from this laboratory prefer using hotels proposed by conferences), has a biometric passport and applied for electronic visa. Events such as “JM attending the conference in Newark area while he has not applied for electronic visa” (among “others” in the Table) will simply be considered very exceptional but not impossible. Such interpretations have degree  $\alpha^{10}$ .

	$A$	$H$	$B$	$V$	$\pi(\cdot)$
$\omega_1$	$a$	$h$	$b$	$v$	1
$\omega_2$	$a$	$\neg h$	$b$	$v$	$\alpha$
$\omega_3$	$\neg a$	$\neg h$	$b$	$v$	$\alpha^2$
$\omega_4$	$\neg a$	$\neg h$	$b$	$\neg v$	$\alpha^3$
$\omega_5$	$\neg a$	$\neg h$	$\neg b$	$\neg v$	$\alpha^4$
$\omega_6$	$\neg a$	$\neg h$	$\neg b$	$v$	$\alpha^5$
<i>Others</i>	-	-	-	-	$\alpha^{10}$

Table 1. An example of a possibility distribution

Lastly, conditioning can be extended to possibility theory. In the purely ordinal case, conditioning a WOP by means of a proposition  $\phi$  comes down to restricting the corresponding plausible ordering  $\succeq$  of possible worlds to models of  $\phi$ . Two different but similar types of conditioning [16], in agreement with this conditional ordering and instrumental for revision purposes, have been defined in possibility theory (when  $\Pi(\phi) > 0$ ):

- In the ordinal setting, we have:

$$\begin{aligned}
 \pi(\omega \mid_m \phi) &= 1 \text{ if } \pi(\omega) = \Pi(\phi) \text{ and } \omega \models \phi \\
 &= \pi(\omega) \text{ if } \pi(\omega) < \Pi(\phi) \text{ and } \omega \models \phi \\
 &= 0 \text{ if } \omega \notin [\phi].
 \end{aligned} \tag{1}$$

This is the definition of *minimum-based conditioning*. It can be defined in any ordinal scale especially the QUALFI and DORD environments.

- In numerical settings such as DENUM or DABS, we can define

$$\begin{aligned}\pi(\omega \mid_p \phi) &= \frac{\pi(\omega)}{\Pi(\phi)} \text{ if } \omega \models \phi \\ &= 0 \text{ otherwise}\end{aligned}\tag{2}$$

This is the definition of *product-based conditioning*, which is also a special case of Dempster rule of conditioning restricted to consonant belief functions [28]. In the DENUM setting, it also captures Spohn [29] conditioning of  $\kappa$ -functions [15].

These two definitions of conditioning satisfy an equation of the form

$$\Pi(\psi, \phi) = \Pi(\psi \mid_{\otimes} \phi) \otimes \Pi(\phi),$$

where  $\otimes$  is  $\min$  ( $\mid_{\otimes} = \mid_m$ ) or the product ( $\mid_{\otimes} = \mid_p$ ) respectively, which is similar to Bayesian conditioning. The rule based on the product is much closer to genuine Bayesian conditioning than the qualitative conditioning defined from the minimum, which is purely based on the comparison of levels; product-based conditioning requires more of the structure of the unit interval.

Besides, when  $\Pi(\phi) = 0$ , the above conditioning rules do not really apply. One option we adopt here is to define  $\pi(\omega \mid_m \phi) = \pi(\omega \mid_p \phi) = 1, \forall \omega \models \phi$ , and 0 otherwise by convention. Then complete ignorance among models of  $\phi$  results from conditioning with such an impossible proposition. For another analysis of the zero possibility case, see [8].

### 3. Iterated semantic revision in possibility theory

Belief revision, as understood here, results from the effect of accepting a new piece of information called the input information, as part of the posterior epistemic state. In this paper, it is assumed that the current epistemic state (represented by a possibility distribution), and the input information are of the same nature (for instance, generic knowledge) but they do not play the same role. Iterated revision is possible since a new epistemic state is obtained as result. The input information takes priority over information contained in the epistemic state, and acts as a constraint on the posterior epistemic state. This asymmetry is expressed by the way the belief change problem is stated, namely the new information alters the epistemic state and not conversely. This asymmetry will appear clearly at the level of belief change operations. This situation is different from the one of uncertain information fusion from several sources (for instance [20]), where no epistemic state dominates *a priori*, and the uncertainty of the input expresses a doubt on the fact that this input is correct, not a constraint on the output epistemic state. In the information fusion context, the use of symmetric merging rules is natural especially when the sources are equally reliable.

#### 3.1. Jeffrey's rule for revising probability distributions

In probability theory, an epistemic state is a probability distribution on possible worlds. There is a natural method for revising a prior probability  $P$  in the presence of new probabilistic information denoted by  $\mu = \{(\phi_i, p_i) \mid i = 1, \dots, m\}$ , where the  $\phi_i$ 's form a set of *mutually exclusive* and *exhaustive* events and represent a partition of the set of possible worlds  $\Omega$  (namely,  $\forall \phi_i, \phi_j$ , and  $i \neq j, \phi_i \cap \phi_j = \emptyset$  and  $\bigcup_{i=1, \dots, m} \phi_i = \Omega$ ). The coefficients  $p_i$  sum to 1 and act as constraints on the posterior probability of

elements  $\phi_i$  of the partition. Such an updating rule is proposed by Jeffrey [19]. It respects the probability kinematics principle, whose objective is to minimize change, usually in the sense of an informational distance between probability distributions [10]. Jeffrey's rule [19] provides an effective means to revise a prior probability distribution  $P$  to a posterior  $P_\mu$ , given an input  $\mu$  with probabilities bearing on elements  $\phi_i$  of a partition of  $\Omega$ . Some axioms guide the revision process:

$$P_\mu(\phi_i) = p_i. \quad (3)$$

This axiom clearly expresses that the probability degree  $p_i$  is part of the input information. The input information and the prior probability are of the same nature, with priority given to the input. It is clear that the coefficient  $p_i$  represents what the probability of  $\phi_i$  should be (it is a *de re* probability), and not (for instance) uncertainty about the reliability of piece of information  $\phi$  (it would then be a *de dicto* probability).

Jeffrey's method also relies on the assumption that, while the probability on a prescribed subalgebra of events is enforced by the input information, the probability of any event  $\psi \subseteq \Omega$  conditional to any uncertain event  $\phi_i$  in this subalgebra is the same in the original and the revised distributions. Namely,

$$\forall \phi_i, \forall \psi, P(\psi|\phi_i) = P_\mu(\psi|\phi_i). \quad (4)$$

The underlying interpretation of minimal change implied by the constraint of Equation 4 is that the revised probability measure  $P_\mu$  must preserve the conditional probability degree of any event  $\psi$  given uncertain event  $\phi_i$ . Jeffrey's rule of conditioning yields the unique distribution that satisfies (3) and (4) (see, e.g., [7]) and takes the following form:

$$P_\mu(\psi) = \sum_{i=1}^m p_i \cdot \frac{P(\psi, \phi_i)}{P(\phi_i)}. \quad (5)$$

One way of justifying the above formula in the spirit of Bayesian nets is as follows (see Pearl [26]). With a slight notation abuse, let  $\mu$  denote the event of receiving the input information  $\{(\phi_i, p_i) \mid i = 1, \dots, m\}$ . On the enlarged frame  $\Omega \times \{\mu, \neg\mu\}$  including the occurrence or not of the input, it always holds that  $P(\psi, \phi_i|\mu) = P(\psi|\phi_i, \mu) \cdot P(\phi_i|\mu)$ . Then assuming conditional independence between  $\psi$  and the fact of receiving the input  $\mu$  in the context of  $\phi_i$ , i.e.  $P(\psi|\phi_i, \mu) = P(\psi|\phi_i)$ , we get  $P(\psi, \phi_i|\mu) = P(\psi|\phi_i) \cdot P(\phi_i|\mu)$ , where, by definition,  $P(\phi_i|\mu) = p_i$ . Then marginalize for  $\psi$ , i.e.  $\sum_{i=1}^m P(\psi|\phi_i) \cdot p_i$ , which yields (5). The posterior probability  $P_\mu$  also minimizes relative entropy with respect to the original distribution under the probabilistic constraints defined by the input  $\mu$  [33].

### 3.2. Axioms for iterated revision in possibility theory

Most existing works on belief revision (both from semantics and axiomatics perspectives) assume that the input information is either a propositional formula, or an epistemic state (namely a plausibility ordering possibly encoded as a possibility distribution). Using possibilistic counterparts to Jeffrey's rule allow us to define general forms of belief revision where the input is simply a partial epistemic state, i.e. a possibility distribution that bears on a partition of the set of possible worlds. Namely, the input is of the form  $\mu = \{(\phi_i, \lambda_i) \mid i = 1, \dots, m\}$  where the  $\phi_i$ 's represent a partition of the set of possible worlds  $\Omega$ , and  $\mu$  is a possibility distribution over the elements of the partition. The only requirement is that there exists

at least one  $\lambda_i$  such that  $\lambda_i = 1$ . In the following,  $\mu$  will be called a partial epistemic state. It is partial in the sense that letting  $\mu(\phi_i) = \lambda_i$  does not amount to the full specification of a possibility distribution on the more detailed frame  $\Omega$ .

Let us first discuss some natural properties that the revision of a possibility distribution  $\pi$  and a new input information  $\mu = \{(\phi_i, \lambda_i) \mid i = 1, \dots, m\}$  leading to a new possibility distribution denoted by  $\pi_\mu$  should satisfy. Natural properties for  $\pi_\mu$  are:

**A<sub>1</sub> (Consistency):**  $\pi_\mu$  should be normalized.

**A<sub>2</sub> (Priority to Input):**  $\forall i = 1, \dots, m, \Pi_\mu(\phi_i) = \lambda_i$ .

**A<sub>3</sub> (Faithfulness):**  $\forall \omega_1, \omega_2 \models \phi_i$  if  $\pi(\omega_1) \geq \pi(\omega_2)$  then  $\pi_\mu(\omega_1) \geq \pi_\mu(\omega_2)$ .

**A<sub>4</sub> (Inertia):**  $\forall i = 1, \dots, m$ , if  $\Pi(\phi_i) = \lambda_i$  then  $\forall \omega \models \phi_i : \pi(\omega) = \pi_\mu(\omega)$ .

**A<sub>5</sub> (Impossibility preservation):** If  $\pi(\omega) = 0$  then  $\pi_\mu(\omega) = 0$ .

**A<sub>1</sub>** means that the new epistemic state is consistent. **A<sub>2</sub>** confirms that the input  $\mu$  is interpreted as a constraint which forces  $\pi_\mu$  to satisfy  $\Pi_\mu(\phi_i) = \lambda_i, \forall i = 1, \dots, m$ . It is the success postulate. Note that

**Proposition 1.** **A<sub>2</sub>** implies **A<sub>1</sub>**.

*Proof:* this is simply due to the fact that one of the values  $\lambda_i$  is 1, by definition of an input epistemic state.  $\square$

**A<sub>3</sub>** means that the new possibility distribution should not reverse the previous relative order between models of each  $\phi_i$ . A stronger version of **A<sub>3</sub>** can be defined:

**SA<sub>3</sub> (Strong Faithfulness):**  $\forall \omega_1, \omega_2 \models \phi_i$  then:  $\pi(\omega_1) \geq \pi(\omega_2)$  if and only if  $\pi_\mu(\omega_1) \geq \pi_\mu(\omega_2)$ ,

**A<sub>4</sub>** means that when the partial epistemic state is in agreement with prior possibility levels of  $\phi_i$ , namely  $\lambda_i$ , then revision does not affect  $\pi$ . The axiom enforces this inertia locally. Indeed if  $\Pi(\phi_i) = \lambda_i$ , there is no reason to alter the degrees of possibility inside  $\phi_i$  even if  $\Pi(\phi_j) \neq \lambda_j$ , for  $j \neq i$ . **A<sub>5</sub>** stipulates that impossible worlds remain impossible after revision. Note that if  $\exists i, \lambda_i > 0$  and  $\Pi(\phi_i) = 0$ , there is a conflict between **A<sub>5</sub>** and **A<sub>2</sub>**, since the latter requires that prior impossible worlds become possible while the former forbids it. So, axioms **A<sub>1</sub>-A<sub>5</sub>** have a solution if and only if  $(\pi, \mu)$  is such that  $\forall i = 1, \dots, m, \Pi(\phi_i) = 0$  implies  $\lambda_i = 0$  and moreover it must be true that  $\exists i, \Pi(\phi_i) > 0$  and  $\lambda_i > 0$ . To eliminate such difficulties, we can first relax **A<sub>2</sub>** (excluding situations where  $\Pi(\phi_i) = 0$ ) into a weak form:

**WA<sub>2</sub> (Weak A<sub>2</sub>):**  $\forall i = 1, \dots, m, \Pi_\mu(\phi_i) = \lambda_i$  provided that  $\Pi(\phi_i) \neq 0$ ,

or we restrict the study to positive possibility distributions (then **A<sub>5</sub>** is irrelevant). Note that if  $\forall i = 1, \dots, m, \min(\lambda_i, \Pi(\phi_i)) = 0$ , then **A<sub>5</sub>** and **WA<sub>2</sub>** enforce  $\pi_\mu = 0$  which now conflicts with **A<sub>1</sub>**. In other words, the possibility setting excludes the situation of a total conflict between the input and the prior epistemic state, due to the strong interpretation of zero possibility here.

Apart from the above restrictions, that there are no further constraints which relate models of different  $\phi_i$  in the new epistemic state. Axioms **A<sub>1</sub>-A<sub>5</sub>** express the idea of minimal change.

In the ORDFI setting, Darwiche and Pearl [9] propose four postulates to characterize iterated belief revision operators that transform a given plausibility ordering on possible worlds, denoted by  $\succeq$  into a new ordering, denoted by  $\succeq_\phi$  in the presence of the new information taking the form of a proposition  $\phi$ . These postulates are:

**CR<sub>1</sub>** : If  $\omega_1 \models \phi$  and  $\omega_2 \models \phi$  then  $\omega_1 \succeq \omega_2$  if and only if  $\omega_1 \succeq_\phi \omega_2$

**CR<sub>2</sub>** : If  $\omega_1 \models \neg\phi$  and  $\omega_2 \models \neg\phi$  then  $\omega_1 \succeq \omega_2$  if and only if  $\omega_1 \succeq_\phi \omega_2$

**CR<sub>3</sub>** : If  $\omega_1 \models \phi$  and  $\omega_2 \models \neg\phi$  then  $\omega_1 \succ \omega_2$  only if  $\omega_1 \succ_\phi \omega_2$

**CR<sub>4</sub>** : If  $\omega_1 \models \phi$  and  $\omega_2 \models \neg\phi$  then  $\omega_1 \succeq \omega_2$  only if  $\omega_1 \succeq_\phi \omega_2$

**CR<sub>1</sub>** (resp. **CR<sub>2</sub>**) simply says that the relative ordering between models (resp. countermodels) of  $\phi$  should be preserved after the revision process. Clearly, Total Faithfulness **SA<sub>3</sub>** reduces to **CR<sub>1</sub>**, **CR<sub>2</sub>** when  $\mu$  only encodes a binary partition of the set of possible worlds  $\Omega$ , namely when  $\mu = \{(\phi, 1), (\neg\phi, \lambda)\}$ , with  $\lambda < 1$ . The other postulates **CR<sub>3</sub>** and **CR<sub>4</sub>** are not required in our framework where the relative plausibility of possible worlds in different elements of the partition is unconstrained in the resulting epistemic state, except if the input is redundant. Our postulates basically apply to the case where possibility degrees belong to a scale  $L$ . **A<sub>1</sub>**, **A<sub>5</sub>** have no direct counterparts in the Darwiche-Pearl framework.

In the following, we denote by  $\succeq_\Pi$  the relation between propositions induced by plausibility relation  $\succeq$  on possible worlds, such that  $\phi \succeq_\Pi \psi \iff \Pi(\phi) \geq \Pi(\psi)$ , where  $\Pi$  is the possibility measure induced by any possibility distribution  $\pi$  that represents  $\succeq$ . In the purely relational ORDFI setting, an input partial epistemic state is a WOP of the form  $WOP_{in} = (\phi_1, \dots, \phi_m)$ , and the prior epistemic state forms another WOP:  $WOP_{prior} = (\psi_1, \dots, \psi_p)$  induced by  $\succeq$ . Only **A<sub>2</sub>**, **A<sub>3</sub>**, **A<sub>4</sub>** (and **SA<sub>3</sub>**) make sense in the following form

**A<sub>2</sub> (Consistency)**: The final possibilistic ordering of events should coincide with the input information ordering on the input subalgebra.

**A<sub>3</sub> (Faithfulness)**:  $\forall \omega_1, \omega_2 \in [\phi_i]$ , if  $\omega_1 \succeq \omega_2$  then  $\omega_1 \succeq_\phi \omega_2$

**A<sub>4</sub> (Inertia)**: If the input  $WOP_{in}$  is such that  $\phi_1 \succ_\Pi \dots \succ_\Pi \phi_m$ , where  $\succeq$  is the prior epistemic state, then  $\succeq = \succeq_\phi$ .

### 3.3. Two forms of possibilistic revision based on Jeffrey's rule

Neither in the qualitative nor in the quantitative cases do the previous properties guarantee a unique definition of revision. **A<sub>3</sub>** suggests that the possibilistic revision process by a partial epistemic input can be achieved using several parallel changes with a sure input on each element of the partition: first, applying a conditioning (using equations 1 or 2, that respect **A<sub>3</sub>**) on each  $\phi_i$  and in order to satisfy **A<sub>2</sub>**, denormalizing the distribution  $\pi(\cdot \mid \phi_i)$  so as to satisfy  $\Pi_\mu(\phi_i) = \lambda_i$ . Therefore, revising with  $\mu$  can be achieved using the following Jeffrey-like revision [15]:

$$\forall(\phi_i, \lambda_i) \in \mu, \forall \omega \models \phi_i, \pi(\omega \mid \mu) = \lambda_i \otimes \pi(\omega \mid \phi_i) \quad (6)$$

where  $\otimes$  is either min or the product, depending on whether conditioning is based on the product or the minimum operator. When  $\otimes =$  product (resp. min) the possibilistic revision will be simply called product-based (resp. minimum-based) conditioning with partial epistemic states. Such possibilistic counterparts to Jeffrey's rule were introduced in [15], without axiomatic considerations, nor emphasizing any counterpart to probability kinematics condition (4). These two natural ways of defining possibilistic revision based on Jeffrey's rule naturally extend the two forms of conditioning that exist in possibility theory, given by the two equations (1) and (2).

Let us see if Jeffrey's rule satisfies the proposed revision axioms.

**Proposition 2.** The Jeffrey-like revision rule (6) satisfies  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ .

*Proof:* For  $\mathbf{A}_1$  to be satisfied we need that  $\exists i, \omega, \lambda_i \otimes \pi(\omega \mid_{\otimes} \phi_i) = 1$ . Note that  $\lambda_1 = 1$ . Then either  $\Pi(\phi_1) > 0$  and  $\exists \omega, \pi(\omega \mid_{\otimes} \phi_1) = 1$ . If  $\Pi(\phi_1) = 0$ , then  $\pi(\omega \mid_{\otimes} \phi_1) = 1$  by convention. For  $\mathbf{A}_2$ ,  $\Pi(\phi_i \mid_{\otimes} \mu) = \lambda_i \otimes \max_{\omega \models \phi_i} \pi(\omega \mid_{\otimes} \phi_i) = \lambda_i$ . For  $\mathbf{A}_3$ , it is obvious, as Jeffrey's rule cannot produce any order reversal on the prior possibility degrees inside the  $\phi_i$ 's. For  $\mathbf{A}_4$ , if,  $\Pi(\phi_i) = \lambda_i$ , then if  $\omega \models \phi_i$ ,

- either  $\otimes = \min$ , then  $\pi(\omega \mid_m \mu) = \min(\lambda_i, \pi(\omega \mid_m \phi_i)) = \pi(\omega)$  since either  $\lambda_i > \pi(\omega \mid_m \phi_i) = \pi(\omega)$  on  $\phi_i$  or  $\pi(\omega \mid_m \phi_i) = 1$ .
- or  $\otimes = \text{product}$ , and  $\pi(\omega \mid_p \mu) = \lambda_i \otimes \frac{\pi(\omega)}{\Pi(\phi_i)} = \pi(\omega)$ .
- if  $\Pi(\phi_i) = 0$ ,  $\pi(\omega \mid_{\otimes} \mu) = \lambda_i \otimes 1 = \lambda_i$ .  $\square$

Note that the revision rule (6) recovers the input if  $\Pi(\phi_i) = 0$ , and even if  $\max_{i=1\dots m} \min(\lambda_i, \Pi(\phi)) = 0$ . In particular, suppose  $\pi(\omega) = 1, \forall \omega \models \phi$ , and 0 otherwise, while  $\mu = \{(\neg\phi, 1), (\phi, 0)\}$ . This is a case of simple conditioning by  $\neg\phi$ . It is clear that  $\pi(\omega \mid_{\otimes} \mu) = 1$  if  $\omega \models \neg\phi$  and 0 otherwise. So axiom  $\mathbf{A}_5$  is violated by Jeffrey's rule, which always enforces the input information. This is because we have assumed that conditioning on an a priori impossible proposition  $\phi$  results in complete ignorance among models of this proposition.

Even if  $\mathbf{A}_3$  obviously hold for Jeffrey's rule, the new possibility degrees of models of  $\phi_i$  depend on the relative position of the *prior* possibility degrees of  $\phi_i$ , the nature of the conditioning rule, and the prescribed posterior possibility degree of  $\phi_i$  (namely,  $\lambda_i$ ):

- If  $\otimes = \min$ , in the qualitative settings, either  $\Pi(\phi_i) > \lambda_i$  and then all possible worlds that were originally more plausible than  $\lambda_i$ , are forced down to level  $\lambda_i$ , which means that some strict ordering between models of  $\phi_i$  may be lost; or  $\Pi(\phi_i) < \lambda_i$ , and then the best models of  $\phi_i$  are raised to level  $\lambda_i$ . Hence  $\mathbf{SA}_3$  is clearly not satisfied.
- When  $\otimes = \text{product}$  in the quantitative settings, and  $\Pi(\phi_i) > \lambda_i$ , all plausibility levels are proportionally shifted down (to the level  $\lambda_i$ ). If  $\Pi(\phi_i) < \lambda_i$ , the plausibility levels of other models are proportionally shifted up (to level  $\lambda_i$ ). Hence  $\mathbf{SA}_3$  is clearly satisfied.

If  $\otimes = \min$  and the scale is ordinal (QUALFI and DORD settings), the qualitative Jeffrey-like revision rule (6) satisfies  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$ , but other revision rules satisfy these axioms. For instance,

the rule  $\forall i = 0, \dots, m, \forall \omega \models \phi_i, \pi_\mu(\omega) = \lambda_i$  obeys  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3$ , but flattens everything, hence clearly violates  $\mathbf{SA}_3$ . To recover  $\mathbf{A}_4$ , modify this revision rule as follows:

$$\begin{aligned} \forall i = 0, \dots, m, \forall \omega \models \phi_i, \pi_\mu(\omega) &= \lambda_i, \text{ if } \Pi(\phi_i) \neq \lambda_i; \\ &= \pi(\omega) \text{ if } \Pi(\phi_i) = \lambda_i. \end{aligned}$$

So there are other revision rules than (6) that satisfy  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$  in QUALFI and DORD settings. One may recover axiom  $\mathbf{A}_5$  by changing the convention on conditioning with impossible events (for instance,  $\pi(\omega \mid_m \phi) = 0$ , if  $\omega \models \phi$  and  $\Pi(\phi) = 0$ ), but then we would violate  $\mathbf{A}_1, \mathbf{A}_2$ .

In numerical settings DENUM and DABS, the revision rules satisfying all axioms  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3, \mathbf{A}_4$  and  $\mathbf{SA}_3$  can be characterized :

**Proposition 3.** In numerical settings DENUM and DABS, a revision rule of  $\pi$  by  $\mu$  satisfies  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{SA}_3, \mathbf{A}_4$  if and only if it is of the form  $\pi_\mu(\omega) = \lambda_i \cdot \frac{f_i(\pi(\omega))}{f_i(\Pi(\phi_i))}, \omega \models \phi_i$  with  $f_i : [0, 1] \rightarrow [0, 1]$ , strictly increasing, such that  $f_i = \text{identity}$  if  $\Pi(\phi_i) = \lambda_i$ .

*Proof :* It is easy to check that this family of revision rules verifies all our revision axioms. For the converse, suppose  $\pi_\mu$  satisfies  $\mathbf{A}_1, \mathbf{WA}_2, \mathbf{A}_4$ , and  $\mathbf{SA}_3$ . Define  $f_i(\pi(\omega)) = \pi_\mu(\omega), \forall \omega \models \phi_i$ , and complete the function in between in a continuous strictly increasing way. Then, function  $f_i$  is a strictly increasing function  $[0, 1] \rightarrow [0, 1]$  due to  $\mathbf{SA}_3$ . Then  $\pi_\mu(\omega) = \lambda_i \cdot \frac{f_i(\pi(\omega))}{f_i(\Pi(\phi_i))}, \omega \models \phi_i$ . Indeed:  $\lambda_i \cdot \frac{f_i(\pi(\omega))}{f_i(\Pi(\phi_i))} = \lambda_i \cdot \frac{\pi_\mu(\omega)}{\max_{\omega_i \models \phi_i} \pi_\mu(\omega)} = \lambda_i \cdot \frac{\pi_\mu(\omega)}{\lambda_i}$  (using  $\mathbf{A}_2$ ) =  $\pi_\mu(\omega)$ . Consider the case where  $\pi(\omega) \neq \lambda_i$  and  $\Pi(\phi_i) = \lambda_i$ . Then axiom  $\mathbf{A}_4$  enforces  $f_i$ 's to be the identity.  $\square$

Suppose now the functions  $f_i$  coincide with a single bijective increasing function  $f : [0, 1] \rightarrow [0, 1]$ , that does not depend on the prior epistemic state  $\pi$ , nor on the input  $\mu$ . Consider the case where  $\Pi(\phi_i) = \lambda_i$ . Then axiom  $\mathbf{A}_4$  imposes  $\pi_\mu(\omega) = \lambda_i \cdot \frac{f(\pi(\omega))}{f(\lambda_i)} = \pi(\omega)$ . The only possible function  $f$  is then the identity since  $f(\lambda_i) = \lambda_i$  must hold, and then  $f(\pi(\omega)) = \pi(\omega), \forall \omega \in \Omega$ . Then the Jeffrey-like rule (6) for  $\otimes = \text{product}$  is obtained.

**Example 3.1.** Let us continue our example. And assume that we have a new piece of information, where the director of laboratory states that now booking the conference hotel is less plausible than booking a non-conference hotel. This new information is represented by the following epistemic input  $\mu = \{(-h, 1), (h, \alpha)\}$ . Table 2 represents the result of revising the possibility distribution of Table 1 using product-based conditioning and min-based conditioning with the partial epistemic states encoded by  $\mu = \{(-h, 1), (h, \alpha)\}$ .

### 3.4. Relationships with Jeffrey's Probability Kinematics Properties

Another way to define possibilistic revision is to simply use Jeffrey's rule axioms [19]. Namely, given an initial possibility distribution  $\pi$  and a partial epistemic state  $\mu = \{(\phi_i, \lambda_i) \mid i = 1, \dots, m\}$  we consider possibility distributions  $\pi_\mu$  that satisfy:

1.  $\Pi_\mu(\phi_i) = \lambda_i$ .
2.  $\forall \phi_i, \forall \psi, \Pi(\psi \mid_{\otimes} \phi_i) = \Pi_\mu(\psi \mid_{\otimes} \phi_i)$ ,

	$A$	$H$	$B$	$V$	$\pi(\cdot)$	$\pi(\cdot \mid_p \mu)$	$\pi(\cdot \mid_m \mu)$
$\omega_1$	$a$	$h$	$b$	$v$	1	$\alpha$	$\alpha$
$\omega_2$	$a$	$\neg h$	$b$	$v$	$\alpha$	<b>1</b>	<b>1</b>
$\omega_3$	$\neg a$	$\neg h$	$b$	$v$	$\alpha^2$	$\alpha$	$\alpha^2$
$\omega_4$	$\neg a$	$\neg h$	$b$	$\neg v$	$\alpha^3$	$\alpha^2$	$\alpha^3$
$\omega_5$	$\neg a$	$\neg h$	$\neg b$	$\neg v$	$\alpha^4$	$\alpha^3$	$\alpha^4$
$\omega_6$	$\neg a$	$\neg h$	$\neg b$	$v$	$\alpha^5$	$\alpha^4$	$\alpha^5$
Other models of $h$	$h$	-	-	-	$\alpha^{10}$	$\alpha^{11}$	$\alpha^{10}$
Other models of $\neg h$	$\neg h$	-	-	-	$\alpha^{10}$	$\alpha^9$	$\alpha^{10}$

Table 2. Jeffrey-like revision of a possibility distribution with input  $\mu = \{(\neg h, 1), (h, \alpha)\}$ 

where  $\otimes$  is either the minimum or the product. Condition 1 is clearly axiom  $\mathbf{A}_2$ , and Condition 2 is in the same spirit as axiom  $\mathbf{SA}_3$  but stronger than the latter since the new condition requires stability of possibility degrees and not just of the ordering between possible worlds inside the elements of the input partition. Condition 2 can be called *Absolute Faithfulness*.

**Proposition 4.** When  $\otimes$  is the product then the Jeffrey-like rule given by (6) is the unique revision rule that satisfies conditions  $\mathbf{A}_2$  and Absolute Faithfulness.

*Proof:* Let  $\omega \models \phi_i$ . Then Condition 2 reads  $\frac{\pi(\omega)}{\prod(\phi_i)} = \frac{\pi_\mu(\omega)}{\prod_\mu(\phi_i)}$ . Using condition 1,  $\frac{\pi(\omega)}{\prod(\phi_i)} = \frac{\pi_\mu(\omega)}{\lambda_i}$ .  $\square$

However, this proposition is not valid when  $\otimes$  is the minimum, for which, in general, Absolute Faithfulness is not satisfied. This is just because axiom  $\mathbf{SA}_3$  is not satisfied by the qualitative Jeffrey-like revision rule, hence Absolute Faithfulness also fails (for more details, see [5]).

The following example illustrates the fact that product-based possibilistic revision satisfies conditions 1 and 2 while min-based possibilistic revision does not.

**Example 3.2.** Let us consider again our example. And assume that the embassy of USA in Paris publicly announces that it is quite exceptional that visitors apply for electronic visa.

This new information is represented by the following epistemic input  $\mu = \{(\neg v, 1), (v, \alpha^3)\}$ .

Table 3 represents the result of revising the possibility distribution Table 1 using product-based conditioning and min-based conditioning with partial epistemic states.

Clearly, from Table 3, one can easily check that product-based possibilistic revision preserves the ranking of models of  $\neg v$  and  $v$  respectively, while min-based possibilistic revisions does not, and hence min-based possibilistic revision does not satisfy the two kinematics properties. The reason is that in the new information  $\mu$ , the new possibility degree of  $v$  should be decreased to  $\alpha^3$ , and hence all models of  $v$  that had a plausibility degree higher than  $\alpha^3$  (namely  $\omega_1, \omega_2, \omega_3$ ) get the same degree  $\alpha^3$ . The original ranking between these three models (namely  $\omega_1, \omega_2, \omega_3$ ) is lost. Hence, one cannot satisfy the second kinematics property.

	$A$	$H$	$B$	$V$	$\pi(\cdot)$	$\pi(\cdot \mid_p \mu_2)$	$\pi(\cdot \mid_m \mu_2)$
$\omega_1$	$a$	$h$	$b$	$v$	$1$	$\alpha^3$	$\alpha^3$
$\omega_2$	$a$	$\neg h$	$b$	$v$	$\alpha$	$\alpha^4$	$\alpha^3$
$\omega_3$	$\neg a$	$\neg h$	$b$	$v$	$\alpha^2$	$\alpha^5$	$\alpha^3$
$\omega_4$	$\neg a$	$\neg h$	$b$	$\neg v$	$\alpha^3$	$\mathbf{1}$	$\mathbf{1}$
$\omega_5$	$\neg a$	$\neg h$	$\neg b$	$\neg v$	$\alpha^4$	$\alpha$	$\alpha^4$
$\omega_6$	$\neg a$	$\neg h$	$\neg b$	$v$	$\alpha^5$	$\alpha^8$	$\alpha^5$
Other models of $v$	-	-	-	$v$	$\alpha^{10}$	$\alpha^{13}$	$\alpha^{10}$
Other models of $\neg v$	-	-	-	$\neg v$	$\alpha^{10}$	$\alpha^7$	$\alpha^{10}$

Table 3. Jeffrey-like revision with input  $\mu = \{(\neg v, 1), (v, \alpha^3)\}$  and Absolute Faithfulness

## 4. Recovering existing belief revision frameworks

Clearly, possibilistic revision with partial epistemic states generalizes possibilistic conditioning on a proposition  $\phi$ . Indeed, applying possibilistic revision given by (6) with a partial epistemic state  $\mu = \{(\phi, 1), (\neg\phi, 0)\}$  gives exactly the same results if one applies equation (1) on  $\phi$  when  $\otimes = \min$  (resp. (2) for  $\otimes = \text{product}$ ). Similarly, the possibilistic revision based on a bipartition, which corresponds to adjustment (see [15, 2]), is a particular case of possibilistic revision with a partial epistemic state, where the input is of the form  $\mu = \{(\phi, 1), (\neg\phi, \lambda)\}$ . In the following, we consider how the Jeffrey-like revision rules in formal settings other than ORDFI capture existing ordinal revision rules modifying epistemic states, such as Boutilier's natural revision (already considered in [15]), Papini's drastic revision [25], a lexicographic approach proposed by the first author and colleagues [4], and finally, the so-called improvement operators [22]. Note that since all these types of revision are defined in the ordinal setting ORDFI, we shall only use positive possibility distributions to represent them. Hence, subtle matters regarding interactions between axioms  $\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_5$  discussed above are irrelevant here. Especially,  $\mathbf{A}_5$  is irrelevant for positive possibility distributions.

### 4.1. Natural belief revision

Let  $\succeq$  be a complete pre-order on the set of possible worlds. Let  $\phi$  be a new piece of information. Natural belief revision of  $\succeq$  by  $\phi$  was proposed in [6], and also hinted by Spohn [29]. It encodes a form of minimal change of  $\succeq$  by letting the most plausible models of  $\phi$  in  $\succeq$  (forming the set denoted by  $\phi^* = \max(\phi, \succeq)$ ) become the most plausible worlds in the revised plausibility relation, denoted by  $\succeq_{N\phi}$ . More precisely,  $\succeq_{N\phi}$  is defined as follows:

- $\forall \omega_1 \in \phi^*, \forall \omega_2 \in \phi^*, \omega_1 \sim_{N\phi} \omega_2$ , where  $\sim_{N\phi}$  is the equivalence relation extracted from  $\succeq_{N\phi}$ .
- $\forall \omega_1 \in \phi^*, \forall \omega_2 \notin \phi^*, \omega_1 \succ_{N\phi} \omega_2$ , where  $\succ_{N\phi}$  is the strict part of  $\succeq_{N\phi}$ .
- $\forall \omega_1 \notin \phi^*, \forall \omega_2 \notin \phi^*, \omega_1 \succeq_{N\phi} \omega_2$  if and only if  $\omega_1 \succeq \omega_2$ .

**Example 4.1.** Let us continue Example 2.1 where  $\succeq$  is simply the complete pre-order on the set of possible worlds induced from Table 1. Namely:

$$\omega_1 \succ \omega_2 \succ \omega_3 \succ \omega_4 \succ \omega_5 \succ \omega_6 \succ \text{others}$$

Consider a new piece of information  $\phi = \neg v$  claiming that it is more likely that JM did not apply for the visa than that he did. Applying natural belief revision to  $\succeq$  produces a new pre-ordering with one change in  $\succeq$  by putting the most preferred models of  $\neg v$  (in our example  $\omega_4$ ) at the top of the new ordering. We get:

$$\omega_4 \succ_{N\phi} \omega_1 \succ_{N\phi} \omega_2 \succ_{N\phi} \omega_3 \succ_{N\phi} \omega_5 \succ_{N\phi} \omega_6 \succ_{N\phi} \text{others}$$

To recover natural belief revision, as a special case of the possibilistic counterpart to Jeffrey's rule, first associate with  $\succeq$  a positive possibility distribution  $\pi$ , on  $L$ , that represents  $\succeq$ .

Suppose we revise by  $\phi$ . The idea conveyed by natural revision, is that the only change that must occur is that the most plausible  $\phi$ -worlds become the most plausible worlds, everything else remaining the same. By default, the enforced certainty of  $\phi$  should be as small as possible, since  $\phi$  is just known to be more likely than  $\neg\phi$ . The input will be modelled as  $\mu = \{(\phi, 1), (\neg\phi, \min(\lambda, \Pi(\neg\phi)))\}$ , where  $1 > \lambda$ , and  $\lambda > \max\{\pi(\omega) : \pi(\omega) \neq 1\}$ <sup>4</sup> to ensure this minimal certainty assumption. Then we let  $\pi_{N\mu}(\omega) = \pi(\omega \mid_m \mu)$  using the Jeffrey-like revision rule (6) for  $\otimes = \min$ , namely:

$$\begin{aligned} \forall \omega \models \phi, \pi(\omega \mid_m \mu) &= \pi(\omega \mid_m \phi), \\ \forall \omega \models \neg\phi, \pi(\omega \mid_m \mu) &= \min(\lambda, \Pi(\neg\phi), \pi(\omega \mid_m \neg\phi)). \end{aligned}$$

Note that the condition  $\lambda > \max\{\pi(\omega) : \pi(\omega) \neq 1\}$  may never be satisfied in the case of a finite scale (QUALFI), that is, if  $\max\{\pi(\omega) : \pi(\omega) \neq 1\} = \alpha_1$ , without augmenting the number of steps in the scale (or shifting down all possibility degrees beforehand). Likewise, the encoding may be impossible in DENUM and  $\otimes = \min$  if  $\max\{\pi(\omega) : \pi(\omega) \neq 1\} = \alpha$  (the next level under 1). However there is no problem encoding natural revision by the min-based possibilistic Jeffrey's rule in a dense possibility scale.

Then it is clear that  $\pi(\omega \mid_m \mu)$  indeed encodes natural belief revision, namely:

**Proposition 5.** If  $\mu = \{(\phi, 1), (\neg\phi, \min(\lambda, \Pi(\neg\phi)))\}$ , then

$$\forall \omega_1, \omega_2 \in \Omega, \pi(\omega_1 \mid_m \mu) \geq \pi(\omega_2 \mid_m \mu) \text{ if and only if } \omega_1 \succeq_{N\phi} \omega_2.$$

*Proof:* There are two cases:

- When  $\phi \in \text{Bel}(\pi)$ , i.e.,  $\Pi(\phi) = 1 > \Pi(\neg\phi)$  then since  $\lambda \geq \Pi(\neg\phi)$ ,

$$\begin{aligned} \forall \omega \models \phi, \pi(\omega \mid_m \mu) &= \pi(\omega \mid_m \phi) = \pi(\omega), \\ \forall \omega \models \neg\phi, \pi(\omega \mid_m \mu) &= \min(\Pi(\neg\phi), \pi(\omega \mid_m \neg\phi)) = \pi(\omega). \end{aligned}$$

Hence there is no change in the initial possibility distribution when  $\phi$  is already accepted.

<sup>4</sup>The convention  $\max \emptyset = 0$  is adopted, so that if the prior possibility expresses ignorance ( $\pi(\omega) = 1, \forall \omega$ ), any  $\lambda \neq 0, 1$  can be chosen.

- When  $\phi \notin \text{Bel}(\pi)$ , i.e.,  $\Pi(\phi) \leq \Pi(\neg\phi) = 1$ , then, since  $\lambda < \Pi(\neg\phi)$ ,

$$\begin{aligned} \forall \omega \models \phi^*, \pi(\omega \mid_m \mu) &= \pi(\omega \mid_m \phi) = 1; \\ \forall \omega \models \phi \cap \neg(\phi^*), \pi(\omega \mid_m \mu) &= \pi(\omega \mid_m \phi) = \pi(\omega); \\ \forall \omega \models (\neg\phi)^*, \pi(\omega \mid_m \mu) &= \min(\lambda, \pi(\omega \mid_m \neg\phi)) = \lambda; \\ \forall \omega \models \neg(\phi \cup (\neg\phi)^*), \pi(\omega \mid_m \mu) &= \min(\lambda, \pi(\omega \mid_m \neg\phi)) = \pi(\omega). \end{aligned}$$

So, the models of  $\phi^*$  become fully possible and the initially most plausible models become the second best models in the revised possibility distribution. These are the only changes made.  $\square$

Note that in [15], it was suggested to recover natural revision by  $\phi$  by means of Jeffrey-like min-based rule with input  $\mu = \{(\phi, 1), (\neg\phi, \lambda)\}$ , which only works if  $\Pi(\phi) \leq \Pi(\neg\phi) = 1$ . Otherwise (if  $\phi$  is already accepted in the initial state), this rule alters the plausibility ordering while natural revision does not. The product-based Jeffrey-like revision rule cannot encode the natural revision ordering, in general since the latter enforces invariance of the ordering between implausible  $\phi$  and  $\neg\phi$  worlds, while the former, by essence, operates changes in  $\phi$  and  $\neg\phi$  worlds independently.

**Example 4.2.** *Let us continue Example 4.1. To recover the total pre-ordering  $\succ_{N\phi}$  resulting from revising  $\succeq$  with  $\neg v$  using natural revision encoded by Jeffrey-like revision rule with  $\oplus = \min$ , we first choose a real number  $\lambda$  such that:  $1 > \lambda > \max\{\pi(\omega) : \pi(\omega) \neq 1\} = \alpha$ . There is no such  $\lambda$  in the DENUM setting adopted. We cannot recover natural revision with the min-based possibilistic Jeffrey's rule using the proposed scale. Now let us use the setting DABS and the enlarged scale  $L = [0, 1]$ . Now we take  $\lambda = \sqrt{\alpha}$ , for instance. Then we define  $\phi^*$  whose models are those satisfying the input information  $\neg v$  and that are maximal in  $\succeq$ , namely:*

$$\phi^* = \max(\neg v, \succeq) = \{\omega_4\}.$$

Lastly, we apply the min-based possibilistic revision with the epistemic input:  $\mu = \{(\phi, 1), (\neg\phi, \sqrt{\alpha})\}$ .

	A	H	B	V	$\pi(\cdot)$	$\pi(\cdot \mid_m \mu)$
$\omega_1$	a	h	b	v	1	$\sqrt{\alpha}$
$\omega_2$	a	$\neg h$	b	v	$\alpha$	$\alpha$
$\omega_3$	$\neg a$	$\neg h$	b	v	$\alpha^2$	$\alpha^2$
$\omega_4$	$\neg a$	$\neg h$	b	$\neg v$	$\alpha^3$	<b>1</b>
$\omega_5$	$\neg a$	$\neg h$	$\neg b$	$\neg v$	$\alpha^4$	$\alpha^4$
$\omega_6$	$\neg a$	$\neg h$	$\neg b$	v	$\alpha^5$	$\alpha^5$
Others	-	-	-	-	$\alpha^{10}$	$\alpha^{10}$

Table 4. Natural revision by input  $\neg v$

From Table 4, one can check that the total pre-ordering associated with  $\pi(\cdot \mid_m \mu)$  is exactly as  $\succeq_{N\phi}$  given in Example 4.1.

## 4.2. Drastic belief revision

Several authors [29, 23, 25, 21] have considered a stronger constraint by imposing that each model of  $\phi$  should be strictly preferred to each countermodel of  $\neg\phi$ , and moreover the relative ordering between models (resp. countermodels) of  $\phi$  should be preserved. More formally, let us denote by  $\succeq_{D\phi}$  be result of applying drastic belief revision by  $\phi$  to  $\succeq$ . The revised plausibility ordering  $\succeq_{D\phi}$  is defined as follows:

- $\forall \omega_1, \omega_2 \models \phi, \omega_1 \succeq_{D\phi} \omega_2$  if and only if  $\omega_1 \succeq \omega_2$ .
- $\forall \omega_1, \omega_2 \models \neg\phi, \omega_1 \succeq_{D\phi} \omega_2$  if and only if  $\omega_1 \succeq \omega_2$ .
- $\forall \omega_1 \models \phi, \forall \omega_2 \models \neg\phi, \omega_1 \succ_{D\phi} \omega_2$ .

**Example 4.3.** Let us continue Example 2.1 where  $\succeq$  is simply the complete pre-order on the set of possible worlds induced from Table 1, i.e.  $\omega_1 \succ \omega_2 \succ \omega_3 \succ \omega_4 \succ \omega_5 \succ \omega_6 \succ$  others. Assume that we have a new piece of information where JM did not book a room at the hotel conference. The reliability of the information is very high (the new piece of information is basically fully sure).

Let us revise  $\succeq$  with the new input  $\neg h$  using the drastic revision which is appropriate in this case. Applying, drastic belief revision on  $\succeq$  produces a new pre-ordering where all models  $\neg h$  are put at the top, namely:

$$\omega_2 \succ_{D\phi} \omega_3 \succ_{D\phi} \omega_4 \succ_{D\phi} \omega_5 \succ_{D\phi} \omega_6 \succ_{D\phi} \text{others models of } \neg h \succ_{D\phi} \omega_1 \succ_{D\phi} \text{others models of } h.$$

To recover drastic belief revision in terms of possibilistic Jeffrey's rules, first represent  $\succeq$  with a positive possibility distribution  $\pi$ , as defined above, on a numerical scale, DENUM or DABS. Let  $\Delta(\phi) = \min\{\pi(\omega) : \omega \models \phi\}$ . The set-function  $\Delta$  is known in possibility theory as a *guaranteed* possibility measure (see for instance, [13, 3]).

Then define  $\pi_{\succeq_{D\phi}}(\cdot) = \pi(\cdot \mid_p \mu)$  where  $\mu = \{(\phi, 1), (\neg\phi, \lambda)\}$ ,  $\pi(\cdot \mid_p \mu)$  is the result applying possibilistic revision given by equation (6) with  $\otimes =$  product. To recover drastic revision it is enough that  $\Delta(\phi \mid_p \mu) = \min\{\pi(\omega \mid_p \mu) : \omega \models \phi\} > \Pi(\neg\phi \mid_p \mu) = \max\{\pi(\omega \mid_p \mu) : \omega \models \neg\phi\}$ . Namely,

- If  $\Pi(\phi) = 1$ , then  $\pi(\omega \mid_p \mu) = \pi(\omega)$  if  $\omega \models \phi$ , and  $\pi(\omega \mid_p \mu) = \lambda \cdot \frac{\pi(\omega)}{\Pi(\neg\phi)}$  if  $\omega \models \neg\phi$ ; then  $\Delta(\phi \mid_p \mu) > \Pi(\neg\phi \mid_p \mu)$  requires the condition  $\Delta(\phi) > \lambda$ .
- If  $\Pi(\phi) < 1$ , then  $\pi(\omega \mid_p \mu) = \frac{\pi(\omega)}{\Pi(\phi)}$  if  $\omega \models \phi$ , and  $\pi(\omega \mid_p \mu) = \lambda \cdot \pi(\omega)$  if  $\omega \models \neg\phi$ ; then  $\Delta(\phi \mid_p \mu) > \Pi(\neg\phi \mid_p \mu)$  requires the condition  $\frac{\Delta(\phi)}{\Pi(\phi)} > \lambda$ .

Then, provided that  $\lambda < \frac{\Delta(\phi)}{\Pi(\phi)}$ ,  $\pi_{\succeq_{D\phi}}$  indeed encodes drastic belief revision, namely:

$$\forall \omega_1, \omega_2 \in \Omega, \pi(\omega_1 \mid_p \mu) \geq \pi(\omega_2 \mid_p \mu) \text{ if and only if } \omega_1 \succeq_{D\phi} \omega_2.$$

Clearly, this encoding requires numerical frameworks DENUM or DABS. Drastic revision cannot be encoded using possibilistic Jeffrey's rule in qualitative settings, because Axiom **SA**<sub>3</sub> is requested.

**Example 4.4.** Let us continue Example 4.3. To recover the total pre-ordering  $\succeq_{D\phi}$  resulting from revising  $\succeq$  with  $\neg h$ , we first define a real number  $\lambda = \alpha^n$  such that:  $\frac{\min\{\pi(\omega) : \omega \models \neg h\}}{\max\{\pi(\omega) : \omega \models \neg h\}} = \frac{\alpha^{10}}{\alpha} > \lambda > 0$ . In this example, we take  $\lambda = \alpha^{10}$ . Then it is enough to apply product-based revision with the epistemic input:  $\mu = \{(\neg h, 1), (h, \alpha^{10})\}$  (see Table 5 for the result).

	$A$	$H$	$B$	$V$	$\pi(\cdot)$	$\pi_{\succeq_{D\phi}}(\cdot)$
$\omega_1$	$a$	$h$	$b$	$v$	1	$\alpha^{10}$
$\omega_2$	$a$	$\neg h$	$b$	$v$	$\alpha$	1
$\omega_3$	$\neg a$	$\neg h$	$b$	$v$	$\alpha^2$	$\alpha$
$\omega_4$	$\neg a$	$\neg h$	$b$	$\neg v$	$\alpha^3$	$\alpha^2$
$\omega_5$	$\neg a$	$\neg h$	$\neg b$	$\neg v$	$\alpha^4$	$\alpha^3$
$\omega_6$	$\neg a$	$\neg h$	$\neg b$	$v$	$\alpha^5$	$\alpha^4$
Other models of $\neg h$	$\neg h$	-	-	-	$\alpha^{10}$	$\alpha^9$
Other models of $h$	$h$	-	-	-	$\alpha^{10}$	$\alpha^{20}$

Table 5. Drastic revision with input  $\neg h$ 

### 4.3. Revising by refinement

In [4] an extension of drastic revision of an epistemic state as a plausibility ordering  $\succ$ , by an input in the form of another plausibility ordering, denoted here by  $\succeq_\mu$ , is defined. The obtained result is a new epistemic state, denoted by  $\succeq_{l\mu}$  ( $l$  for lexicographic ordering), and defined as follows:

- $\forall \omega_1, \omega_2 \in \Omega$ , if  $\omega_1 \succ_\mu \omega_2$  then  $\omega_1 \succ_{l\mu} \omega_2$ .
- $\forall \omega_1, \omega_2 \in \Omega$ , if  $\omega_1 \sim_\mu \omega_2$  then  $\omega_1 \succeq_{l\mu} \omega_2$  if and only if  $\omega_1 \succ \omega_2$ .

Namely,  $\succeq_{l\mu}$  is obtained by refining  $\succeq_\mu$  by means of the initial ordering  $\succ$  used for breaking ties in  $\succeq_\mu$ . Note that drastic revision can be recovered by defining  $\succeq_\mu$  as follows :  $\omega_1 \succ_\mu \omega_2 \iff \omega_1 \models \phi$  and  $\omega_2 \models \neg\phi$  and  $\omega_1 \sim \omega_2$  otherwise.

**Example 4.5.** Let us continue Example 2.1 where the initial epistemic state is:  $\omega_1 \succ \omega_2 \succ \omega_3 \succ \omega_4 \succ \omega_5 \succ \omega_6 \succ$  others. Assume that the new information is of the form:  $\omega_5 \succ_\mu \omega_4 \succ_\mu$  other models of  $\neg v \succ_\mu$  models of  $v$ . The revision of  $\succeq$  by input  $\succeq_\mu$  leads to:

$\omega_5 \succ_{l\mu} \omega_4 \succ_{l\mu}$  other models of  $\neg v \succ_{l\mu} \omega_1 \succ_{l\mu} \omega_2 \succ_{l\mu} \omega_3 \succ_{l\mu} \omega_6 \succ_{l\mu}$  other models of  $v$ .

For our purpose, we denote  $\{\phi_0, \dots, \phi_n\}$  the well-ordered partition of  $\Omega$  induced by  $\succeq_\mu$ . Namely:

- $\forall i = 0, \dots, n, \forall \omega_1, \omega_2 \models \phi_i, \omega_1 \sim_\mu \omega_2$ ,
- $\forall \omega_1, \omega_2 \in \Omega, \omega_1 \succeq_\mu \omega_2$  if and only if  $\omega_1 \models \phi_i, \omega_2 \models \phi_j$  and  $i < j$ .

To recover this kind of revision by refinement as a form of possibilistic Jeffrey's rule, first define a positive possibility distribution on  $[0, 1]$  representing  $\succeq$ , let  $\mu = \{(\phi_i, \lambda_i) : i = 0, \dots, m\}, \lambda_0 = 1 > \lambda_1 > \dots > \lambda_m > 0$  and define  $\forall \omega \models \phi_i, \pi_{\succeq_{l\mu}}(\omega) = \pi(\omega \mid_p \mu) = \lambda_i \frac{\pi(\omega)}{\Pi(\phi_i)}$ , the result of applying possibilistic revision given by equation (6) with  $\otimes$  =product. We need to make sure that

$$\forall i = 0, m-1, \forall \omega_1 \models \phi_i, \forall \omega_2 \models \phi_{i+1}, \lambda_i \frac{\pi(\omega_1)}{\Pi(\phi_i)} > \lambda_{i+1} \frac{\pi(\omega_2)}{\Pi(\phi_{i+1})}$$

In other words,  $\lambda_i \cdot \frac{\Delta(\phi_i)}{\Pi(\phi_i)} > \lambda_{i+1} \frac{\Pi(\phi_{i+1})}{\Pi(\phi_{i+1})} = \lambda_{i+1}$ . This is always possible in the setting DABS, letting  $\lambda_1 < \frac{\Delta(\phi_0)}{\Pi(\phi_0)}$ ,  $\lambda_2 < \frac{\lambda_1 \Delta(\phi_1)}{\Pi(\phi_1)}$ ,  $\dots$ ,  $\lambda_m < \frac{\lambda_{m-1} \Delta(\phi_{m-1})}{\Pi(\phi_{m-1})}$ . It is clear that this approach can as well be applied to the DENUM setting, letting  $\lambda_i = \alpha^{k_i}$ , with  $k_0 = 0$ ,  $k_1 = 1$  and suitable integers  $k_i > 1$ . We can as well use infinitesimals of the form  $\lambda_i = \epsilon^i$ 's (and by convention  $\epsilon^0 = 1$ ), where such values are then independent of the input and the prior epistemic state. But as shown above this is not necessary. Then we can show that  $\pi(\cdot \mid_p \mu)$  indeed encodes  $\succeq_{l\mu}$ , namely:  $\forall \omega_1, \omega_2 \in \Omega, \pi(\omega_1 \mid_p \mu) > \pi(\omega_2 \mid_p \mu)$  if and only if  $\omega_1 \succeq_{l\mu} \omega_2$ . However, as this kind of revision relies on the Strong Faithfulness axiom, it cannot be captured by Jeffrey's rule in qualitative settings.

**Example 4.6.** Let us continue our example where the partition of  $\Omega$  induced by  $\mu$  is

$\mu = \{\phi_0 = \{\omega_5\}, \phi_1 = \{\omega_4\}, \phi_2 = \{\text{other models of } \neg v\}, \phi_3 = \{\{\omega_1\omega_2\omega_3\omega_6\} \cup \{\text{other models of } v\}\}$ . Then  $\frac{\Delta(\phi_0)}{\Pi(\phi_0)} = 1$ ,  $\frac{\Delta(\phi_1)}{\Pi(\phi_1)} = 1$ ,  $\frac{\Delta(\phi_2)}{\Pi(\phi_2)} = 1$ . Hence we can define  $\lambda_1 = \alpha$ ,  $\lambda_2 = \alpha^2$ ,  $\lambda_3 = \alpha^3$ . One can check that the plausibility ordering induced by  $\pi(\cdot \mid_p \mu)$  given in Table 6 is the same as the one computed in Example 4.5

	A	H	B	V	$\pi(\cdot)$	$\pi(\cdot \mid_p \mu)$
$\omega_1$	a	h	b	v	1	$\alpha^3$
$\omega_2$	a	$\neg h$	b	v	$\alpha$	$\alpha^4$
$\omega_3$	$\neg a$	$\neg h$	b	v	$\alpha^2$	$\alpha^5$
$\omega_4$	$\neg a$	$\neg h$	b	$\neg v$	$\alpha^3$	$\alpha$
$\omega_5$	$\neg a$	$\neg h$	$\neg b$	$\neg v$	$\alpha^4$	1
$\omega_6$	$\neg a$	$\neg h$	$\neg b$	v	$\alpha^5$	$\alpha^8$
Other models of $\neg v$	-	-	-	$\neg v$	$\alpha^{10}$	$\alpha^2$
Other models of v	-	-	-	v	$\alpha^{10}$	$\alpha^{13}$

Table 6. Revision by refinement

#### 4.4. Improvement operator

The last approach to be recovered by Jeffrey-like revision scheme is called a *reinforcement* or *improvement* operator, recently proposed in [22]. This is when the revision of  $\succeq$  by a proposition  $\phi$  only results in a small increase of the plausibility of  $\phi$ , namely the result makes  $\phi$  “one unit” more plausible. It would correspond to a learning process, whereby repeating  $\phi$  makes it gradually increase in plausibility.

The new epistemic state, denoted by  $\succeq_{R\phi}$ , obtained after reinforcing  $\phi$  is defined as follows. Let  $G_\omega = \{\omega_1 \in \Omega, \omega_1 \succ \omega \text{ and } \nexists \omega_2 \in \Omega \text{ such that } \omega_1 \succ \omega_2 \succ \omega\}$ .  $G_\omega$  represents the set of possible worlds which are next strictly more plausible than  $\omega$ , in the sense of  $\succeq$ . Then the following minimal change assumptions are made:

- The relative ordering between models (resp. counter-models) of  $\phi$  is preserved by  $\succeq_{R\phi}$ .

- $\forall \omega_1 \models \phi, \omega_2 \models \neg\phi$ , if  $\omega_1 \succeq \omega_2$  then  $\omega_1 \succeq_{R\phi} \omega_2$
- $\forall \omega_1 \models \phi, \omega_2 \models \neg\phi$ , if  $\omega_2 \succ \omega_1$  then, if  $\omega_2 \in G_{\omega_1}$  then  $\omega_1 \sim_{R\phi} \omega_2$  else  $\omega_2 \succ_{R\phi} \omega_1$

In other words we upgrade by one unit all models of  $\phi$ , so that if the model next more plausible than a given model of  $\phi$  is a countermodel of  $\phi$ , these two models become equally plausible after revision.

**Example 4.7.** Let us apply this method to the plausibility ordering of Example 2.1 induced from Table 1:  $\omega_1 \succ \omega_2 \succ \omega_3 \succ \omega_4 \succ \omega_5 \succ \omega_6 \succ \text{others}$ . Assume that we have a new piece of information claiming that JM is not attending the conference in Newark area, nor in the hotel, nor did he apply for electronic visa, namely  $\phi = \neg a \neg h \neg v$ . But we find it hard to accept it right away. Let us revise  $\succeq$  with  $\phi$  using the improvement operator.

There are two models of the new information  $\phi$ , namely  $\omega_4$  and  $\omega_5$ . Using the construction of the improvement operator, where we have  $G_{\omega_5} = \{\omega_4\}$  and  $G_{\omega_4} = \{\omega_3\}$ , gives the following total pre-ordering on the set of possible worlds:

$$\omega_1 \succ_{R\phi} \omega_2 \succ_{R\phi} \omega_3 \sim_{R\phi} \omega_4 \succ_{R\phi} \omega_5 \succ_{R\phi} \omega_6 \succ_{R\phi} \text{others}.$$

To recover the improvement operator by Jeffrey's rule, we first define  $\pi$  to be a positive possibility distribution associated with  $\succeq$ , as follows. Let  $\{\psi_0, \dots, \psi_k\}$  be the WOP associated to  $\succeq$  and let  $\pi(\omega) = \alpha^i$  if  $\omega \models \psi_i$ . The chosen setting for the possibility scale is DENUM. Upgrading by one unit then means moving from  $\alpha^i$  to  $\alpha^{i-1}$ , and conversely for downgrading.

The way the reinforcement of  $\phi$  is implemented depends on the position of the best models of  $\phi$  with respect to the best models of  $\neg\phi$ .

- If  $\neg\phi \in \text{Bel}(\pi)$ , i.e.,  $\Pi(\neg\phi) > \Pi(\phi)$  then we simply shift up "by one unit" possibility degrees of models of  $\phi$ , turning  $\pi(\omega)$  into  $\frac{\pi(\omega)}{\alpha}$ , which makes sense since  $\Pi(\phi) \leq \alpha$  in the DENUM setting.
- Conversely, if  $\neg\phi \notin \text{Bel}(\pi)$ , i.e.,  $\Pi(\neg\phi) \leq \Pi(\phi)$  then we simply shift down "by one unit" possibility degrees of countermodels of  $\phi$ , turning  $\pi(\omega)$  into  $\pi(\omega) \cdot \alpha$

It is possible to define a partial epistemic state  $\mu$  such that the revision of  $\pi$  by  $\mu$  in the sense of Jeffrey-like rule with  $\otimes =$  product encodes the improvement operator. Namely consider

$$\mu = \left\{ \left( \phi, \min\left(1, \frac{\Pi(\phi)}{\alpha}\right) \right), \left( \neg\phi, \frac{\alpha \Pi(\neg\phi)}{\max(\alpha, \Pi(\phi))} \right) \right\} \quad (7)$$

Note that this is a proper partial epistemic state in the sense that one of the degrees attached to the proposition  $\phi$  or its complement in  $\mu$  is 1. Indeed if  $\Pi(\phi) = 1$  then  $\min\left(1, \frac{\Pi(\phi)}{\alpha}\right) = 1$ . Otherwise,  $\Pi(\neg\phi) = 1$  and  $\Pi(\phi) \leq \alpha$ , hence  $\frac{\alpha \Pi(\neg\phi)}{\max(\alpha, \Pi(\phi))} = 1$ .

**Proposition 6.** The possibility distribution  $\pi(\cdot \mid_p \mu)$ , where  $\mu$  is the input defined by eq. (7), encodes the revised plausibility ordering  $\succeq_{R\phi}$ , i.e.,  $\forall \omega_1, \omega_2 \in \Omega, \pi(\omega_1 \mid_p \mu) \geq \pi(\omega_2 \mid_p \mu)$  if and only if  $\omega_1 \succeq_{R\phi} \omega_2$ .

*Proof:* Suppose  $\Pi(\phi) < 1$ , hence  $\Pi(\phi) \leq \alpha$ . We must apply Jeffrey's rule with  $\mu = \left\{ \left( \phi, \frac{\Pi(\phi)}{\alpha} \right), (\neg\phi, 1) \right\}$ . Hence

- If  $\omega \models \phi$ ,  $\pi(\omega \mid_p \mu) = \frac{\Pi(\phi)\pi(\omega)}{\alpha\Pi(\phi)} = \frac{\pi(\omega)}{\alpha}$ ;
- If  $\omega \models \neg\phi$ ,  $\pi(\omega \mid_p \mu) = \frac{1 \cdot \pi(\omega)}{\Pi(\neg\phi)} = \pi(\omega)$  since  $\Pi(\neg\phi) = 1$ .

Suppose  $\Pi(\phi) = 1$ . We must apply Jeffrey's rule with  $\mu = \{(\phi, 1), (\neg\phi, \alpha\Pi(\neg\phi))\}$ . Hence

- If  $\omega \models \phi$ ,  $\pi(\omega \mid_p \mu) = \pi(\omega)$ ;
- If  $\omega \models \neg\phi$ ,  $\pi(\omega \mid_p \mu) = \frac{\alpha\Pi(\neg\phi)\pi(\omega)}{\Pi(\neg\phi)} = \alpha\pi(\omega)$ .

We do upgrade by one unit the models of  $\phi$  if  $\neg\phi$  is the most plausible a priori, and downgrade by one unit the models of  $\neg\phi$  if  $\phi$  is the most plausible a priori.  $\square$

**Example 4.8.** In Example 2.1, we use the same DENUM encoding of the  $\succ$  as previously, except that we use  $\alpha^6$  for "other models". We are in a situation where  $\Pi(\neg\phi) > \Pi(\phi)$  hence we simply shift up "by one unit" possibility degrees of models  $\omega_4$  and  $\omega_5$  of  $\phi$ . The revised possibility distribution after applying the reinforcement operator is given by Table 7.

	<i>A</i>	<i>H</i>	<i>B</i>	<i>V</i>	$\pi(\cdot)$	$\pi_{\succeq_t \mu}(\cdot)$
$\omega_1$	<i>a</i>	<i>h</i>	<i>b</i>	<i>v</i>	1	1
$\omega_2$	<i>a</i>	$\neg h$	<i>b</i>	<i>v</i>	$\alpha$	$\alpha$
$\omega_3$	$\neg a$	$\neg h$	<i>b</i>	<i>v</i>	$\alpha^2$	$\alpha^2$
$\omega_4$	$\neg a$	$\neg h$	<i>b</i>	$\neg v$	$\alpha^3$	$\alpha^2$
$\omega_5$	$\neg a$	$\neg h$	$\neg b$	$\neg v$	$\alpha^4$	$\alpha^3$
$\omega_6$	$\neg a$	$\neg h$	$\neg b$	<i>v</i>	$\alpha^5$	$\alpha^5$
Other models	$\neg h$	-	-	-	$\alpha^6$	$\alpha^6$

Table 7. Improvement revision with input  $\phi = \neg a \neg h \neg v$

## 5. Conclusion

In this paper, we have proposed a new axiomatic framework for iterated belief revision, that generalizes the Darwiche-Pearl framework and accommodates more complex inputs than single propositions. Input information consists in partial epistemic states defined on a partition of the set of possible worlds. We show how Jeffrey's rule, once adapted to possibility theory, can be used to define such revision operations changing an epistemic state into another one. The obtained family of revision operations obey the proposed axioms to a large extent. The importance of choosing a proper measurement scale for plausibility has been highlighted. It leads to two possible families of revision rules, one being qualitative and the other being quantitative. We show how possibilistic Jeffrey's rules can encode several important existing approaches to iterated belief revision, including the more recent notion of reinforcement operator. All these methods can be used to enhance the belief management capabilities of intelligent agents.

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