FUZZY-SET BASED LOGICS — AN HISTORY-ORIENTED PRESENTATION OF THEIR MAIN DEVELOPMENTS

Didier Dubois, Francesc Esteva, Lluís Godo, Henri Prade

1 INTRODUCTION: A HISTORICAL PERSPECTIVE

The representation of human-originated information and the formalization of commonsense reasoning has motivated different schools of research in Artificial or Computational Intelligence in the second half of the 20th century. This new trend has also put formal logic, originally developed in connection with the foundations of mathematics, in a completely new perspective, as a tool for processing information on computers. Logic has traditionally put emphasis on symbolic processing at the syntactical level and binary truth-values at the semantical level. The idea of fuzzy sets introduced in the early sixties [Zadeh, 1965] and the development of fuzzy logic later on [Zadeh, 1975a] has brought forward a new formal framework for capturing graded imprecision in information representation and reasoning devices. Indeed, fuzzy sets membership grades can be interpreted in various ways which play a role in human reasoning, such as levels of intensity, similarity degrees, levels of uncertainty, and degrees of preference.

Of course, the development of fuzzy sets and fuzzy logic takes its roots in concerns already encountered in non-classical logics in the first half of the century, when the need for intermediary truth-values and modalities emerged. We start by briefly surveying some of the main issues raised by this research line before describing the historical development of fuzzy sets, fuzzy logic and related issues.

Jan Łukasiewicz (1878-1956) and his followers have developed three-valued logics, and other many-valued systems, since 1920 [Łukasiewicz, 1920]. This research was motivated by philosophical concerns as well as some technical problems in logic but not so much by issues in knowledge representation, leaving the interpretation of intermediate truth-values unclear. This issue can be related to a misunderstanding regarding the law of excluded middle and the law of non-contradiction, and the connections between many-valued logics and modal logics. The principle of bivalence,

Every proposition is either true or false,

Handbook of the History of Logic. Volume 8
Dov M. Gabbay and John Woods (Editors)

formulated and strongly defended by Chrisippus and his school in antique Greece, was for instance questioned by Epicureans, and even rejected by them in the case of propositions referring to future contingencies.

Let us take an example considered already by Aristotle, namely the proposition:

"There will be a sea battle to-morrow (p) and
there will not be a sea battle to-morrow ($\neg p$)"

This proposition "p and $\neg p$" is ever false, because of the non-contradiction law and the proposition "p or $\neg p$" is ever true, because *tertium non datur*. But we may fail to know the truth of both propositions "there will be a sea battle to-morrow" and "there will not be a sea battle to-morrow". In this case, at least intuitively, it seems reasonable to say that it is *possible* that there will be a sea battle to-morrow but at the same time, it is *possible* that there will not be a sea battle to-morrow. There has been a recurrent tendency, until the twentieth century many-valued logic tradition, to claim the failure of the bivalence principle on such grounds, and to consider the modality *possible* as a third truth value. This was apparently (unfortunately) the starting motivation of Łukasiewicz for introducing his three-valued logic. Indeed, the introduction of a third truth-value was interpreted by Łukasiewicz as standing for *possible*. However the proposition "*possible* p" is not the same as p, and "*possible* $\neg p$" is not the negation of "*possible* p". Hence the fact that the proposition

$$\text{“}possible\ p\text{”} \wedge \text{“}possible\ \neg p\text{”}$$

may be true does not question the law of non-contradiction since "*possible* p" and "*possible* $\neg p$" are not mutually exclusive. This situation leads to interpretation problems for a fully truth-functional calculus of possibility, since even if p is "possible" and $\neg p$ is "possible", still $p \wedge \neg p$ is ever false.

On the contrary, vague or fuzzy propositions are ones such that, due to the gradual boundary of their sets of models, proposition "p and $\neg p$" is not completely false in some interpretations. This is why Moisil [1972] speaks of fuzzy logics as Non-Chrisippean logics.

A similar confusion seems to have prevailed in the first half of the century between probability and partial truth. Trying to develop a quantitative concept of truth, H. Reichenbach [1949] proposed his *probability logic* in which the alternative true-false is replaced by a continuous scale of truth values. In this logic he introduces *probability propositions* to which probabilities are assigned, interpreted as grades of truth. In a simple illustrative example, he considers the statement "I shall hit the center". As a measure of the degree of truth of this statement, Reichenbach proposes to measure the distance r of the hit to the center and to take the truth-value as equal to $1/(1 + r)$. But, of course, this can be done only after the shot. However, quantifying the proposition after the hit is not a matter of belief assessment when the distance to the center is known. It is easy to figure out retrospectively that this method is actually evaluating the fuzzy proposition "I hit close to the center". Of course we cannot evaluate the truth of the above sentence *before* the shot, because now it is a matter of belief assessment, for which probability can be suitable.

Very early, when many-valued logics came to light, some scholars in the foundations of probability became aware that probabilities differ from what logicians call truth-values. De Finetti [1936], witnessing the emergence of many-valued logics (especially the works of Łukasiewicz, see [Łukasiewicz, 1970]), pointed out that uncertainty, or partial belief, as captured by probability, is a meta-concept with respect to truth degrees, and goes along with the idea that a proposition, in its usual acceptance, is a binary notion. On the contrary, the notion of partial truth (i.e. allowing for intermediary degrees of truth between *true* -1- and *false* -0-) as put forward by Łukasiewicz [1930], leads to changing the very notion of proposition. Indeed, the definition of a proposition is a matter of convention. This remark clearly points out the fact that many-valued logics deal with many-valuedness in the logical status of propositions (as opposed to Boolean status), not with belief or probability of propositions. On the contrary, uncertainty pertains to the beliefs held by an agent, who is not totally sure whether a proposition of interest is true or false, without questioning the fact that ultimately this proposition cannot be but true or false.

Probabilistic logic, contrary to many-valued logics, is not a substitute of binary logic. It is only superposed to it. However this point is not always clearly made by the forefunners of many-valued logics. Carnap [1949] also points out the difference in nature between truth-values and probability values (hence degrees thereof), precisely because "true" (resp: false) is not synonymous to "known to be true" (resp: known to be false), that is to say, verified (resp: falsified). He criticizes Reichenbach on his claim that probability values should supersede the two usual truth-values.

In the same vein, H. Weyl [1946] introduced a calculus of vague predicates treated as functions defined on a fixed universe of discourse U, with values in the unit interval. Operations on such predicates $f : U \to [0, 1]$ have been defined as follows:

$$f \cap g = \min(f, g) \text{ (conjunction);}$$
$$f \cup g = \max(f, g) \text{ (disjunction);}$$
$$f^c = 1 - f \text{ (negation).}$$

Clearly, this is one ancestor of the fuzzy set calculus. However, one of the approaches discussed by him for interpreting these connectives again considers truth values as probabilities. As shown above, this interpretation is dubious, first because probability and truth address different issues, and especially because probabilities are not compositional for all logical connectives (in fact, only for negation).

The history of fuzzy logic starts with the foundational 1965 paper by Lotfi Zadeh entitled "Fuzzy Sets" [Zadeh, 1965]. In this paper, motivated by problems in pattern classification and information processing, Zadeh proposes the idea of fuzzy sets as generalized sets having elements with intermediary membership grades. In this view, a fuzzy set is characterized by its membership function, allocating a membership grade to any element of the referential domain. The unit interval is usually taken as the range of these membership grades, although any suitable partially ordered set could also be used (typically: a complete lattice [Goguen, 1967]. Then, extended set theoretic operations on membership functions are defined by means of many-valued connectives, such as minimum and maximum for the intersection and the union respectively. Later, due to other researchers, it has been recognised that the appropriate connectives for defining generalized intersection and union operations was a class of associative monotonic connectives known as triangular norms (t-norms for short), together with their De Morgan dual triangular co-norms (t-conorms for short) (see Section 2.1). These operations are at the basis of the semantics of a class of mathematical fuzzy logical systems that have been thoroughly studied in the recent past, as it will be reported later in Section 3.

While the many-valued logic stream has mainly been developed in a mathematical logic style, the notion of fuzzy set-based approximate reasoning as imagined by Zadeh in the seventies is much more related to information processing: he wrote in 1979 that "the theory of approximate reasoning is concerned with the deduction of possibly imprecise conclusions from a set of imprecise premises" [Zadeh, 1979a]. Fuzzy logic in Zadeh's sense, as it can be seen in the next section, is both a framework allowing the representation of vague (or gradual) predicates and a framework to reason under incomplete information. By his interest in modeling vagueness, Zadeh strongly departs from the logical tradition that regards vague propositions as poor statements to be avoided or to be reformulated more precisely [Russell, 1923]. Moreover, the view of *local* fuzzy truth-values emphasized by Bellman and Zadeh [1977] really means that in fuzzy logic, what is called *truth* is evaluated with respect to a description of a state of (vague, incomplete) knowledge, and not necessarily with respect to an objective, completely and precisely known state of the world.

Many-valued logics are a suitable formalism to deal with an aspect of vagueness, called fuzziness by Zadeh, pertaining to gradual properties. It should be emphasized that the fuzziness of a property is not viewed as a defect in the linguistic expression of knowledge (e.g., lack of precision, sloppiness, limitation of the natural languages), but rather as a way of expressing gradedness. In that sense, fuzzy sets do not have exactly the same concern as other approaches to vagueness. For instance, K. Fine [1975] proposes that statements about a vague predicate be taken to be true if and only if they hold for all possible ways of making the predicate clear-cut. It enables classical logic properties to be preserved, like the mutual exclusiveness between a vague predicate A and its negation not-A. In contrast, the fuzzy set view maintains that in some situations there is no clear-cut predicate underlying a fuzzy proposition due to the smooth transition from one class to another induced by its gradual nature. In particular, A and not-A will have a limited overlap; see [Dubois *et al.*, 2005] for a detailed discussion. The presence of this overlap leads to a logical view of interpolative reasoning [Klawonn and Novák, 1996; Dubois *et al.*, 1997].

However, when only imprecise or incomplete information is available, truth-values (classical or intermediate) become ill-known. Then belief states can be modeled by sets of truth-values. Actually, what are called *fuzzy truth-values* by Zadeh turn out to be ill-known truth-values in this sense. They are fuzzy sets of truth-values and not so much an attempt to grasp the linguistic subtleties of the word *true* in natural languages.

Strictly speaking, fuzzy set theory deals with classes with unsharp boundaries and gradual properties, but it is not concerned with uncertainty or partial belief. The latter is rather due to a lack of precise (or complete) information, then making truth-values ill-known. This is the reason why Zadeh [1978a] introduced possibility theory, which naturally complements fuzzy set theory for handling uncertainty induced by fuzzy and incomplete pieces of information. Possibility theory turns out to be a non-probabilistic view of uncertainty aiming at modeling states of partial or complete ignorance rather than capturing randomness. Based on possibility theory, a logical formalism has been developed in the last twenty years under the name of *possibilistic logic* (see Section 4.1).

Therefore we can distinguish:

- states with Boolean information from states with gradual information (leading to intermediate uncertainty degrees) and,
- statements that can be only true or false from statements that may have an intermediate truth-values because they refer to vague or gradual properties.

This analysis leads us to four noticeable classes of formalisms: (i) classical logic where both truth and belief (understood as the status of what can be inferred from available information) are Boolean, (ii) many-valued logics where truth is a matter of degree but consequencehood is Boolean, (iii) possibilistic logic for graded belief about Boolean statements, and (iv) the general case of non-Boolean statements leading to graded truth and imprecise information leading to graded beliefs, which motivated Zadeh's proposal.

In the last twenty years, while researchers have been developing formal many-valued logics and uncertainty logics based on fuzzy sets, Zadeh rather emphasized computational and engineering issues by advocating the importance of *soft computing* (a range of numerically oriented techniques including fuzzy rules-based control systems, neural nets, and genetic algorithms [Zadeh, 1994b]) and then introduced new paradigms about computational intelligence like *granular computing* [Zadeh, 1997], *computing with words* [Zadeh, 1995] and *perception-based reasoning* [Zadeh, 1999], trying to enlarge his original motivation for a computational approach to the way humans handle information.

Since fuzzy sets, fuzzy logic, possibility theory, and soft computing have the same father, Zadeh, these notions are too often confused although they refer to quite different tasks and have been developed in sometimes opposite directions. On the one hand, the term *fuzzy logic*, understood in the narrow/technical sense refers to many-valued logics that handle gradual properties (that are a matter of degree, e.g. "large", "old", "expensive", ...). These logics are developed by logicians or artificial intelligence theoreticians. Technicaly speaking, they are compositional w.r.t. to all logical connectives, while uncertainty logics (like possibilistic logic) cannot never be compositional w.r.t. to all logical connectives. On the other hand, "fuzzy logics", in the broad sense, is a generic expression that most of the time refers to that part of soft computing where fuzzy sets and fuzzy rules are used. Lastly, "soft computing" is a buzz-word sometimes referring to the same research trend as "computational intelligence" (viewed as an alternative problem solving paradigm to classical artificial intelligence methods that are found to be too symbolically-oriented).

The remaining part of the chapter is structured as follows. Section 2 provides a detailed account of the fuzzy set-based approach to approximate reasoning. It starts with a review of fuzzy set connectives and the possibility theory-based representation of information under the form of flexible constraints. Then the approximate reasoning methodology based on the combination and projection of such flexible constraints is described, before providing a detailed discussion on the specially important notion of fuzzy truth value in this setting. The last part of this section is devoted to the representation of different types of fuzzy if-then rules and to the discussion of the generalized modus ponens and some related issues such as basic inference patterns.

Section 3 contains a survey of the main many-valued logical systems more recently developed in relation to the formalization of fuzzy logic in narrow sense. The so-called t-norm based fuzzy logics are first introduced, providing Hilbert-style axiomatizations of main systems, their algebraic semantics as well as analytical proof calculi based on hypersequents for some of these logics. Extensions of these logics with truth-constants and additional connectives are also reported. Then, an overview of other systems of many-valued logic with deduction based on resolution-style inference rules is presented. A more abstract point of view, the consequence operators approach to fuzzy logic, is also surveyed. Finally, a many-valued logic encoding of major approximate reasoning patterns is described.

Section 4 is devoted to fuzzy set-based logical formalisms handling uncertainty and similarity, including possibilistic logic, its extension to deal with fuzzy constants, similarity-based inference, modal fuzzy theories of uncertainty, and logics handling fuzzy truth values in their syntax.

2 A GENERAL THEORY OF APPROXIMATE REASONING

Zadeh proposed and developed the theory of approximate reasoning in a long series of papers in the 1970's [1973; 1975a; 1975b; 1975c; 1976; 1978b; 1979a], at the same time when he introduced possibility theory [Zadeh, 1978a] as a new approach to uncertainty modeling. His original approach is based on a fuzzy set-based representation of the contents of factual statements (expressing elastic restrictions on the possible values of some parameters) and of if-then rules relating such fuzzy statements.

The phrase *fuzzy logic* appears rather early [Zadeh, 1973]: "[...] the pervasiveness of fuzziness in human thought processes suggests that much of the logic behind human reasoning is not the tradidional two-valued or even multivalued logic, but a logic with fuzzy truths, fuzzy connectives and fuzzy rules of inference. In our view, it is this *fuzzy*, and as yet not well-understood, *logic*[1] that plays a basic role in what may well be one of the most important facets of human thinking [...]". Clearly, after its founder, fuzzy logic strongly departs at first glance from the standard view of logic where inference does not depend on the contents of propositions. Indeed from p and $p' \to q$ one always infers q whenever $p \vdash p'$ for any propositions p, p' and q, while in Zadeh's generalized modus ponens, which is a typical pattern of approximate reasoning, from "X is A^*" and "if X is A then Y is B", one deduces "Y is B^*" where $B^* = f(A^*, A, B)$ depends on the implication chosen, and may differ from B while being non-trivial. Thus, in this approach, the content of an inference result does depend on the semantic contents of the premises.

Strictly speaking, the presentation in retrospect, below, of Zadeh's theory of approximate reasoning does not contain anything new. Still, we emphasize how altogether its main features contribute to a coherent theory that turns to encompass several important particular cases of extensions of classical propositional logic, at the semantic level. Moreover, we try to point out the importance of the idea of *fuzzy truth as compatibility*, and of the converse notion of *truth qualification*, two key issues in the theory of approximate reasoning which have been often overlooked or misunderstood, as well as the role of the minimal specificity principle in the representation of information in possibility theory. The section below can be viewed as a revised and summarized version of [Bouchon-Meunier *et al.*, 1999], where more details can be also found about various approaches that are more loosely inspired from Zadeh's proposal.

2.1 Fuzzy sets

This section provides basic definitions of fuzzy set theory and its main connectives. The emphasis is also put here on the various representations of a fuzzy set, that are instrumental when extending formal notions from sets to fuzzy sets.

Membership Functions

L. A. Zadeh has given in his now famous paper [Zadeh, 1965] the following definition: A fuzzy set is a class with a continuum of membership grades. So, a fuzzy set (class) F in a referential U is characterized by a membership function which associates with each element $u \in U$ a real number in the interval $[0, 1]$. The value of the membership function at element u represents the "grade of membership"of u in F. A fuzzy set F is thus defined as a mapping

$$F : U \to [0, 1],$$

[1]Italics are ours

and it is a kind of generalization of the traditional characteristic function of a subset $A : U \to \{0, 1\}$.There is a tendency now to identify the theory of fuzzy sets with a theory of generalized characteristic functions[2]. In particular, $F(u)$ =1 reflects full membership of u in F, while $F(u) = 0$ expresses absolute non-membership in F. Usual sets can be viewed as special cases of fuzzy sets where only full membership and absolute non-membership are allowed. They are called crisp sets, or Boolean sets. When $0 < F(u) < 1$, one speaks of partial membership. For instance, the term *young* (for ages of humans) may apply to a 30-year old individual only at degree 0.5.

A fuzzy set can be also denoted as a set of pairs made of an element of U and its membership grade when positive: $\{(u, F(u)), u \in (0, 1]\}$. The set of fuzzy subsets of U is denoted $\mathcal{F}(U)$. The membership function attached to a given word (such as *young*) depends on the contextual intended use of the word; a young retired person is certainly older than a young student, and the idea of what a young student is also depends on the user. However, in the different contexts, the term *young* will be understood as a gradual property generally. Membership degrees are fixed only by convention, and the unit interval as a range of membership grades, is arbitrary. The unit interval is natural for modeling membership grades of fuzzy sets of real numbers. The continuity of the membership scale reflects the continuity of the referential. Then a membership degree $F(u)$ can be viewed as a degree of proximity between element u and the prototypes of F, that is, the elements v such that $F(v) = 1$. The membership grade decreases as elements are located farther from such prototypes. This representation points out that there is no precise threshold between ages that qualify as *young* and ages that qualify as *not young*. More precisely there is a gap between protopypes of *young* and proptypes of *not young*. It is clear that fuzzy sets can offer a natural interface between linguistic representations and numerical representations. Of course, membership grades never appear as such in natural languages. In natural language, gradual predicates are those to which linguistic hedges such as *very* can be applied. Such linguistic hedges are the trace of gradual membership in natural language. Clearly the numerical membership grade corresponding to *very* is itself ill-defined. It is a fuzzy set of membership degrees as suggested by Zadeh [1972]. He suggested to build the membership function of *very young* from the one of *young* and the one of *very*, by letting *very-young*$(\cdot)$ = *very*(*young*$(\cdot)$). So, fuzzy subsets of membership grades (represented by a function from $[0, 1]$ to itself) model linguistic hedges that can modify membership functions of fuzzy predicates.

However if the referential set U is a finite set of objects then the use of the unit interval as a set of membership grades is more difficult to justify. A finite totally ordered set L will then do. It results from a partitioning of elements of U with respect to a fuzzy set F, each class in the partition gathering elements with equal membership, and the set of classes being ordered from full membership to non-membership.

[2]This is why in the following we shall equivalently denote the membership grade of u to a fuzzy set F as $F(u)$ or the more usual $\mu_F(u)$, according to best convenience and clarity

Parikh [1983] questions the possibility of precisely assessing degrees of truth for a vague predicate. In practice, however membership degrees have mainly an ordinal meaning. In other words it is the ordering induced by the membership degrees between the elements that is meaningful, rather than the exact value of the degrees. This is in agreement with the qualitative nature of the most usual operations that are used on these degrees ($\min, \max$ and the complementation to 1 as an order-reversing operation in $[0, 1]$, as recalled below).

Obviously a fuzzy membership function will depend on the context in various ways. First, the universe of discourse (i.e., the domain of the membership function) has to be defined (e.g., *young* is not the same thing for a man or for a tree). Second, it may depend on the other classes which are used to cover the domain. For instance, with respect to a given domain, *young* does not mean exactly the same thing if the remaining vocabulary includes only the word *old*, or is richer and contains both *mature* and *old*. Lastly, a fuzzy membership function may vary from one person to another. However, what is really important in practice is to correctly represent the pieces of knowledge provided by an expert and capture the meaning he intends to give to his own words. Whether there can be a universal consensus on the meaning of a linguistic expression like *young* man is another matter.

Level Cuts

Another possible and very convenient view of fuzzy set is that of a nested family of classical subsets, via the notion of level-cut. The α-level cut F_α of a fuzzy set F is the set $\{u \in U : F(u) \geq \alpha\}$, for $1 \geq \alpha > 0$. The idea is to fix a positive threshold α and to consider as members of the set the elements with membership grades above the threshold. Moving the threshold in the unit interval, the family of crisp sets $\{F_\alpha : 1 \geq \alpha > 0\}$ is generated. This is the horizontal view of a fuzzy set. For $\alpha = 1$, the *core* of F is obtained. It gathers the prototypes of F. Letting α vanish, the *support* $s(F)$ of F is obtained. It contains elements with positive membership grades, those which belong to some extent to F. Note that the support is different from $F_0 = U$. Gentilhomme[1968]'s “ensembles flous” were fuzzy sets with only a core and a support.

The set of level-cuts of F is nested in the sense that :

(1) $\alpha < \beta$ implies $F_\beta \subseteq F_\alpha$

Going from the level-cut representation to the membership function and back is easy. The membership function can be recovered from the level-cut as follows:

(2) $F(u) = \sup\{\alpha : u \in F_\alpha\}$

Conversely, given an indexed nested family $\{A_\alpha : 1 \geq \alpha > 0\}$ such that $A_0 = U$ and condition (1) (plus a continuity requirement in the infinite case) holds, then there is a unique fuzzy set F whose level-cuts are precisely $F_\alpha = A_\alpha$ for each $\alpha \in [0, 1]$. This representation theorem was obtained by Negoita and Ralescu [1975].

Fuzzy Connectives: Negations, Conjunctions and Disjunctions

The usual set-theoretic operations of complementation, intersection and union were extended by means of suitable operations on $[0,1]$ (or on some weaker ordered structure), that mimic, to some extent, the properties of the Boolean connectives on $\{0,1\}$ used to compute the corresponding characteristic functions. Namely, denoting $(\cdot)^c, \cap, \cup$, the fuzzy set complementation, intersection and union, respectively, these connectives are usually understood as follows:

(3) $A^c(u) = n(A(u))$

(4) $(A \cap B)(u,v) = T(A(u), B(v))$

(5) $(A \cup B)(u,v) = S(A(u), B(v))$

where A is a fuzzy subset of a universe U, B a fuzzy subset of a universe V, and where n is a so-called negation function, T is a so-called *triangular norms* and S a *triangular conorms*, whose characteristic properties are stated below. Note that strictly speaking, equations 4-5 define the intersection and union of fuzzy sets only if $U = V$ and $u = v$; otherwise they define the Cartesian product of A and B and the dual co-product. All these connective operations are actually extensions of the classical ones, i.e., for the values 0 and 1, they behave classically, and give rise to different multiple-valued logical systems when they are taken as truth-functions for connectives (see Section 3 of this chapter).

It is worth noticing that in his original paper, acknowledgedly inspired in part by Kleene's many-valued logics [Kleene, 1952], Zadeh proposed to interpret complementation, intersection and union by means of $1-(\cdot)$, min and max operations respectively. These operations are the only ones that are compatible with the level cuts view of fuzzy sets. Zadeh also mentioned the possibility of using other operations, namely the algebraic product for intersection-like, and its De Morgan dual as well as algebraic sum (when not greater than 1) for union-like fuzzy set theoretic operations. Axioms for fuzzy set operations were proposed as early as 1973, starting with [Bellman and Giertz, 1973] and later Fung and Fu [Fung and Fu, 1975]. However the systematic study of fuzzy set connectives was only started in the late seventies by several scholars, like Alsina, Trillas, Valverde [1980; 1983], Hoehle [1979], Klement[1980], Dubois and Prade[1979a; 1980] (also [Dubois, 1980], [Prade, 1980]) and many colleagues, and led to a general framework outlined below.

A negation n is a unary operation in $[0,1]$ [Trillas, 1979] satisfying the following properties:

$$n(0) = 1; \tag{6}$$

$$n(1) = 0; \tag{7}$$

$$n(a) \geq n(b), \quad \text{if } a \leq b; \tag{8}$$

$$n(n(a)) \geq a. \tag{9}$$

Furthermore, if $n(n(a)) = a$, i.e., if n is an involution, n is called a *strong negation*. The most typical strong negation is $n(a) = 1 - a$, for all $a \in [0,1]$.

Gödel's negation, defined as $n(0) = 1$ and $n(a) = 0$ for all $a \in (0, 1]$, is an example of non-strong negation.

Triangular norms (t-norms for short) and triangular conorms (t-conorms for short) were invented by Schweizer and Sklar [1963; 1983], in the framework of probabilistic metric spaces, for the purpose of expressing the triangular inequality. They also turn out to the most general binary operations on [0, 1] that meet natural and intuitive requirements for conjunction and disjunction operations. Namely, a t-norm T is a binary operation on $[0, 1]$, i.e., $T : [0, 1]\times : [0, 1] \to [0, 1]$, that satisfies the following conditions:

- commutative : $T(a, b) = T(b, a)$;
- associative: $T(a, T(b, c)) = T(T(a, b), c)$;
- non-decreasing in both arguments: $T(a, b) \leq T(a', b')$ if $a \leq a'$ and $b \leq b'$;
- boundary conditions: $T(a, 1) = T(1, a) = a$.

It can be proved that $T(a, 0) = T(0, a) = 0$. The boundary conditions and the latter conditions respectively express the set-theoretic properties $A \cap U = A$ and $A \cap \emptyset = \emptyset$. It is known that the minimum operation is the greatest t-norm, i.e., for any t-norm T, $T(a, b) \leq \min(a, b)$ holds for all $a, b \in [0, 1]$. Typical basic examples of t-norms are

- the minimum : $T(a, b) = \min(a, b)$,
- the product: $T(a, b) = a \cdot b$
- the linear t-norm: $T(a, b) = \max(0, a + b - 1)$

The linear t-norm is often referred to as Łukasiewicz's t-norm[3] . Note the inequalities,

$$\max(0, a + b - 1) \leq a \cdot b \leq \min(a, b).$$

The De Morgan-like dual notion of a t-norm (w.r.t. negation $n(a) = 1 - a$, or a more general strong negation) is that of a t-conorm. A binary operation S on [0, 1] is called a t-conorm if it satisfies the same properties as the ones of a t-norm except for the boundary conditions, namely, here 0 is an identity and 1 is absorbent. Namely the following conditions express that $A \cup \emptyset = A$:

$$\text{boundary conditions: } S(0, a) = S(a, 0) = a.$$

Hence $S(a, 1) = S(1, a) = 1$, expressing that $A \cup U = U$. Dually, the maximum operation is the smallest t-conorm ($S(a, b) \geq \max(a, b)$).

T-norms and t-conorms are dual with respect to strong negations in the following sense: if T is a (continuous) t-norm and n a strong negation then the function

[3] because it is closely related to the implication connective $\min(1, 1-a+b)$ originally introduced by Łukasiewicz

S defined as $S(a,b) = n(T(n(a), n(b)))$ is a (continuous) t-conorm, and conversely, if S is a t-conorm, then the function T defined as $T(a,b) = n(S(n(a), n(b)))$ is a t-norm.

Typical basic examples of t-conorms are the duals of minimum, product and Łukasiewicz' t-norms, namely the maximum $S(a,b) = \max(a,b)$, the so-called probabilistic sum $S(a,b) = a+b-ab$ and the bounded sum $S(a,b) = \min(1, a+b)$. Note now the inequalities

$$\max(a,b) \le a + b - a \cdot b \le \min(1, a+b).$$

A t-norm (resp. a t-conorm) is said to be continuous if it is a continuous mapping from $[0,1]^2$ into $[0,1]$ in the usual sense. For continuous t-norms commutativity is a consequence of the other properties (see Theorem 2.43 in [Klement *et al.*, 2000]). All the above examples are continuous. An important example of non-continuous t-norm is the so-called *nilpotent minimum* [Fodor, 1995] defined as

$$T(a,b) = \begin{cases} \min(a,b), & \text{if } a+b \ge 1 \\ 0, & \text{otherwise.} \end{cases}$$

See the monographs by Klement, Mesiar and Pap [2000] and by Alsina, Frank and Schweizer [2006] for further details on triangular norms, conorms and negation functions.

Fuzzy Implications

Most well-known fuzzy implication functions $I : [0,1] \times [0,1] \to [0,1]$, are generalizations, to multiple-valued logical systems, of the classical implication function. In classical logic the deduction theorem states the equivalence between the entailments $r \wedge p \models q$ and $r \models p \to q$, and this equivalence holds provided that $p \to q \equiv \neg p \vee q$. In terms of conjunction and implication functions, this can be expressed as

$$c \le I(a,b) \iff T(a,c) \le b$$

where $a, b, c \in \{0,1\}$. In the Boolean setting it is easy to see that $I(a,b) = S(n(a), b)$, where S coincide with disjunction and n with classical negation.

However these two interpretations give rise to distinct families of fuzzy implications, extending the set $\{0,1\}$ to the unit interval. The *strong* and *residuated* implication functions (S-implications and R-implications for short) are respectively defined as follows [Trillas and Valverde, 1981].

1. S-implications are of the form $I_S(a,b) = S(n(a), b)$, where S is a t-conorm and n is a strong negation function, hence the name of strong implication, also due to the fact that when $S = \max$, or probabilistic sum, it refers to a strong fuzzy set inclusion requiring that the support of one fuzzy set be included into the core of the other one).

2. R-implications are of the form $I_R(a,b) = \sup\{z \in [0,1] : T(a,z) \le b\}$, where T is a t-norm. This mode of pseudo-inversion of the t-norm is a generalization of the traditional residuation operation in lattices, e.g. [Galatos *et al.*, 2007] for a recent reference.

Residuated implications make sense if and only if the generating t-norm is left-continuous. Both kinds of implication functions share the following reasonable properties:

- Left-decreasingness: $I(a,b) \ge I(a',b)$ if $a \le a'$;
- Right-increasingness: $I(a,b) \le I(a,b')$ if $b \le b'$;
- Neutrality: $I(1,b) = b$;
- Exchange: $I(a, I(b,c)) = I(b, I(a,c))$.

Notice that another usual property like

$$\text{Identity: } I(a,1) = 1$$

easily comes from the neutrality and monotonicity properties. The main difference between strong and residuated implications lies in the fact that the contraposition property, i.e.

$$\text{Contraposition: } I(a,b) = I(n(b), n(a)),$$

symbol n being some negation function, holds for all strong implications but fails for most residuated implications. In contrast, the following property

$$\text{Ordering: } I(a,b) = 1 \text{ iff} a \le b,$$

which establishes the fact that implication defines an ordering, holds for all residuated implications but fails for most strong ones. The failure of the contraposition property for the residuated implications enables a third kind of implication functions to be defined, the so-called reciprocal R-implications, in the following way:

$$I_C(a,b) = I_R(n(b), n(a))$$

for some residuated implication I_R and negation n. The above monotonicity and exchange properties still hold for these reciprocal implications, but now the neutrality principle is no longer valid for them. However, the following properties do hold for them:

- Negation: $I_C(a,0) = n(a)$
- Ordering: $I_C(a,b) = 1$ iff $a \le b$

generating t-norm	S-implication $n(a) = 1 - a$	R-implication	Reciprocal R-implication $n(a) = 1 - a$
$\min(a, b)$	$\max(1 - a, b)$ *Kleene-Dienes*	$\begin{cases} 1, \text{ if } a \le b \\ b, \text{ otherwise} \end{cases}$ *Gödel*	$\begin{cases} 1, & \text{if } a \le b \\ 1 - a, & \text{otherwise} \end{cases}$
$a \cdot b$	$1 - a + a \cdot b$ *Reichenbach*	$\begin{cases} 1, & \text{if } a \le b \\ b/a, & \text{otherwise} \end{cases}$ *Goguen*	$\begin{cases} 1, & \text{if } a \le b \\ \frac{1-a}{1-b}, & \text{otherwise} \end{cases}$
$\max(0, a + b - 1)$	$\min(1, 1 - a + b)$ *Łukasiewicz*	$\min(1, 1 - a + b)$ *Łukasiewicz*	$\min(1, 1 - a + b)$ *Łukasiewicz*

Table 1. Main multiple-valued implications

Notice that the first one also holds for strong implications while the second, as already noticed, holds for the residuated implications as well. Table 1 shows the corresponding strong, residuated and reciprocal implications definable from the three main t-norms and taking the usual negation $n(a) = 1 - a$. Notice that the well-known Łukasiewicz implication $I(a, b) = \min(1, 1 - a + b)$ is both an S-implication and an R-implication, and thus a reciprocal R-implication too. The residuated implication induced by the nilpotent minimum is also an S-implication defined by:

$$I_R(a, b) = \begin{cases} 1, & \text{if } a \le b \\ \max(1 - a, b), & \text{otherwise.} \end{cases}$$

More generally all R-implications such that $I_R(a, 0)$ define an involutive negation are also S-implications.

Considering only the core of R-implications gives birth to another multiple-valued implication of interest, usually named Gaines-Rescher implication, namely

$$I_R(a, b) = \begin{cases} 1, & \text{if } a \le b \\ 0, & \text{otherwise.} \end{cases}$$

Let us observe that this implication fails to satisfy the neutrality property, we only have $I(1, b) \le b$, since $I(1, b) = 0$ when $b < 1$. Moreover, by construction, this connective is all-or-nothing although it has many-valued arguments.

For more details the reader is referred to studies of various families of fuzzy implication functions satisfying some sets of required properties, for instance see [Baldwin and Pilsworth, 1980; Domingo *et al.*, 1981; Gaines, 1976; Smets and Magrez, 1987; Trillas and Valverde, 1985a; Weber, 1983]. See also [Fodor and Yager, 2000] for a more extensive survey of fuzzy implications.

Remark: Non-Commutative Conjunctions. Dubois and Prade[1984a] have shown that S-implications and R-implications could be merged into a single family, provided that the class of triangular norms is enlarged to non-commutative conjunction operators. See [Fodor, 1989] for a systematic study of this phenomenon. For instance, the Kleene-Dienes S-implication $a \to b = \max(1 - a, b)$ can be obtained by residuation from the non-commutative conjunction

$$T^{\star}(a, b) = \begin{cases} 0, & \text{if } a + b \leq 1 \\ b, & \text{otherwise} \end{cases} .$$

Note that the nilpotent minimum t-norm value for the pair (a, b) is the minimum of $T^{\star}(a, b)$ and $T^{\star}(b, a)$.

2.2 The possibility-theoretic view of reasoning after Zadeh

The core of Zadeh's approach to approximate reasoning [Zadeh, 1979a] can retrospectively be viewed as relying on two main ideas: i) the possibility distribution-based representation of pieces of knowledge, and ii) a combination / projection method that makes sense in the framework of possibility theory. This what is restated in this section.

Possibility distributions and the minimal specificity principle

Zadeh's knowledge representation framework is based on the idea of expressing restrictions on the possible values of so-called *variables*. These variables are more general than the notion of propositional variable in logic, and refer to parameters or single-valued attributes used for describing a situation, such as for instance, the pressure, the temperature of a room, the size, the age, or the sex for a person. Like in the case of random variables and probability distributions, the ill-known value of these variables can be associated with distributions mapping the domain of the concerned parameter or attribute to the real unit real interval $[0, 1]$. These distributions are named *possibility distributions*. Thus, what is known about the value of a variable x, whose domain is a set U, is represented by a possibility distribution π_x. A value $\pi_x(u)$ is to be understood as the degree of possibility that $x = u$ (variable x takes value u). When $\pi_x(u) = 0$, it means that the value u (in U) is completely impossible for x, while $\pi_x(u)$ is all the larger as u is considered to be a more possible (or in fact, less impossible) value for x; $\pi_x(u) = 1$ expresses that absolutely nothing forbids to consider u as a possible value for x, but there may exist other values u' such $\pi_x(u') = 1$. In that sense, π_x expresses potential possibility.

Since knowledge is often expressed linguistically in practice, Zadeh uses fuzzy sets as a basis for the possibilistic representation setting that he proposes. Then a fuzzy set E is used to represent an incomplete piece of information about the value of a single-valued variable x, the membership degree attached to a value expresses the level of possibility that this value is indeed *the* value of the variable. This is what happens if the available information is couched in words, more precisely in fuzzy statements S of the form "x is E", like in, e.g."Tom is young". Here the fuzzy set "young" represents the set of possible values of the variable x = *age of Tom*. The fuzzy set E is then interpreted as a *possibility distribution* [Zadeh, 1978a], which expresses the levels of plausibility of the possible values of the ill-known variable x. Namely if the *only available* knowledge about x is that "x lies in E" where E is a fuzzy subset of U, then the possibility distribution of x is defined by the equation:

(10) $\pi_x(u) = \mu_E(u), \forall u \in U,$

where E (with membership function μ_E) is considered as the fuzzy set of (more or less) possible values of x and where π_x ranges on $[0, 1]$. More generally, the range of a possibility distribution can be any bounded linearly ordered scale (which may be discrete, with a finite number of levels). Fuzzy sets, viewed as possibility distributions, act as flexible constraints on the values of variables referred to in natural language sentences. The above equation represents a statement of the form "x lies in E" or more informally "x is E". It does not mean that possibility distributions are the same as membership functions, however. The equality $\pi_x = \mu_E$ is an assignment statement since it means: given that the only available knowledge is "x lies in E", the degree of possibility that $x = u$ is evaluated by the degree of membership $\mu_E(u)$.

If two possibility distributions pertaining to the same variable x, π_x and π'_x are such that $\pi_x < \pi'_x$, π_x is said to be more *specific* than π'_x in the sense that no value u is considered as less possible for x according to π'_x than to π_x. This concept of specificity whose importance has been first stressed by Yager [1983a] underlies the idea that any possibility distribution π_x is provisional in nature and likely to be improved by further information, when the available one is not complete. When $\pi_x < \pi'_x$, the information π'_x is redundant and can be dropped.

When the available information stems from several reliable sources, the possibility distribution that accounts for it is the least specific possibility distribution that satisfies the set of constraints induced by the pieces of information given by the different sources. This is the *principle of minimal specificity.* Particularly, it means that given a statement "x is E", then any possibility distribution π such that $\pi(u) \leq \mu_E(u)$, $\forall u \in U$, is in accordance with "x is E". However, in order to represent our knowledge about x, choosing a particular π such that $\exists u$, $\pi(u) < \mu_E(u)$ would be arbitrarily too precise. Hence the equality $\pi_x = \mu_E$ is naturally adopted if "x is E" is the only available knowledge, and already embodies the principle of minimal specificity.

Let x and y be two variables taking their values on domains U and V respectively. Any relation R, fuzzy or not, between them can be represented by a joint possibility distribution, $\pi_{x,y} = \mu_R$, which expresses a (fuzzy) restriction on the Cartesian product $U \times V$. Common examples of such fuzzy relations R between two variables x and y are representations of "approximately equal" (when $U = V$), "much greater than" (when $U = V$ is linearly ordered), or function-like relations such that the one expressed by the fuzzy rule “if x is small then y is large” (when U and V are numerical domains). Joint possibility distributions can be easily extended to more than two variables. Generally speaking, we can thus represent fuzzy statements S of the form “$(x_1, \ldots, x_n)$ are in relation R” (where R may be itself defined from more elementary fuzzy sets, as seen later in the case of fuzzy rules).

Possibility and necessity measures

The extent to which the information “x is E”, represented by the possibility distribution $\pi_x = \mu_E$, is consistent with a statement like “the value of x is in subset A” is estimated by means of the possibility measure Π, defined by Zadeh[1978a]:

(11) $\Pi(A) = \sup_{u \in A} \pi_x(u).$

where A is a *classical subset* of U. The value of $\Pi(A)$ corresponds to the element(s) of A having the greatest possibility degree according to π_x; in the finite case, “sup” can be changed into “max” in the above definition of $\Pi(A)$ in eq. (11). $\Pi(A) = 0$ means $x \in A$ is impossible knowing that “x is E” . $\Pi(A)$ estimates the *consistency* of the statement “$x \in A$” with what we know about the possible values of x. It corresponds to a logical view of possibility. Indeed, if π_x models a non-fuzzy piece of incomplete information represented by an ordinary subset E, the definition of a possibility measure reduces to

(12) $\Pi_E(A) = \begin{cases} 1, & \text{if } A \cap E \neq \emptyset \qquad (x \in A \text{ and } x \in E \text{ are consistent}) \\ 0, & \text{otherwise} \qquad (A \text{ and } E \text{ are mutually exclusive}). \end{cases}$

Any possibility measure Π satisfies the following max-decomposability characteristic property

(13) $\Pi(A \cup B) = \max(\Pi(A), \Pi(B)).$

Among the features of possibility measures that contrast with probability measures, let us point out the weak relationship between the possibility of an event A and that of its complement A^c ('not A'). Either A or A^c must be possible, that is $\max(\Pi(A), \Pi(A^c)) = 1$ due to $A \cup A^c = U$ and $\Pi(U) = 1$ (normalization of Π). The normalization of Π requires that $\sup_{u \in U} \pi_x(u) = 1$; if U is finite, it amounts to requiring the existence of some $u_0 \in U$ such that $\pi_x(u_0) = 1$. This normalization expresses consistency of the information captured by π_x (it will be even clearer when discussing possibilistic logic). $\Pi(U)$ estimates the consistency of the statement “$x \in U$” (it is a tautology if U is an exhaustive set of possible values), with what we know about the possible values of x. Indeed, it expresses that not all the values u are somewhat impossible for x (to a degree $1 - \pi_x(u) > 0$) and that at least one value u_0 will be fully possible. In case of total ignorance, $\forall u \in U, \pi(u) = 1$. Then, all contingent events are fully possible: $\Pi(A) = 1 = \Pi(A^c), \forall A \neq \emptyset, U$. Note that this leads to a representation of ignorance ($E = U$ and $\forall A \neq \emptyset, \Pi_E(A) = 1$) which presupposes nothing about the number of elements in the reference set U (elementary events), while the latter aspect plays a crucial role in probabilistic modeling. The case when $\Pi(A) = 1, \Pi(A^c) > 0$ corresponds to partial ignorance about A. Besides, only $\Pi(A \cap B) \leq \min(\Pi(A), \Pi(B))$ holds. It agrees with the fact that in case of total ignorance about A, $\Pi(A) = \Pi(A^c) = 1$, while for $B = A^c$, $\Pi(A \cap B) = 0$ since $\Pi(\emptyset) = 0$.

The index $1 - \Pi(A^c)$ evaluates to the impossibility of 'not A', hence about the *certainty* (or necessity) of occurrence of A since when 'not A' is impossible then A is certain. It is thus natural to use this duality and define the degree of necessity of A [Dubois and Prade, 1980; Zadeh, 1979b] as

$$(14)\quad N(A) = 1 - \Pi(A^c) = \inf_{u \notin A} 1 - \pi_x(u).$$

Clearly, a necessity measure N satisfies $N(A \cap B) = \min(N(A), N(B))$. In case of a discrete linearly ordered scale, the mapping $s \mapsto 1 - s$ would be replaced by the order-reversing map of the scale. The above duality relation is clearly reminiscent of modal logics that handle pairs of modalities related by a relation of the form $\Box p \equiv \neg\Diamond\neg p$. But here possibility and necessity are graded. Note that the definitions of possibility and necessity measures are qualitative in nature, since they only require a bounded linearly ordered scale. Modal accounts of possibility theory involving conditional statements have been proposed in [Lewis, 1973b] (this is called the VN conditional logic), [Fariñas and Herzig, 1991; Boutilier, 1994; Fariñas *et al.*, 1994; Hájek *et al.*, 1994; Hájek, 1994]. Before Zadeh, a graded notion of possibility was introduced as a full-fledged approach to uncertainty and decision in the 1940-1970's by the English economist G. L. S. Shackle [1961], who called *degree of potential surprise* of an event its degree of impossibility, that is, the degree of necessity of the opposite event. It makes the point that possibility, in possibility theory, is understood as being potential, not actual. Shackle's notion of possibility is basically epistemic, it is a "character of the chooser's particular state of knowledge in his present." Impossibility is then understood as disbelief. Potential surprise is valued on a disbelief scale, namely a positive interval of the form $[0, y^*]$, where y^* denotes the absolute rejection of the event to which it is assigned. The Shackle scale is thus reversed with respect to the possibility scale. In case everything is possible, all mutually exclusive hypotheses have zero surprise (corresponding to the ignorant possibility distribution where $\pi(u) = 1, \forall u$). At least one elementary hypothesis must carry zero potential surprise (the normalization condition $\pi(u) = 1$, for some u). The degree of surprise of an event, a set of elementary hypotheses, is the degree of surprise of its least surprising realization (the basic "maxitivity" axiom of possibility theory). The disbelief notion introduced later by Spohn [1990] employs the same type of convention as potential surprise, but using the set of natural integers as a disbelief scale.

Inference in approximate reasoning

Inference in the framework of possibility theory as described by Zadeh [1979a] is a four-stepped procedure that can be respectively termed i) representation; ii) combination; iii) projection; iv) interpretation. Namely, given a set of n statements $S_1, \ldots, S_n$ expressing fuzzy restrictions that form a knowledge base, inference proceeds in the following way:

i) *Representation.* Translate $S_1, \ldots, S_n$ into possibility distributions $\pi^1, \ldots, \pi^n$ restricting the values of involved variables. In particular, facts of the form $S_t =$ "x is F" translate into $\pi^t_x = \mu_F$. Statements of rules of the form $S_t =$ "if x is F then y is G" translate into possibility distributions $\pi^t_{x,y} = \mu_R$ with $\mu_R = f(\mu_F, \mu_G)$ where f depends on the intended semantics of the rule, as explained below in section 2.4. Let $\overline{x} = (x_1, \ldots, x_k, \ldots, x_m)$ be a vector made of all the variables involved in statements $S_1, \ldots, S_n$. Assume S_t only involves variables $x_1, \ldots, x_k$, then its possibility distribution can be cylindrically extended to x as

$$\pi^t_{\overline{x}}(u_1, \ldots, u_k, u_{k+1}, \ldots, u_m) = \pi^t(u_1, \ldots, u_k), \forall u_{k+1}, \ldots, u_m$$

which means that the possibility that $x_1 = u_1, \ldots, x_k = u_k$ according to S_t does not depend on the values $u_{k+1}, \ldots, u_m$ taken by the other variables $x_{k+1}, \ldots, x_m$.

ii) *Combination.* Combine the possibility distributions $\pi^1_{\overline{x}}, \ldots, \pi^n_{\overline{x}}$ obtained at step (i) in a conjunctive way in order to build a joint possibility distribution $\pi_{\overline{x}}$ expressing the contents of the whole knowledge base, namely,

$$\pi_{\overline{x}} = \min(\pi^1_{\overline{x}}, \ldots, \pi^n_{\overline{x}}).$$

Indeed each granule of knowledge "$\overline{x}$ is E_i", for $i = 1, \ldots, n$, as already said, translates into the inequality constraint

(15) $\forall u, \pi_{\overline{x}}(u) \leq \mu_{E_i}(u).$

Thus given several pieces of knowledge of the form "x is E_i", for $i = 1, \ldots, n$, we have

(16) $\forall i, \pi_{\overline{x}} \leq \mu_{E_i},$ or equivalently $\pi_{\overline{x}} \leq \min_{i=1,\ldots,n} \mu_{E_i}.$

Taking into account all the available pieces of knowledge $S_1 =$ "$\overline{x}$ is E_1",..., $S_n =$ "$\overline{x}$ is E_n", the minimal specificity principle is applied. It is a principle of minimal commitment that stipulates that anything that is not explicitly declared impossible should remain possible (in other words, one has not to be more restrictive about the possible situations than what is enforced by the available pieces of knowledge). Thus, the available information should be represented by the possibility distribution:

(17) $\pi_{\overline{x}}(u) = \min_{i=1,\ldots,n} \mu_{E_i}$.

iii) *Projection.* Then $\pi_{\overline{x}}$ is projected on the domain(s) corresponding to the variable(s) of interest, i.e., the variable(s) for which one wants to know the restriction that can be deduced from the available information. Given a joint possibility distribution $\pi_{x,y}$ involving two variables defined on $U \times V$ (the extension to n variables is straightforward), its projection π_y on V is obtained [Zadeh, 1975b]:

(18) $\pi_y(v) = \sup_{u \in U} \pi_{x,y}(u, v)$.

Clearly, what is computed is the possibility measure for having $y = v$ given $\pi_{x,y}$. Generally, $\pi_{x,y} \leq \min(\pi_x, \pi_y)$ where $\pi_x(u) = \Pi(\{u\} \times V)$. When equality holds, $\pi_{x,y}$ is then said to be *min-separable*, and the variables x and y are said to be *non-interactive* [Zadeh, 1975b]. It is in accordance with the principle of minimal specificity, since $\pi_y(v)$ is calculated from the highest possibility value of pairs (x, y) where $y = v$. When modeling incomplete information, non-interactivity expresses a lack of knowledge about potential links between x and y. Namely, if we start with two pieces of knowledge represented by π_x and π_y, and if we do not know if x and y are interactive or not, i.e., $\pi_{x,y}$ is not known, we use the upper bound $\min(\pi_x, \pi_y)$ instead, which is less informative (but which agrees with the available knowledge). The combination and projection steps are also in agreement with Zadeh's *entailment principle*, which states that if "x is E" then "x is F", as soon as the fuzzy set inclusion $E \subseteq F$ holds, i.e.,$\forall u,\ \mu_E(u) \leq \mu_F(u)$, where x denotes a variable or a tuple of variables, and u any instantiation of them. Indeed, if F is entailed by the knowledge base, i.e., $\min_{i=1,\ldots,n} \mu_{E_i} \leq \mu_F$, F can be added to the knowledge base without changing anything, since $\pi_x = \min(\min_{i=1,\ldots,n} \mu_{E_i}, \mu_F) = \min_{i=1,\ldots,n} \mu_{E_i}$.

iv) *Interpretation.* This last step, which is not always used, aims at providing conclusions that are linguistically interpretable [Zadeh, 1978b]. Indeed, at step (i) one starts with linguistic-like statements of the form "x_i is E_i", and at step (iii) what is obtained is a possibility distribution π_y (or $\pi_{\overline{y}}$ in case of a subset of variables), and not something of the form "y is F". F as the best *linguistic approximation* of the result of step (iii) should obey three conditions:

(a) F belongs to some subsets of fuzzy sets (defined on the domain V of y) that represent linguistic labels or some combinations of them that are authorized (e.g. "not very young and not very old", built from the elementary linguistic labels "young" and "old");

(b) F should agree with the entailment principle, i.e. obey the constraint $\pi_y \leq \mu_F$;

(c) F should be *maximally* specific, i.e. as small as possible (in the sense of fuzzy set inclusion); in order to have a conclusion that is meaningful for the end-user (condition a), valid (condition b), and as precise as permitted (condition c), see, e.g. [Baldwin, 1979] for a solution to this optimization problem.

Observe that if the pieces of knowledge are not fuzzy but clear-cut, this four steps procedure reduces to classical deduction, since a classical logic knowledge base is generally viewed as equivalent to the logical conjunction of the logical formulas p_i that belong to the base. Moreover, in the case of propositional logic, asserting p_i, where p_i is a proposition, amounts to saying that any interpretation (situation) that falsifes p_i is impossible, because it would not be compatible with the state of knowledge. So, at the semantic level, p_i can be represented by the possibility distribution $\pi^i = \mu_{[p_i]}$, where $[p_i]$ is the set of models of p_i, and $\mu_{[p_i]}$ its characteristic function.

It also encompasses possibilistic logic (see section 4.1) as a particular case [Dubois and Prade, 1991a], where pieces of knowledge are semantically equivalent to prioritized crisp constraints of the form $N(E_i) \geq \alpha_i$ and N is a necessity measure. Such an inequality has a unique minimally specific solution, namely the possibility distribution $\pi_{x_i} = \max(\mu_{E_i}, 1 - \alpha_i)$. Propositional logic corresponds to the case where $\forall i$, $\alpha_i = 1$ (and $E_i = [p_i]$).

The combination and projection steps applied to a fact S_1 = "x is F'", and a rule S_2 = "if x is F then y is G", yields

$$\pi_y(v) = \sup_{u \in U} \min(\mu_{F'}(u), \mu_R(u, v)),$$

where μ_R represents the rule S_2. Then, the fact "y is G'" is inferred such that $\mu_{G'}(v) = \pi_y(v)$. This is called the generalized modus ponens, first proposed by Zadeh[1973]. However, $\mu_{G'} = \mu_G$ follows from $\mu_{F'} = \mu_F$ only for a particular choice of f in $\mu_R = f(\mu_F, \mu_G)$, as discussed below in Section 2.5.

2.3 Fuzzy truth-values - Degree of truth vs. degree of uncertainty

Zadeh [1978b; 1979a] also emphasizes that his theory of approximate reasoning can be interpreted in terms of what he calls "fuzzy truth-values" (see also [Bellman and Zadeh, 1977]). This terminology has led to many misunderstandings (e.g., [Haack, 1979]), that brings us back to the often made confusion (already mentioned in the introduction) between intermediate truth and uncertainty, hence between degree of truth and degree of belief. This is the topic of this section.

Fuzzy truth-values as compatibility profiles

It was emphasized earlier that Zadeh's approach to approximate reasoning is based on a representation of the contents of the pieces of information. This led Bellman and Zadeh [1977] to claim that the notion of truth is local rather than absolute:

a statement can be true only with respect to another statement held for sure. In other words, truth is viewed as the compatibility between a statement and "what is known about reality", understood as the description of some actual state of facts as stored in a database. Namely, computing the degree of truth of a statement S comes down to estimating its conformity with the description D of what is known about the actual state of facts. This point of view is in accordance with the test-score semantics for natural languages of Zadeh [1981]. It does not lead to scalar degrees of truth, but to fuzzy sets of truth-values in general.

Bellman and Zadeh [1977] define the fuzzy truth-value of a fuzzy statement $S =$ "x is A" given that another one, $D =$ "x is B", is taken for granted. When $B = \{u_0\}$, i.e. $D =$ "x is (equal to) u_0", the degree of truth of S is simply $\mu_A(u_0)$, the degree of membership of u_0 to the fuzzy set A. More generally, the information on the degree of truth of S given D will be described by a fuzzy set $\tau(S; D)$ (or simply τ for short) of the unit interval [0, 1], understood as the compatibility $COM(A; B)$ of the fuzzy set A with respect to the fuzzy set B, with membership function:

$$(19) \quad \tau(\alpha) = \mu_{COM(A;B)}(\alpha) \begin{cases} \sup\{B(u) \mid A(u) = \alpha\}, & \text{if } A^{-1}(\alpha) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

for all $\alpha \in [0, 1]$. As can be checked, $\tau(S; D)$ is a fuzzy subset of truth-values and $\tau(\alpha)$ is the degree of possibility, according to the available information D, that there exists an interpretation that makes S true at degree α. In fact, $\tau(S; D)$ is an epistemic state. As a consequence, truth evaluation comes down to a semantic pattern matching procedure. Six noticeable situations can be encountered [Dubois and Prade, 1988b], [Dubois *et al.*, 1991c]. In each situation, a particular case of $\tau(S; D)$ is obtained.

a) Boolean statement evaluated under complete information: S is a classical statement and D is a precise (i.e., complete) description of the actual state of facts. Namely A is not fuzzy and $B = \{u_0\}$. Either D is compatible with S and S is true (this is when $u_0 \in A$) and $\tau(S; D) = \{1\}$; or D is not compatible with S and S is false (this is when $u_0 \notin A$) and $\tau(S; D) = \{0\}$. This situation prevails for any Boolean statement S. When B is the set of models of a classical knowledge base K, then this situation is when K is logically complete.

b) Fuzzy statement evaluated under complete information: In that case D is still of the form $x = u_0$ but the conformity of S with respect to D becomes a matter of degree, because A is a fuzzy set. The actual state of facts $B = \{u_0\}$ can be borderline for A. For instance, the statement S to evaluate is "John is tall" and it is known that $D =$ "John's height is 1.75 m". Then $\tau(S; D) = \{A(u_0)\}$, a precise value in $[0, 1]$. Then what can be called *a degree of truth* can be attached to the statement S (in our example $\tau(S; D) = tall(1.75)$); by convention $\tau(S; D) = \{1\}$ implies that S is true, and $\tau(S; D) = \{0\}$ implies that S is false. But S can be *half-false* as well. In any case, the truth-value of S is precisely known. This situation is captured by truth-functional many-valued logics.

c) Fuzzy statement; incomplete non-fuzzy information: In this case, the information D does not contain fuzzy information but is just incomplete, and A is a fuzzy set. Then, it can be checked that $\tau(S;D)$ is a crisp set of truth values $\{A(u) : u \in B\}$. This set is lower bounded by $\inf_{u \in B} A(u)$ and upper bounded by $\sup_{u \in B} A(u)$ and represents the potential truth-values of S.

d) Boolean statement evaluated under incomplete non-fuzzy information: In that case, S and D are representable in classical logic, neither A nor B are fuzzy, and the conformity of S with respect to D is still an all-or-nothing matter but may be ill-known due to the fact that D does not precisely describe the actual state of facts, i.e., there may be two distinct states of facts u and u' that are both compatible with D such that u is compatible with S but u' is compatible with "not S". Hence the truth-value of S, which is either true or false (since A is not fuzzy), may be unknown. Namely, either D classically entails S, so S is certainly true (this is when $B \subseteq A$), and $\tau(S;D) = \{1\}$; or D is not compatible with S, so S is certainly false (this is when $B \cap A = \emptyset$) and $\tau(S;D) = \{0\}$. But there is a third case, namely when D neither classically entails S nor does it entail its negation (this is when $B \cap A \neq \emptyset$ and $B \cap A^c \neq \emptyset$). Then the (binary) truth-value of S is unknown. This corresponds to the fuzzy truth-value $\tau(S;D) = \{0,1\}$. This situation is fully described in classical logic. The logical view of possibility is to let $\Pi_B(A) = 1$ when $B \cap A \neq \emptyset$, $\Pi_B(A) = 0$ otherwise. It can be checked that, generally:

$$\tau(S;D)(0) = \mu_{COM(A;B)}(0) = \Pi_B(A^c)$$
$$\tau(S;D)(1) = \mu_{COM(A;B)}(1) = \Pi_B(A).$$

Equivalently, $N_B(A) = 1 - \Pi_B(A^c) = 1$ is interpreted as the assertion of the certainty of S. Hence the fuzzy truth-value provides a complete description of the partial belief of S. So, fuzzy truth-values describe uncertainty as much as truth (see also Yager[1983b]).

e) Boolean statement evaluated under fuzzy information: In that case, S is a classical logic statement (A is an ordinary set) but D contains fuzzy information. The conformity of S with respect to the actual state of facts is still an all-or-nothing matter but remains ill-known as in the previous case. The presence of fuzzy information in D leads to qualify the uncertainty about the truth-value of S in a more refined way. A grade of possibility $\Pi(A)$, intermediary between 0 and 1, can be attached to S. This grade is interpreted as the level of consistency between S and D. The dual level $N_B(A) = 1 - \Pi_B(A^c) = 1$ is interpreted as the degree the certainty of S and expresses the extent to which S is a consequence of D. These are standard possibility and necessity measures as recalled above.

Clearly these numbers are *not* degrees of truth, but only reflect a state of belief about the truth or the falsity of statement S. In such a situation, the fuzzy truth-value $\tau(S; D)$ reduces to a fuzzy set τ of $\{0, 1\}$, such that $\tau(0) = \Pi_B(A^c)$ and $\tau(1) = \Pi_B(A)$. Moreover, if the fuzzy sets A and B are normalized, we have $\max(\tau(0), \tau(1)) = 1$, i. e., τ is a normalized fuzzy set of $\{0, 1\}$.

f) Fuzzy statement evaluated under fuzzy incomplete information: When both S and D can be expressed as fuzzy sets, the fuzzy truth-value $\tau(S; D)$ is a genuine fuzzy subset of $[0, 1]$. It restricts the more or less possible values of the degree of truth. Indeed, in this case, truth may altogether be a matter of degree and may be ill-known. In other words, to each truth-value $\alpha = \tau(S; u)$ representing the degree of conformity of the fuzzy statement S with some precise state of facts u compatible with D, a degree of possibility $\tau(\alpha)$ that S has truth-value α is assigned. It reflects the uncertainty that u be the true state of facts. This is the most complex situation.

In the particular case where $S =$ "x is A" and $D =$ "x is A" (i.e., $B = A$), the compatibility $COM(A; A)$ reduces to

$$(20)\quad \tau(\alpha) = \mu_{COM(A;A)}(\alpha) \begin{cases} \alpha, & \text{if } A^{-1}(\alpha) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

When $A^{-1}(\alpha) \neq \emptyset$ for all α, $\tau(\alpha) = \alpha, \forall \alpha \in [0, 1]$. This particular fuzzy truth value corresponds to the idea of "certainly true" ("u-true" in Zadeh's original terminology) . In case $A^{-1}(\alpha) = \emptyset, \forall \alpha$ except 0 and 1, i.e., A is non-fuzzy, "certainly true" enforces standard Boolean truth (our case (a) above), since then $COM(A; A) = \{1\}$, whose membership function is $\mu_{COM(A;A)}(1) = 1$ and $\mu_{COM(A;A)}(0) = 0$ on the truth set $\{0, 1\}$.

The fuzzy truth-value $COM(A; B)$ thus precisely describes the relative position of fuzzy set A (involved in statement S) with respect to fuzzy set B (involved in statement D). It can be summarized, by means of two indices, the possibility and necessity of fuzzy events, respectively expressing degree of consistency of S with respect to D, and the degree of entailment of S from D, namely:

$$\begin{aligned} \Pi_B(A) &= \sup_{u \in U} \min(A(u), B(u)), \\ N_B(A) &= \inf_{u \in U} \max(A(u), 1 - B(u)). \end{aligned}$$

Indeed, $\Pi_B(A)$ and $N_B(A)$ can be directly computed from the fuzzy truth-value $COM(A; B)$. Namely, as pointed out in [Baldwin and Pilsworth, 1979; Prade, 1982; Yager, 1983b; Dubois and Prade, 1985a] :

$$\Pi_B(A) = \sup_{\alpha \in [0,1]} \min(\alpha, \mu_{COM(A;B)}(\alpha)) \tag{21}$$

$$N_B(A) = \inf_{\alpha \in [0,1]} \max(\alpha, 1 - \mu_{COM(A;B)}(\alpha)). \tag{22}$$

Truth qualification

This view of local truth leads Zadeh [1979a] to reconstruct a statement "x is B" from a fuzzy truth-qualified statement of the form "(x is A) is τ-true", where τ is a fuzzy subset of $[0, 1]$ (that may mean for instance "almost true", "not very true"...), according to the following equivalence:

$$(x \text{ is } A) \text{ is } \tau \Leftrightarrow \ x \text{ is } B$$

So, given that "(x is A) is τ-true", the fuzzy set B such that "(x is A) is τ-true given that x is B" is any solution of the following functional equation:

$$\forall \alpha \in [0, 1],\ \tau(\alpha) = \mu_{COM(A;B)}(\alpha)$$

where τ and A are known. The principle of minimal specificity leads us to consider the greatest solution B to this equation, defined as, after [Bellman and Zadeh, 1977; Sanchez, 1978]:

(23) $B(u) = \tau(A(u)), \forall u.$

This is also supported by an equivalent definition of COM(A; B)[Godo, 1990] which is

$$\mu_{COM(A;B)} = \inf\{f \mid f : [0, 1] \to [0, 1], f \circ A \geq B\}$$

where inf and $\geq$ refer respectively to the point-wise infimum and inequality, that is, $COM(A; B)$ represents the minimal functional modification required for the fuzzy subset A in order to include the fuzzy subset B, in agreement with the entailment principle. The similarity of $B(u) = \tau(A(u))$ with the modeling of linguistic modifiers [Zadeh, 1972], such as "very" ($very_A(u) = (A(u))^2$) has been pointed out. Indeed, linguistic hedges can can be viewed as a kind of truth-qualifiers. This is not surprising since in natural language, truth-qualified sentences like "It is almost true that John is tall" stand for "John is almost tall".

Using this representation, fuzzy sets of $[0, 1]$ can be interpreted in terms of fuzzy truth-values [Bellman and Zadeh, 1977; Baldwin, 1979; Yager, 1985b]. Especially

- "It is true that x is A" must be equivalent to "x is A" so that the fuzzy set of $[0, 1]$ with membership function $\tau(\alpha) = \alpha$ has been named *true* in the literature (while it really means "certainly true").
- "It is false that x is A" is often equivalent to the negative statement "x is not-A", that is, "x is A^c" with $A^c(\cdot) = 1 - A(\cdot)$, hence the fuzzy set of $[0, 1]$ with membership function $\tau(\alpha) = 1 - \alpha$ has been named *false* (while it really means "certainly false").
- "It is unknown if x is A" must be equivalent to "x is U" where U is the whole domain of x. Hence, the set $[0, 1]$ ifself corresponds to the case of a totally *unknown* truth-value. This is a clear indication that what Zadeh calls a fuzzy truth value is not a genuine truth-value: *unknown* is not a truth-value, it expresses a state of (lack of) knowledge.

It clearly appears now that what is called a fuzzy truth-value above is not a genuine truth-value. In the Boolean setting, what this discussion comes down to is to distinguish between an element of $\{0,1\}$, where 0 means false and 1 means true, from a singleton in $2^{\{0,1\}}$, where the set $\{0\}$ means certainly false and $\{1\}$ means certainly true. So, fuzzy truth-values *true* and *false* are misnomers here. The natural language expression "it is true that x is A" really means "it is *certainly true* that x is A", and "it is false that x is A" really means "it is *certainly false* that x is A". One thus may argue that the fuzzy set with membership function $\tau(\alpha) = \alpha$ could be better named *certainly true*, and is a modality, the fuzzy set with membership function $\tau(\alpha) = 1 - \alpha$ could be named *certainly false*; this is in better agreement with the representation of "unknown" by the set $[0,1]$ itself, not by a specific element of the truth set.

In a nutshell, Zadeh's fuzzy truth-values are epistemic states modeled by (fuzzy) subsets of the truth-sets.

The term *fuzzy truth-value* could wrongly suggest a particular view of Fuzzy Logic as a fuzzy truth-valued logic, i.e., a logic where truth-values are fuzzy sets (represeting linguistic labels). Viewed as such, fuzzy logic would be just another multiple-valued logic whose truth set is a family of fuzzy sets. This view is not sanctioned by the above analysis of fuzzy truth-values. Zadeh's fuzzy logic is a logic where truth-qualified statements can be expressed using (linguistic) values represented by fuzzy sets of the unit interval. That is, the truth set is just the unit interval, and fuzzy truth-values described here express uncertainty about precise truth-values. The situation where a fuzzy set of the unit interval could be viewed as a genuine truth-value would be in the case of a fuzzy statement S represented by a type 2 fuzzy set (a fuzzy set with fuzzy set-valued membership grades, [Mizumoto and Tanaka, 1976; Dubois and Prade, 1979b] and a reference statement D expressing complete information $x = u_0$. Then $A(u_0)$ is a fuzzy set of the unit interval which could be interpreted as a genuine (fuzzy) truth-value. Type 2 fuzzy logic, and especially the particular case of interval-valued fuzzy logic, have been developed at a practical level in the last ten years for trying to cope with engineering needs [Mendel, 2000].

As seen above, $COM(A;B)$ has its support in $\{0,1\}$ if A is not fuzzy. It makes no sense, as a consequence, to assert "it is τ-true that x is A" using a fuzzy (linguistic) truth-value τ, namely a fuzzy set τ whose support extends outside $\{0,1\}$. This is because one is not entitled, strictly speaking, to attach intermediary grades of truth to Boolean statements, e.g., formulas in classical logic. However it is possible to give a meaning to sentences such as "it is almost true that $x =$ 5". It clearly intends to mean that "x is *almost* equal to 5". This can be done by equipping the set of interpretations of the language with fuzzy proximity relations R such that saying "x is A" means in fact "x is $R \circ A$" (see [Prade, 1985], p. 269), where the composition $R \circ A$ (defined by $(R \circ A)(u) = \sup_{v \in A} R(u,v)$ is a *fuzzy* subset which is larger than A, while A may be Boolean. Then $R \circ A$ corresponds to an upper approximation of A which gathers the elements in A and those which are close to them. This indicates a dispositional use of Boolean statements that need to be fuzzified before their meaning can be laid bare. This view has been specially advocated by Ruspini [1991]. This latter dispositional use of Boolean statements contrasts with the one related to *usuality* described by Zadeh [1987], for whom "snow is white" is short for "usually, snow is white"", which is in the spirit of default rules having potential exceptions, as studied in nonmonotonic reasoning (see also section 4.1).

This fuzzification of Boolean concepts is related to Weston [1987]'s idea of approximate truth as reflecting a distance between a statement and the ideal truth, since fuzzy proximity relations are closely related to distances. Niskanen [1988] also advocates in favor of a distance view of approximate truth where the degree of truth of a statement S with respect to the available information D is computed as a relative distance between the (fuzzy) subsets representing S and D (by extending to fuzzy sets a relative distance which is supposed to exist on the referential). This distance-based approach corresponds to an "horizontal view" directly related to the distance existing between elements of the referential corresponding to D and S, and completely contrasts with the "vertical view" of the information system approach presented here where membership functions of the representations of S and D are compared, in terms of degrees of inclusion and non-empty intersection.

Truth qualification and R.C.T. Lee's fuzzy logic

An interesting particular case of truth qualification is the one of statements of the form "(x is A) is at least γ-true", where $\gamma \in [0,1]$. This means that "(x is A) is τ^γ-true", with $\tau^\gamma(\alpha) = 0$ if $\alpha < \gamma$ and $\tau^\gamma(\alpha) = 1$ if $\alpha \geq \gamma$. This is a truth-qualified fuzzy proposition "p is at least γ-true" with p = "x is A". Applying Zadeh's view, it precisely means that the truth-qualified statement is equivalent to "x is A_γ", where A_γ is the γ-level cut of the fuzzy set A, a classical subset defined by $A_\gamma = \{u \mid \mu_A(u) \geq \gamma\}$. This enables us to retrieve a noticeable particular case of multiple-valued logics of Lee [1972] and Yager [1985], see [Dubois *et al.*, 1991c] for a survey.

Assume we have the two statements "(x is A or B) is at least γ_1-true" and "(x is not A or C) is at least γ_2-true". First, note that in Zadeh's approach, the disjunction "(x is A) or (x is B)" is represented by the disjunction of constraints "($\pi_x \leq \mu_A$) or ($\pi_x \leq \mu_B$)", which entails $\pi_x \leq \max(\mu_A, \mu_B)$. This leads to take $\pi_x = \max(\mu_A, \mu_B)$ as a representation of the disjunction "(x is A) or (x is B)", in agreement with the spirit of the minimal specificity principle (since there is no μ such that "($\pi_x \leq \mu_A$) or ($\pi_x \leq \mu_B$)" entails $\pi_x \leq \mu$, with $\mu < \max(\mu_A, \mu_B)$). Then, taking $\mu_{AorB} = \max(\mu_A, \mu_B)$, which is the most commonly used definition of the union of fuzzy sets, "x is A or B" is equivalent to"(x is A) or (x is B)", while observe that "[(x is A) is at least γ-true] or [(x is B) is at least γ-true]" only entails "(x is A or B) is at least γ-true" (since $\mu_A(u) \geq \gamma$ or $\mu_B(u) \geq \gamma$ implies $\max(\mu_A(u), \mu_B(u)) \geq \gamma$). Moreover, "$x$ is not A" is assumed to be represented by the constraint $\pi_x \leq \mu_{notA} = 1 - \mu_A$. Thus, the two statements "(x is A or B) is at least γ_1-true" and "(x is not A or C) is at least γ_2-true" are respectively represented by the constraints

$$\gamma_1 \leq \max(\mu_A, \mu_B)$$
$$\gamma_2 \leq \max(1 - \mu_A, \mu_C),$$

and thus

$$\min(\gamma_1, \gamma_2) \leq \min(\max(\mu_A, \mu_B), \max(1 - \mu_A, \mu_C))$$

which implies $\min(\gamma_1, \gamma_2) \leq \max(\min(\mu_B, \mu_C), \min(\mu_A, 1 - \mu_A))$ and also $\min(\gamma_1, \gamma_2) \leq \max(\min(\mu_B, \mu_C), 0.5)$, since $\min(\mu_A, 1 - \mu_A)) \leq 0.5$. Thus, assuming $\min(\gamma_1, \gamma_2) > 0.5$, we get $\min(\gamma_1, \gamma_2) \leq \max(\mu_B, \mu_C)$.

Hence the following inference pattern (where p, q, and r are fuzzy propositions) is again in agreement with Zadeh's theory of approximate reasoning:

$$\frac{v(p \vee q) \geq \gamma_1, \quad v(\neg p \vee r) \geq \gamma_2}{v(q \vee r) \geq \min(\gamma_1, \gamma_2)}, \quad \text{if } 0.5 < \min(\gamma_1, \gamma_2)$$

with $v(p \vee q) = \max(v(p), v(q))$ and $v(\neg p) = 1 - v(p)$, as in [Lee, 1972].

Truth qualification and possibilistic logic

This corresponds to situation (e) above of a Boolean statement in the face of fuzzy information. But now, the fuzzy information "x is B" should be retrieved from the equations $\tau(0) = \Pi_B(A^c) = 1 - N_B(A)$ and $\tau(1) = \Pi_B(A)$ with $\max(\tau(0), \tau(1)) = 1$, where A is an ordinary subset and thus $p =$ "x is A" is a *classical* proposition. Assume $\tau(1) = 1$. It means that $p =$ "(x is A) is certain to degree $1 - \tau(0)$" (or if we prefer that "it is certain to degree $1 - \tau(0)$ that p is true"), since $N_B(A) = 1 - \tau(0)$ (with B unknown), which is then equivalent to the fuzzy statement "x is B" represented by

$$\forall u, \pi_x(u) = B(u) = \max(A(u), \tau(0)),$$

by application of the minimal specificity principle. If $\tau(0) = 1$, it means that $p =$ "(x is not A) is certain at degree $1 - \tau(1)$", then one obtains $\forall u, \pi_x(u) = B(u) = \max(A^c(u), \tau(1))$. As can be seen, if $\tau(1) = 1 = \tau(0)$, then we are in the situation of complete ignorance, i.e. $\forall u, B(u) = 1$ (neither A nor 'not A' are somewhat certain). The latter particular case of certainty qualification of Boolean statements corresponds to the semantical side of possibilistic logic, as explained in section 4.1. The distinction between thresholding degrees of truth and thresholding degrees of certainty is first emphasized in [Dubois *et al.*, 1997], further elaborated in [Lehmke, 2001b], where a more general logical framework is proposed that attaches fuzzy truth-values τ to fuzzy propositions.

Certainty qualification of fuzzy propositions

Informally, asserting "It is *true* that x is A" is viewed as equivalent to "x is A". Then what is considered as true, stricto sensu, is that $\pi_x = A(\cdot)$ is certain.

Interpreting *true* in a very strong way as the certainty that the truth value is maximal, i.e., $\tau'(\alpha) = 0$ if $\alpha < 1$ and $\tau'(1) = 1$, would come down to postulating that "It is *true* that x is A" is equivalent to "x is in $core(A)$", where $core(A) = \{u \mid A(u) = 1\}$, or in other words, "$A(x) = 1$". So, as already said, the fuzzy set of $[0, 1]$ with membership function $\tau(\alpha) = \alpha$ modeling 'true' here means more than pointing to a single truth value, it is the invariant operator in the set of (linguistic) modifiers of membership functions, such that $true(A(u)) = A(u)$. Similarly, "it is false that x is A" is understood as "x is A^c" (equivalent to "'it is true that x is A^c"). It follows the linguistic exchange rule between linguistic modifiers and fuzzy truth-values, with $false(\alpha) = 1 - \alpha$, and $false(A(u)) = 1 - A(u) = A^c(u)$. It is not the same as asserting that "(x is in $core(A)$) is false", nor that "$A(x) = 0$", although all these views coincide in the non-fuzzy case. It also differs from the (meta) negation, bearing on the equality, of the assertion $\pi_x = A(\cdot)$.

More generally, it is natural to represent the certainty-qualified statement "it is certain at degree α that x is A", when A is fuzzy, by $\pi_x = \max(1-\alpha, A(\cdot))$[Dubois and Prade, 1990]. Indeed, first consider the simpler case of "x is A is certain", where A is fuzzy. Clearly the formula gives back $\pi_x = A(\cdot)$ for $\alpha = 1$. Let us observe that "x is A" is equivalent to say that "(x is A_λ) is $1 - \lambda$ certain", for $\lambda \in [0, 1)$, where A_λ is the strict λ-cut of A, i.e., $A_\lambda = \{u \in U \mid A(u) > \lambda\}$, since $N(A_\lambda) \geq 1 - \lambda$ where N is the necessity measure defined from $\pi_x = A(\cdot)$. In the general case of statements of the form "(x is A) is (at least) α-certain", it is natural to forbid the certainty of any level cut to overpass α. It amounts to stating that $\forall\lambda$, "(x is A_λ) is (at least) $\min(\alpha, 1 - \lambda)$-certain". This is satisfied by keeping $\pi_x = \max(1 - \alpha, A(\cdot))$.

Observe, however, that π_x cannot be retrieved as the least specific solution of equation $N(A) \geq \alpha$ using the definition of the necessity of a fuzzy event given by $N(A) = \inf_{u\in U} \max(A(u), 1 - \pi(u)) = 1 - \sup u \in U \min(1 - A(u), \pi(u)) = 1 - \Pi(A^c)$, since $N(A)$ is then not equal to 1 for $\pi_x = A(\cdot)$. Nevertheless, $\pi_x = \max(1 - \alpha, A(\cdot))$ is still the least specific solution of an equation of the form $C(A) \geq \alpha$, where $C(A)$ is defined by

$$C(A) = \inf_u \pi_x(u) \to A(u)$$

where $\alpha \to \beta$ is the reciprocal of Gödel's implication, namely $\alpha \to \beta = 1$ if $\alpha \leq \beta$ and $\alpha \to \beta = 1-\alpha$ otherwise. The equivalence $C(A) \geq \alpha \Leftrightarrow \pi_x \leq \max(1-\alpha, A(\cdot))$ is easy to prove using the equivalence $\gamma \leq \max(1 - \alpha, \beta) \Leftrightarrow \gamma \to \beta \geq \alpha$. $C(A)$ is a particular case of a degree of inclusion of B (with $\pi_x = \mu_B(\cdot)$) into A. Then $C(A) = 1$ yields $\pi_x \leq A(\cdot)$, while $N(A) = 1$ would yield $\pi_x \leq \mu_{core(A)}(\cdot)$ (since $N(A) = 1$ if and only if $\{u \in U \mid \pi_x(u) > 0\} \subseteq core(A)$). As expected, "it is true that x is A", represented by $\pi_x = A(\cdot)$, indeed means "it is certain that (x is A) is true", since then $C(A) = 1$, and "it is false that x is A", represented by $\pi_x = 1 - A(\cdot)$, indeed means "it is certain that (x is A) is false" since then $C(A^c) = 1$. While if 'true' refers to the usual truth-value (represented here by $\tau^1(\alpha) = 0$ if $\alpha < 1$ and $\tau^1(1) = 1$), "it is true that x is A" is represented by $\pi_x = mu_{core(A)}(\cdot)$, and $N(A) = 1$. Moreover, note that both N

and C still enjoy the characteristic properties $N(A \cap B) = \min(N(A), N(B))$ and $C(A \cap B) = \min(C(A), C(B))$, when the intersection of two fuzzy sets is defined by combining pointwisely their membership functions by the operation min.

Graded truth versus degrees of uncertainty: the compositionality problem

The frequent confusion pervading the relationship between truth and (un)certainty in the approximate reasoning literature is apparently due to the lack of a dedicated paradigm for interpreting partial truth and degrees of uncertainty in a single framework, although the distinction between the two concepts has been made a long time ago e.g. [Carnap, 1949; de Finetti, 1936]. Such a paradigm has been provided above. An important consequence of our information-based interpretation of truth is that degrees of uncertainty cannot be compositional for all connectives [Dubois and Prade, 1994; Dubois and Prade, 2001]. Let g stand for a $[0, 1]$-valued function that intends to estimate degrees of confidence in propositions. Let A be the set of situations where proposition S is true. It corresponds to assuming a fuzzy truth value in Zadeh's sense, defined on $\{0, 1\}$, letting $\tau(1) = g(A)$ and $\tau(0) = g(A^c)$. Then, A^c, $A_1 \cap A_2$, $A_1 \cup A_2$, respectively denote the set of situations where the propositions "not-S", "S_1 and S_2", S_1 or S_2" hold, $g(A)$ is the degree of confidence in proposition S. It can be proved that there cannot exist operations $\otimes$ and $\oplus$ on $[0, 1]$, nor negation functions f such that the following identities simultaneously hold for all propositions whose meaning is described by crisp sets A_1, A_2, A:

(i) $g(A^c) = f(g(A))$;

(ii) $g(A_1 \cap A_2) = g(A_1) \otimes g(A_2)$;

(iii) $g(A_1 \cup A_2) = g(A_1) \oplus g(A_2)$.

More precisely, (i)-(ii)-(iii) entail that for any $A, g(A) \in \{0, 1\}$, and either $g(A) = 0$ or $g(A) = 1$, i.e., this the case of complete information, where all statements are either certainly true or certainly false and g is isomorphic to a classical truth-assignment function. This result is proved independently in [Weston, 1987], [Dubois and Prade, 1988b].

However weak forms of compositionality are allowed; for instance $\Pi(A_1 \cup A_2) = \max(\Pi(A_1), \Pi(A_2))$ in possibility theory, but generally, $\Pi(A_1 \cap A_2) < \min(\Pi(A_1), \Pi(A_2))$; the equality $\Pi(A_1 \cap A_2) = \min(\Pi(A_1), \Pi(A_2))$ holds for propositions "x_1 is A_1" and "x_2 is A_2" that refer to non-interactive variables x_1 and x_2 (see the previous section 2.2). Similarly, with grades of probability $P(A) = 1 - P(A^c)$ but $P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$ holds only in situations of stochastic independence between A_1 and A_2. The above impossibility result is another way of stating a well-known fact, i.e., that the unit interval cannot be equipped with a Boolean algebra structure.

This result is based on the assumption that the propositions to evaluate are not fuzzy and thus belong to a Boolean algebra. By contrast, confidence values of fuzzy (or non-fuzzy) propositions may be assumed to be compositional when these propositions are evaluated under complete information, since then, sets of possible truth-values reduce to singletons. The possibility of having $g(A) \notin \{0,1\}$ is because sets of fuzzy propositions are no longer Boolean algebras. For instance, using max, min, $1-(\cdot)$ for expressing disjunction, conjunction and negation of fuzzy propositions, sets of such propositions are equipped with a distributive lattice structure weaker than a Boolean algebra, which is compatible with the unit interval. Sometimes, arguments against fuzzy set theory rely on compositionality issues, (e.g.,[Weston, 1987], [Elkan, 1994]). These arguments are based either on the wrong assumptions that the algebra of propositions to be evaluated is Boolean, or that intermediate degrees of truth can model uncertainty.

As a consequence, fuzzy truth-values à la Zadeh are not truth-functional, generally, since they account for uncertainty. Namely $COM(A_1 \cap A_2; B)$ is not a function of $COM(A_1; B)$ and $COM(A_2; B)$; $COM(A_1 \cup A_2; B)$ is not a function of $COM(A_1; B)$ and $COM(A_2; B)$. This lack of compositionality is one more proof that fuzzy truth-values are not intermediate truth-values in the sense of a compositional many-valued logic. Neither is Zadeh's fuzzy logic a type 2 fuzzy logic in the sense of [Dubois and Prade, 1979b], who use $2^{[0,1]}$ as a truth set, and define compositional connectives by extending those of multiple valued logic to fuzzy set-valued arguments.

The presence or absence of compositional rules is a criterion to distinguish between the problem of defining truth tables in logics with gradual propositions, and the problem of reasoning under uncertainty (logics that infer from more or less certainly true classical propositions under incomplete information). However it does not mean that all logics of graded truth are compositional (for instance, similarity logics using crisp propositions fuzzified by a fuzzy proximity relation (as done in [Ruspini, 1991]), are not compositional [Dubois and Prade, 1998b]. The information system paradigm underlying Zadeh's view of fuzzy truth values nevertheless questions the comparison made in [Gaines, 1978] between probabilistic logics which are not compositional, and a particular (max-min) many-valued logic which is truth-functional. The setting in which this comparison takes place (i.e., abstract distributive lattices equipped with a valuation) does not allow for a proper conceptual discrimination between graded truth and uncertainty. The meaning of valuations attached to propositions is left open, so that grades of probability and degrees of truth in fuzzy logic are misleadingly treated as special cases of such abstract valuations. As a consequence Gaines' comparison remains at an abstract level and has limited practical significance. Moreover the chosen abstract setting is not general enough to encompass all many-valued logics. For instance Gaines "standard uncertainty logic" (SUL) assumes that conjunction and disjunction are idempotent; this assumption rules out most of the compositional many-valued calculi surveyed in Section 3 of this chapter, where operations other than min and max are used to represent conjunctions and disjunctions of fuzzy predicates.

Moreover, when the SUL is compositional, it suffers from the above trivialization.

Alternative views of fuzzy truth

The above approach to truth and uncertainty has been tailored for a special purpose, i.e., that of dealing with knowledge-based reasoning systems. It suggests fuzzy matching techniques between the meaning of a proposition and a state of knowledge as natural procedures for effectively computing degrees of uncertainty, modeled as fuzzy truth-values in the presence of fuzziness. Clearly other empirical settings for defining truth-values exist. Gaines [1978] suggests a systematic way of generating valuations in a SUL by resolving paradoxes (such as the Barber Paradox). This approach, also advocated by Smets and Magrez [1988], does not make a clear distinction between graded truth and uncertainty; moreover its relevance and practical usefulness for dealing with knowledge-based systems is questionable.

Another view of truth is the one proposed in [Giles, 1988a; Giles, 1988b]. Namely the truth of a vague statement S in a supposedly known state of fact $D = \{u\}$ reflects the "gain in prestige" an individual would get by asserting S in front of a society of people. This gain is expressed as a pay-off function. When the state of facts is ill-known, Giles assumes that it can be represented by a subjective probability distribution and the degree of truth of S is viewed as the expected pay-off for asserting S. Giles' metaphor provides a nice device to elicit degrees of membership in terms of utility values. His view is in accordance with our data base metaphor where only probability distributions would be admitted to represent uncertainty. However the distinction between truth-values and degrees of belief (viewed by Giles as "the subjective form of degrees of truth") is again hard to make. Especially the expected pay-off of S is the probability $P(S)$ of the fuzzy event S, i.e., a grade of uncertainty; but it is also an expected truth-value. The use of expectations mixes truth-values and degrees of belief. Note that the two equations (21) and (22) consider possibility and necessity as a kind of qualitative expected values of the compatibility. So, expectation-based evaluations, summarizing distributions over truth values, are not compositional.

2.4 Fuzzy if-then rules

Fuzzy if-then rules are conditional statements of the form "if x is A then y is B", or more generally "if x_1 is A_1 and ... and x_n is A_n then y is B", where A, A_i, B are fuzzy sets. They appear originally in [Zadeh, 1973], that provides an outline of his future theory of approximate reasoning. From this initial proposal, a huge amount of literature was produced aiming at proposing different encoding of fuzzy rules or some mechanisms for processing them, often motivated by some engineering concerns such as fuzzy rules-based control, e.g., [Mamdani, 1977; Sugeno and Takagi, 1983; Sugeno, 1985]. It is out of the scope of the present chapter to review all the approximate reasoning literature in detail (see [Bouchon-Meunier *et al.*, 1999] for a detailed overview). In the following, we first provide the representation of different kinds of fuzzy rules that make sense in the possibility theory-based setting presented above, and then discuss how drawing inferences in this setting. Understanding the semantics of the different models of fuzzy rules is a key issue for figuring out their range of applicability and their proper processing. For the sake of clarity we start the presentation with non-fuzzy rules and we then extend the discussion to the general case of fuzzy rules.

Two understandings of if-then rules

Consider the rule "if $x \in A$ then $y \in B$" where x and y are variables ranging on domains U and V, and A and B are ordinary (i.e., non fuzzy) subsets of U and V respectively. The partial description of a relationship R between x and y that the rule provides can be equivalently formulated in terms of (Boolean) membership functions as the condition:

$$\text{if } A(u) = 1 \text{ then } B(v) = 1.$$

If we think of this relationship as a binary relation R on $U \times V$, then clearly pairs (u, v) of values of the variables (x, y) such that $A(u) = B(v) = 1$ must belong to the relation R, while pairs such that $A(u) = 1$ and $B(v) = 0$ cannot belong to R. However, this condition says nothing about pairs (u, v) for which $A(u) = 0$. That is, these pairs may or may not belong to the relation R. Therefore, the only constraints enforced by the rule on relation R are the following ones:

$$\min(A(u), B(v)) \leq R(u, v) \leq \max(1 - A(u), B(v)).$$

In other words, R contains at least all the pairs (u, v) such that $A(u) = B(v) = 1$ and at most those pairs (u, v) such that either $A(u) = 0$ or $B(v) = 1$. Thus, the above inequalities express that any representation of the rule "if $x \in A$ then $y \in B$" is lower bounded by the representation of the conjunction "$x \in A$ and $y \in B$" and upper bounded by the representation of the material implication "$x \in A$ implies $y \in B$", i.e.," $x \notin A$ or $y \in B$". In set notation, it reads $A \times B \subseteq R \subseteq (A^c \times V) \cup (U \times B)$.

Thus, in terms of the constraints induced on the joint possibility distribution $\pi_{x,y}$ restricting the possible values of the two-dimensional variable (x, y), the above inequalities lead to the two following types of constraints:

- the inequality
$$\pi_{x,y}(u, v) \leq \max(1 - A(u), B(v))$$
expresses that values outside B are impossible values for y when x takes value in A (i.e., $\pi_{x,y}(u, v) = 0$ if $A(u) = 1$ and $B(v) = 0$), while the possible values for y are unrestricted ($\pi_{x,y}(u, v) \leq 1$) when x does not take value in A. Thus, the meaning of this inequality can be read: if $x \in A$, it is certain that the value of y is in B.

- the inequality
$$\pi_{x,y}(u, v) \geq \min(A(u), B(v))$$

means that all values $v \in B$ are possible when x takes value in A (that is, $\pi_{x,y}(u,v) = 1$ if $A(u) = B(v) = 1$), while no constraint is provided for the values of y when x does not take value in A. Thus, the semantics of the latter inequality reads: if $x \in A$, all the values in B are possible (admissible, feasible) for y.

We immediately recognize in the right-hand side of the two above inequalities a (binary) implication and a (binary) conjunction respectively. They respectively define the conjunction-based and the implication-based models of rules. But even if they are of different nature, both models stem from considering a rule as a (partial) specification of a binary relation R on the product space $U \times V$. Note that $R \subseteq (A^c \times V) \cup (U \times B)$ is equivalent to $A \circ R \subseteq B$ in the Boolean case, $A \circ R$ being the usual image of A via R ($A \circ R = \{v \in V \mid \exists u \in U, A(u) = 1, R(u,v) = 1\}$).

Implication-based models of rules correspond to a type of constraints that we have already encountered when introducing the possibility theory setting. Conjunction-based models of rules cannot be processed using the minimal specificity principle. As we shall see they correspond to another type of information than the one usually considered in classical logical reasoning and involve a notion of possibility different from the one estimated by Π.

The existence and the proper use of implication-based and conjunction-based representations of fuzzy rules has been often misunderstood in various fields of applications. As pointed out in a series of papers by Dubois and Prade [1989; 1991a; 1992a; 1992b; 1996a], there are several types of fuzzy rules with different semantics, corresponding to several types of implications or conjunctions. As seen above, the meaning of a rule of the form "if x is A then y is B" is significantly different when modeled using a genuine implication $A \to B$ or using a Cartesian product $A \times B$.

Implication-based fuzzy rules

Let us consider the rule "if x is A then y is B" where A and B are now fuzzy subsets of U and V respectively. In this case, the intuitive idea underlying such a rule is to say that if the value of x is no longer in the core of A, but still close to it, the possible values of y lie in some fuzzy subset not too much different from B. The ways B can be modified in order to accommodate the possible values of y depend on the intended meaning of the fuzzy rule, as expressed by the connective relating A and B. In this subsection a fuzzy rule is viewed as a constraint $\pi_{x,y}(u,v) \leq I(A(u), B(v))$ for some many-valued implication$I$. However, contrary to the Boolean case, $R \subseteq (A^c \times V) \cup (U \times B)$ is no longer equivalent to $A \circ R \subseteq B$, due to the difference between two types of multiple-valued implications: S-implications and R-implications. It gives birth to two types of fuzzy rules.

Certainty rules. A first way of relaxing the conclusion B is to attach some level of certainty to it, independently of whether B is fuzzy or not, in such a way that the possibility degrees of the values outside the support of B become strictly positive. This corresponds to rules of the type "the more x is A, the more certain y is B" and they are known in the literature as *certainty rules*. A simple translation of this type of constraint is the inequality

$$\forall u, A(u) \leq C(B)$$

where $C(B)$ stands for the certainty of B under the unknown possibility distribution $\pi_{x,y}$ (as for certainty-qualified fuzzy statements), i. e.

$$C(B) = \inf_v I(\pi_{x,y}(u,v), B(v)),$$

where the implication I is the reciprocal of an R-implication I_R (the previous definition of the certainty of a fuzzy statement introduced in Section 2.3 is here enlarged to any reciprocal of an R-implication). Then, in agreement with the minimal specificity principle, the greatest solution of this certainty-qualification problem provides the solution to the problem of representing certainty rules, namely

$$\pi_{x,y}(u,v) \leq I_S(A(u), B(v)) = S(n(A(u)), B(v))$$

where the right hand side of the inequality corresponds to the strong implication defined from the negation function n and the t-conorm S which is n-dual of the t-norm T generator of I_R. In particular, if $n(\alpha) = 1 - \alpha$, $T(\alpha, \beta) = \min(\alpha, \beta)$, $S(\alpha, \beta) = \max(\alpha, \beta)$, we obtain

$$\pi_{x,y}(u,v) \leq \max(1 - A(u), B(v))$$

where Kleene-Dienes implication $\alpha \to \beta = \max(1 - \alpha, \beta)$ can be recognized.

Gradual Rules. The second way of relaxing the conclusion amounts to enlarging the core of B, in such a way that if x takes value in the α-cut of A, then the values in the α-cut of B become fully possible for y. This interpretation, which requires B to be fuzzy, corresponds to the so-called *gradual rules*, i.e., rules of the type "the more x is A, the more y is B", as in the piece of knowledge "the bigger a truck, the slower its speed". (Statements involving "the less" are easily obtained by duality, using the fuzzy set complementation). The name 'gradual rule' was coined by Prade [1988]; see also [Dubois and Prade, 1992b]. The intended meaning of a gradual rule, understood as "the greater the membership degree of the value of x to the fuzzy set A, the greater the membership degree of the value of y to the fuzzy set B should be" is captured by the following inequality:

$$\min(A(u), \pi_{x,y}(u,v)) \leq B(v)$$

or equivalently,

$$\pi_{x,y}(u,v) \to A(u) \to B(v),$$

where $\to$ denotes Gödel's implication. The above inequality can be relaxed by introducing a triangular norm T, i.e.,

$$T(A(u), \pi_{x,y}(u,v)) \leq B(v).$$

Then $\rightarrow$ will be replaced by the corresponding R-implication generated by T. Clearly, in this type of rules the degree of truth of the antecedent constrains the degree of truth of the consequent, since $A(u) \rightarrow B(v) = 1$ if and only if $A(u) \leq B(v)$ for R-implications.

Impossibility rules. A third category of implication-based rule is obtained by writing a constraint expressing that "the more x is A, the less possible the complement of B is a range for y". Such rules are interpreted as saying, "if $x = u$ then the complement of B is at most $(1 - A(u))$-possible". This corresponds to the following inequality as interpretation of the fuzzy rule (where the usual definition of the possibility of a fuzzy event is extended using a triangular norm T instead of the minimum operation only):

$$\Pi(B^c) = \sup_v T(1 - B(v), \pi_{x,y}(u,v)) \leq 1 - A(u),$$

this reads "the more x is A, the more impossible not-B". It leads to the following equivalent inequality

$$\pi_{x,y}(u,v) \leq (1 - B(v)) \leq (1 - A(u))$$

where $\rightarrow$ is the R-implication associated with T. If $T = \min$, then we get the following constraint $\pi_{x,y}(u,v) \leq 1 - A(u)$ if $A(u) > B(v)$. If $T = \textit{product}$, the upper bound of $\pi_{x,y}(u,v)$ is the reciprocal of Goguen implication from $A(u)$ to $B(v)$. In practice these rules are close to certainty rules since they coincide when B is a non-fuzzy set (as expected from the semantics). However, when B is fuzzy, impossibility rules combine the main effects of certainty and gradual rules: apparition of a level of uncertainty and widening of the core of B: the more x is A, the more certain y is in a smaller subset of values around the core of B. Thus, they could also be named *certainty-gradual rules* so as to account for this double effect.

Note that in the implication-based models, $\pi_{x,y}$ is always upper bounded; then applying the minimal specificity principle leads to a possibility distribution which is normalized (if B is normalized).

The three types of implication-based fuzzy rules correspond to the three basic types of implication functions recalled above. In the fuzzy logic literature, other models of implication functions have been considered. For instance, let us mention QL-implications [Trillas and Valverde, 1981]. They are based on interpreting $p \rightarrow q$ as $\neg p \vee (p \wedge q)$, which is used in quantum logic (in classical logic it obviously reduces to material implication). This view leads to implication functions of the form $I(\alpha, \beta) = S(n(\alpha), T(\alpha, \beta))$ where S is a t-conorm, n a strong negation and T is the n-dual t-norm of S. The so-called Zadeh's implication [Zadeh, 1973] corresponds to taking $S = \max$, i.e.,$I(\alpha, \beta) = \max(1 - \alpha, \min(\alpha, \beta))$, and is the basis for another type of fuzzy rules.

Conjunction-based fuzzy rules

Conjunction-based fuzzy rules first appear as an ad hoc proposal in the first fuzzy rule-based controllers [Mamdani, 1977]. Later, they were reinterpreted in the setting of possibility theory, using a new type of possibility evaluation. Namely, interpreting "x is A is (at least) β-possible" as "all elements in A are possible values for x, at least with degree β", i.e., $\Delta(A) = \inf_{u \in A} \pi_x(u) \geq \beta$, leads to state the following constraint on π_x:

$$\forall u, \pi_x(u) \geq \min(A(u), \beta).$$

This approach is actually in the spirit of a proposal also briefly discussed in [Zadeh, 1978b] and more extensively in [Sanchez, 1978]. See [Dubois and Prade, 1992b] for the introduction of the measure of *guaranteed possibility* Δ, and [Dubois *et al.*, 2000], [Dubois *et al.*, 2003] for the development of a bipolar view of possibility theory allowing for the representation of positive and negative pieces of information. Constraints enforcing lower bounds on a possibility distribution, as above, are positive pieces of information, since it guarantees a minimum level of possibility for some values or interpretations. This contrasts with constraints enforcing upper bounds on a possibility distribution, which are negative pieces of information, since they state that some values are to some extent impossible (those values whose degree of possibility is strictly less than 1 and may be close to 0). Note that classical logic handles negative information in the above sense. Indeed, knowing a collection of propositional statements of the form "x is A_i" (where the A_i's are classical subsets of a universe U) is equivalent to saying that values for x outside $\cap_i A_i$ are impossible.

Note that positive information should obey a maximal specificity principle that states that only what is reported as being actually possible should be considered as such (and to a degree that is not higher than what is stated). This means that we only know that $\forall u, \pi_x(u) \geq \min(A(u), \beta)$, as far as positive information is concerned, then the positive part of the information will be represented by the smallest possibility distribution obeying the constraint, here, $\forall u, \pi_x(u) = \min(A(u), \beta)$. In case of several pieces of positive information stating that "x is A_i" is guaranteed to be possible, then we can conclude from $\pi_x(u) \geq A_i(u)$, that $\pi_x(u) \geq \max_i A_i(u)$, ("$x$ is $\cup_i A_i$" in case of classical subsets), which corresponds to a disjunctive combination of information. Note that both the minimal specificity principle for negative information and the maximal specificity principle for positive information are the two sides of the same coin. They are in fact minimal commitment principles. Together they state that potential values for x cannot be considered as more impossible (in the Π-sense), nor as more possible (in the Δ-sense) than what follows from the constraints representing the available negative or positive information.

In the case where A is a fuzzy set, the representation of statements of the form "x is A is (at least) β-possible" by $\pi_x(u) \geq \min(A(u), \beta)$, is still equivalent to $\Delta(A) \geq \beta$, provided that $\Delta(A)$ is extended to fuzzy events by

$$\Delta(A) = \inf_u A(u) \to \pi_x(u)$$

where $\rightarrow$ is Gödel 's implication. This can be easily shown using the equivalence $\alpha \rightarrow \beta \geq \gamma \Leftrightarrow \beta \geq \min(\alpha, \gamma)$. This is the basis for defining *possibility rules*.

Possibility rules. They correspond to rules of the form "the more x is A, the more possible y is B", understood as if $x = u$, any value compatible with "y is B" is all the more guaranteed as being possible for y as $A(u)$ is higher, in agreement with the sense of the set function Δ. Thus, the representation of such possibility-qualified statements obey the constraint:

$$A(u) \leq \Delta(B).$$

Hence the constraint on the conditional possibility distribution $\pi_{x,y}(u, \cdot)$ for y is

$$\min(A(u), B(v)) \leq \pi_{x,y}(u, v)$$

or, more generally $T(A(u), B(v)) \leq \pi_{x,y}(u, v)$ if we allow the use of any t-norm T in place of min. As already mentioned, this type of rules (using $T = \min$ or *product*) pervades the literature on fuzzy control, since it is in accordance with viewing fuzzy rules as partial descriptions of a fuzzy graph R relating x and y, in the sense that "if x is A then y is B" says nothing but the fuzzy set $A \times B$ belongs to the graph of R, i.e., $A \times B \subseteq R$. This interpretation helps us understand why the fuzzy output of fuzzy rules-based controllers is generally subnormalized: the obtained output is nothing but a lower bound on the actual possibility distribution. When A and B become fuzzy, the equivalence between $A \times B \subseteq R$ and $A \circ R^c \subseteq B^c$ no longer holds. This leads to a new conjunction-based kind of fuzzy rules, called *antigradual rules*, where the guaranteed possible range of values for y is reduced when x moves away from the core of A.

Antigradual rules. They correspond to a rule of the type "the more x is A and the less y is related to x, the less y is B", and to the corresponding constraint

$$T(A(u), 1 - \pi_{x,y}(u, v)) \leq 1 - B(v)$$

where T is a triangular norm. This can be equivalently written

$$T^*(A(u), B(v)) =_{def} 1 - (A(u) \rightarrow_R (1 - B(v))) \leq \pi_{x,y}(u, v)$$

where $\rightarrow_R$ is the residuated implication based on T. T^* is a non-commutative conjunction that is the right adjoint of a strong implication, i. e., the strong implication $a \rightarrow_S b = n(T(a, n(b))$ can be obtained from T^* by residuation, starting with a continuous t-norm T [Dubois and Prade, 1984a]. It can be checked that $\pi_{x,y}(u, v) \geq B(v)$ if and only if $A(u) > 1 - B(v)$ for $T = \min$. Thus, the values v such that $B(v) > 1 - A(u)$ are guaranteed to be possible for y, and the larger $A(u)$, the larger the subset of values v for y guaranteed as possible (at degree $B(v)$). In other words, the subset of values for y with some positive guaranteed possibility becomes smaller as x moves away from the core of A.

Note that the same non-commutative conjunction where A and B are permuted corresponds to a third kind of rules expressed by the constraint "the more y is B and the less y is related to x, the less x is A", i.e.

$$T(B(v), 1 - \pi_{x,y}(u, v)) \leq 1 - A(u).$$

For $T = \min$, it leads to the constraint

$$\pi_{x,y}(u, v) \geq T^*(B(v), A(u)).$$

Viewed as a rule from A to B, this is very similar to a possibility rule, since both types of rules coincide when B is non-fuzzy. When B is fuzzy, the behaviour of the above inequality is somewhat similar to the ones of both possibility and anti-gradual rules: truncation of B and skrinking of its support. Namely, the more x is A, the more possible a larger subset of values around the core of B. However this is not really a different kind of rule: it is an antigradual rule of the form "if y is B then x is A".

Remark. The different fuzzy rules surveyed above can be understood in terms of the modification applied to the conclusion part "y is B", when a precise input $x = u_0$ matches the condition "x is A" at the level $A(u_0) = \alpha$. For min-based models of fuzzy rules, B is modified into B' such that $B'(v) = \tau(B(v)), \forall v$ where τ is a modifier (or equivalently a fuzzy truth value in Zadeh'sense) defined by $\forall t \in [0, 1]$,

$\tau(\theta) = 1$ if $\theta \geq \alpha$; $\tau(\theta) = \theta$ if $\theta < \alpha$ (gradual rule);
$\tau(\theta)) = \max(\theta, 1 - \alpha)$ (certainty rule);
$\tau(\theta) = \min(\theta, \alpha)$ (possibility rule);
$\tau(\theta) = 0$ if $\theta \leq 1 - \alpha$; $\tau(\theta) = \theta > 1 - \alpha$ (antigradual rule).

It can be seen that some modifiers introduce a level of uncertainty, while others rather provide a variation around the fuzzy set B by increasing high degrees of membership or decreasing low degrees.

Meta-rules

Besides the relational view presented in the two above subsections, we can think of a rule "if x is A then y is B" as specifying some constraints between the marginal possibility distributions π_x and π_y describing the available knowledge about the variables x and y. Indeed, the meanings of the individual components of the rule, in terms of their induced constraints, are $\pi_x \leq \mu_A$ and $\pi_y \leq \mu_B$. Therefore, a possible understanding of the rule is just the following condition

$$\text{if } \pi_x \leq \mu_A \text{ then } \pi_y \leq \mu_B$$

which, in turn, has the following easy possibilistic interpretation in case A and B are not fuzzy: "if A is certain ($N_x(A) = 1$) then B is certain ($N_y(B) = 1$)", where N_x and N_y denote the necessity and possibility measures generated by the possibility distributions π_x and π_y respectively.

Having in mind the logical equivalence in classical logic of the material implication $p \to q$ with the disjunction $\neg p \vee q$, one could yet think of another interpretation of the fuzzy rule "if x is A then y is B" as "(x is A^c) or (y is B)", that is "$\pi_x \leq 1 - \mu_A$ or $\pi_y \leq \mu_B$", or, put it in another way,

$$\text{if not}(\pi_x \leq 1 - \mu_A) \text{ then } \pi_y \leq \mu_B$$

In possibilistic terms it also reads (since A is non-fuzzy) "if A is possible ($\Pi_x(A) = 1$) then B is certain ($N_y(B) = 1$)". The difference between the two readings can be seen as relying on the two types of negation at work here, namely not$(\pi_x \leq \mu_A)$ and $\pi_x \leq 1 - \mu_A$ respectively. With such meta-level models, we no longer need to apply the combination/projection principle on their representations because π_y is directly assessed once the condition part of the rule is satisfied.

In the fuzzy case, the two above readings can be generalized, turning them respectively into the inequalities

$$\inf_u \pi_x(u) \to \mu_A(u) \leq \inf_v \pi_y(v) \to \mu_B(v),$$
$$\sup_u T^*(\pi_x(u), \mu_A(u)) \leq \inf_v \pi_y(v) \to \mu_B(v),$$

where $T^*(\alpha, \beta) = 1 - (\alpha \to (1 - \beta))$ is the non-commutative conjunction adjoint of t-norm T. Observe that $C_x(A) = \inf_u \pi_x(u) \to \mu_A(u)$ and $C_y(B) = \inf_v \pi_y(v) \to \mu_B(v)$ are certainty-like indices, while $Pos_x(A) = \sup_x T^*(\pi_x(u), \mu_A(u)) = 1 - C_x(A^c)$ is a possibility-like index. Certainty rules described in the previous section mean that "y is B" is certain as much as "x is A ($\mu_A(u) = 1$), while the first meta rule reading states here that "y is B" is certain as much as "x is A" is certain. Its fuzzy extension above expresses that "the more certain x is A, the more certain y is B", while the second one means "the more possible x is A, the more certain y is B". Solving the above inequalities yields respectively

$$\pi_y(v) \leq C_x(A) \to \mu_B(v), \text{ and}$$
$$\pi_y(v) \leq Pos_x(A) \to \mu_B(v),$$

where $\to$ is a R-implication, which lays bare the behavior of such models. Namely, they modify the output by widening the core of B on the basis of some amount of uncertainty α, thus producing less restrictive outputs (since $\alpha \to \mu_B(v) \geq \mu_B(v)$, $\forall \alpha$). Notice that as soon as the uncertainty degree is as low as $\mu_B(v)$, $\pi_y(v)$ is unrestricted. The two considered meta-level models of fuzzy rule coincide for a precise input $x = u_0$ with gradual rules due to the use of R-implications in the approach.

This meta-level view has been less investigated than the other ones (see [Esteva *et al.*, 1997a]). However it underlies the so-called compatibility-modification inference of Cross and Sudkamp [1994].

2.5 Inference with fuzzy if-then rules

This section does not aim at providing a survey of the different fuzzy logic mechanisms that have been proposed in the literature in the eighties and in the nineties, nor an overview of the problems raised by their practical use and implementation. See [Bouchon-Meunier *et al.*, 1999] in that respect. We focus our interest on a local pattern of inference of particular importance, usually called *generalized modus ponens*, which sufficiently illustrates the main issues. As we shall see, the properties of this pattern of inference heavily depend on the connective used for modeling the if-then rule. Moreover, classical modus ponens can be retrieved as a particular case for fuzzy premises only for appropriate choices of the implication in the fuzzy rule and of the operation for combining the two premises in the pattern. We shall discuss the meaning of this state of fact.

The generalized modus ponens

The generalized modus ponens can be viewed as a particular case of a more general rule, the *compositional rule of inference*, introduced by Zadeh [1979a]:

From:	$S =$ "(x, y) is F" $S' =$ "(y, z) is G"
Infer:	$S'' =$ "(x, z) is $F \circ G$".

where:

1. x, y and z are linguistic variables taking values in U, V and W respectively,
2. F is a fuzzy subset of $U \times V$, and G a fuzzy subset of $V \times W$, and
3. $F \circ G$ is the fuzzy subset of $U \times W$ defined by sup-min composition of F and G, i.e., $F \circ G(u, w) = \sup_{v \in V} \min(F(u, v), G(v, w))$.

This is a direct consequence of the combination-projection method underlying the possibility theory-based treatment of inference. Indeed S and S' translate into the constraints $\pi_{x,y}(u, v) \leq F(u, v)$ and $\pi_{y,z}(u, v) \leq G(u, v)$. So, by combining them, after a cylindrical extension, we get

$$\pi_{x,y,z}(u, v, w) \leq \min(F(u, v), G(v, w)).$$

Finally, projecting this constraint on the joint variable (x, z) we get

$$\pi_{x,z}(u, w) \leq \sup_{v \in V} \min(F(u, v), G(v, w)),$$

which yields, after application of the minimal specificity principle, the representation of the statement S'' in the above rule. This rule has found various applications. For instance, assume $F =$ "approximately equal to", $G =$ "much greater than",

S = "x is approximately equal to y", S' = "y is somewhat greater than z". Using parameterized representations of F and G, one can compute the parameters underlying $F \circ G$, and then interpret it [Dubois and Prade, 1988a].

The generalized modus ponens inference pattern proposed by Zadeh [1973] is of the form:

From:	S = "x is A^*"
	S' = "if x is A then y is B"
Infer:	S'' = "y is B^*".

It is a particular case of the compositional rule of inference where A and A^* are fuzzy subsets of U, B is a fuzzy subset of V, and where statement S is represented by $\pi_x(u) \leq A^*(u)$, and S' is interpreted as a statement of the form "(x, y) is R", represented by $\pi_{x,y}(u, v) \leq R(u, v)$, where R is the fuzzy relation defined by $R(u, v) = I(A(u), B(v))$, I being some suitable implication connective. Then

$$B^* = A^* \circ R.$$

Speaking in an informal way, the idea is that the closer A^* is to A, the closer the conclusion "y is B^*". is to the consequent "y is B" (however the underlying notion of *closeness* varies according to the modeling of the rule). For instance, when I is Kleene-Dienes implication, i.e., when we interpret the fuzzy rule as a certainty rule (see section 3.1), we get

$$B^*(v) = \sup_u \min(A^*(u), \max(1 - A(u), B(v))) = \max(1 - N_{A^*}(A), B(v)),$$

where $N_{A^*}(A) = \inf_u \max(A(u), 1 - A^*(u))$ is the usual necessity measure of A, computed with $\pi(u) = A^*(u)$. B^* means that "y is B" is certain to the degree $N_{A^*}(A)$. This agrees with the understanding of certainty rules as "the more certain x is A, the more certain y is B" in the presence of a fuzzy input "x is A^*". It is also very similar to what is obtained in the meta-rule view (where there is more freedom left in the evaluation of certainty degrees when A is fuzzy).

Note that with Kleene-Dienes implication (i.e., with certainty rules), we have $A \circ I(A, B) = B^*$, where $B^* = \max(1 - N_A(A), B)$, and when A is fuzzy it is only guaranteed that $N_A(A) \geq 1/2$, so the output B^* corresponds to "(y is B) is $N_A(A)$-certain" and not to "y is B" (which is however obtained when A is not fuzzy). This means that the coincidence with classical modus ponens is lost. However, it holds that $core(A) \circ I(A, B) = B$, which is well in agreement with the intended meaning of certainty rules. Indeed "y is B" is obtained only if $N_{A^*}(A) = 1$, which requires that the support of A^* contains only typical elements of A ($A^* \subseteq core(A)$). For instance, if A = "bird" (here a fuzzy set, the set of more or less typical birds) and B = "able to fly" (B is non-fuzzy), then B follows for sure only if x designates a typical bird.

This contrasts with the situation encountered with gradual rules and Gödel implication, for which it holds that $A \circ I(A, B) = B$ in any case. In fact, it has been noticed quite early that the use of the min operation in the combination step of the inference process (as stipulated by the possibilistic framework) is not compatible with the requirement that $B^* = B$ can be derived when $A^* = A$, except for Gödel implication.

More generally, if we require that classical modus ponens continue to hold for fuzzy premises, more solutions are found if a combination operation T other than min (thus departing from the possibility theory setting) is allowed. Namely, we start with the functional equation expressing this requirement

$$\sup_u T(A(u), I(A(u), B(v))) = B(v).$$

This problem has been addressed from two slightly different points of view in [Trillas and Valverde, 1985a; Valverde and Trillas, 1985] and [Dubois and Prade, 1984b; Dubois and Prade, 1985b]. Solutions to the above equation are provided by choosing T as a continuous t-norm and I as its associated residuated implication.

Apart from the perfect coincidence with classical modus ponens, other *natural* or desirable requirements have been proposed for the generalized modus ponens by different authors who have looked for the appropriate implications (and possibly combination operations) that ensure these required properties (see e.g., [Baldwin and Pilsworth, 1980; Fukami *et al.*, 1980; Mizumoto and Zimmermann, 1982; Whalen and Schott, 1983; Whalen and Schott, 1985; Whalen, 2003]). Some of these requirements like monotonicity ($A_1^* \subseteq A_2^*$ implies $B_1^* \subseteq B_2^*$, where fuzzy set inclusion is pointwisely defined by an inequality between membership degrees) are always satisfied, while some other "natural" ones, like $B^* \supseteq B$ (nothing more precise than what the rule says can be inferred) may sometimes be debatable (e.g., if we are modeling interpolative reasoning), and are violated by some implications such as Rescher-Gaines implication which is defined by $I(\alpha, \beta) = 1$ if $\alpha \leq \beta$ and $I(\alpha, \beta) = 0$ if $\alpha > \beta$, and which corresponds to the core of Gödel implication.

Systems of parallel fuzzy if-then rules

Let us now briefly consider the case of a system of parallel implication-based fuzzy if-then rules { "if x is A_i then y is B_i" $\}_{i=1,n}$. Each rule i is represented by the inequality

$$\forall i, \pi_{x,y}(u, v) \leq I(A_i(u), B_i(v)).$$

This leads to

$$\pi_{x,y}(u, v) \leq \min_i I(A_i(u), B_i(v)).$$

By projection and applying the minimal specificity principle, the inference from the set of parallel implication-based rules, and a fact "x is A^*", produces "y is B^*" defined by

$$B^*(v) = \sup_u \min(A^*(u), \min_i I(A_i(u), B_i(v))).$$

Denoting the above inference $B^* = A^* \circ [\cap_i (A_i \to B_i)]$, the following inclusion can be easily established

$$A^* \circ [\cap_{i=1,n}(A_i \to B_i)] \subseteq \cap_{i=1,n}[A^* \circ (A_i \to B_i)].$$

This expresses that the combination/projection principle should be performed globally (which can be computationally heavy), if one wants to obtain an exact result rather than a valid but imprecise result. In other words, it might be rather uninformative to perform each inference $B_i^* = A^* \circ (A_i \to B_i)$ separately and then combine the B_i^*'s in a conjunctive manner. For instance if $A^* = A_i \cup A_j$ for some i and j such that $A_i \cap A_j = \emptyset$ then $A^* \circ (A_i \to B_i) = V$ (nothing is inferred) while $(A_i \cup A_j) \circ [(A_i \to B_i) \cap (A_j \to B_j)] = B_i \cup B_j$, for Gödel implication. This property points out a major weakness in the traditional rule by rule strategy used in many expert system inference engines (that prescribe to trigger rules separately), in the presence of fuzziness, or even incomplete Boolean information. Techniques for reasoning with parallel fuzzy implication-based rules in the presence of imprecise outputs have been little studied in the literature (see [Ughetto *et al.*, 1997] for gradual rules, and a more general theoretical study in [Morsi and Fahmy, 2002]).

Inference with fuzzy conjunctive rules

Let us examine the situation with a conjunction-based model for fuzzy rules (see Section 2.4). For an input "x is A^*" and a fuzzy rule "if x is A then y is B" assumed to be represented by $\pi_{x,y}(u,v) = \min(A(u), B(v))$, the combination/projection method yields the output

$$B^*(v) = \sup_{u \in U} \min(A^*(u), \min(A(u), B(v)))$$

This expression, which corresponds to Mamdani[1977]'s model, can be simplified into

$$B^*(v) = \min(\Pi_{A^*}(A), B(v))$$

where $\Pi_{A^*}(A) = \sup_{u \in U} \min(A^*(u), A(u))$ is the possibility of A computed with $\pi = A^*(\cdot)$. Let us denote this fuzzy inference $A^* \circ (A \times B) = B^*$. Note that $A \circ (A \times B) = B$ if A is normalized. However, we should go back to the understanding of such rules as positive pieces of information (see section 3.1) for explaining why parallel conjunction-based fuzzy rules should be combined disjunctively, as in Mamdani's model of fuzzy control inference.

Indeed from a bipolar possibility theory point of view, a system of conjunction-based rules (where each rule is modeled by the Cartesian product $A_i \times B_i$, i. e., $\forall i, \pi_{x,y}(u,v) \geq \min(A_i(u), B_i(v))$ leads to the inequality

$$\pi_{x,y}(u,v) \geq \max_i \min(A_i(u), B_i(v)).$$

Then, given a set of fuzzy if-then rules {"if x is A_i then y is B_i" : $i = 1, n$} and an input "x is A^*", Mamdani's method consists in three steps:

(i) The output B_i^* for each rule is computed as follows:

$$B_i^*(v) = \sup_u \min(A^*(u), \min(A_i(u), B_i(v))) = \min(\Pi_{A^*}(A_i), B_i(v)).$$

(ii) The global output B^* is then the disjunctive combination of the outputs of each rule, which allows for a rule by rule computation. Indeed applying the maximal specificity principle to the representation of the set of rules, and then the combination/projection method, we get

$$B^*(v) = \sup_u \min(A^*(u), \max_i \min(A_i(u), B_i(v))) = \max_i B_i^*(v).$$

(iii) Finally, there is a defuzzification process in order to come up with a single value $v_0 \in B^*$ for y. This defuzzication step is out of the scope of logic and then of this paper.

Still, problems remain with the inference with conjunction-based rules in case of a fuzzy input. Indeed, the above approach is questionable because adding a rule then may lead to a more imprecise conclusion (before defuzzification), and $A_j \circ \cup_i(A_i \times B_i) \neq B_j$ except if the A_i's are disjoint as pointed out in [Di Nola *et al.*, 1989].

To overcome these difficulties, it is useful to consider the fuzzy relation obtained from a set of conjunction-based rules for what it really is, namely, positive information, as proposed in [Dubois *et al.*, 2003]. A conjunctive rule base actually is a memory of fuzzy cases. Then, what appeared to be anomalies under the negative information view, becomes natural. It is clear that adding a new conjunctive rule to a fuzzy case memory should expand the possibilities, not reduce them. The fuzzy input still consists in a restriction on the values of the input variable and thus is of a different nature. It is in some sense negative information. So, the question is "how to exploit a set of fuzzy cases, which for each input value describes the fuzzy set of guaranteed possible output values, on the basis of negative imprecise information on the input?" In fact, what has to be computed, via an appropriate projection, are the output values that are guaranteed possible for y, for all values of x compatible with the restriction A^* on the input value. The expected conclusion, in terms of guaranteed possible values, is given for a nonfuzzy input A^* by

$$B_*(v) = \inf_{u \in A*} \max_i \min(A_i(u), B_i(v)).$$

What is computed is the intersection of the sets of images of precise inputs compatible with A^*. Any value $y = v$ in this intersection is guaranteed possible, by any input value compatible with A^*. The term B_* is the lower image of A^* via the fuzzy relation aggregating the fuzzy cases conjunctively. In the case where none of the sets are fuzzy,

$$B_* = \{v \in V \mid \forall u \in A^*, \exists i \text{ s.t. } u \in A_i \text{ and } v \in B_i\} = A^* \to \cup_i(A_i \times B_i)$$

In the case where A^* is fuzzy, B^* is defined by

$$B_*(v) = \inf_u \{A^*(u) \to \max_i \min(A_i(u), B_i(v))\}$$

where $\to$ is Gödel implication. Indeed, starting that, from the constraints

$$\begin{aligned} \pi_{x,y}(u,v) &\geq \max_{i=1,n} \min(A_i(u), B_i(v)), \text{for } i = 1, \ldots, n \\ \pi_x(u) &\leq A^*(u) \end{aligned}$$

representing respectively the positive information given by the set of conjunctive-based fuzzy rules and the negative information corresponding to the input, one can derive by simple computations the following further constraint

$$\pi_y(v) \geq \inf_u \{A^*(u) \to \max_{i=1,n} \min(A_i(u), B_i(v))\},$$

provided that π_x is normalized (i.e. $\sup_u \pi_x(u) = 1$). For any fixed value v of y, $B_*(v)$ is nothing but the guaranteed possibility measure $\Delta_v(A^*)$ of A^* as being in relation with v through the fuzzy relation aggregating the fuzzy cases (while $B^*(v)$ was the possibility measure $\Pi_v(A^*)$ of the same event). It can be checked that for usual fuzzy partitions (such that $A_i(u) = 1 \Rightarrow A_j(u) < 1$ for $j \neq i$), if $A^* = A_i$, then $B_* = B_i$, a result that cannot be obtained using the sup-min composition.

Di Nola *et al.* [1985] have pointed out that, when the rule is modeled by means of a t-norm T, $R(u,v) = T(A(u), B(v))$ is the least solution of the fuzzy relational equation

$$\inf_u I(A(u), R(u,v)) = B(v),$$

where I is the residuated implication associated with T. When $T = \min$, I is Gödel implication. Note that this definition of R as a least solution is well in accordance with the interpretation of the possibility rules, to which a principle of maximal specificity must be applied.

Capturing interpolation in approximate reasoning

Many authors, including Zadeh [1992], have pointed out that approximate reasoning techniques in fuzzy control, such as Mamdani's method, perform an interpolation between the conclusions of the rules of the fuzzy controller, on the basis of the degrees of matching of the (usually precise) input measurements (describing the current state of the system to be controlled), with the condition parts of these rules. However the interpolative effect is achieved by defuzzification and is not part of the logical inference step. Klawonn and Novak [1996] have contrasted fuzzy interpolation on the basis of an imprecisely known function (described by fuzzy points $A_i \times B_i$) and logical inference in the presence of fuzzy information. Besides, Sudkamp [1993] discusses the construction of fuzzy rules from sets of pairs of precise values (a_i, b_i) and similarity relations on U and V.

Sugeno and Takagi [1983]'s fuzzy modeling method (see also [Sugeno, 1985] for control) can be viewed as a special case of Mamdani's and a generalization thereof. It starts from n rules with precise numerical conclusion parts, of the form "if x_1 is $A_1^{(i)}$ and ... and x_p is $A_p^{(i)}$ then y is $b^{(i)}(x)$", where $x = (x_1, \ldots, x_p)$. Here the conclusions in the rules depend on the input value, contrary to the fuzzy rules in Mamdani's approach. Let $\alpha_i(u) = \min(A_1^{(i)}(u_1), \ldots, A_p^{(i)}(u_p))$ be the level of matching between the input and the conditions of rule i. Sugeno and Takagi define the relation between x and y to be the following function:

$$y = \frac{\Sigma_i \alpha_i(x) \cdot b^{(i)}(x)}{\Sigma_i \alpha_i(x)}$$

which indeed performs a weighted interpolation. This result can be retrieved using Mamdani's method, noticing that in this case $B^* = \{b^{(i)}(u)/\alpha_i(u) : i = 1, n\}$, where b/μ indicates that element b has membership value μ, and applying the center of gravity method for selecting a value representing B^*.

When the conclusions $b^{(i)}(x) = b_i$ do not depend on x, and assuming single condition rules, this interpolation effect can be obtained within the inference step, by applying Zadeh's approximate reasoning combination and projection approach. For this purpose, consider the rules as pure gradual rules (based on Rescher-Gaines implication rather than on Gödel's), expressing that "the closer x is to a_i, the closer y is to b_i", where (a_i, b_i), $i = 1, n$ are pairs of scalar values, where we assume $a_1 < \ldots < a_{i-1} < a_i < a_{i+1} < \ldots < a_n$. The first problem is to represent "close to a_i", by means of a fuzzy set A_i. It seems natural to assume that $A_i(a_{i-1}) = A_i(a_{i+1}) = 0$ since there are special rules adapted to the cases $x = a_{i-1}, x = a_{i+1}$. Moreover if $u \neq a_i$, $A_i(u) < 1$ for $u \in (a_{i-1}, a_{i+1})$, since information is only available for $u = a_i$. Hence A_i should be a fuzzy interval with support (a_{i-1}, a_{i+1}) and core $\{a_i\}$. Since the closer x is to a_{i-1}, the farther it is from a_i, A_{i-1} should decrease when A_i increases, and by symmetry, $A_i((a_i + a_{i+1})/2) = A_{i-1}((a_{i-1} + a_i)/2) = 1/2$. The simplest way of achieving this is to let $\forall u \in [a_{i-1}, a_i], A_{i-1}(u) + A_i(u) = 1$, an example of which are triangular-shaped fuzzy sets. Clearly the conclusion parts of the rules should involve fuzzy sets B_i whose meaning is "close to b_i", with similar conventions.

In other words, each rule is understood as "the more x is A_i, the more y is B_i". Pure gradual rules are modeled by inequality constraints of the form $A_i(u) \leq B_i(v)$. Then the subset of V obtained by combining the results of the rules for the input $x = u_0$ is given by

$$B^*(v) = \min_{i=1,n} A_i(u_0) \to B_i(v)$$

where the implication is the one of Rescher-Gaines, defined by $a \to b = 1$ if $a \leq b$ and $a \to b = 0$ if $a > b$. In that case the output associated with the precise input u_0 where $a_{i-1} < u_0 < a_i$, is

$$B^* = (\alpha_{i-1} \to B_{i-1}) \cap (\alpha_i \to B_i) = [B_{i-1}]_{\alpha_{i-1}} \cap [B_i]_{\alpha_i}$$

since $\alpha \to B(\cdot)$ corresponds to the level cut $[B]_\alpha$, $\alpha_{i-1} = A_{i-1}(u_0)$, $\alpha_i = A_i(u_0)$, and $\alpha_{i-1} + \alpha_i = 1$. Due to the latter assumption it can be easily proved (without the assumption of triangular shaped fuzzy sets), that there exists a unique value $y = b$ such that $B^*(b) = 1$, which exactly corresponds to the result of the linear interpolation, i.e., $b = \alpha_{i-1} \cdot b_{i-1} + \alpha_i \cdot b_i$. The conclusion thus obtained is nothing but the singleton value computed by Sugeno and Takagi's method. It is a theoretical justification for this inference method in the one-dimensional case. Hence reasoning with gradual rules does model interpolation, linear interpolation being retrieved as a particular case. The more complicated case of gradual rules with compound conditions, i.e., rules of the form "the more x_1 is $A_1, \ldots,$ and the more x_p is A_p, the more y is B" is also studied in detail in [Dubois *et al.*, 1994]. Then provided that the rules satisfy a coherence condition, the output of a system of pure gradual rules, where conditions and conclusions are fuzzy intervals, is an interval.

2.6 Concluding remarks on approximate reasoning

The presentation has emphasized the basic ideas underlying Zadeh's original proposal, showing their consistency, their close relation to the representation setting of possibility theory. Various inference machineries can be handled at the semantic level. Still many issues of interest considered elsewhere in the literature (see [Bouchon-Meunier *et al.*, 1999]), like computational tractability, coherence of a set of fuzzy rules, special applications to temporal or to order-of-magnitude reasoning, the handling of fuzzy quantifiers (viewed as imprecisely known conditional probabilities) in reasoning patterns, fuzzy analogical reasoning, interpolative reasoning with sparse fuzzy rules, etc, have been left apart, let alone more practically oriented research works. This framework can express pieces of information with rich contents. The important but sometimes misleading, notion of fuzzy truth-value, encompassing both notions of intermediate degrees of truth and (degrees of) uncertainty about truth has been discussed at length. It is crucial for a proper appraisal of the line of thought followed by the founder of fuzzy logic, and in order to situate the role of fuzzy logic in the narrow sense, mainly developed in the nineties and summarized in the remainder of this paper, for the purpose of knowledge representation. In the meantime, in the last twenty years, Zadeh [1988; 1989; 1997; 1999; 2001; 2005] has continued to elaborate his semantic, non-linear optimization approach to human fuzzy and uncertain reasoning, to precisiate as well as to enlarge it, to propose new perspectives, emphasizing the importance of key-notions like computing with words and perceptions as opposed to numbers and measurements, and information granulation.

3 MANY VALUED LOGICAL SYSTEMS BASED ON FUZZY SET CONNECTIVES

In the preface of the book [Zadeh, 1994a], Zadeh made a very clear distinction between the two main meanings of the term *fuzzy logic*. Indeed, he writes:

> The term "fuzzy logic" has two different meanings: wide and narrow. In a narrow sense it is a logical system which aims a formalization of approximate reasoning. In this sense it is an extension of multivalued logic. However the agenda of fuzzy logic (FL) is quite different from that of traditional many-valued logic. Such key concepts in FL as the concept of linguistic variable, fuzzy if-then rule, fuzzy quantification and defuzzification, truth qualification, the extension principle, the compositional rule of inference and interpolative reasoning, among others, are not addressed in traditional systems. In its wide sense, FL, is fuzzily synonymous with the fuzzy set theory of classes of unsharp boundaries.

Hájek, in the introduction of his monograph [Hájek, 1998a] makes the following comment to Zadeh's quotation:

> Even if I agree with Zadeh's distinction (...) I consider formal calculi of many-valued logic to be the kernel of fuzzy logic in the narrow sense and the task of explaining things Zadeh mentions by means of this calculi to be a very promising task.

On the other hand, Novák *et al.*, also in the introduction of their monograph [Novák *et al.*, 1999], write:

> Fuzzy logic in narrow sense is a special many-valued logic which aims at providing formal background for the graded approach to vagueness.

According to Hájek and Novák *et al.*'s point of view, this section is devoted to the formal background of fuzzy logic in narrow sense, that is, to formal systems of many-valued logics having the real unit interval as set of truth values, and truth functions defined by fuzzy connectives that behave classically on extremal truth values (0 and 1) and satisfy some natural monotonicity conditions. Actually, these connectives originate from the definition and algebraic study of set theoretical operations over the real unit interval, essentially developed in the eighties, when this field had a great development. It was in that period when the use of t-norms and t-conorms as operations to model fuzzy set conjunction and disjunction respectively was adopted, and related implication and negation functions were studied, as reported in Section 2.1. Therefore, the syntactical issues of fuzzy logic have followed the semantical ones.

The main many-valued systems described in this section are the so-called *t-norm based fuzzy logics*. They correspond to $[0, 1]$-valued calculi defined by a conjunction and an implication interpreted respectively by a (left-continuous) t-norm and its residuum, and have had a great development over the past ten years and from many points of view (logical, algebraic, proof-theoretical, functional representation, and complexity), as witnessed by a number of important monographs that have appeared in the literature, see [Hájek, 1998a; Gottwald, 2001; Novák *et al.*, 1999]. Actually, two prominent many-valued logics that fall in this class, namely Łukasiewicz and Gödel infinitely-valued logics [Łukasiewicz, 1930; Gödel, 1932], were defined much before fuzzy logic was born. They indeed correspond to the calculi defined by Łukasiewicz and min t-norms respectively. Łukasiewicz logic Ł has received much attention from the fifties, when completeness results were proved by Rose and Rosser [1958], and by algebraic means by Chang [1958; 1959], who developed the theory of MV-algebras largely studied in the literature. Moreover McNaughton theorem [McNaughton, 1951] provides a functional description of its logical functions. Many results about Łukasiewicz logic and MV-algebras can be found in the book [Cignoli *et al.*, 1999]. On the other hand, a completeness theorem for Gödel logic was already given in the fifties by Dummett [1959]. Note that the algebraic structures related to Gödel logic are linear Heyting algebras (known as Gödel algebras in the context of fuzzy logics), that have been studied in the setting of intermediate or superintuitionistic logics, i.e. logics between intuitionistic and classical logic.

The key ideas of these logical systems are described in the first three subsections. Then, in the next two subsections, more complex systems resulting from the addition of new connectives, as well as a number of further issues related to t-norm based fuzzy logics, are briefly surveyed. The sixth subsection shows how to embed the main patterns of approximate reasoning inside a residuated fuzzy logic. The following subsection is devoted to variants of fuzzy logic systems, including clausal and resolution-based fuzzy logics. The former are mainly systems related to the logical calculi on the real unit interval defined by a De Morgan triple: a t-norm for conjunction, a strong negation and the dual t-connorm for disjunction. The section concludes with a subsection dealing with notions of graded consequence and their relationship to closure operators, in a Tarski-style. This is a different approach to formalize a form of fuzzy logic which, in particular, has been the topic of Gerla's monograph [2001] and partially also in [Bělohlávek, 2002b].

Even if the set of topics addressed in this section is very wide, we acknowledge the fact that we do not cover for sure all the approaches and aspects of formal systems of fuzzy logic that have been proposed in the literature. This is the case for instance of a whole research stream line on fuzzifying modal logics, started indeed very early by Schotch [1975], and then enriched by a number of significant contributions, like Gabbay's general fibring method for building modal fuzzy logics [Gabbay, 1996; Gabbay, 1997] or the introduction of various types of modalities in the frame of the above mentioned t-norm based fuzzy logics [Hájek, 1998a, Chap. 8], to cite only a very few of them.

3.1 BL and related logics

Probably the most studied and developed many-valued systems related to fuzzy logic are those corresponding to logical calculi with the real interval $[0,1]$ as set of truth-values and defined by a conjunction & and an implication $\rightarrow$ interpreted respectively by a (left-continuous) t-norm $*$ and its residuum $\Rightarrow$, and where negation is defined as $\neg\varphi = \varphi \rightarrow \overline{0}$, with $\overline{0}$ being the truth-constant for falsity.

In the framework of these logics, called *t-norm based fuzzy logics*, each (left continuous) t-norm $*$ uniquely determines a semantical (propositional) calculus $PC(*)$ over formulas defined in the usual way from a countable set of propositional variables, connectives $\wedge$, & and $\rightarrow$ and truth-constant $\overline{0}$ [Hájek, 1998a]. Further connectives are defined as follows:

$$
\begin{array}{lll}
\varphi \vee \psi & \text{is} & ((\varphi \rightarrow \psi) \rightarrow \psi) \wedge ((\psi \rightarrow \varphi) \rightarrow \varphi), \\
\neg\varphi & \text{is} & \varphi \rightarrow \overline{0}, \\
\varphi \equiv \psi & \text{is} & (\varphi \rightarrow \psi) \& (\psi \rightarrow \varphi).
\end{array}
$$

Evaluations of propositional variables are mappings e assigning each propositional variable p a truth-value $e(p) \in [0,1]$, which extend univocally to compound formulas as follows:

$$
\begin{aligned}
e(\overline{0}) &= 0 \\
e(\varphi \wedge \psi) &= \min(e(\varphi), e(\psi)) \\
e(\varphi \& \psi) &= e(\varphi) * e(\psi) \\
e(\varphi \rightarrow \psi) &= e(\varphi) \Rightarrow e(\psi)
\end{aligned}
$$

Note that, from the above defintions, $e(\varphi \vee \psi) = \max(e(\varphi), e(\psi))$, $\neg\varphi = e(\varphi) \Rightarrow 0$ and $e(\varphi \equiv \psi) = e(\varphi \rightarrow \psi) * e(\psi \rightarrow \varphi)$. A formula φ is a said to be a 1-tautology of $PC(*)$ if $e(\varphi) = 1$ for each evaluation e. The set of all 1-tautologies of $PC(*)$ will be denoted as $TAUT(*)$.

Three outstanding examples of (continuous) t-norm based fuzzy logic calculi are:

Gödel logic calculus: defined by the operations

$$
\begin{array}{rcl}
x *_G y & = & \min(x, y) \\
x \Rightarrow_G y & = & \begin{cases} 1, & \text{if } x \leq y \\ y, & \text{otherwise.} \end{cases}
\end{array}
$$

Łukasiewicz logic calculus: defined by the operations

$$
\begin{array}{rcl}
x *_{Ł} y & = & \max(x + y - 1, 0) \\
x \Rightarrow_{Ł} y & = & \begin{cases} 1, & \text{if } x \leq y \\ 1 - x + y, & \text{otherwise.} \end{cases}
\end{array}
$$

Product logic calculus: defined by the operations

$$\begin{array}{rcl} x *_{\Pi} y & = & x \cdot y \quad \text{(product of reals)} \\ x \Rightarrow_{\Pi} y & = & \begin{cases} 1, & \text{if } x \leq y \\ y/x, & \text{otherwise.} \end{cases} \end{array}$$

These three cases are important since each *continuous* t-norm is definable as an *ordinal sum* of copies of Łukasiewicz, Minimum and Product t-norms (see e.g. [Klement *et al.*, 2000]), and the min and max operations are definable from $*$ and $\Rightarrow$. Indeed, for each continuous t-norm $*$ and its residuated implication $\Rightarrow$, the following identities are true:

$$\begin{array}{rcl} \min(x, y) & = & x * (x \Rightarrow y), \\ \max(x, y) & = & \min((x \Rightarrow y) \Rightarrow y, (y \Rightarrow x) \Rightarrow x). \end{array}$$

Actually, two of these logics correspond to many-valued systems already studied before fuzzy logic was born. These are the well-known infinitely-valued Łukasiewicz [1930] and Gödel [1932] logics[4] which are the logical systems corresponding to the so-called Łukasiewicz and minimum t-norms and their residuated implications respectively (see, for example, [Cignoli *et al.*, 1999; Gottwald, 2001] for excellent descriptions of these logics). Much later, already motivated by research on fuzzy logic, Product logic, the many-valued logic corresponding to Product t-norm and its residuum, was also axiomatized in [Hájek *et al.*, 1996]. All these logics enjoy *standard* completeness, that is, completeness with respect to interpretations over the algebra on the unit real interval $[0, 1]$ defined by the corresponding t-norm and its residuum. Namely, it holds that:

φ is provable in Łukasiewicz logic iff $\varphi \in TAUT(*_{\text{Ł}})$
φ is provable in Gödel logic iff $\varphi \in TAUT(*_G)$
φ is provable in Product logic iff $\varphi \in TAUT(*_{\Pi})$.

A main step in the formalization of fuzzy logic in narrow sense is Hájek's monograph [Hájek, 1998a], where the author introduced the Basic Fuzzy logic BL as a common fragment of the above mentioned three outstanding many-valued logics, and intending to syntactically capture the common tautologies of all propositional calculi $PC(*)$ for $*$ being a continuous t-norm. The language of BL logic is built (in the usual way) from a countable set of propositional variables, a conjunction &, an implication $\rightarrow$ and the constant $\overline{0}$. Since for a continuous t-norm $*$ and its residuum $\Rightarrow$ we have $\min(x, y) = x * (x \Rightarrow y)$, in BL the connective $\wedge$ is taken as definable from & and $\rightarrow$:

[4]Gödel logic is also known as Dummett logic, referring to the scholar who proved its completeness.

$\varphi \wedge \psi$ is $\varphi \& (\varphi \to \psi)$

Other connectives ($\vee, \neg, \equiv$) are defined as in $PC(*)$.

The following formulas are the *axioms*[5] of BL:

(A1) $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$
(A2) $(\varphi \& \psi) \to \varphi$
(A3) $(\varphi \& \psi) \to (\psi \& \varphi)$
(A4) $(\varphi \& (\varphi \to \psi) \to (\psi \& (\psi \to \varphi))$
(A5a) $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$
(A5b) $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$
(A6) $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$
(A7) $\bar{0} \to \varphi$

The *deduction rule* of BL is modus ponens.

Axiom (A1) captures the transitivity of the residuum, axioms (A2) and (A3) stand for the weakening and commutativity properties of the conjunction, axiom (A4) forces the commutativity of the defined $\wedge$ connective and it is related to the divisibility and the continuity of the &, axioms (A5a) and (A5b) stand for the residuation property of the pair $(\&, \to)$, axiom (A6) is a form of proof-by-cases property and is directly related to the pre-linearity axiom $(\varphi \to \psi) \vee (\psi \to \varphi)$, which is an equivalent formulation of (A6), and finally axiom (A7) establishes that 0 is the least truth-value.

These axioms and deduction rule defines a notion of proof, denoted $\vdash_{BL}$, in the usual way. As a matter of fact, Łukasiewicz, Gödel and Product logics are axiomatic extensions of BL. Indeed, it is shown in [Hájek, 1998a] that Łukasiewicz logic is the extension of BL by the axiom

(Ł) $\neg\neg\varphi \to \varphi$,

forcing the negation to be involutive, and Gödel logic is the extension of BL by the axiom

(G) $\varphi \to (\varphi \& \varphi)$.

forcing the conjunction to be idempotent. Finally, product logic is just the extension of BL by the following two axioms:

(Π1) $\neg\neg\chi \to (((\varphi \& \chi) \to (\psi \& \chi)) \to (\varphi \to \psi))$,
(Π2) $\varphi \wedge \neg\varphi \to \bar{0}$.

The first axiom indicates that if $c \neq 0$, the cancellation of c on both sides of the inequality $a \cdot c \leq b \cdot c$ is possible, hence the strict monotony of the conjunction

[5] These are the original set of axioms proposed by Hájek in [Hájek, 1998a]. Later Cintula showed [Cintula, 2005a] that (A3) is redundant.

on $(0,1]$. The last axiom is due to the fact that negation in product logic behaves such that $n(a) = a \to 0 = 0$ if $a > 0$.

From a semantical point of view, if one takes a continuous t-norm $*$ for the truth function of & and the corresponding residuum $\Rightarrow$ for the truth function of $\to$ (and evaluating $\overline{0}$ by 0) then all the axioms of BL become 1-tautologies (have identically the truth value 1). And since modus ponens preserves 1-tautologies, all formulas provable in BL are 1-tautologies, i.e. if $\vdash_{BL} \varphi$ then $\varphi \in \cap\{TAUT(*) : *$ is a continuous t-norm$\}$. This shows that BL is sound with respect to the *standard* semantics, i.e. with respect to evaluations on $[0,1]$ taking as truth-functions continuous t-norms and their residua.

Actually, standard semantics is a particular case of a more general algebraic semantics. Indeed, the algebraic counterpart of BL logic are the so-called BL-algebras. A *BL-algebra* is an algebra $\mathbf{L} = \langle L, *, \Rightarrow, \wedge, \vee, 0, 1\rangle$ with four binary operations and two constants such that:

(i) $(L, \wedge, \vee, 0, 1)$ is a lattice with the largest element 1 and the least element 0 (with respect to the lattice ordering $\leq$),

(ii) $(L, *, 1)$ is a commutative semigroup with the unit element 1, i.e. $*$ is commutative, associative and $1 * x = x$ for all x,

(iii) the following conditions hold:

(1) $z \leq (x \Rightarrow y)$ iff $x * z \leq y$ for all x, y, z. (residuation)

(2) $x \wedge y = x * (x \Rightarrow y)$ (divisibility)

(3) $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$. (pre-linearity)

Thus, in other words, a BL-algebra is a bounded, integral commutative *residuated lattice* satisfying (2) and (3). The class of all BL-algebras forms a variety. Due to (3), each BL-algebra can be decomposed as a subdirect product of linearly ordered BL-algebras. BL-algebras defined on the real unit interval $[0,1]$, called *standard BL-algebras*, are determined by continuous t-norms, i.e. any standard BL-algebra is of the form $[0,1]_* = \langle [0,1], *, \Rightarrow, \min, \max, 0, 1\rangle$ for some continuous t-norm $*$, where $\Rightarrow$ is its residuum.

By defining $\neg x = x \Rightarrow 0$, it turns out that the algebraic semantics of Łukasiewicz logic, defined by the class of *MV-algebras* (or Wajsberg algebras), correspond to the subvariety of BL-algebras satisfying the additional condition $\neg\neg x = x$, while the algebraic semantics of Gödel logic, defined by the class of G-*algebras*, corresponds to the subvariety of BL-algebras satisfying the additional condition $x * x = x$. Finally, *Product algebras*, which define the algebraic semantics for Product logic, are just BL-algebras further satisfying

$$x \wedge \neg x = 0,$$
$$\neg\neg z \Rightarrow ((x * z = y * z) \Rightarrow x = y) = 1.$$

Given a BL-algebra $\mathbf{L}$, one can define $\mathbf{L}$-evaluations of formulas in the same way as in $[0,1]$ just by taking as truth-functions the operations of $\mathbf{L}$. An $\mathbf{L}$-evaluation e is called a *model* of a formula φ when $e(\varphi) = 1$ (1 being the top element of the algebra), and it is a model of a set of formulas Γ if it is a model of every formula of Γ. A $\mathbf{L}$-tautology is then a formula getting the value 1 for each $\mathbf{L}$-evaluation, i.e. any $\mathbf{L}$-evaluation is a model of the formula. In particular, when $\mathbf{L} = [0,1]_*$, the set of $\mathbf{L}$-tautologies is the set $TAUT(*)$ introduced before. Then, the logic BL is sound with respect to $\mathbf{L}$-tautologies: if φ is provable in BL then φ is an $\mathbf{L}$-tautology for each BL-algebra $\mathbf{L}$. Moreover, Hájek proved the following completeness results for BL, namely the following three conditions are proved in [Hájek, 1998a] to be equivalent:

(i) $\Gamma \vdash_{BL} \varphi$,

(ii) for each BL-algebra $\mathbf{L}$, any $\mathbf{L}$-evaluation which is a model of Γ, it is a model of φ as well,

(iii) for each linearly ordered BL-algebra $\mathbf{L}$, any $\mathbf{L}$-evaluation which is a model of Γ, it is a model of φ as well,

Hájek's conjecture was that BL captured the 1-tautologies common to all many-valued calculi defined by a continuous t-norm. In fact this was proved [Hájek, 1998b; Cignoli *et al.*, 2000] to be the case soon after, that is, it holds that

$$\varphi \text{ is provable in BL} \quad \text{iff} \quad \varphi \in \bigcap\{TAUT(*) : * \text{ is a continuous t-norm}\}$$

This is the so-called *standard completeness* property for BL. More than that, a stronger completeness property holds: if Γ is a *finite* set of formulas, then $\Gamma \vdash_{BL} \varphi$ if and only if for each *standard BL-algebra* $\mathbf{L}$, any $\mathbf{L}$-evaluation which is a model of Γ, it is a model of φ. This result is usually referred as *finite strong* standard completeness of BL.

On the other hand, in [Esteva *et al.*, 2004] the authors provide a general method to get a finite axiomatization, as an extension of BL, of each propositional calculus $PC(*)$, for $*$ being a continuous t-norm. Therefore, for each of these logics, denoted L_*, one has that a formula φ is provable in L_* iff $\varphi \in TAUT(*)$. Note that L_* is equivalent to Gödel logic G when $* = \min$, to Łukasiewicz logic Ł when $*$ is the Łukasiewicz t-norm $*_{\text{Ł}}$ and to Product logic when $*$ is the product of real numbers.

Actually, the book [Hájek, 1998a] was the starting point of many fruitful and deep research works on BL logic and their extensions, as well as on its algebraic counterpart, the variety of BL-algebras. See the special issue [Esteva and Godo (eds.), 2005] for a quite exhaustive up-to-date overview on recent results on BL-algebras and BL-logics.

The well-known result that a t-norm has residuum if and only if the t-norm is left-continuous makes it clear that BL is not the most general t-norm-based logic (in the setting of residuated fuzzy logics). In fact, a weaker logic than BL, called Monoidal t-norm-based Logic, MTL for short, was defined in [Esteva and Godo, 2001] and proved in [Jenei and Montagna, 2002] to be the logic of left-continuous t-norms and their residua. Thus MTL is indeed the most general residuated t-norm-based logic. The basic difference between BL and MTL is the divisibility axiom (or algebraically the equality $x \wedge y = x * (x \Rightarrow y)$), which characterizes the continuity of the t-norm and which is not satisfied in MTL. This means that the min-conjunction $\wedge$ is not definable in MTL and, as opposed to BL, it has to be introduced as a primitive connective into the language together with BL primitive connectives (strong conjunction &, implication $\rightarrow$ and the truth constant $\overline{0}$). Axioms of MTL are obtained from those of BL by replacing axiom (A4) by the three following ones:

(A4a) $\varphi \wedge \psi \rightarrow \varphi$
(A4b) $\varphi \wedge \psi \rightarrow \psi \wedge \varphi$
(A4c) $\varphi \& (\varphi \rightarrow \psi) \rightarrow \varphi \wedge \psi$

Most of well-known fuzzy logics (among them Łukasiewicz logic, Gödel logic, Hájek's BL logic and Product logic)—as well as the Classical Propositional Calculus[6]—can be presented as axiomatic extensions of MTL. Tables 2 and 3 collect some axiom schemata[7] and the axiomatic extensions of MTL they define[8]. Notice that in extensions of MTL with the divisibility axiom (Div), i.e. in extensions of BL, the additive conjunction $\wedge$ is in fact definable and therefore it is not considered as a primitive connective in their languages. For the sake of homogeneity we will keep $\mathcal{L} = \{\&, \rightarrow, \wedge, \overline{0}\}$ as the common language for all MTL extensions.

The algebraic counterpart of MTL logic is the class of the so-called *MTL-algebras*. MTL-algebras are in fact pre-linear residuated lattices (understood as commutative, integral, bounded residuated monoids). Of particular interest are the MTL-algebras defined on the real unit interval [0, 1], which are defined in fact by left-continuous t-norms and their residua. Jenei and Montagna proved that MTL is (strongly) complete with respect to the class of MTL-algebras defined on the real unit interval. This means in particular that

$$\varphi \text{ is provable in MTL} \quad \text{iff} \quad \varphi \in \bigcap \{TAUT(*) : * \text{ is a left-continuous t-norm}\}.$$

One common property of all MTL extensions is that they enjoy a *local* form of the *deduction theorem*, namely, for any MTL axiomatic extension L it holds that

$$\Gamma \cup \{\varphi\} \vdash_L \psi \text{ iff there exists } n \in \mathbb{N} \text{ such that } \Gamma \vdash_L \varphi^n \rightarrow \psi,$$

[6]Indeed, Classical Propositional Calculus can be presented as the extension of MTL (and of any of its axiomatic extensions) with the excluded-middle axiom (EM).

[7]Axioms of pseudo-complementation (PC) and n-contraction (C_n) are also known respectively by the names of *weak contraction* and *n-potence*, see e.g. [Galatos *et al.*, 2007].

[8]Of course, some of these logics were known well before MTL was introduced. We only want to point out that it is possible to present them as the axiomatic extensions of MTL obtained by adding the corresponding axioms to the Hilbert style calculus for MTL given above. Moreover, these tables only collect some of the most prominent axiomatic extensions of MTL, even though many other ones have been studied in the literature (see e.g. [Noguera, 2006], [Wang *et al.*, 2005b] and [Wang *et al.*, 2005a]).

Axiom schema	Name
$\neg\neg\varphi \rightarrow \varphi$	Involution (Inv)
$\neg\varphi \vee ((\varphi \rightarrow \varphi\&\psi) \rightarrow \psi)$	Cancellation (C)
$\neg(\varphi\&\psi) \vee ((\psi \rightarrow \varphi\&\psi) \rightarrow \varphi)$	Weak Cancellation (WC)
$\varphi \rightarrow \varphi\&\varphi$	Contraction (Con)
$\varphi \wedge \psi \rightarrow \varphi\&(\varphi \rightarrow \psi)$	Divisibility (Div)
$\varphi \wedge \neg\varphi \rightarrow \overline{0}$	Pseudo-complementation (PC)
$\varphi \vee \neg\varphi$	Excluded Middle (EM)
$(\varphi\&\psi \rightarrow \overline{0}) \vee (\varphi \wedge \psi \rightarrow \varphi\&\psi)$	Weak Nilpotent Minimum (WNM)
$\varphi^{n-1} \rightarrow \varphi^n$	n-Contraction (C_n)

Table 2. Some usual axiom schemata in fuzzy logics.

where φ^n stands for $\varphi\& \overset{n}{\ldots} \&\varphi$. It is local in the sense that n depends on particular formulas involved Γ, φ and ψ. It turns out that the only axiomatic extension of MTL for which the *classical* (global) deduction theorem

$$\Gamma \cup \{\varphi\} \vdash_L \psi \text{ iff } \Gamma \vdash_L \varphi \rightarrow \psi$$

holds is for L being Gödel fuzzy logic. This fact clearly indicates, that in general, syntactic inference $\varphi \vdash_L \psi$ in BL, MTL and any of their extensions L does not implement Zadeh's entailment principle of approximate reasoning in the semantics (except in Gödel logic). For Zadeh, the inference of a fuzzy proposition ψ from φ means that ψ is always at least as true as φ in all interpretations. At the syntactic level, it generally corresponds to proving $\vdash_L \varphi \rightarrow \psi$, not $\varphi \vdash_L \psi$. At the semantic level, the latter only corresponds to the inclusion of cores of the corresponding fuzzy sets (that is, the preservation of the highest membership value 1).

Regarding this issue, the NM logic can be considered the closest to Gödel logic, since it also enjoys a global form of deduction theorem, but with $n = 2$ in the above deduction theorem expression, i.e. it holds that

$$\Gamma \cup \{\varphi\} \vdash_{NM} \psi \text{ iff } \Gamma \vdash_{NM} \varphi\&\varphi \rightarrow \psi \ .$$

for all Γ, φ, ψ. Actually, NM is a genuine MTL-extension (i.e. it is not a BL-extension) that axiomatizes the calculus defined by the nilpotent minimum t-norm $*_{NM}$ (see Section 2.1), and satisfies the following standard completeness property:

$$\varphi \text{ is provable in NM} \quad \text{iff} \quad \varphi \in TAUT(*_{NM})$$

where $x *_{NM} y = \min(x, y)$ if $x > 1 - y$, $x *_{NM} y = 0$ otherwise. This logic, introduced in [Esteva and Godo, 2001], has very nice logical properties besides the above global deduction theorem, as having an involutive negation (like Łukasiewicz logic), or being complete for deduction from arbitrary theories (not only for theorems). Indeed, this logic has received much attention by the Chinese school leaded

Logic	Additional axiom schemata	References
SMTL	(PC)	[Hájek, 2002]
ΠMTL	(C)	[Hájek, 2002]
WCMTL	(WC)	[Montagna *et al.*, 2006]
IMTL	(Inv)	[Esteva and Godo, 2001]
WNM	(WNM)	[Esteva and Godo, 2001]
NM	(Inv) and (WNM)	[Esteva and Godo, 2001]
C_nMTL	(C_n)	[Ciabattoni *et al.*, 2002]
C_nIMTL	(Inv) and (C_n)	[Ciabattoni *et al.*, 2002]
BL	(Div)	[Hájek, 1998a]
SBL	(Div) and (PC)	[Esteva *et al.*, 2000]
Ł	(Div) and (Inv)	[Hájek, 1998a]
Π	(Div) and (C)	[Hájek *et al.*, 1996]
G	(Con)	[Hájek, 1998a]

Table 3. Some axiomatic extensions of MTL obtained by adding the corresponing additional axiom schemata and the references where they have been introduced (in the context of fuzzy logics).

by G.J. Wang. It turns out that he independently introduced in [Wang, 1999; Wang, 2000] a logic in the language $(\neg, \vee, \rightarrow)$, called $\mathcal{L}^*$, with an algebraic semantics consisting of a variety of algebras called R_0-algebras. Pei later showed [Pei, 2003] that both R_0 algebras and NM were in fact definitionally equivalent, and hence that logics NM and $\mathcal{L}^*$ were equivalent as well. A similar relation was also found for IMTL and weaker version of $\mathcal{L}^*$.

In the tradition of substructural logics, both BL and MTL are logics without contraction (see Ono and Komori's seminal work [1985]). The weakest residuated logic without contraction is Höhle's Monoidal Logic ML [Höhle, 1995], equivalent to FL_{ew} (Full Lambek calculus with exchange and weakening)[9] introduced by Kowalski and Ono [2001] as well as to Adillon and Verdú's IPC$^*\backslash c$ (Intuitionistic Propositional Calculus without contraction) [Adillon and Verdú, 2000], and that is the logic corresponding to the variety of (bounded, integral and commutative) residuated lattices. From them, MTL can be obtained by adding the prelinearity axiom and from there, a hierarchy of all t-norm-based fuzzy logics can be considered as different schematic extensions [Kowalski and Ono, 2001; Esteva *et al.*, 2003]. Figure 1 shows a diagram of this hierarchy with the main logics involved.

The issue of completeness of these and other t-norm based fuzzy logics extending of MTL has been addressed in the literature. In fact, several kinds of algebraic completeness have been considered, depending on the number of premises. Here we

[9] Also known as aMALL or aMAILL (affine Multiplicative Additive fragment of (propositional) Intuitionistic Linear logic or H_{BCK} [Ono and Komori, 1985].

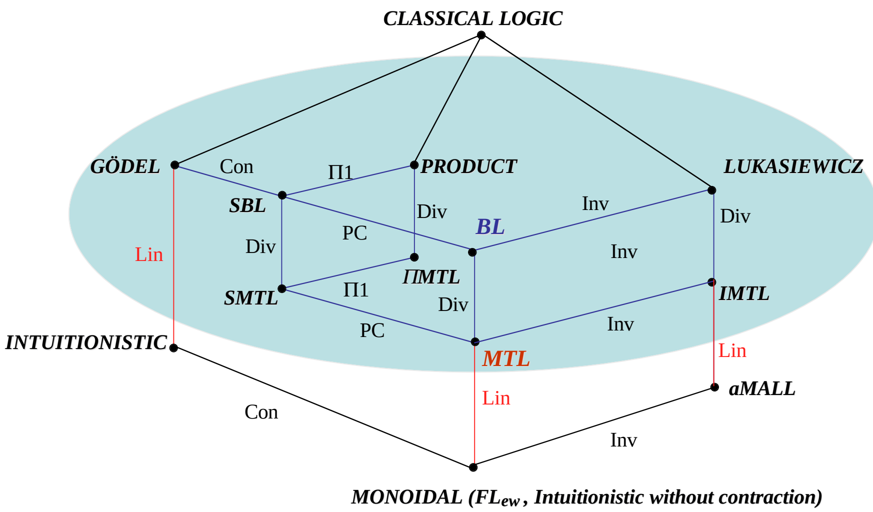

Figure 1. Hierarchy of some substructural and fuzzy logics.

will only refer to the completeness properties with respect to the usually intended semantics (standard semantics) on the real unit interval $[0, 1]$.

For any L axiomatic extension of MTL and for every set of L-formulas $\Gamma \cup \{\varphi\}$, we write $\Gamma \models_L \varphi$ when for every evaluation e of formulas on the any standard L-algebra (L-chain on $[0, 1]$) one has $e(\varphi) = 1$ whenever $e(\psi) = 1$ for all $\psi \in \Gamma$. Then:

- L has the property of *strong standard completeness*, SSC for short, when for every set of formulae Γ, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_L \varphi$.

- L has the property of *finite strong standard completeness*, FSSC for short, when for every *finite* set of formulae Γ, $\Gamma \vdash_{\mathrm{L}} \varphi$ iff $\Gamma \models_L \varphi$.

- L has the property of (weak) *standard completeness*, SC for short, when for every formula φ, $\vdash_{\mathrm{L}} \varphi$ iff $\models_L \varphi$.

Of course, the SSC implies the FSSC, and the FSSC implies the SC. Table 4 gathers the different standard results for some of the main t-norm based logics. Note that for some of these logics one may restrict to check completeness with respect to a single standard algebra defined by a distinguished t-norm, like in the cases of G, Ł, Π and NM logics.

In the literature of t-norm based logics, one can find not only a number of axiomatic extensions of MTL but also extensions by means of expanding the language with new connectives. Some of these expansions (like those with Baaz[1996]'s Δ

Logic	SC	FSSC	SSC	References
MTL	Yes	Yes	Yes	[Jenei and Montagna, 2002]
IMTL	Yes	Yes	Yes	[Esteva *et al.*, 2002]
SMTL	Yes	Yes	Yes	[Esteva *et al.*, 2002]
ΠMTL	Yes	Yes	No	[Horčík, 2005b; Horčík, 2007]
BL	Yes	Yes	No	[Hájek, 1998a; Cignoli *et al.*, 2000]
SBL	Yes	Yes	No	[Esteva *et al.*, 2000]
Ł	Yes	Yes	No	see [Hájek, 1998a]
Π	Yes	Yes	No	[Hájek, 1998a]
G	Yes	Yes	Yes	see [Hájek, 1998a]
WNM	Yes	Yes	Yes	[Esteva and Godo, 2001]
NM	Yes	Yes	Yes	[Esteva and Godo, 2001]

Table 4. Standard completeness properties for some axiomatic extensions of MTL and their references. For the negative results see [Montagna *et al.*, 2006].

connective, an involutive negation, with other conjunction or implication connectives, or with intermediate truth-constants) will be addressed later in Sections 3.3 and 3.4. All of MTL extensions and most of its expansions defined elsewhere share the property of being complete with respect to a corresponding class of linearly ordered algebras. To encompass all these logics and prove general results common to all of them, Cintula introduced the notion of *core fuzzy logics*[10] in [Cintula, 2006]. Namely, a finitary logic L in a countable language is a *core fuzzy logic* if:

(i) L expands MTL;

(ii) L satisfies the congruence condition: for any φ, ψ, χ, $\varphi \equiv \psi \vdash_{\mathrm{L}} \chi(\varphi) \equiv \chi(\psi)$;

(iii) L satisfies the following local deduction theorem:
$\Gamma, \varphi \vdash_{\mathrm{L}} \psi$ iff there a is natural number n such that $\Gamma \vdash_{\mathrm{L}} \varphi \& \overset{n}{\ldots} \& \varphi \to \psi$.

Each core fuzzy logic L has a corresponding notion of L-algebra (defined as usual) and a corresponding class $\mathbb{L}$ of L-algebras, and enjoys many interesting properties. Among them we can highlight the facts that L is algebraizable in the sense of Blok and Pigozzi [1989] and $\mathbb{L}$ is its equivalent algebraic semantics, that $\mathbb{L}$ is indeed a variety, and that every L-algebra is representable as a subdirect product of L-chains, and hence L is (strongly) complete with respect to the class of L-chains.

Predicate fuzzy logics Predicate logic versions of the propositional t-norm based logics described above have also been defined and studied in the literature. Following [Hájek and Cintula, 2007] we provide below a general definition of the predicate logic L∀ for any core fuzzy logic L.

[10] Actually, Cintula also defines the class of Δ-*core fuzzy logics* to capture all expansions having the Δ connective (see Section 3.4), since they have slightly different properties.

As usual, the propositional language of L is enlarged with a set of predicates $Pred$, a set of object variables Var and a set of object constants $Const$, together with the two classical quantifiers $\forall$ and $\exists$. The notion of formula trivially generalizes taking into account that now, if φ is a formula and x is an object variable, then $(\forall x)\varphi$ and $(\exists x)\varphi$ are formulas as well.

In first-order fuzzy logics it is usual to restrict the semantics to L-chains only. For each L-chain $\mathcal{A}$ an L-interpretation for a predicate language $\mathcal{PL} = (Pred, Const)$ of L$\forall$ is a structure

$$\mathbf{M} = (M, (r_P)_{P \in Pred}, (m_c)_{c \in Const})$$

where $M \neq \emptyset$, $r_P : M^{ar(P)} \to A$ and $m_c \in M$ for each $P \in Pred$ and $c \in Const$. For each evaluation of variables $v : Var \to M$, the truth-value $\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v}$ of a formula (where $v(x) \in M$ for each variable x) is defined inductively from

$$\|P(x, \cdots, c, \cdots)\|^{\mathcal{A}}_{\mathbf{M},v} = r_P(v(x), \cdots, m_c \cdots),$$

taking into account that the value commutes with connectives, and defining

$$\|(\forall x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \inf\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables, except } x\}$$
$$\|(\exists x)\varphi\|^{\mathcal{A}}_{\mathbf{M},v} = \sup\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v'} \mid v(y) = v'(y) \text{ for all variables, except } x\}$$

if the infimum and supremum exist in $\mathcal{A}$, otherwise the truth-value(s) remain undefined. An structure $\mathbf{M}$ is called $\mathcal{A}$-safe if all infs and sups needed for definition of the truth-value of any formula exist in $\mathcal{A}$. Then, the truth-value of a formula φ in a safe $\mathcal{A}$-structure $\mathbf{M}$ is just

$$\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = \inf\{\|\varphi\|^{\mathcal{A}}_{\mathbf{M},v} \mid v : Var \to M\}.$$

When $\|\varphi\|^{\mathcal{A}}_{\mathbf{M}} = 1$ for a $\mathcal{A}$-safe structure $\mathbf{M}$, the pair $(\mathbf{M}, \mathcal{A})$ is said to be a model for φ, written $(\mathbf{M}, \mathcal{A}) \models \varphi$.

The axioms for L$\forall$ are the axioms resulting from those of L by substitution of propositional variables with formulas of $\mathcal{PL}$ plus the following axioms on quantifiers (the same used in [Hájek, 1998a] when defining BL$\forall$):

($\forall$1) $(\forall x)\varphi(x) \to \varphi(t)$ (t substitutable for x in $\varphi(x)$)
($\exists$1) $\varphi(t) \to (\exists x)\varphi(x)$ (t substitutable for x in $\varphi(x)$)
($\forall$2) $(\forall x)(\nu \to \varphi) \to (\nu \to (\forall x)\varphi)$ (x not free in ν)
($\exists$2) $(\forall x)(\varphi \to \nu) \to ((\exists x)\varphi \to \nu)$ (x not free in ν)
($\forall$3) $(\forall x)(\varphi \vee \nu) \to ((\forall x)\varphi \vee \nu)$ (x not free in ν)

Rules of inference of MTL$\forall$ are modus ponens and generalization: from φ infer $(\forall x)\varphi$.

A completeness theorem for first-order BL was proven in [Hájek, 1998a] and the completeness theorems of other predicate fuzzy logics defined in the literature have been proven in the corresponding papers where the propositional logics were introduced. The following general formulation of completeness for predicate core and Δ-core fuzzy logics is from the paper [Hájek and Cintula, 2006]: for any be a (Δ-)core fuzzy logic L over a predicate language $\mathcal{PL}$, it holds that

$$T \vdash_{L\forall} \varphi \text{ iff } (\mathbf{M}, \mathcal{A}) \models \varphi \text{ for each model } (\mathbf{M}, \mathcal{A}) \text{ of } T,$$

for any set of sentences T and formula φ of the predicate language $\mathcal{PL}$.

For some MTL axiomatic extensions L there are postive and negative results of standard completeness of the corresponding predicate logic L∀. For instance, for L being either Gödel, Nilpotent Minimum, MTL, SMTL or IMTL logics, the corresponding predicate logics G∀, NM∀, MTL∀, SMTL∀ and IMTL∀ have been proved to be standard complete for deductions from arbitrary theories (see [Hájek, 1998a; Esteva and Godo, 2001; Montagna and Ono, 2002]). However, the predicate logics Ł∀, Π∀, BL∀, SBL∀ and ΠMTL∀ are not standard complete [Hájek, 1998a; Montagna *et al.*, 2006; Horčík, 2007].

For more details on predicate fuzzy logics, including complexity results and model theory, the interested reader is referred to [Hájek and Cintula, 2006] and to the excellent survey [Hájek and Cintula, 2007].

3.2 Proof theory for t-norm based fuzzy logics

From a proof-theoretic point of view, it is well known that Hilbert-style calculi are not a suitable basis for efficient proof search (by humans or computers). For the latter task one has to develop proof methods that are "analytic"; i.e., the proof search proceeds by step-wise decomposition of the formula to be proved. Sequent calculi, together with natural deduction systems, tableaux or resolution methods, yield suitable formalisms to deal with the above task. In this section we survey some analytic calculi that have been recently proposed for MTL (e.g. see [Gabbay *et al.*, 2004] for a survey) and some of its extensions using *hypersequents*, a natural generalization of Gentzen's sequents introduced by Avron [1991].

Cut-free sequent calculi provide suitable analytic proof methods. Sequents are well-known structures of the form

$$\varphi_1, \ldots, \varphi_n \vdash \psi_1, \ldots, \psi_m$$

which can be intuitively understoof as "φ_1 and ... and φ_n implies ψ_1 or ... ψ_m". Sequent calculi have been defined for many logics, however they have problems with fuzzy logics, namely to cope with the linear ordering of truth-values in [0, 1]. To overcome with this problem when devising a sequent calculus for Gödel logic, Avron [1991] introduced a natural generalization of sequents called *hypersequents*. A hypersequent is an expression of the form

$$\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$$

where for all $i = 1, \ldots n$, $\Gamma_i \vdash \Delta_i$ is an ordinary sequent. $\Gamma_i \vdash \Delta_i$ is called a *component* of the hypersequent. The intended interpretation of the symbol "|" is disjunctive, so the above hypersequent can be read as stating that one of the ordinary sequents $\Gamma_1 \vdash \Delta_1$ holds.

Like in ordinary sequent calculi, in a hypersequent calculus there are axioms and rules which are divided into two groups: *logical* and *structural rules*. The logical rules are essentially the same as those in sequent calculi, the only difference is the presence of dummy contexts G and H, called *side hypersequents* which are used as variables for (possibly empty) hypersequents. The structural rules are divided into *internal* and *external rules*. The internal rules deal with formulas within components. If they are present, they are the usual weakening and contraction rules. The external rules manipulate whole components within a hypersequent. These are external weakening (EW) and external contraction (EC):

$$(EW)\quad \frac{H}{H \mid \Gamma \vdash A} \qquad\qquad (EC)\quad \frac{H \mid \Gamma \vdash A \mid \Gamma \vdash A}{H \mid \Gamma \vdash A}$$

In hypersequent calculi it is possible to define further structural rules which simultaneously act on several components of one or more hypersequents. It is this type of rule which increases the expressive power of hypersequent calculi with respect to ordinary sequent calculi. An example of such a kind of rule is Avron's communication rule:

$$(com)\quad \frac{H \mid \Pi_1, \Gamma_1 \vdash A \qquad G \mid \Pi_2, \Gamma_2 \vdash B}{H \mid G \mid \Pi_1, \Pi_2 \vdash A \mid \Gamma_1, \Gamma_2 \vdash B}$$

Indeed, by adding (*com*) to the hypersequent calculus for intuitionistic logic one gets a cut-free calculus for Gödel logic [Avron, 1991]. Following this approach, a proof theory for MTL has been investigated in [Baaz *et al.*, 2004], where an analytic hypersequent calculus has been introduced. This calculus, called **HMTL**, has been defined by adding the (*com*) rule to the hypersequent calculus for intuitionistic logic without contraction $\mathrm{IPC}^* \setminus c$ (or equivalently Monidal logic ML or Full Lambek with exchange and weaking FL_{ew}). More precisely, axioms and rules of **HMTL** are those of Table 3.2.

In fact, in [Baaz *et al.*, 2004] it is shown that **HMTL** is sound and complete for MTL and that **HMTL** admits cut-elimination. Cut-free hypersequent calculi have also been obtained by Ciabattoni *et al.* [Ciabattoni *et al.*, 2002] for IMTL and SMTL.

Elegant hypersequent calculi have also been defined by Metcalfe, Olivetti and Gabbay for Łukasiewicz logic [Metcalfe *et al.*, 2005] and Product logic [Metcalfe *et al.*, 2004a], but using different rules for connectives. A generalization of both hypersequents and sequents-of-relations, called *relational hypersequents* is introduced in [Ciabattoni *et al.*, 2005]. Within this framework, they are able to provide logical rules for Łukasiewicz, Gödel and Product logics that are uniform i.e., identical for all three logics and then purely syntactic calculi with very simple initial relational hypersequents are obtained by introducing structural rules reflecting the characteristic properties of the particular logic. Such a framework is also used by Bova and Montagna in a very recent paper [Bova and Montagna, 2007] to provide a proof system for BL, a problem which has been open for a long time.

Finally, let us comment that other proof search oriented calculi include a tableaux calculus for Łukasiewicz logic [Hähnle, 1994], decomposition proof systems for

$$A \vdash A \qquad \overline{0} \vdash A \qquad\qquad (cut)\ \frac{H \mid \Gamma \vdash A \quad G \mid A, \Gamma' \vdash C}{H \mid G \mid \Gamma, \Gamma' \vdash C}$$

Internal and External Structural Rules :

$$(iw)\ \frac{H \mid \Gamma \vdash C}{H \mid \Gamma, B \vdash C} \qquad\qquad (EC), \quad (EW), \quad (com)$$

Multiplicative fragment :

$$(\&, l)\ \frac{H \mid \Gamma, A, B \vdash C}{H \mid \Gamma, A\&B \vdash C} \qquad\qquad (\&, r)\ \frac{H \mid \Gamma \vdash A \quad G \mid \Gamma' \vdash B}{H \mid G \mid \Gamma, \Gamma' \vdash A\&B}$$

$$(\rightarrow, l)\ \frac{G \mid \Gamma \vdash A \quad H \mid \Gamma', B \vdash C}{G \mid H \mid \Gamma, \Gamma', A \rightarrow B \vdash C} \qquad\qquad (\rightarrow, r)\ \frac{H \mid \Gamma, A \vdash B}{H \mid \Gamma \vdash A \rightarrow B}$$

Additive fragment :

$$(\wedge, l_i)_{i=1,2}\ \frac{H \mid \Gamma, A_i \vdash C}{H \mid \Gamma, A_1 \wedge A_2 \vdash C} \qquad\qquad (\wedge, r)\ \frac{G \mid \Gamma \vdash A \quad H \mid \Gamma \vdash B}{G \mid H \mid \Gamma \vdash A \wedge B}$$

$$(\vee, l)\ \frac{H \mid \Gamma, A \vdash C \quad G \mid \Gamma, B \vdash C}{H \mid G \mid \Gamma, A \vee B \vdash C} \qquad\qquad (\vee, r_i)_{i=1,2}\ \frac{H \mid \Gamma \vdash A_i}{H \mid \Gamma \vdash A_1 \vee A_2}$$

Table 5. Axioms and rules of the hypersequent calculus **HMTL**.

Gödel logic [Avron and Konikowska, 2001], and goal-directed systems for Łukasiewicz and Gödel logics [Metcalfe *et al.*, 2004b; Metcalfe *et al.*, 2003]. Also, a general approach is presented in [Aguzzoli, 2004] where a calculus for any logic based on a continuous t-norm is obtained via reductions to suitable finite-valued logics, but not very suitable for proof search due to a very high branching factor of the generated proof trees. For an exhaustive survey on proof theory for fuzzy logics, the interested reader is referred to the forthcoming monograph [Metcalfe *et al.*, to appear].

3.3 Dealing with partial truth: Pavelka-style logics with truth-constants

The notion of deduction in t-norm based fuzzy logics is basically crisp, in the sense it preserves the distinguished value 1. Indeed, a deduction

$$T \vdash_L \psi$$

in a complete logic L actually means that ψ necessarily takes the truth-value 1 in all evaluations that make all the formulas in T 1-true. However, from another point of view, more in line with Zadeh's approximate reasoning, one can also consider t-norm based fuzzy logics as logics of *comparative truth*. In fact, the residuum $\Rightarrow$ of a (left-continuous) t-norm $*$ satisfies the condition $x \Rightarrow y = 1$ if, and only if, $x \leq y$ for all $x, y \in [0, 1]$. This means that a formula $\varphi \rightarrow \psi$ is a logical consequence of a theory T, i.e. if $T \vdash_L \varphi \rightarrow \psi$, if the truth degree of φ is at most as high as the truth degree of ψ in any interpretation which is a model of the theory T. Therefore, implications indeed implicitly capture a notion of comparative truth. This is fine, but in some situations one might be also interested to explicitly represent and reason with *partial degrees* of truth. For instance, in any logic L_* of a left-continuous t-norm $*$, any truth-evaluation e satisfying $e(\varphi \to \psi) \geq \alpha$ and $e(\varphi) \geq \beta$, necessarily satisfies $e(\psi) \geq \alpha * \beta$ as well. Therefore, having this kind of graded (semantical) form of modus ponens inside the logic (as many applied fuzzy systems do [Dubois *et al.*, 1991c]) may seem useful when trying to devise mechanisms for allowing deductions from partially true propositions.

One convenient and elegant way to allow for an explicit treatment of degrees of truth is by introducing truth-constants into the language. In fact, if one introduces in the language new constant symbols $\overline{\alpha}$ for suitable values $\alpha \in [0,1]$ and stipulates that

$$e(\overline{\alpha}) = \alpha$$

for all truth-evalutations, then a formula of the kind $\overline{\alpha} \to \varphi$ becomes 1-true under any evaluation e whenever $\alpha \leq e(\varphi)$.

This approach actually goes back to Pavelka [1979] who built a propositional many-valued logical system PL which turned out to be equivalent to the expansion of Łukasiewicz Logic by adding into the language a truth-constant $\overline{r}$ for each *real* $r \in [0,1]$, together with a number of additional axioms. The semantics is the same as Łukasiewicz logic, just expanding the evaluations e of propositional variables in $[0,1]$ to truth-constants by requiring $e(\overline{r}) = r$ for all $r \in [0,1]$. Although the resulting logic is not strong standard complete (SSC in the sense defined in Section 3.1) with respect to that intended semantics, Pavelka proved that his logic is complete in a different sense. Namely, he defined the *truth degree* of a formula φ in a theory T as

$$\|\varphi\|_T = \inf\{e(\varphi) \mid e \text{ is a PL-evaluation model of } T\},$$

and the *provability degree* of φ in T as

$$|\varphi|_T = \sup\{r \in [0,1] \mid T \vdash_{\mathrm{PL}} \overline{r} \to \varphi\}$$

and proved that these two degrees coincide. This kind of completeness is usually known as Pavelka-style completeness, and strongly relies on the continuity of Łukasiewicz truth functions. Note that $\|\varphi\|_T = 1$ is not equivalent to $T \vdash_{PL} \varphi$, but only to $T \vdash_{PL} \overline{r} \to \varphi$ for all $r < 1$. Novák extended Pavelka's approach to Łukasiewicz first order logic [Novák, 1990a; Novák, 1990b].

Later, Hájek [1998a] showed that Pavelka's logic PL could be significantly simplified while keeping the completeness results. Indeed he showed that it is enough to extend the language only by a countable number of truth-constants, one for each *rational* in $[0,1]$, and by adding to the logic the two following additional axiom schemata, called *book-keeping axioms*:

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r *_L s}$$
$$\overline{r} \to \overline{s} \leftrightarrow \overline{r \Rightarrow_L s}$$

for all $r \in [0,1] \cap \mathbb{Q}$, where $*_L$ and $\Rightarrow_L$ are the Łukasiewicz t-norm and its residuum respectively. He called this new system Rational Pavelka Logic, RPL for short. Moreover, he proved that RPL is strong standard complete for finite theories (FSSC in the usual sense). He also defined the logic RPL∀, the extension of RPL to first order, and showed that RPL∀ enjoys the same Pavelka-style completeness.

Similar *rational* expansions for other t-norm based fuzzy logics can be analogously defined, but unfortunately Pavelka-style completeness cannot be obtained since Łukasiewicz Logic is the only fuzzy logic whose truth-functions (conjunction and implication) are continuous functions.

However, several expansions with truth-constants of fuzzy logics different from Łukasiewicz have been studied, mainly related to the other two outstanding continuous t-norm based logics, namely Gödel and Product logic. We may cite [Hájek, 1998a] where an expansion of G_Δ (the expansion of Gödel Logic G with Baaz's projection connective Δ) with a finite number of rational truth-constants, [Esteva *et al.*, 2000] where the authors define logical systems obtained by adding (rational) truth-constants to $G_\sim$ (Gödel Logic with an involutive negation) and to Π (Product Logic) and $\Pi_\sim$ (Product Logic with an involutive negation). In the case of the rational expansions of Π and $\Pi_\sim$ an infinitary inference rule (from $\{\varphi \to \overline{r} : r \in \mathbb{Q} \cap (0,1]\}$ infer $\varphi \to \overline{0}$) is introduced in order to get Pavelka-style completeness. Rational truth-constants have been also considered in some stronger logics (see Section 3.4) like in the logic $Ł\Pi\frac{1}{2}$ [Esteva *et al.*, 2001], a logic that combines the connectives from both Łukasiewicz and Product logics plus the truth-constant $\overline{1/2}$, and in the logic PŁ [Horčík and Cintula, 2004], a logic which combines Łukasiewicz Logic connectives plus the Product Logic conjunction (but not implication), as well as in some closely related logics.

Following this line, Cintula gives in [Cintula, 2005d] a definition of what he calls *Pavelka-style extension* of a particular fuzzy logic. He considers the Pavelka-style extensions of the most popular fuzzy logics, and for each one of them he defines an axiomatic system with infinitary rules (to overcome discontinuities like in the case of Π explained above) which is proved to be Pavelka-style complete. Moreover he also considers the first order versions of these extensions and provides necessary conditions for them to satisfy Pavelka-style completeness.

Recently, a systematic approach based on traditional algebraic semantics has been considered to study completeness results (in the usual sense) for expansions of t-norm based logics with truth-constants. Indeed, as already mentioned, only the case of Łukasiewicz logic was known according to [Hájek, 1998a]. Using this algebraic approach the expansions of the other two distinguished fuzzy logics, Gödel and Product logics, with countable sets of truth-constants have been reported in [Esteva *et al.*, 2006] and in [Savický *et al.*, 2006] respectively. Following [Esteva *et al.*, 2007a; Esteva *et al.*, 2007b], we briefly describe in the rest of this section the main ideas and results of this general algebraic approach.

If L_* is a logic of (left-continuous) t-norm $*$, and $\mathcal{C} = \langle C, *, \Rightarrow, \min, \max, 0, 1\rangle$ is a countable subalgebra of the standard L_*-algebra $[0,1]_*$, then the logic $L_*(\mathcal{C})$ is defined as follows:

(i) the language of $\mathrm{L}_*(\mathcal{C})$ is the one of L_* expanded with a new propositional variable $\overline{r}$ for each $r \in C$

(ii) the axioms of $\mathrm{L}_*(\mathcal{C})$ are those of L_* plus the bookeeping axioms

$$\overline{r} \& \overline{s} \leftrightarrow \overline{r * s}$$
$$\overline{r} \rightarrow \overline{s} \leftrightarrow \overline{r \Rightarrow_* s}$$

for each $r, s \in C$. The algebraic counterpart of the $\mathrm{L}_*(\mathcal{C})$ logic consists of the class of $\mathrm{L}_*(\mathcal{C})$-algebras, defined as structures

$$\mathcal{A} = \langle A, \&, \rightarrow, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$$

such that:

1. $\langle A, \&, \rightarrow, \wedge, \vee, \overline{0}^{\mathcal{A}}, \overline{1}^{\mathcal{A}}\rangle$ is an L_*-algebra, and

2. for every $r, s \in C$ the following identities hold:

$\overline{r}^{\mathcal{A}} \& \overline{s}^{\mathcal{A}} = \overline{r * s}^{\mathcal{A}}$

$\overline{r}^{\mathcal{A}} \rightarrow \overline{s}^{\mathcal{A}} = \overline{r \Rightarrow s}^{\mathcal{A}}$.

A $\mathrm{L}_*(\mathcal{C})$-chain defined over the real unit interval $[0, 1]$ is called standard. Among the standard chains, there is one which reflects the intended semantics, the so-called *canonical* $\mathrm{L}_*(\mathcal{C})$-chain

$$[0, 1]_{\mathrm{L}_*(\mathcal{C})} = \langle [0, 1], *, \Rightarrow, \min, \max, \{r : r \in C\}\rangle,$$

i. e. the one where the truth-constants are interpreted by themselves. Note that, for a logic $\mathrm{L}_*(\mathcal{C})$ there can exist multiple standard chains, as soon as there exist different ways of interpreting the truth-constants on $[0, 1]$ respecting the book-keeping axioms. For instance, for the case of Gödel logic, when $* = \min$ and $C = [0, 1] \cap \mathbb{Q}$, the algebra $\mathcal{A} = \langle [0, 1], \&, \rightarrow, \wedge, \vee, \{\overline{r}^{\mathcal{A}} : r \in C\}\rangle$ where

$$\overline{r}^{\mathcal{A}} = \begin{cases} 1, & \text{if } r \geq \alpha \\ 0, & \text{otherwise} \end{cases}$$

is a standard $\mathrm{L}_*(\mathcal{C})$ algebra for any $\alpha > 0$.

Since the additional symbols added to the language are 0-ary, $\mathrm{L}_*(\mathcal{C})$ is also an algebraizable logic and its equivalent algebraic semantics is the variety of $\mathrm{L}_*(\mathcal{C})$-algebras, This, together with the fact that $\mathrm{L}_*(\mathcal{C})$-algebras are representable as a subdirect product of $\mathrm{L}_*(\mathcal{C})$-chains, leads to the following general completeness result of $\mathrm{L}_*(\mathcal{C})$ with respect to the class of $\mathrm{L}_*(\mathcal{C})$-chains: for any set $\Gamma \cup \{\varphi\}$ of $\mathrm{L}_*(\mathcal{C})$ formulas,

$$\Gamma \vdash_{\mathrm{L}_*(\mathcal{C})} \varphi \text{ if, and only if,}$$
$$\text{for each } \mathrm{L}_*(\mathcal{C})\text{-chain } \mathcal{A},\ e(\varphi) = \overline{1}^{\mathcal{A}} \text{ for all } \mathcal{A}\text{-evaluation } e \text{ model of } \Gamma.$$

The issue of studying when a logic $\mathrm{L}_*(\mathcal{C})$ is also complete with respect to the class of standard $\mathrm{L}_*(\mathcal{C})$-chains (called *standard completeness*) or with respect to the canonical $\mathrm{L}_*(\mathcal{C})$-chain (called *canonical completeness*) has been addressed in the literature for some logics L_*. Hájek already proved in [Hájek, 1998a] the canonical completeness of the expansion of Łukasiewicz logic with rational truth-constants for finite theories. More recently, the expansions of Gödel (and of some t-norm based logic related to the nilpotent minimum t-norm) and of Product logic with countable sets of truth-constants have been proved to be canonical complete for theorems in [Esteva *et al.*, 2006] and in [Savický *et al.*, 2006] respectively. A rather exhaustive description of completeness results for the logics $L_*(\mathcal{C})$ can be found in [Esteva *et al.*, 2007a; Esteva *et al.*, 2007b] and about complexity in [Hájek, 2006b].

One negative result for many of these logics (with the exception of Łukasiewicz logic) is that they are not canonical complete for deductions from non-empty theories. However, such canonical completeness can be recovered in some cases (see e.g. [Esteva *et al.*, 2007a]) when the one considers the fragment of formulas of the kind

$$\overline{r} \to \varphi,$$

where φ is a formula *without* additional truth-constants. Actually, this kind of formulas, under the notation as a pair $(\overline{r}, \varphi)$, have been extensively considered in other frameworks for reasoning with partial degrees of truth, like in Novák's formalism of fuzzy logic with evaluated syntax based on Łukasiewicz Logic (see e.g. [Novák *et al.*, 1999]), in Gerla's framework of abstract fuzzy logics [Gerla, 2001] or in fuzzy logic programming (see e.g. [Vojtáš, 2001]).

3.4 More complex residuated logics

Other interesting kinds of fuzzy logics are those expansions obtained by joining the logics of different t-norms or by adding specific t-norm related connectives to certain logics. In this section we describe some of them, in particular expansions with Baaz's Δ connective, expansions with an involutive negation, and the logics ŁΠ, ŁΠ$\frac{1}{2}$ and PŁ combining connectives from Łukasiewicz and Product logics.

Logics with Δ Here below we describe L_Δ, the expansion of an axiomatic extension L of MTL with Baaz's Δ connective. The intended semantics for the Δ unary connective, introduced in [Baaz, 1996], is that $\Delta\varphi$ captures the *crisp* part of a fuzzy proposition φ (similar to the core of a fuzzy set). This is done by extending the truth-evaluations e on formulas with the additional requirement:

$$e(\Delta\varphi) = \begin{cases} 1, & \text{if } e(\varphi) = 1 \\ 0, & \text{otherwise} \end{cases}$$

Therefore, for any formula φ, $\Delta\varphi$ behaves as a classical (two-valued) formula. At the syntactical level, axioms and rules of L_Δ are those of L plus the following additional set of axioms:

(Δ1) $\Delta\varphi \vee \neg\Delta\varphi$,
(Δ2) $\Delta(\varphi \vee \psi) \to (\Delta\varphi \vee \Delta\psi)$,

($\Delta 3$) $\Delta\varphi \to \varphi$,
($\Delta 4$) $\Delta\varphi \to \Delta\Delta\varphi$,
($\Delta 5$) $\Delta(\varphi \to \psi) \to (\Delta\varphi \to \Delta\psi)$.

and the *Necessitation* rule for Δ: from φ derive $\Delta\varphi$[11]. The notion of proof in L_Δ is the usual one.

Notice that in general the local deduction theorem for MTL and its extensions L fails for the logics L_Δ. Indeed, $\varphi \vdash_{L_\Delta} \Delta\varphi$, but for each n it may be the case $\not\vdash_{L_\Delta} \varphi^n \to \Delta\varphi$. Take, for example, a strict continuous t-norm $*$, hence isomorphic to the product. Then for all $0 < x < 1$, $x^n > 0$. However, every logic L_Δ satisfies another form of deduction theorem, known as [Hájek, 1998a]:

$$\Gamma \cup \{\varphi\} \vdash \psi \text{ iff } \Gamma \vdash \Delta\varphi \to \psi.$$

The algebraic semantics of L_Δ is given by L_Δ-algebras, i.e. L-algebras expanded with a unary operator δ, satisfying the following conditions for all x, y:

($\delta 1$) $\delta(x) \vee \neg\delta(x) = 1$
($\delta 2$) $\delta(x \vee y) \leq (\delta(x) \vee \delta(y))$
($\delta 3$) $\delta(x) \leq x$
($\delta 4$) $\delta(x) \leq \delta(\delta(x))$
($\delta 5$) $\delta(x \Rightarrow y) \leq (\delta(x) \Rightarrow \delta(y))$
($\delta 6$) $\delta(1) = 1$

Notice that in any linearly ordered L_Δ-algebra $\delta(x) = 1$ if $x = 1$, and $\delta(x) = 0$ otherwise. The notions of evaluation, model and tautology are obviously adapted from the above case. Then the following is the general completeness results for L_Δ logics [Hájek, 1998a; Esteva and Godo, 2001]: for each set of L_Δ-formulas Γ and each L_Δ-formula φ the following are equivalent:

1. $\Gamma \vdash_{L_\Delta} \varphi$,
2. for each L_Δ-chain $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$,
3. for each L_Δ-algebra $\mathcal{A}$ and each $\mathcal{A}$-model e of Γ, $e(\varphi) = 1$.

Standard completeness for L_Δ logics have been proved in the literature whenever the logic L has been shown to be standard complete, like e.g. it is the case for all the logics listed in Table 3.

Logics with an involutive negation Basic strict fuzzy logic SBL was introduced in [Esteva *et al.*, 2000] as the axiomatic extension of BL by the single axiom (PC) $\neg(\varphi \wedge \neg\varphi)$, and SMTL in an analogous way as extension of MTL [Hájek, 2002]. Note that Gödel logic G and Product logic Π are extensions of SBL (and

[11]Note that this rule holds because syntactic derivation only preserves the maximal truth value, contrary to Zadeh's entailment principle

thus of SMTL as well). In any extension of SMTL, the presence of the axiom (PC) forces the negation $\neg$ to be *strict*, i.e. any evaluation e model of (PC) one has

$$e(\neg\varphi) = \begin{cases} 1, & \text{if } e(\varphi) = 0 \\ 0, & \text{otherwise} \end{cases}$$

This kind of "two-valued" negation is also known in the literature as *Gödel negation*. In the logics with Gödel negation, one cannot define a meaningful (strong) disjunction $\underline{\vee}$ by duality from the conjunction &, i.e. to define $\varphi \underline{\vee} \psi$ as $\neg(\neg\varphi \& \neg\psi)$, as well as a corresponding S-implication $\varphi \rightarrow_S \psi$ as $\neg\varphi\underline{\vee}\psi$. It seems therefore natural to introduce in these logics an involutive negation $\sim$ as an extra connective. To do so, and noticing that a suitable combination of both kinds of negations behaves like the Δ connective, i.e.

$$e(\neg \sim \varphi) = \left\{ \begin{array}{ll} 1, & \text{if } e(\varphi) = 1 \\ 0, & \text{otherwise} \end{array} \right\} = e(\Delta\varphi),$$

the logic $\mathrm{SBL}_\sim$, where the Δ connective is in fact a derivable connective ($\Delta\varphi$ is $\neg \sim \varphi$) was introduced in [Esteva *et al.*, 2000] as an axiomatic extension of the logic SBL_Δ by the following two axioms:

(∼1) $\sim\sim \varphi \equiv \varphi$
(∼2) $\Delta(\varphi \rightarrow \psi) \rightarrow (\sim \psi \rightarrow \sim \varphi)$

Axiom (∼1) forces the negation $\sim$ to be involutive and axiom (∼2) to be order reversing. Similar extensions have been defined for Gödel logic ($G_\sim$), Product logic ($\Pi_\sim$) and SMTL ($\mathrm{SMTL}_\sim$).

Standard completeness for these logics was proved but, interestingly enough, these two axioms are not enough to show completeness of SBL (Π, SMTL, resp.) with respect to SBL-algebras (Π-algebras, SMTL-algebras resp.) on $[0, 1]$ expanded only by the standard negation $n(x) = 1 - x$, one needs to consider all possible involutive negations in $[0, 1]$, even though all of them are isomorphic. This was noticed in [Esteva *et al.*, 2000], and has been deeply studied by Cintula *et al.* in [2006] where the expansions of SBL with an involutive negation are systematically investigated. The addition of an involutive negation in the more general framework of MTL has also been addressed by Flaminio and Marchioni in [2006].

The logic $\text{Ł}\Pi\frac{1}{2}$. $\text{Ł}\Pi\frac{1}{2}$ is a logic "putting Łukasiewicz and Product logics together", introduced an studied in [Esteva and Godo, 1999; Montagna, 2000; Esteva *et al.*, 2001] and further developed by Cintula in [2001a; 2001c; 2003; 2005b]. The language of the ŁΠ logic is built in the usual way from a countable set of propositional variables, three binary connectives $\rightarrow_L$ (Łukasiewicz implication), $\odot$ (Product conjunction) and $\rightarrow_\Pi$ (Product implication), and the truth constant $\bar{0}$. A truth-evaluation is a mapping e that assigns to every propositional variable a real number from the unit interval [0, 1] and extends to all formulas as follows:

$$e(\bar{0}) = 0, \qquad e(\varphi \to_L \psi) = \min(1 - e(\varphi) + e(\psi), 1),$$

$$e(\varphi \odot \psi) = e(\varphi) \cdot e(\psi), \qquad e(\varphi \to_\Pi \psi) = \begin{cases} 1, & \text{if } e(\varphi) \le e(\psi) \\ e(\psi)/e(\varphi), & \text{otherwise} \end{cases}.$$

The truth constant $\bar{1}$ is defined as $\varphi \to_L \varphi$. In this way we have $e(\bar{1}) = 1$ for any truth-evaluation e. Moreover, many other connectives can be defined from those introduced above:

$$\begin{array}{rcl@{\qquad}rcl}
\neg_L \varphi & \text{is} & \varphi \to_L \bar{0}, & \neg_\Pi \varphi & \text{is} & \varphi \to_\Pi \bar{0}, \\
\varphi \wedge \psi & \text{is} & \varphi \& (\varphi \to_L \psi), & \varphi \vee \psi & \text{is} & \neg_L(\neg_L \varphi \wedge \neg_L \psi), \\
\varphi \oplus \psi & \text{is} & \neg_L \varphi \to_L \psi, & \varphi \& \psi & \text{is} & \neg_L(\neg_L \varphi \oplus \neg_L \psi), \\
\varphi \ominus \psi & \text{is} & \varphi \& \neg_L \psi, & \varphi \equiv \psi & \text{is} & (\varphi \to_L \psi) \& (\psi \to_L \varphi), \\
\Delta \varphi & \text{is} & \neg_\Pi \neg_L \varphi, & \nabla \varphi & \text{is} & \neg_\Pi \neg_\Pi \varphi,
\end{array}$$

with the following interpretations:

$$\begin{array}{rl@{\qquad}rl}
e(\neg_L \varphi) = & 1 - e(\varphi), & e(\neg_\Pi \varphi) = & \begin{cases} 1, & \text{if } e(\varphi) = 0 \\ 0, & \text{otherwise} \end{cases}, \\
e(\varphi \wedge \psi) = & \min(e(\varphi), e(\psi)), & e(\varphi \vee \psi) = & \max(e(\varphi), e(\psi)), \\
e(\varphi \oplus \psi) = & \min(1, e(\varphi) + e(\psi)), & e(\varphi \& \psi) = & \max(0, e(\varphi) + e(\psi) - 1), \\
e(\varphi \ominus \psi) = & \max(0, e(\varphi) - e(\psi)), & e(\varphi \equiv \psi) = & 1 - |e(\varphi) - e(\psi)|, \\
e(\Delta \varphi) = & \begin{cases} 1, & \text{if } e(\varphi) = 1 \\ 0, & \text{otherwise} \end{cases}, & e(\nabla \varphi) = & \begin{cases} 1, & \text{if } e(\varphi) > 0 \\ 0, & \text{otherwise} \end{cases}.
\end{array}$$

The logical system ŁΠ is the logic whose axioms are[12]:

(Ł) Axioms of Łukasiewicz logic (for $\to_Ł, \&, \bar{0}$);
(Π) Axioms for product logic (for $\to_\Pi, \odot, \bar{0}$);
(¬) $\neg_\Pi \varphi \to_Ł \neg_Ł \varphi$
(Δ) $\Delta(\varphi \to_Ł \psi) \equiv_Ł \Delta(\varphi \to_\Pi \psi)$
(ŁΠ5) $\varphi \odot (\psi \ominus \chi) \equiv_Ł (\varphi \odot \psi) \ominus (\varphi \odot \chi)$

and whose inference rules are modus ponens (for $\to_L$) and necessitation for Δ: from φ infer $\Delta\varphi$.

The logic Ł$\Pi\frac{1}{2}$ is then obtained from ŁΠ by adding a truth constant $\overline{\frac{1}{2}}$ together with the axiom:

(Ł$\Pi\frac{1}{2}$) $\overline{\frac{1}{2}} \equiv \neg_L \overline{\frac{1}{2}}$

Obviously, a truth-evaluation e for ŁΠ is easily extended to an evaluation for Ł$\Pi\frac{1}{2}$ by further requiring $e(\overline{\frac{1}{2}}) = \frac{1}{2}$.

The notion of proof in Ł$\Pi\frac{1}{2}$ is as usual and it is indeed strongly complete for finite theories with respect to the given semantics. That is, if T is a finite set of formulas, then $T \vdash_{Ł\Pi\frac{1}{2}} \varphi$ iff $e(\varphi) = 1$ for any Ł$\Pi\frac{1}{2}$-evaluation e model of T.

It is interesting to remark that ŁΠ and Ł$\Pi\frac{1}{2}$ are indeed very powerful logics. Indeed ŁΠ conservatively extends Łukasiewicz, Product and Gödel logics (note that

[12]This definition, proposed in [Cintula, 2003], is actually a simplified version of the original definition of $L\Pi$ given in [Esteva *et al.*, 2001].

Gödel implication $\rightarrow_G$ is also definable by putting $\varphi \rightarrow_G \psi$ as $\Delta(\varphi \rightarrow \psi) \vee \psi$). Moreover, as shown in [Esteva *et al.*, 2001], rational truth constants $\bar{r}$ (for each rational $r \in [0,1]$) are definable in Ł$\Pi\frac{1}{2}$ from the truth constant $\overline{\frac{1}{2}}$ and the connectives. Therefore, in the language of Ł$\Pi\frac{1}{2}$ there is a truth-constant for each rational in $[0,1]$, and due to completeness of Ł$\Pi\frac{1}{2}$, the following book-keeping axioms for rational truth constants are provable:

$$\begin{array}{llllllll} (RL\Pi1) & \neg_L \overline{r} & \equiv & \overline{1-r}, & (RL\Pi2) & \overline{r} \rightarrow_L \overline{s} & \equiv & \overline{\min(1, 1-r+s)}, \\ (RL\Pi3) & \overline{r} \odot \overline{s} & \equiv & \overline{r \cdot s}, & (RL\Pi4) & \overline{r} \rightarrow_\Pi \overline{s} & \equiv & \overline{r \Rightarrow_P s}, \end{array}$$

where $r \Rightarrow_P s = 1$ if $r \leq s$, $r \Rightarrow_P s = s/r$ otherwise. Moreover, Cintula [2003] shows (see also [Marchioni and Montagna, 2006]) that, for each continuous t-norm $*$ that is an ordinal sum of finitely many copies of Łukasiewicz, product and minimum t-norms, L_* (the logic of the t-norm $*$) is interpretable in Ł$\Pi\frac{1}{2}$. Indeed, he defines a syntactical translation of L_*-formulas into Ł$\Pi\frac{1}{2}$-formulas, say $\varphi \mapsto \varphi'$, such that L_* proves φ if and only if Ł$\Pi\frac{1}{2}$ proves φ'. Connections between the logics ŁΠ and $\Pi_\sim$ (the extension of product logic Π with an involutive negation, see above) have been also investigated in [Cintula, 2001c].

The predicate ŁΠ and Ł$\Pi\frac{1}{2}$ logics have been studied in [Cintula, 2001a], showing in particular that they conservatively extend Gödel predicate logic.

To conclude, let us remark that the so-called Ł$\Pi\frac{1}{2}$-algebras, the algebraic counterpart of the logic Ł$\Pi\frac{1}{2}$, are in strong connection with ordered fields. Indeed, Montagna has shown [Montagna, 2000; Montagna, 2001], among other things, that Ł$\Pi\frac{1}{2}$-algebras are substructures of fields extending the field of rational numbers. Morever, as recently shown in [Marchioni and Montagna, 2006; Marchioni and Montagna, to appear], that the theory of real closed fields is faithfully interpretable in Ł$\Pi\frac{1}{2}$. See also [Montagna and Panti, 2001; Montagna, 2005] for further deep algebraic results regarding ŁΠ-algebras.

The logic PŁ. Starting from algebraic investigations on MV-algebras with additional operators by Montagna [2001; 2005], the logic PŁ, for Product-Łukasiewicz, was introduced by Horčík and Cintula in [2004]. Basically, PŁ is an expansion of Łukasiewicz logic by means of the product conjunction, and its language is built up from three binary connectives, & (Łukasiewicz conjunction), $\rightarrow$ (Łukasiewicz implication), $\odot$ (Product conjunction), and the truth constant $\overline{0}$. The axioms of PŁ are those of Łukasiewicz logic, plus the following additional axioms:

(PŁ1) $\varphi \odot (\psi \& (\chi \rightarrow \overline{0})) \leftrightarrow (\varphi \odot \psi) \& ((\varphi \odot \chi) \rightarrow \overline{0})$,

(PŁ2) $\varphi \odot (\chi \odot \psi) \leftrightarrow (\varphi \odot \psi) \odot \chi$,

(PŁ3) $\varphi \rightarrow \varphi \odot \overline{1}$,

(PŁ4) $\varphi \odot \psi \rightarrow \varphi$,

(PŁ5) $\varphi \odot \psi \rightarrow \psi \odot \varphi$

They also consider the logic PŁ′ as the extension of PŁ by the deduction rule:

(ZD) from $\neg(\varphi \odot \varphi)$, derive $\neg\varphi$.

PŁ′ is shown to be standard complete with respect to the standard Łukasiewicz algebra expanded with the product (of reals) operation (see also [Montagna, 2001]), hence w. r. t. the intended semantics, while PŁ is not. In fact, it is the inference rule (ZD) that makes the difference, forcing the interpretation of the product $\odot$ connective to have no zero divisors. At the same time, in contrast to all the other algebraic semantics surveyed so far, the class of algebras associated to the PŁ′ does not form a variety but a quasi-variety.

In [Horčík and Cintula, 2004], the authors also study expansions of these logics by means of Baaz's Δ connective and by rational truth constants, as well as their predicate versions.

A logic which is very related to these systems is Takeuti and Titani's logic [1992]. It is a predicate fuzzy logic based on the Gentzen's system LJ of intuitionistic predicate logic. The connectives used by this logic are just the connectives of the predicate PŁ logic with a subset of rational truth-constants but Takeuti and Titani's logic has two additional deduction rules and 46 axioms and it is sound and complete w.r.t. the standard PŁ$_\Delta$-algebra (cf. [Takeuti and Titani, 1992, Th. 1.4.3]). In [Horčík and Cintula, 2004] it is shown it exactly corresponds to the expansion of predicate PŁ$_\Delta$ logic with truth-constants which are of the form $k/2^n$, for natural numbers k and n.

3.5 Further issues on residuated fuzzy logics

The aim in the preceding subsections has been to survey main advances in the logical formalization of residuated many-valued systems underlying fuzzy logic in narrow sense. This field has had a great development in the last 10-15 years, and many scholars from different disciplines like algebra, logic, computer science or artificial intelligence joined efforts. Hence, our presentation is not exhaustive by far. A lot of aspects and contributions have not been covered by lack of space reasons, although they deserve to be commented. At the risk of being again incomplete, we briefly go through some of them in the rest of this subsection.

A. Other existing expansions and fragments of MTL and related logics

Hoop fuzzy logics: In [Esteva *et al.*, 2003] the positive (falsehood-free) fragments of BL and main extensions (propositional and predicate calculi) are axiomatized and they are related 0-free subreducts of the corresponding algebras, which turn out to be a special class of algebraic structures known as *hoops* (hence the name of hoop fuzzy logic). Similar study is carried for MTL and extensions, introducing the related algebraic structures which are called *semihoops*. Issues of completeness, conservativeness and complexity are also addressed. The class of the so-called *basic hoops*, hoops corresponding to BLH, the hoop variant of BL, have an important role in the algebraic study of linearly ordered chains [Aglianó *et al.*, to appear].

Rational Łukasiewicz logic and DMV-algebras: A peculiar kind of expansion which allows the representation of rational truth-constants is given by the indexRational Łukasiewicz logicRational Łukasiewicz logic RŁ introduced by Gerla [2001b]. RŁ is obtained by extending Łukasiewicz logic by the unary connectives δ_n, for each $n \in \mathbb{N}$, plus the following axioms:

$$\text{(D1) } \delta_n\varphi \oplus \overset{n}{\ldots} \oplus \delta_n\varphi \leftrightarrow \varphi \qquad \text{(D2) } \neg\delta_n\varphi \oplus \neg(\delta_n\varphi \oplus \overset{n}{\ldots} \oplus \delta_n\varphi).$$

where $\oplus$ is Łukasiewicz strong disjunction. The algebraic semantics for RŁ is given by DMV-algebras (divisible MV-algebras). A Łukasiewicz logic evaluation e into the real unit interval is extended to the connectives δ_n by $e(\delta_n\varphi) = e(\varphi)/n$. In this way one can define in RŁ all rationals in $[0, 1]$. RŁ was shown to enjoy both finite strong standard completeness and Pavelka-style completeness (see [Gerla, 2001b] for all details). In particular, Hájek's Rational Pavelka logic can be faithfully interpreted in RŁ.

Fuzzy logics with equality: The question of introducing the (fuzzy) equality predicate in different systems of fuzzy logic has been dealt with in several papers, see e.g. [Liau and Lin, 1988; Bělohlávek, 2002c; Hájek, 1998a; Novák *et al.*, 1999; Novák, 2004; Bělohlávek and Vychodil, 2005] . Actually, in most of the works, fuzzy equality is a generalization of the classical equality because it is subject to axioms which are formally the same as the equality axioms in classical predicate logic. Semantically, fuzzy equality is related to the characterization of graded similarity among objects, with the meaning that the more similar are a couple of objects, the higher is the degree of their equality.

B. About computational complexity

The issue of complexity of t-norm based logics has also been studied in a number of papers starting with Mundici's [1994] pioneering work regarding NP-completeness of Łukasiewicz logic and flourishing during the nineties, with some problems still left open. It has to be pointed out that the dichotomy of the SAT and TAUT problems in classical logic, where checking the tautologicity of φ is equivalent to check that $\neg\varphi$ is not satisfiable and vice-versa, is no longer at hand in many-valued logics. Unlike in classical logic, for a many-valued semantics there need not be a simple relationship between its TAUT and SAT problems. This is the reason why, given a class $\mathcal{K}$ of algebras of the same type, it is natural to distinguish the following sets of formulas (as suggested in [Baaz *et al.*, 2002] for the SAT problems):

$$TAUT_1^{\mathcal{K}} = \{\varphi \mid \forall A \in \mathcal{K}, \forall e_A, e_A(\varphi) = 1\}$$
$$TAUT_{pos}^{\mathcal{K}} = \{\varphi \mid \forall A \in \mathcal{K}, \forall e_A, e_A(\varphi) > 0\}$$
$$SAT_1^{\mathcal{K}} = \{\varphi \mid \exists A \in \mathcal{K}, \exists e_A, e_A(\varphi) = 1\}$$

$$SAT^{\mathcal{K}}_{pos} = \{\varphi \mid \exists A \in \mathcal{K}, \exists e_A, e_A(\varphi) > 0\}$$

The interested reader is referred to two excellent surveys on complexity results and methods used: the one by Aguzzoli, Gerla and Hanniková [2005] concerning a large family of propositional fuzzy logics (BL and several of its expansions) as well as some logics with the connective Δ; and the one by Hájek's [2005b] for the case of prominent predicate fuzzy logics.

C. Weaker systems of fuzzy logic

Non commutative fuzzy logics: Starting from purely algebraic motivations (see [Di Nola *et al.*, 2002]), several authors have studied generalizations of BL and MTL (and related t-norm based logics) with a non-commutative conjunction &, e.g. [Hájek, 2003a; Hájek, 2003b; Jenei and Montagna, 2003]. These logics have two implications, corresponding to the left and right residuum of the conjunction. The algebraic counterpart are the so-called pseudo-BL and pseudo-MTL algebras. Interestingly enough, while there are pseudo-MTL algebras over the real unit interval $[0, 1]$, defined by left continuous pseudo-t-norms (i.e. operations satisfying all properties of t-norms but the commutativity), there are not pseudo-BL algebras, since continuous pseudo-t-norms are necessarily commutative. Still a weaker fuzzy logic, the so-called *flea logic* is investigated in [Hájek, 2005c], which is a common generalization of three well-known generalizations of the fuzzy (propositional) logic BL, namely the monoidal t-norm logic MTL, the hoop logic BHL and the non-commutative logic pseudo-BL.

Weakly implicative fuzzy logics: Going even further on generalizing systems of fuzzy logic, Cintula [2006] has introduced the framework of weakly implicative fuzzy logics. The main idea behind this class of logics is to capture the notion of comparative truth common to all fuzzy logics. Roughly speaking, they are logics close to Rasiowa's implicative logics [Rasiowa, 1974] but satisfying a proof-by-cases property. This property ensures that these logics have a semantics based on linearly ordered sets of truth-values, hence allowing a proper notion of comparative truth. The interested reader is referred to [Behounek and Cintula, 2006b] where the authors advocate for this view of fuzzy logic.

D. Functional representation issues

McNaughton famous theorem [McNaughton, 1951], establishing that the class of functions representable by formulas of Łukasiewicz logic is the class of piecewise linear functions with integer coefficients, has been the point of departure of many research efforts trying to generalize it for other important fuzzy logics, i.e. trying to describe the class of real functions which can be defined by the truth tables of formulas of a given fuzzy logic. For instance we may cite [Gerla, 2000; Gerla, 2001a; Wang *et al.*, 2004; Aguzzoli *et al.*, 2005; Aguzzoli *et al.*, 2006] for the case of Gödel, Nilpotent Minimum and related logics, [Cintula and Gerla, 2004] for the case of product logic, [Montagna and Panti, 2001] for the case of the Łukasiewicz expansions like $Ł_\Delta$, $PŁ_\Delta$, ŁΠ, ŁΠ$\frac{1}{2}$ logics. It is interesting to notice that the problem of whether the class of functions (on $[0,1]$) defined by formulas of Product Łukasiewicz logic PŁ (see Section 3.4) amounts to the famous Pierce-Birkhoff conjecture: "Is every real-valued continuous piecewise polynomial function on real affine n-space expressible using finitely many polynomial functions and the operations of (pointwise) supremum and infimum?" This has been actually proved true for the case of functions of three variables, but it remains an open problem for the case of more variables.

3.6 T-norm based fuzzy logic modelling of approximate reasoning

We have already referred in previous sections to the distinction between fuzzy logic in a narrow sense and in a broad sense. In Zadeh's opinion [1988], fuzzy logic in the narrow sense is an extension of many-valued logic but having a different agenda, in particular including the approximate reasoning machinery described in Section 2 (flexible constraints propagation, generalized modus ponens, etc.) and other aspects not covered there, such as linguistic quantifiers, modifiers, etc. In general, linguistic and semantical aspects are mainly stressed.

The aim of this section is to show that fuzzy logic in Zadeh's narrow sense can be presented as classical deduction in the frame of the t-norm based fuzzy logics described in previous subsections, and thus bridging the gap between the contents of Section 2 and Section 3.

In the literature one can find several approaches to cast main Zadeh's approximate reasoning constructs in a formal logical framework. In particular, Novák and colleagues have done much in this direction, using the model of fuzzy logic with evaluated syntax, fully elaborated in the monograph [Novák *et al.*, 1999] (see the references therein and also [Dvořák and Novák, 2004]), and more recently he has developed a very powerful and sophisticated model of fuzzy type theory [Novák, 2005; Novák and Lehmke, 2006]. In his monograph, Hájek [1998a] also has a part devoted to this task.

In what follows, we show a simple way of how to capture at a syntactical level, namely in a many-sorted version of predicate fuzzy logic calculus, say MTL∀, some of the basic Zadeh's approximate reasoning patterns, basically from ideas in [Hájek, 1998a; Godo and Hájek, 1999]. It turns out that the logical structure becomes rather simple and the fact that fuzzy inference is in fact a (crisp) deduction becomes rather apparent. The potential advantges of this presentation are several. They range from having a formal framework which can be common or very similar for various kinds of fuzzy logics to the availability of well-developed proof theoretical tools of many-valued logic.

Consider the simplest and most usual expressions in Zadeh's fuzzy logic of the form

$$\text{“}x \text{ is } A\text{”},$$

discussed in Section 2.2, with the intended meaning the variable x takes the value in A, represented by a fuzzy set μ_A on a certain domain U. The representation of this statement in the frame of possibility theory is the constraint

$$(\forall u)(\pi_x(u) \leq \mu_A(u))$$

where π_x stands for the possibility distribution for the variable x. But such a constraint is very easy to represent in MTL$\forall$ as the

$$(\forall x)(X(x) \to A(x))$$

(Caution!: do not confuse the logical variable x in this logical expression from the linguistic (extra-logical) variable x in "x is A") where A and X are many-valued *predicates* of the same sort in each particular model $\mathbf{M}$. Their interpretations (as fuzzy relations on their common domain) can be understood as the membership function $\mu_A : U \longrightarrow [0,1]$ and the possibility distribution π_x respectively. Indeed, one can easily observe that $\| (\forall x)(X(x) \to A(x)) \|_M = 1$ if and only if $\| X(x) \|_{M,e} \leq \| A(x) \|_{M,e}$, for all x and any evaluation e. From now on, variables ranging over universes will be x, y; "x is A" becomes $(\forall x)(X(x) \to A(x))$ or just $X \subseteq A$; if z is 2-dimmensional variable (x, y), then an expression "z is R" becomes $(\forall x, y)(Z(x, y) \to R(x, y))$ or just $Z \subseteq R$.

In what follows, only two (linguistic) variables will be involved x, y and $z = (x, y)$. Therefore we assume that X, Y (corresponding to the possibility distributions π_x and π_y) are projections of a binary binary fuzzy predicate Z (corresponding to the joint possibility distribution $\pi_{x,y}$). The axioms we need to state in order to formalize this asumption are:

$$\Pi 1: \quad (\forall x, y)(Z(x, y) \to X(x)) \ \& \ (\forall x, y)(Z(x, y) \to Y(y))$$

$$\Pi 2: \quad (\forall x)(X(x) \to (\exists y)Z(x, y)) \ \& \ (\forall y)(Y(y) \to (\exists x)Z(x, y))$$

Condition $\Pi 1$ expresses the monotonicity conditions $\pi_{x,y}(u, v) \leq \pi_x(u)$ and $\pi_{x,y}(u, v) \leq \pi_y(v)$, whereas both conditions $\Pi 1$ and $\Pi 2$ used together express the marginalization conditions $\pi_x(u) = sup_v \pi_{x,y}(u, v)$ and $\pi_y(v) = sup_v \pi_{x,y}(u, v)$. These can be equivalently presented as the only one condition $Proj$, as follows:

$$\textit{Proj:} \quad (\forall x)(X(x) \equiv (\exists y)Z(x, y)) \ \& \ (\forall y)(Y(y) \equiv (\exists x)Z(x, y))$$

Next we shall consider several approximate reasoning patterns described in Section 2, and for each pattern we shall present a corresponding tautology and its derived deduction rule, which will automatically be sound.

1. *Entailment Principle:* From "x is A" infer "x is A^*", whenever $\mu_A(u) \leq \mu_{A^*}(u)$ for all u.

   Provable tautology:

$$(A \subseteq A^*) \to (X \subseteq A \to X \subseteq A^*)$$

Sound rule:

$$\frac{A \subseteq A^*, X \subseteq A}{X \subseteq A^*}$$

2. *Cylindrical extension:* From "x is A" infer "(x, y) is A^+", where $\mu_{A^+}(u, v) = \mu_A(u)$ for each v.

   Provable tautology:

$$\Pi 1 \to [(X \subseteq A) \to ((\forall xy)(A^+(x, y) \leftrightarrow A(x)) \to (Z \subseteq A^+))]$$

   Sound rule:

$$\frac{\Pi 1, X \subseteq A, (\forall xy)(A^+(x, y) \leftrightarrow A(x))}{Z \subseteq A^+}$$

3. *min–Combination*: From "x is A_1" and "x is A_2" infer "x is $A_1 \cap A_2$", where $\mu_{A_1 \cap A_2}(u) = \min(\mu_{A_1}(u), \mu_{A_2}(u))$.

   Tautology:

$$(X \subseteq A_1) \to ((X \subseteq A_2) \to (X \subseteq (A_1 \wedge A_2)))$$

   Rule:

$$\frac{X \subseteq A_1, X \subseteq A_2}{X \subseteq (A_1 \wedge A_2)}$$

   where $(A_1 \wedge A_2)(x)$ is an abbreviation for $A_1(x) \wedge A_2(x)$.

4. *Projection*: From "(x, y) is R" infer "y is R_Y", where $\mu_{R_Y}(y) = \sup_u \mu_R(u, v)$ for each v.

   Provable tautology:

$$\Pi 2 \to ((Z \subseteq R) \to (\forall y)(Y(y) \to (\exists x)R(x, y)))$$

   Sound rule:

$$\frac{\Pi 2, Z \subseteq R}{(\forall y)(Y(y) \to (\exists x)R(x, y))}$$

Note that the formalization of the *max–min composition rule* (from "x is A" and "(x, y) is R" infer "y is B", where $\mu_B(y) = \sup_u \min(\mu_A(u), \mu_R(u, v))$)

$$\frac{Cond, Proj, (X \subseteq A), (Z \subseteq R)}{Y \subseteq B},$$

where $Cond$ is the formula $(\forall y)(B(y) \equiv (\exists x)(A(x) \wedge R(x, y)))$, is indeed a derived rule from the above ones.

More complex patterns like those related to inference with fuzzy if-then rules "if x is A then y is B" can also be formalized. As we have seen in Section 2, there are several semantics for the fuzzy if-then rules in terms of the different types constraints on the joint possibility distribution $\pi_{x,y}$ it may induce. Each particular semantics will obviously have a different representation. We will describe just a couple of them.

Within the implicative interpretations of fuzzy rules, gradual rules are interpreted by the constraint $\pi_{x,y}(u,v) \leq A(u) \Rightarrow B(v)$, for some residuated implication $\Rightarrow$. According to this interpretation, the folllowing is a derivable (sound) rule

$$\frac{Cond,\ Proj,\ X \subseteq A^*,\ Z \subseteq A \to B}{Y \subseteq B^*},$$

where $(A \to B)(x,y)$ stands for $A(x) \to B(y)$ and $Cond$ is $(\forall y)[B^*(y) \equiv (\exists x)(A^*(x) \wedge(A(x) \to B(y)))]$. If one wants to strengthen this rule as to force to derive $(\forall y)(B^*(y) \equiv B(y))$ when adding the condition $(\forall x)(A^*(x) \equiv A(x))$ to the premises, then one has to move to another generalized modus ponens rule that is also derivable

$$\frac{Cond,\ \Pi2',\ X \subseteq A^*,\ Z \subseteq A \to B}{Y \subseteq B^*},$$

where $Cond$ is now $(\forall y)(B^*(y) \equiv (\exists x)[A^*(x)\,\&\,(A(x) \to B(y)))]$ and where condition $\Pi2'$ is $(\forall y)(Y(y) \to (\exists x)(X(x)\,\&\,Z(x,y)))$, a slightly stronger condition than $\Pi2$.

Finally, within the conjunctive model of fuzzy rules, where a rule "if x is A then y is B" is interpreted by the constraint $\pi_{x,y}(u,v) \geq A(u) \wedge B(v)$, and an observation "$x$ is A^*" by a positive constraint $\pi_x(u) \geq A^*(u)$, one can easily derive the Mamdani model (here with just one rule)

$$\frac{Cond,\ Proj,\ X \supseteq A^*,\ Z \supseteq A \wedge B}{Y \supseteq B^*},$$

where $Cond$ is $(\forall y)[B^*(y) \equiv (\exists x)(A^*(x) \wedge A(x)) \wedge B(y)]$. Interestingly enough, if the observation is instead modelled as a negative constraint $\pi_x(u) \leq A^*(u)$, then one can derive the following rule,

$$\frac{Cond,\ Proj,\ (\exists x)X(x),\ X \subseteq A^*,\ Z \supseteq A \wedge B}{Y \supseteq B^*},$$

where $Cond$ is now $(\forall y)[B^*(y) \equiv (\forall x)(A^*(x) \to (A(x)) \wedge B(y))]$, which is in accordance with the discussion in Section 2.5.

3.7 Clausal and resolution-based fuzzy logics

S-fuzzy logics. Another family of fuzzy logics, very different from the class of logics presented in the previous subsections, can be built by taking as basic connectives a conjunction $\sqcap$, a disjunction $\sqcup$ and a negation $\neg$, rather than a conjunction and a (residuated) implication. These connectives are to be interpreted in $[0,1]$ by the triple $(\max, \min, 1-\cdot)$, or more generally by a De Morgan triple

(T, S, N) where T is a t-norm, N a strong negations function and S is the N-dual t-conorm, i.e. $S(x, y) = N(T(N(x), N(y))$. See [Klement and Navara, 1999] for a comparison of these two fuzzy logic traditions.

Butnariu and Klement [1995] introduced the so-called *S-fuzzy logics*, associated to the family of Frank t-norms. This is a parametrized family of continuous t-norms $\{T_\lambda\}_{\lambda \in [0,\infty]}$, strictly decreasing with respect to the parameter λ, and which has three interesting limit cases $\lambda = 0, 1, \infty$ corresponding to the three well known t-norms: $T_0 = \min$, $T_1 = *_\Pi$ (product t-norm) and $T_\infty = *_Ł$ (Łukasiewicz t-norm). For $\lambda \in (0, \infty)$,

$$T_\lambda(x, y) = \log_\lambda(1 + \frac{(\lambda^x - 1)(\lambda^y - 1)}{\lambda - 1})$$

is a t-norm isomorphic to $*_\Pi$.

The language of S-fuzzy logics $\mathcal{L}_\lambda$ is built over a countable set of propositional variables and two connectives $\sqcap$ and $\neg$. Disjunction $\sqcup$ and implication $\rightarrow$ are defined conenctives, $\varphi \sqcup \psi$ is $\neg(\neg\varphi \sqcap \neg\psi)$ and $\varphi \rightarrow \psi$ is $\neg(\varphi \sqcap \neg\psi)$. Semantics of $\mathcal{L}_\lambda$ is defined by evaluations of propositonal variables into $[0, 1]$ that extend to arbitrary propositions by defining

$$e(\varphi \sqcap \psi) = T_\lambda(e(\varphi), e(\psi)), \quad e(\neg\varphi) = 1 - e(\varphi).$$

Notice that the interpretation of the implication is given by

$$e(\varphi \rightarrow \psi) = I_{S_\lambda}(e(\varphi), e(\psi)),$$

where $I_{S_\lambda}(x, y) = S_\lambda(1 - x, y)$ is an S-implication (see Section 2.1), with S_λ being the dual t-conorm of T_λ. This is the main reason why these logics are called S-fuzzy logics. When $\lambda = 0$, $\mathcal{L}_0$ is the so-called max-min S-logic, while for $\lambda = \infty$, $\mathcal{L}_\infty$ corresponds to Łukasiewicz logic Ł.

In S-fuzzy logics $\mathcal{L}_\lambda$ for $\lambda \neq \infty$ there are no formulas that take the value 1 under all truth-evaluations, but on the other hand, the set of formulas which are always evaluated to an strictly positive value is closed by modus ponens. This leads to define that a formula φ is a $\mathcal{L}_\lambda$-tautology whenever $e(\varphi) > 0$ for all $\mathcal{L}_\lambda$-evaluation e. Then the authors prove the following kind of completeness: the set of $\mathcal{L}_\lambda$-tautologies coincide with classical (two-valued) tautologies. This is in accordance with the well-known fact that, in the frame of Product logic Π (and more generally in SMTL), the fragment consisting of the double negated formulas $\neg\neg\varphi$ is indeed equivalent to classical logic.

Fuzzy logic programming systems. Many non-residuated logical calculi that have early been developed in the literature as extensions of classical logic programming systems are related to some form of S-fuzzy logic, and a distinguishing feature is that the notion of proof is based on a kind of resolution rule, i.e. computing the truth value $\|\psi \sqcup \chi\|$ from $\|\varphi \sqcup \psi\|$ and $\|\neg\varphi \sqcup \chi\|$.

The first fuzzy resolution method was defined by [Lee, 1972] and it is related to the max-min S-fuzzy logic mentioned above. At the syntactic level, formulas are *classical first-order formulas* (thus we write below $\wedge$, and $\vee$ instead of $\sqcap$ and $\sqcup$ resp.) but at the semantic level, formulas have a *truth value* which may be intermediary between 0 and 1. An *interpretation* $\mathbf{M}$ is defined by an assignment of a truth value to each atomic formula, from which truth values of compound formulas are computed in the following way:

$$\begin{aligned}\|\neg\varphi\|_{\mathbf{M}} &= 1 - \|\varphi\|_{\mathbf{M}},\\ \|\varphi \wedge \psi\|_{\mathbf{M}} &= \min(\|\varphi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}}),\\ \|\varphi \vee \psi\|_{\mathbf{M}} &= \max(\|\varphi\|_{\mathbf{M}}, \|\psi\|_{\mathbf{M}}).\end{aligned}$$

The notions of validity, consistency and inconsistency are generalized to fuzzy logic: Let φ be a fuzzy formula. φ is *valid* iff $\|\varphi\|_{\mathbf{M}} \geq 0.5$ for each interpretation $\mathbf{M}$, i.e the set of designated truth values is $[0.5, 1]$. φ is *inconsistent* iff $\|\varphi\|_{\mathbf{M}} \leq 0.5$ for each interpretation $\mathbf{M}$. And, φ *entails* another formula ψ, denoted $\varphi \models \psi$, if $\|\psi\|_{\mathbf{M}} \geq 0.5$ for each interpretation $\mathbf{M}$ such that $\|\varphi\|_{\mathbf{M}} \geq 0.5$. [Lee and Chang, 1971] proved that a fuzzy formula is valid (respec. inconsistent) iff the formula is classically valid (respectively, inconsistent), i.e. considering the involved predicates and propositions as crisp; and that $\varphi \models \psi$ in fuzzy logic iff $\varphi \models \psi$ in classical logic. The resolvent of two clauses C_1 and C_2 is defined as in classical first-order logic. [Lee, 1972] proved that provided that C_1 and C_2 are ground clauses, and if $\min(\|C_1\|, \|C_2\|) = a > 0.5$ and $\max(\|C_1\|, \|C_2\|) = b$, then $a \leq \|R(C_1, C_2)\| \leq b$ for each resolvent $R(C_1, C_2)$ of C_1 and C_2 (see the discussion in section 2.3). This is generalized to resolvents of a set of ground clauses obtained by a number of successive applications of the resolution principle. Hence, Lee's resolution is *sound*. This result also holds for intervals of truth values with a lower bound greater than 0.5. Lee's proof method does not deal with refutation, hence it is not complete (since resolution is not complete for deduction). Many subsequent works have been based on Lee's setting. In [Shen *et al.*, 1988; Mukaidono *et al.*, 1989] Lee's resolution principle was generalized by introducing a *fuzzy resolvent*. Let C_1 and C_2 be two clauses of fuzzy logic and let $R(C_1, C_2)$ be a classical resolvent of C_1 and C_2. Let l be the literal on the basis of which $R(C_1, C_2)$ has been obtained. Then, the *fuzzy resolvent* of C_1 and C_2 is $R(C_1, C_2) \vee (l \wedge \neg l)$ with the truth value $\max(\|R(C_1, C_2)\|, \|(l \wedge \neg l)\|)$. It is proved that a fuzzy resolvent is always a logical consequence of its parent clauses, which generalizes Lee's result. See also [Chung and Schwartz, 1995] for a related approach.

One of the drawbacks of these and other early approaches is that they are based on the language of classical logic, and thus, does not make it possible to deal with intermediate truth values at the syntactic level. Nevertheless, the trend initiated by [Lee, 1972] blossomed in the framework of logic programming giving birth to a number of fuzzy logic programming systems. An exhaustive survey on fuzzy logic programming before 1991 is in [Dubois *et al.*, 1991c, Sec. 4.3]. Most of them are mainly heuristic-based and not with a formal logical background. This is in part due to the difficulty of adapting resolution-based proof methods to fuzzy logics with residuated implication, with the exception of Łukasiewicz logic (whose implication is also an S-implication). Indeed, a resolution rule for Łukasiewicz-based logics has been proposed in [Thiele and Lehmke, 1994; Lehmke, 1995; Klawonn and Kruse, 1994; Klawonn, 1995]. Lehmke and Thiele defined a resolution system for so-called *weighted bold clauses*. Clauses are of the form $C = l_1 \sqcup \cdots \sqcup l_n$, where l_i are literals in classical way (they consider only propositional logic) and $\sqcup$ is the Łukasiewicz (strong) disjunction (i.e. $\|C_1 \sqcup C_2\| = \min(\|C_1\| + \|C_2\|, 1)$). They introduce the resolution rule as follows:

$$\frac{\begin{array}{c} T \vdash C_1, \text{ and } p \text{ occurs in } C_1 \\ T \vdash C_2, \text{ and } \neg p \text{ occurs in } C_2 \end{array}}{T \vdash ((C_1 \sqcup C_2)\backslash p)\backslash \neg p},$$

where $\backslash$ denotes the operation of omitting the corresponding literal. Then, they get the following result:

$$\text{If } T \vdash C \text{ then } T \models C, \text{ and if } T \text{ has no } \mathbf{1}\text{-model then } T \vdash \bot.$$

Klawonn and Kruse [1994] turned to predicate fuzzy logic in the setting of finitely-valued Łukasiewicz logics. They introduce special implication clauses of the form $(\forall x_1 \ldots x_n)(\varphi \Rightarrow A)$ and $(\forall x_1 \ldots x_n)\varphi$, where A is an atomic formula and φ contains only "and" and "or" types of connectives and no quantifiers. In this framework they define a prolog system (called LULOG) with a complete proof procedure for deriving the greatest lower bound for the truth-value of implication clauses, and based on the following graded resolution rule: from $(\neg\varphi \sqcup \psi, \alpha)$ and $(\neg\psi \sqcup \chi, \beta)$ derive $(\neg\varphi \sqcup \chi, \max(\alpha + \beta - 1, 0)$.

Soundness and completeness results can be also found in the literature for fuzzy prolog systems where rules (without negation) are interpreted by as formulas $p_1 \& \ldots \& p_n \to q$ of genuine residuated logic. For instance we may cite [Mukaidono and Kikuchi, 1993] for the case of Gödel semantics, and [Vojtáš, 1998] for the general case where & and $\to$ are interpreted by a left-continous t-norm and its residdum. Moreover, Vojtáš [2001] presented a soundness and completeness proof for fuzzy logic programs without negation and with a wide variety of connectives, and generalized in the framework of multi-adjoint residuated lattices by Medina *et al.* [2001].

3.8 Graded consequence and fuzzy consequence operators

The systems of t-norm-based logics discussed in the previous sections aim at formalizing the logical background for fuzzy set based approximate reasoning, and their semantics are based on allowing their formulas to take intermediary degrees of truth. But, as already pointed out in Section 3.3, they all have crisp notions of consequence, both of logical entailment and of provability. It is natural to ask whether it is possible to generalize these considerations to the case that one starts from *fuzzy sets of formulas*, and that one gets from them, as logical consequence, fuzzy sets of formulas.

One form of attacking this problem is by extending the logic with truth-constants as described in Section 3.3. However, there is also another approach, more algebraically oriented toward consequence operations for the classical case, originating from Tarski [1930], see also [Wójcicki, 1988]. This approach treats consequence operations as closure operators. Many works have been devoted to extend the notions of closure operators, closure systems and consequence relations from two-valued logic to many-valued / fuzzy logics.

Actually, both approaches have the origin in the work of Pavelka. Although one of the first works on fuzzy closure operators, was done by Michálek [1975] in the framework of Fuzzy Topological Spaces, the first and best well-known approach to fuzzy closure operators in the logical setting is due to Pavelka [1979] and the basic monograph elaborating this approach is Novák, Perfilieva and Močkoř's [1999]. In this approach, closure operators (in the standard sense of Tarski) are defined as mappings from fuzzy sets of formulas to fuzzy sets of formulas. In some more detail (following [Gottwald and Hájek, 2005]'s presentation), let $\mathcal{L}$ be a propositional language, $\mathcal{P}(\mathcal{L})$ be its power set and $\mathcal{F}(\mathcal{L})$ the set of L-fuzzy subsets of $\mathcal{L}$, where $\mathbf{L} = (L, *, \Rightarrow, \wedge, \vee, \leq, 0, 1)$ is a complete MTL-algebra. Propositions of $\mathcal{L}$ will be denoted by lower case letters p,q, …, and fuzzy sets of propositions by upper case letters A,B, etc. For each $A \in \mathcal{F}(\mathcal{L})$ and each $p \in \mathcal{L}$, $A(p) \in L$ will stand for the membership degree of p to A. Moreover, the lattice structure of L induces a related lattice structure on $\mathcal{F}(\mathcal{L})$, $(\mathcal{F}(\mathcal{L}), \cap, \cup, \subseteq, \bar{0}, \bar{1})$, which is complete and distributive as well, where $\cap, \cup$ are the pointwise extensions of the lattice operations $\wedge$ and $\vee$ to $\mathcal{F}(\mathcal{L})$, i.e.

$$\begin{array}{rcl} (A \cap B)(p) & = & A(p) \wedge B(p), \text{ for all } p \in \mathcal{L} \\ (A \cup B)(p) & = & A(p) \vee B(p), \text{ for all } p \in \mathcal{L}, \end{array}$$

and where the lattice (subsethood) ordering and top and bottom elements are defined respectively by

$$\begin{array}{rcl} A \subseteq B & \text{iff} & A(p) \leq B(p) \text{ for all } p \in \mathcal{L} \\ \bar{0}(p) = 0 & \text{and} & \bar{1}(p) = 1, \text{ for all } p \in \mathcal{L}\ . \end{array}$$

For any $k \in L$, we shall also denote by $\bar{k}$ the constant fuzzy set defined by $\bar{k}(p) = k$ for all $p \in \mathcal{L}$. The Pavelka-style approach is an easy matter as long as the semantic consequence is considered. An $\mathbf{L}$-evaluation e is a model of a fuzzy set of formulas $A \in \mathcal{F}(\mathcal{L})$ if and only if

$$A(p) \leq e(p)$$

holds for each formula p. This leads to define as semantic consequence of A the following fuzzy sets of formulas:

$$C^{sem}(A)(p) = \bigcap \{e(p) \mid e \text{ model of } A\}, \text{ for each } p \in \mathcal{L}$$

For a syntactic characterization of this consequence relation it is necessary to have some logical calculus $\mathcal{K}$ which treats formulas of the language together with truth degrees. So the language of this calculus has to extend the language of the basic logical system by having also symbols for the truth degrees (truth-constants) denoted $\overline{r}$ for each $r \in L$, very similar to what has been described in Section 3.3. Once this is done, one can consider evaluated formulas, i.e. pairs $(\overline{r}, p)$ consisting of a truth constant and a formula. Using this notion, one can understand in a natural way each fuzzy set of formulas A as a (crisp) set of evaluated formulas $\{(\overline{A(p)}, p) \mid p \in \mathcal{L}\}$. Then, assuming the calculus $\mathcal{K}$ has a suitable notion of derivation for evaluated formulas $\vdash_{\mathcal{K}}$, then each $\mathcal{K}$-derivation of an evaluated formula $(\overline{r}, p)$ can be understood as a derivation of p to the degree $r \in L$. Since p can have multiple derivations, it is natural to define the provability degree of p as the supremum of all these degrees. This leads to the following definition of fuzzy syntactical consequence of a fuzzy set of formulas A:

$$C^{syn}(A)(p) = \bigcup \{r \in L \mid \{(\overline{A(q)}, q) \mid q \in \mathcal{L}\} \vdash_{\mathcal{K}} (\overline{r}, p)\}$$

This is in fact an *infinitary* notion of provability, that can be suitably handled by Łukasiewicz logic Ł since it has their truth-functions continuous. Indeed, by defining the derivation relation $\vdash_{\mathcal{K}}$ from the set of axioms of Ł written in the form $(\overline{1}, \varphi)$, and having as inference rule the following kind of evaluated modus ponens

$$\frac{(\overline{r}, p) \ (\overline{s}, p \to q)}{(\overline{r * s}, q)},$$

where $*$ is Łukasiewicz t-norm, it can be shown (see e.g. [Novák *et al.*, 1999]) that one gets the following strong completeness result:

$$C^{sem}(A)(p) = C^{syn}(A)(p)$$

for any formula p and any fuzzy set of formulas A, that establishes the equivalence between the semantical and syntactical definitions of the consequence operators in the setting of Łukasiewicz logic.

Thus Pavelka's fuzzy consequence operators map each fuzzy set of formulas A (i.e. each set of evaluated formulas) to a fuzzy set of formulas denoted generically $\tilde{C}(A)$ (i.e. a set of evaluated formulas) that corresponds to the set of evaluated formulas that are consequences of the initial set represented by A. And this mapping fulfills the properties of a fuzzy closure operator as defined by Pavelka [1979]. Namely, a *fuzzy closure operator* on the language $\mathcal{L}$ is a mapping $\tilde{C}$: $\mathcal{F}(\mathcal{L}) \mapsto \mathcal{F}(\mathcal{L})$ fulfilling, for all $A, B \in \mathcal{F}(\mathcal{L})$, the following properties:

$\tilde{C}$1) *fuzzy inclusion*: $A \subseteq \tilde{C}(A)$

$\tilde{C}$2) *fuzzy monotony*: if $A \subseteq B$ then $\tilde{C}(A) \subseteq \tilde{C}(B)$

$\tilde{C}$3) *fuzzy idempotence*: $\tilde{C}(\tilde{C}(A)) \subseteq \tilde{C}(A)$.

This generalization of the notion of consequence operators leads to study closure operators and related notions like closure systems and consequence relations in other, more general fuzzy logic settings. In the rest of this section we review some of the main contributions.

Gerla [1994a] proposes a method to extend any classical closure operator C defined on $\mathcal{P}(\mathcal{L})$, i.e. on classical sets of formulas, into a fuzzy closure operator

$\tilde{C}^*$ defined in $\mathcal{F}(\mathcal{L})$, i.e. on fuzzy sets of formulas. This approach is further delevoped in [Gerla, 2001, Chap. 3]. In the following, we assume $\mathcal{F}(\mathcal{L})$ to be fuzzy sets of formulas valued on a complete linearly-ordered Gödel BL-algebra $\mathbf{L}$, i.e. a BL-chain $(L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$ where $\otimes = \wedge$. Then, given a closure operator $C : \mathcal{P}(\mathcal{L}) \longmapsto \mathcal{P}(\mathcal{L})$, the *canonical extension of* C is the fuzzy operator $\tilde{C}^* : \mathcal{F}(\mathcal{L}) \longmapsto \mathcal{F}(\mathcal{L})$ defined by

$$\tilde{C}^*(A)(p) = \sup\{\alpha \in L \mid p \in C(A_\alpha)\},$$

where A_α stands for the α-cut of A, i.e. $A_\alpha = \{p \in \mathcal{L} \mid A(p) \geq \alpha\}$. According to this definition, the canonical extension $\tilde{C}^*$ is a fuzzy closure operator such that $\tilde{C}^*(A)(p) = 1$ if $p \in C(\emptyset)$ and $\tilde{C}^*(A)(p) \geq \sup\{A(q_1) \wedge \ldots \wedge A(q_n) \mid p \in C(\{q_1, \ldots, q_n\})\}$. If C is compact, then the latter inequality becomes an equality. It also follows that a fuzzy set A is closed by $\tilde{C}^*$ then any α-cut of A is closed by $\tilde{C}$. Canonical extensions of classical closure operators were characterized in [Gerla, 2001] in the following terms: a fuzzy closure operator $\tilde{C}$ is the canonical extension of a closure operator if, and only if, for every meet-preserving function $f : L \longmapsto L$ such that $f(1) = 1$, if $\tilde{C}(A) = A$ then $\tilde{C}(f \circ A) = f \circ A$. In other words, this characterization amounts to requiring that if A belongs to the closure system defined by $\tilde{C}$, then so does $f \circ A$.

As regards the generalization of the notion of consequence relation, Chakraborty [1988; 1995] introduced the notion of *graded consequence relation* as a fuzzy relation between crisp sets of formulas and formulas. To do this, he assumes to have a monoidal operation $\otimes$ in L such that $(L, \otimes, 1, \leq, \Rightarrow)$ is a complete residuated lattice. Then a fuzzy relation $gc : \mathcal{P}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ is called a *graded consequence relation* by Chakraborty if, for every $A, B \in \mathcal{P}(\mathcal{L})$ and $p, q \in \mathcal{L}$, gc fulfills:

$gc1$) *fuzzy reflexivity*: $gc(A, p) = 1$ for all $p \in A$

$gc2$) *fuzzy monotony*: if $B \subseteq A$ then $gc(B, p) \leq gc(A, p)$

$gc3$) *fuzzy cut*: $[\inf_{q \in B} gc(A, q)] \otimes gc(A \cup B, p) \leq gc(A, p)$.[13]

Links between fuzzy closure operators and graded consequence relations were examined by Gerla [1996] and by Castro Trillas and Cubillo [1994]. In particular Castro *et al.* point out that several methods of approximate reasoning used in Artificial Intelligence, such as Polya's models of plausible reasoning [Polya, 1954] or Nilsson's probabilistic logic [Nilsson, 1974], are not covered by the formalism of graded consequence relations, and they introduce a new concept of consequence relations, called *fuzzy consequence relations* which, unlike Chakraborty's graded consequence relation, apply over fuzzy sets of formulas. Namely, a fuzzy relation $fc : \mathcal{F}(\mathcal{L}) \times \mathcal{L} \longmapsto L$ is called a *fuzzy consequence relation* in [Castro *et al.*, 1994] if the following three properties hold for every $A, B \in \mathcal{F}(\mathcal{L})$ and $p, q \in \mathcal{L}$:

$fc1$) *fuzzy reflexivity*: $A(p) \leq fc(A, p)$

[13] By residuation, this axiom is equivalent to $[\inf_{q \in B} gc(A, q)] \leq gc(A \cup B, p) \Rightarrow gc(A, p)$

$fc2$) *fuzzy monotony*: If $B \subseteq A$ then $fc(B,p) \leq fc(A,p)$

$fc3$) *fuzzy cut*: if for all p, $B(p) \leq fc(A,p)$, then for all q, $fc(A \cup B, q) \leq fc(A,q)$

However, it is worth noticing that fuzzy consequence relations as defined above, when restricted over crisp sets of formulas, become only a particular class of graded consequence relations. Namely, regarding the two versions of the fuzzy cut properties, ($gc3$) and ($fc3$), it holds that for $A, B \in \mathcal{P}(\mathcal{L})$, if $B(p) \leq fc(A,p)$ for all $p \in \mathcal{L}$, it is clear that $\inf_{q \in B} fc(A,q) = 1$.

Let us point out that, in the classical setting, there are well known relationships of interdefinability among closure operators, consequence relations and closure systems. In the fuzzy framework, fuzzy closure operators and fuzzy consequence relations are related in a analogous way, as proved in [Castro *et al.*, 1994]:

- if $\tilde{C}$ is a fuzzy closure operator then fc, defined as $fc(A,p) = \tilde{C}(A)(p)$, is a fuzzy consequence relation.
- if fc is a fuzzy consequence relation then $\tilde{C}$, defined as $\tilde{C}(A) = fc(A, \cdot)$, is a fuzzy closure operator.

Therefore, via these relationships, the fuzzy idempotence property ($\tilde{C}3$) for closure operators and the fuzzy cut property ($fc3$) for consequence relations become equivalent.

In the context of MTL-algebra $\mathbf{L} = (L, \wedge, \vee, \otimes, \Rightarrow, 0, 1)$, using the notation of closure operators and the notion of degree of inclusion between L-fuzzy sets of formulas defined as as

$$[A \subseteq_{\otimes} B] = \inf_{p \in \mathcal{L}} A(p) \Rightarrow B(p),$$

the relation between Chakraborty 's graded consequence and Castro et. al.'s fuzzy consequence relation becomes self evident. As already mentioned, the former is defined only over classical sets while the latter is defined over fuzzy sets, but both yield a fuzzy set of formulas as output. Nevertheless, having this difference in mind, the two first conditions of both operators become syntactically the same as $\tilde{C}1$ and $\tilde{C}2$ of Pavelka's definition of fuzzy closure operators while the fuzzy cut properties (the third ones) become very close one to another:

$gc3$) *fuzzy cut*: $([B \sqsubseteq_{\otimes} \tilde{C}(A)] \otimes \tilde{C}(A \cup B)) \subseteq \tilde{C}(A)$,
where $[B \sqsubseteq_{\otimes} \tilde{C}(A)] = \inf_{q \in B} \tilde{C}(A)(q)$ (recall that B is a classical set).

$fc3$) *fuzzy cut*: if $B \subseteq \tilde{C}(A)$ then $\tilde{C}(A \cup B) \subseteq \tilde{C}(A)$

In [Rodríguez *et al.*, 2003] a new class of fuzzy closure operators is introduced, the so-called *implicative closure operators*, as a generalization of Chakraborty's graded consequence relations over fuzzy sets of formulas. The adjective *implicative* is due to the fact that they generalize the Fuzzy Cut property ($gc3$) by means of the above defined degree of inclusion, which in turn depends on the implication operation $\Rightarrow$ of the algebra $\mathbf{L}$. More precisely, a mapping $\tilde{C} : \mathcal{F}(\mathcal{L}) \longmapsto \mathcal{F}(\mathcal{L})$ is called an *implicative closure operator* if, for every $A, B \in \mathcal{F}(\mathcal{L})$, $\tilde{C}$ fulfills:

$\tilde{C}1$) *fuzzy inclusion*: $A \subseteq \tilde{C}(A)$

$\tilde{C}2$) *fuzzy monotony*: If $B \subseteq A$ then $\tilde{C}(B) \subseteq \tilde{C}(A)$

$\tilde{C}3$) *fuzzy cut*[14] $[B \sqsubseteq_{\otimes} \tilde{C}(A)] \leq [\tilde{C}(A \cup B) \sqsubseteq_{\otimes} \tilde{C}(A)]$

The corresponding implicative consequence relation, denoted by I_c, is defined as $I_c(A, p) = \tilde{C}(A)(p)$. The translation of the properties of Implicative closure operators to implicative consequence relations read as follows:

$ic1$) *fuzzy reflexivity*: $A(p) \leq I_c(A, p)$

$ic2$) *fuzzy monotony*: If $B \subseteq A$ then $ic(B, p) \leq I_c(A, p)$

$ic3$) *fuzzy cut*: $[B \sqsubseteq_{\otimes} \tilde{C}(A)] \leq I_c(A \cup B, p) \Rightarrow I_c(A, p)$.

Now, it is easy to check that the restriction of implicative consequence relations over classical sets of formulas are exactly Chakraborty's graded consequence relations, since if B is a crisp set, $[B \sqsubseteq_{\otimes} \tilde{C}(A)] = \inf_{p \in B} I_c(A, p)$. On the other hand, fuzzy consequence relations are implicative as well, since property ($ic3$) clearly implies ($fc3$). Therefore, implicative consequence relations generalize both graded and fuzzy consequence relations.

The relationship of implicative consequence operators to deduction in fuzzy logics with truth constants (as reported in Section 3.3) is also addressed in [Rodríguez *et al.*, 2003]. An it turns out that, although implicative closure operators are very general and defined in the framework of BL-algebras, strangely enough, they do not capture graded deduction (Pavelka-style) in any of the extensions of BL, except for Gödel's logic.

Belohlávek [2001; 2002a] proposes yet another notion of closure operator over fuzzy sets with values in a complete residuated lattice $\mathbf{L}$, with the idea of capturing what he calls *generalized monotonicity condition* that reads as "'if A is *almost* a subset of B then the closure of A is *almost* a subset of the closure of B". Using the degree of inclusion defined before[15], for every order filter K of $\mathbf{L}$, a new closure operator is defined as follows. An $\mathbf{L}_K$*-closure operator* on $\mathcal{F}(\mathcal{L})$ is a mapping $\tilde{C} : \mathcal{F}(\mathcal{L}) \to \mathcal{F}(\mathcal{L})$ satisfying for all $A, A_1, A_2 \in \mathcal{F}(\mathcal{L})$ the conditions:

($\tilde{B}1$) $A \subseteq \tilde{C}(A)$

($\tilde{B}2$) $[A_1 \sqsubseteq_{\otimes} A_2] \leq [\tilde{C}(A_1) \sqsubseteq_{\otimes} \tilde{C}(A_2)]$ whenever $[A_1 \sqsubseteq_{\otimes} A_2] \in K$.

($\tilde{B}3$) $\tilde{C}(A) = \tilde{C}(\tilde{C}(A))$

[14] The original and equivalent presentation of this property in [Rodríguez *et al.*, 2003] is $[B \sqsubseteq_{\otimes} \tilde{C}(A)] \otimes \tilde{C}(A \cup B) \subseteq \tilde{C}(A)$, directly extending ($g_c3$).

[15] Actually, in Belohlávek's paper it is considered as a fuzzy relation denoted as $S(A_1, A_2)$, instead of $[A_1 \sqsubseteq_{\otimes} A_2]$ used above.

It is clear that for $\mathbf{L} = \{0,1\}$, $\mathbf{L}_{\{1\}}$-closure operators are classical closure operators and for $\mathbf{L} = [0,1]_G$, $\mathbf{L}_{\{1\}}$-closure operators are precisely the fuzzy closure operators studied by Gerla.

In fact, although introduced independently, this notion is very close to implicative closure operators. Indeed, it is shown in [Bělohlávek, 2001] that conditions ($\tilde{B}2$) and ($\tilde{B}3$) can be equivalently replaced by the following condition:

($\tilde{B}4$) $[A_1 \sqsubseteq_\otimes \tilde{C}(A_2)] \leq [\tilde{C}(A_1) \sqsubseteq_\otimes \tilde{C}(A_2)]$ whenever $[A_1 \sqsubseteq_\otimes \tilde{C}(A_2)] \in K$.

Notice the similarity between ($\tilde{B}4$) and ($\tilde{C}3$). Indeed, when $K = L$, ($\tilde{C}3$) alone is slightly stronger than ($\tilde{B}4$), this shows that in that case implicative closure operators are L_K-closure operators. But in [Rodríguez *et al.*, 2003] it is proved that in the presence of ($\tilde{C}1$) and ($\tilde{C}2$), ($\tilde{C}3$) is actually equivalent to ($\tilde{B}4$). Therefore, when $K = L$, both implicative operators and L_K-closure operators are exactly the same, as also witnessed by the very similar characterizations of these two kinds of fuzzy closure operators provided in [Rodríguez *et al.*, 2003] and [Bělohlávek, 2001; Bělohlávek, 2002a] in terms of their associated fuzzy closure systems.

The study of the relationships between fuzzy closure operators and fuzzy similarities and preorders have also received some attention in the literature. In classical logic it is clear that the relation $R(\varphi,\psi)$ iff $\varphi \vdash \psi$ defines a preorder in the set of formulas and $E(\varphi,\psi) = R(\psi,\varphi) \wedge R(\varphi,\psi)$ defines and equivalence relation. This is not the case in the fuzzy setting, but there exist some relations that have been analyzed in several papers, e.g. [Castro and Trillas, 1991; Gerla, 2001; Rodríguez *et al.*, 2003; Elorza and Burillo, 1999; Bělohlávek, 2002a].

Finally, let us briefly comment that in the literature, different authors have studied the so-called *fuzzy operators* defined by fuzzy relations. Given a L-fuzzy relation $R: \mathcal{L} \times \mathcal{L} \longmapsto L$ on a given logical language $\mathcal{L}$, the associated *fuzzy operator* $\tilde{C}_R$ over $\mathcal{F}(\mathcal{L})$ is defined by:

$$\tilde{C}_R(A)(q) = \bigvee_{p\in\mathcal{L}}\{A(p) \otimes R(p,q)\}$$

for all $A \in \mathcal{F}(\mathcal{L})$, that is $\tilde{C}_R$ computes the image of fuzzy sets by sup $-\otimes$ composition with R. Properties of these operators have been studied for instance when R is a fuzzy preorder [Castro and Trillas, 1991] or when is a fuzzy similarity relation [Castro and Klawonn, 1994; Esteva *et al.*, 1998]. A special class of fuzzy operators appearing in the context of approximate reasoning patterns has been studied by Boixader and Jacas [Boixader and Jacas, 1998]. These operators, called *extensional inference operators*, are required to satisfy a extensionality condition which is very similar to condition (B2) above, and they can be associated to particular models of fuzzy if-then rules.

3.9 Concluding remarks: what formal fuzzy logic is useful for?

From the contents of the section it will probably become clear that the concept of fuzzy logic, even understood as a formal system of many-valued logic, admits of multiple formalizations and interpretations. This may be felt as a shortcoming but it can also be thought as an indication of the richness and complexity of the body of existing works. It may be particularly interesting for the reader to consult a recent special issue [Novák, 2006] of the journal Fuzzy Sets and Systems devoted to discuss the question of *what fuzzy logic is*. So far no definitive answer exists.

The other important conceptual question is: *what formal fuzzy logic is useful for?*. The use of fuzzy logic (in narrow sense) to model linguistic vagueness would seem to be the most obvious application, however it is not generally accepted yet within the philosophic community. In fact vagueness often refers to semantic ambiguity and this is often confused with the gradual nature of linguistic categories. Fuzzy logic clearly accounts for the latter, but it is true as well that linguistic categories can be both gradual and semantically ambiguous. Also, fuzzy logic is not often used for knowledge representation in Artificial Intelligence (AI) because of the lack of epistemic concepts in it, and because there is a strong Boolean logic tradition in AI. However, introducing many-valuedness in AI epistemic logics can be handled in fuzzy logic as explained in next section. Fuzzy logic may prove on the other hand to be very useful for the synthesis of continuous functions, like Karnaugh tables were used for the synthesis of Boolean functions. This problem has no relationship to approximate reasoning, but this topic is close to fuzzy rule-based systems used as neuro-fuzzy universal approximators of real functions.

New uses of first order logic related to the Semantic Web, such as description logics, can also benefit from the framework of fuzzy logic, so as to make formal models of domain ontologies more flexible, hence more realistic. This subject-matter may well prove to be a future prominent research trend, as witnessed by the recent blossoming of publications in this area, briefly surveyed below.. Description logics [Baader *et al.*, 2003], initially named "terminological logics", are tractable fragments of first-order logic representation languages that handle the notions of concepts (or classes), of roles (and properties), and of instances or objects, thus directly relying at the semantic level on the notions of set, binary relations, membership, and cardinality. They are especially useful for describing ontologies that consist in hierarchies of concepts in a particular domain.

Since Yen's [1991] pioneering work, many proposals have been made for introducing fuzzy features in description logic [Tresp and Molitor, 1998; Straccia, 1998; Straccia, 2001; Straccia, 2006a], and in semantic web languages, since fuzzy sets aim at providing a representation of classes and relations with gradual membership, which may be more suitable for dealing with concepts having a somewhat vague or elastic definition. Some authors have recently advocated other settings for a proper handling of fuzzy concepts, such as the fuzzy logic BL [Hájek, 2005a; Hájek, 2006a], or an approach to fuzzy description logic programs under the answer set semantics [Lukasiewicz, 2006].

Moreover, some authors [Hollunder, 1994; Straccia, 2006b; Straccia, 2006c] have also expressed concern about handling uncertainty and exceptions in description logic. Hollunder [1994] has introduced uncertainty in terminological logics using possibilistic logic (see Section 4.1). Recently, Dubois, Mengin and Prade [2006] have discussed how to handle both possibilistic uncertainty and fuzziness practically in description logic (by approximating fuzzy classes by finite families of nested ordinary classes).

4 FUZZY SET-BASED LOGICAL HANDLING OF UNCERTAINTY AND SIMILARITY

Fuzzy logics as studied in the previous section can be viewed as abstract formal machineries that can make syntactic inferences about gradual notions, as opposed to classical logic devoted to binary notions. As such it does not contain any epistemic ingredient, as opposed to Zadeh's approximate reasoning framework. Indeed, a fuzzy set, viewed as a possibility distribution, can model graded incomplete knowledge, hence qualifies as a tool for handling uncertainty that differs from a probability distribution. However, it should be clear that a fuzzy set can capture incomplete knowledge because it is a set, not because it is fuzzy (i.e. gradual). Hence no surprise if some logics of uncertainty can be devised on the basis of fuzzy set theory and the theory of approximate reasoning. This is naturally the case of possibilistic logic and its variants, which bridge the gap with knowledge representation concerns in artificial intelligence, such as non-monotonic reasoning. The gradual nature of fuzzy sets also lead to logics of graded similarity. Moreover, being abstract machines handling gradual notions, fuzzy logic can embed uncertainty calculi because belief is just another (usually) gradual notion. This section surveys the application of fuzzy logic to current trends in reasoning about knowledge and beliefs.

4.1 Possibilistic logic

Zadeh's approach to approximate reasoning can be particularized to offer proper semantics to reasoning with a set of classical propositions equipped with a complete pre-ordering that enable reliable propositions to be distinguished from less reliable ones. Conclusions are all the safer as they are deduced from more reliable pieces of information. The idea of reasoning from sets of (classical) logic formulas stratified in layers corresponding to different levels of confidence is very old. Rescher [1976] proposed a deductive machinery on the basis of the principle that the strength of a conclusion is the strength of the weakest argument used in its proof, pointing out that this idea dates back to Theophrastus (372-287 BC)[16]. However, Rescher did not provide any semantics for his proposal. The contribution of the possibilistic logic setting is to relate this idea (measuring the validity of an inference chain by its weakest link) to fuzzy set-based necessity measures in the framework of Zadeh [1978a]'s possibility theory, since the following pattern, first pointed out by Prade [1982], then holds

$$N(\neg p \vee q) \geq \alpha \text{ and } N(p) \geq \beta \text{ imply } N(q) \geq \min(\alpha, \beta),$$

[16]A disciple of Aristotle, who was also a distinguished writer and the creator of the first botanic garden!

where N is a necessity measure; see section 2.2 equation (14). This interpretative setting provides a semantic justification to the claim that the weight attached to a conclusion should be the weakest among the weights attached to the formulas involved in a derivation.

Basic formalism

Possibilistic logic (Dubois and Prade [1987; 2004]; Dubois, Lang and Prade [2002; 1994b], Lang[1991; 2001]) manipulates propositional or first order logical formulas weighted by lower bounds of necessity measures, or of possibility measures. A first-order possibilistic logic formula is essentially a pair made of a classical first order logic formula and a weight expressing certainty or priority. As already said, in possibilistic logic [Dubois *et al.*, 1994a; Dubois *et al.*, 1994b; Dubois and Prade, 1987], weights of formulas p are interpreted in terms of lower bounds $\alpha \in (0, 1]$ of necessity measures, i.e., the possibilistic logic expression (p, α) is understood as $N(p) \geq \alpha$, where N is a necessity measure.

Constraints of the form $\Pi(p) \geq \alpha$ could be also handled in the logic but they correspond to very poor pieces of information [Dubois and Prade, 1990; Lang *et al.*, 1991], while constraint $N(p) \geq \alpha \Leftrightarrow \Pi(\neg p) \leq 1-\alpha$ expresses that $\neg p$ is somewhat impossible, which is much more informative. Still, both kinds of constraints can be useful for expressing situations of partial or complete ignorance about p by stating both $\Pi(p) \geq \alpha$ and $\Pi(\neg p) \geq \alpha'$ and then propagating this ignorance to be able to determine what is somewhat certain and what cannot be such due to acknowledged ignorance (to be distinguished from a simple lack of knowledge when no information appears in the knowledge base). A mixed resolution rule [Dubois and Prade, 1990]

$$N(\neg p \vee q) \geq \alpha \text{ and } \Pi(p \vee r) \geq \beta \text{ imply } \Pi(q \vee r) \geq \beta \text{ if } \alpha > 1 - \beta$$
$$(\text{if } \alpha \leq 1 - \beta,\ \Pi(q \vee r) \geq 0)$$

is at the basis of the propagation mechanism for lower possibility bound information in a logic of graded possibility and certainty (Lang, Dubois, and Prade [1991]). In the following, we focus on the fragment of possibilistic logic handling only lower necessity bound information.

Syntax

An axiomatisation of 1st order possibilistic logic is provided by Lang [1991]; see also [Dubois *et al.*, 1994a]. In the propositional case, the axioms consist of all propositional axioms with weight 1. The inference rules are:

- $\{(\neg p \vee q, \alpha), (p, \beta)\} \vdash (q, \min(\alpha, \beta))$ (modus ponens)
- for $\beta \leq \alpha$, $(p, \alpha) \vdash (p, \beta)$ (weight weakening),

where $\vdash$ denotes the syntactic inference of possibilistic logic. The min-decomposability of necessity measures allows us to work with weighted *clauses* without lack of generality, since $N(\wedge_{i=1,n}p_i) \geq \alpha$ iff $\forall i,\ N(p_i) \geq \alpha$. It means that possibilistic logic expressions of the form $(\wedge_{i=1,n}p_i, \alpha)$ can be interpreted as a set of n formulas (p_i, α). In other words, any weighted logical formula put in Conjunctive Normal Form is equivalent to a set of weighted clauses. This feature considerably simplifies the proof theory of possibilistic logic. The basic inference rule in possibilistic logic put in clausal form is the resolution rule:

$$(\neg p \vee q, \alpha); (p \vee r, \beta) \vdash (q \vee r, \min(\alpha, \beta)).$$

Classical resolution is retrieved when all the weights are equal to 1. Other valid inference rules are for instance:

- if p classically entails q, $(p, \alpha) \vdash (q, \alpha)$ (formula weakening)
- $((\forall x)p(x), \alpha) \vdash (p(s), \alpha)$ (particularization)
- $(p, \alpha); (p, \beta) \vdash (p, \max(\alpha, \beta))$ (weight fusion).

Observe that since $(\neg p \vee p, 1)$ is an axiom, formula weakening is a particular case of the resolution rule (indeed $(p, \alpha); (\neg p \vee p \vee r, 1) \vdash (p \vee r, \alpha)$). Formulas of the form $(p, 0)$ that do not contain any information ($\forall p, N(p) \geq 0$ always holds), are not part of the possibilistic language.

Refutation can be easily extended to possibilistic logic. Let K be a knowledge base made of possibilistic formulas, i.e., $K = \{(p_i, \alpha_i)\}_{i=1,n}$. Proving (p, α) from K amounts to adding $(\neg p, 1)$, put in clausal form, to K, and using the above rules repeatedly until getting $K \cup \{(\neg p, 1)\} \vdash (\perp, \alpha)$. Clearly, we are interested here in getting the empty clause with the greatest possible weight [Dubois *et al.*, 1987]. It holds that $K \vdash (p, \alpha)$ if and only if $K_\alpha \vdash p$ (in the classical sense), where $K_\alpha = \{p \mid (p, \beta) \in K, \beta \geq \alpha\}$. Proof methods for possibilistic logic are described by Dubois, Lang and Prade [1994a], Liau and Lin [1993], and Hollunder [1995]. See [Lang, 2001] for algorithms and complexity issues.

Remarkably enough, the repeated use of the *probabilistic* counterpart to the possibilistic resolution rule (namely, $Prob(\neg p \vee q) \geq \alpha; Prob(p \vee r) \geq \beta \vdash Prob(q \vee r) \geq \max(0, \alpha + \beta - 1)$) is not in general sufficient for obtaining the best lower bound on the probability of a logical consequence, in contrast to the case of possibilistic logic.

An important feature of possibilistic logic is its ability to deal with inconsistency. The level of inconsistency of a possibilistic logic base is defined as

$$Inc(K) = \max\{\alpha \mid K \vdash (\perp, \alpha)\}$$

where, by convention $\max \emptyset = 0$. More generally, $Inc(K) = 0$ if and only if $K^* = \{p_i \mid (p_i, \alpha_i) \in K)\}$ is consistent in the usual sense. Note that this not true in case α_i would represent a lower bound of the probability of p_i in a probabilistically weighted logic.

Semantics

Semantic aspects of possibilistic logic, including soundness and completeness results with respect to the above syntactic inference machinery, are presented in [Lang, 1991; Lang *et al.*, 1991; Dubois *et al.*, 1994b; Dubois *et al.*, 1994a]. From a semantic point of view, a possibilistic knowledge base $K = \{(p_i, \alpha_i)\}_{i=1,n}$ is understood as the possibility distribution π_K representing the fuzzy set of models of K:

$$\pi_K(\omega) = \min_{i=1,n} \max(\mu_{[p_i]}(\omega), 1 - \alpha_i)$$

where $[p_i]$ denotes the sets of models of p_i such that $\mu_{[p_i]}(\omega) = 1$ if $\omega \in [p_i]$ (i.e. $\omega \models p_i$), and $\mu_{[p_i]}(\omega) = 0$ otherwise). In the above formula, the degree of possibility of ω is computed as the complement to 1 of the largest weight of a formula falsified by ω. Thus, ω is all the less possible as it falsifies formulas of higher degrees. In particular, if ω is a counter-model of a formula with weight 1, then ω is impossible, i.e. $\pi_K(\omega) = 0$. It can be shown that π_K is the largest possibility distribution such that $N_K(p_i) \geq \alpha_i, \forall i = 1, n$, i.e., the possibility distribution which allocates the greatest possible possibility degree to each interpretation in agreement with the constraints induced by K (where N_K is the necessity measure associated with π_K, namely $N_K(p) = \min_{v \in [\neg p]}(1 - \pi_K(v))$). It may be that $N_K(p_i) > \alpha_i$, for some i, due to logical constraints between formulas in K. The possibilistic closure corrects the ranking of formulas for the sake of logical coherence.

Moreover, it can be shown that $\pi_K = \pi_{K'}$ if and only if, for any level α, K_α and K'_α are logically equivalent in the classical sense. K and K' are then said to be semantically equivalent. The semantic entailment is then defined by $K \models (p, \alpha)$ if and only if $N_K(p) \geq \alpha$, i.e., if and only if $\forall \omega, \pi_K(\omega) \leq \max(\mu_{[p]}(\omega), 1 - \alpha)$.

Besides, it can be shown that $Inc(K) = 1 - \max_\omega \pi_K(\omega)$. Soundness and completeness are expressed by

$$K \vdash (p, \alpha) \;\Leftrightarrow\; K \models (p, \alpha).$$

In this form of possibilistic entailment, final weights attached to all formulas are at least equal to the inconsistency level of the base. The inconsistency-free formulas, which are above this level, entail propositions that have higher weights. Biacino and Gerla [1992] provide an algebraic analysis of possibility and necessity measures generated by this form of inference. The closure of a possibilistic knowledge base is an example of canonical extension of the closure operator of classical logic in the sense of [Gerla, 2001, Chap. 3].

To summarize, a possibilistic logic base is associated with a *fuzzy set* of models. This fuzzy set is understood as either the set of more or less plausible states of the world (given the available information), or as the set of more or less satisfactory states, according to whether we are dealing with uncertainty or with preference modeling. Conversely, it can be shown that any fuzzy set F representing a fuzzy piece of knowledge, with a membership function μ_F defined on a *finite* set is semantically equivalent to a possibilistic logic base.

There is a major difference between possibilistic logic and weighted many-valued logics of Pavelka-style [Pavelka, 1979; Hájek, 1998a], especially fuzzy Prolog languages like Lee's fuzzy clausal logic [Lee, 1972], although they look alike syntactically. Namely, in the latter, a weight t attached to a (many-valued) formula p often acts as a truth-value threshold, and (p, t) in a fuzzy knowledge base expresses the requirement that the truth-value of p should be at least equal to t for (p, t) to be valid. So in such fuzzy logics, while truth is many-valued, the validity of a weighted formula is two-valued. For instance, in Pavelka-like languages, (p, t) can be encoded as $\overline{t} \to p$ adding a truth-constant $\overline{t}$ to the language. Using Rescher-Gaines implication, $\overline{t} \to p$ has validity 1 if p has truth-value at least t, and 0 otherwise; then (p, t) is Boolean. Of course, using another many-valued implication, (p, t) remains many-valued. On the contrary, in possibilistic logic, truth is two valued (since p is Boolean), but the validity of (p, α) with respect to classical interpretations is many-valued [Dubois and Prade, 2001]. In some sense, weights in Pavelka style may defuzzify many-valued logics, while they fuzzify Boolean formulas in possibilistic logic. Moreover inferring (p, α) in possibilistic logic can be viewed as inferring p with some certainty, quantified by the weight α, while in standard many valued logics (i.e. with a standard notion of proof) a formula is either inferred or not [Hájek, 1998a].

Since possibilistic logic bases are semantically equivalent to fuzzy sets of interpretations, it makes sense to use fuzzy set aggregation operations for merging the bases. Pointwise aggregation operations applied to fuzzy sets can be also directly performed at the syntactic level. This idea was first pointed out by Boldrin [1995] (see also [Boldrin and Sossai, 1995]), and generalized [Benferhat *et al.*, 1998c] to two possibilistic bases $K_1 = \{(p_i, \alpha_i) \mid i \in I\}$ and $K_2 = \{(q_j, \beta_j) \mid j \in J\}$. It can be, in particular, applied to triangular norm and triangular co-norm operations. Let π_T and π_S be the result of the combination of π_{K_1} and π_{K_2} based on a t-norm operation T, and the dual t-conorm operation $S(\alpha, \beta) = 1 - T(1 - \alpha, 1 - \beta)$ respectively. Then, π_T and π_S are respectively associated with the following possibilistic logic bases:

- $K_T = K_1 \cup K_2 \cup \{(p_i \vee q_j, S(\alpha_i, \beta_j)) \mid (p_i, \alpha_i) \in K_1, (q_j, \beta_j) \in K_2\}$,
- $K_S = \{(p_i \vee q_j, T(\alpha_i, \beta_j)) \mid (p_i, \alpha_i) \in K_1, (q_j, \beta_j) \in K_2\}$.

With $T = \min$, $K_{\min} = K_1 \cup K_2$ in agreement with possibilistic logic semantics. This method also provides a framework where symbolic approaches for fusing classical logic bases [Konieczny and Pino-Pérez, 1998] can be recovered by making the implicit priorities induced from Hamming distances between sets of models, explicit [Benferhat *et al.*, 2002; Konieczny *et al.*, 2002].

Bipolar possibilistic logic

A remarkable variant of possibilistic logic is obtained by no longer interpreting weights as lower bounds of necessity (nor possibility) measures, but as constraints in terms of yet another set function expressing *guaranteed* possibility. Section 2.2

recalled how a possibility measure Π and a necessity measure N are defined from a possibility distribution π. However, given a (non-contradictory, non-tautological) proposition p, the qualitative information conveyed by π pertaining to p can be assessed not only in terms of possibility and necessity measures, but also in terms of two other functions. Namely, $\Delta(p) = \min_{\omega \in [p]} \pi(\omega)$ and $\nabla(p) = 1 - \Delta(\neg p)$. Δ is called a *guaranteed possibility* function [Dubois and Prade, 1992c][17]. Thus a constraint of the form $\Delta(p) \geq \alpha$ expresses the guarantee that all the models of p are possible at least at degree α. This is a form of positive information, which contrasts with constraints of the form $N(p) \geq \alpha$ ($\Leftrightarrow \Pi(\neg p) \leq 1 - \alpha$) that rather expresses negative information in the sense that counter-models are then (somewhat) impossible [Dubois *et al.*, 2000].

Starting with a set of constraints of the form $\Delta(p_j) \geq \beta_j$ for $j = 1, \ldots, n$, expressing that (all) the models of p_j are guaranteed to be possible at least at level β_j, and applying a principle of maximal specificity that minimizes possibility degrees, the most informative possibility distribution π_* such that the constraints are satisfied is obtained. Note that this principle is the converse of the one used for defining π_K, and is in the spirit of a closed-world assumption: only what is said to be (somewhat) guaranteed possible is considered as so. Namely

$$\pi_*(\omega) = \max_{j=1,n} \min(\mu_{[p_j]}(\omega), \beta_j).$$

By contrast with Π and N, the function Δ is non-increasing (rather than non-decreasing) w. r. t. logical entailment. Fusion of guaranteed possibility-pieces of information is disjunctive rather than conjunctive (as expressed by π_* by contrast with the definition of π_K). Δ satisfies the characteristic axiom

$$\Delta(p \vee q) = \min(\Delta(p), \Delta(q)),$$

and the basic inference rules, in the propositional case, associated with Δ are

- $[\neg p \wedge q, \alpha], [p \wedge r, \beta] \vdash [q \wedge r, \min(\alpha, \beta)]$ (resolution rule)
- if p entails q classically, $[q, \alpha] \vdash [p, \alpha]$ (formula weakening)
- for $\beta \leq \alpha$, $[p, \alpha] \vdash [p, \beta]$ (weight weakening)
- $[p, \alpha]; [p, \beta] \vdash [p, \max(\alpha, \beta)]$ (weight fusion).

where $[p, \alpha]$ stands for $\Delta(p) \geq \alpha$. The first two properties show the reversed behavior of Δ-based formulas w. r. t. usual entailment. Indeed, if all the models of q are guaranteed to be possible, then it holds as well to any subset of models, e.g. the models of p, knowing that p entails q. Besides, observe that the formula $[p \wedge q, \alpha]$ is semantically equivalent to $[q, \min(v(p), \alpha)]$, where $v(p) = 1$ if p is true and $v(p) = 0$ if p is false. This means that $p \wedge q$ is guaranteed to be possible at least to the level α, if q is guaranteed to be possible to this level when p is true.

[17]Not to be confused with Baaz Δ operator in Section 3.4.

This remark can be used in hypothetical reasoning, as in the case of standard possibilistic formulas. So, Δ-based formulas behave in a way that is very different and in some sense opposite to the one of standard (N-based) formulas (since the function Δ is non-increasing).

When dealing with uncertainty, this leads to a twofold representation setting distinguishing between

- what is not impossible because not ruled out by our beliefs; this is captured by constraints of the form $N(p_i) \geq \alpha_i$ associated with a possibility distribution π^* expressing the semantics of a standard possibilistic knowledge base,
- and what is known as feasible because it has been observed; this is expressed by constraints of the form $\Delta(q_j) \geq \beta_j$ associated with π_*.

In other words, it offers a framework for reasoning with rules and cases (or examples) in a joint manner. Clearly, some consistency between the two types of information (what is guaranteed possible cannot be ruled out as impossible) should prevail, namely

$$\forall\omega, \pi_*(\omega) \leq \pi^*(\omega)$$

and should be maintained through fusion and revision processes [Dubois *et al.*, 2001]. The idea of a separate treatment of positive information and negative information has been also proposed by Atanassov [1986; 1999] who introduces the so-called *intuitionistic fuzzy sets* [18] as a pair of membership and non-membership functions constrained by a direct counterpart of the above inequality (viewing $1 - \pi^*$ as a non-membership function). However, apart from the troublesome use of the word 'intuitionistic' here, the logic of intuitionistic fuzzy sets (developed at the semantic level) strongly differs from bipolar possibilitic logic. See [Dubois *et al.*, 2005] for a discussion. A proposal related to Atanassov's approach, and still different from bipolar possibilitic logic (in spite of its name) can be found in [Zhang and Zhang, 2004].

Possibilistic logic can be used as a framework for qualitative reasoning about preference [Liau, 1999; Benferhat *et al.*, 2001d; Dubois *et al.*, 1999]. When modeling preferences, bipolarity enables us to distinguish between positive desires encoded using Δ, and negative desires (states that are rejected) where N-based constraints describe states that are not unacceptable [Benferhat *et al.*, 2002c]. Deontic reasoning can also be captured by possibilistic logic as shown by Liau [1999]. Namely, necessity measures encode obligation and possibility measures model implicit permission. Dubois *et al.* [2000] have pointed out that Δ functions may account for explicit permission.

4.2 Extensions of possibilistic logic

Possibilistic logic is amenable to different extensions. A first idea is to exploit refined or generalized scales, or yet allows weights to have unknown, or variable

[18] This is a misleading terminology as the underlying algebra does not obey the properties of intuitionistic logic; see [Dubois *et al.*, 2005]

values, while preserving classical logic formulas and weights interpreted in terms of necessity measures. Variable weights enables a form of hypothetical reasoning to be captured, as well as accounting for some kinds of fuzzy rules as we shall see.

Lattice-valued possibilistic logics

The totally ordered scale used in possibilistic logic can be replaced by a complete distributive lattice. Examples of the interest of such a construct include:

- multiple-source possibilistic logic [Dubois *et al.*, 1992b], where weights are replaced by fuzzy sets of sources that more or less certainly support the truth of formulas;
- timed possibilistic logic [Dubois *et al.*, 1991b] where weights are fuzzy sets of time points where formulas are known as being true with some time-dependent certainty levels
- a logic of supporters [Lafage *et al.*, 2000], where weights are sets of irredundant subsets of assumptions that support formulas.

A formal study of logics where formulas are associated with general "weights" in a complete lattice has been carried out by Lehmke [2001b]. Necessity values attached to formulas can be encoded as a particular case of such "weights". More generally, a partially ordered extension of possibilistic logic whose semantic counterpart consists of partially ordered models has been recently proposed by (Benferhat, Lagrue and Papini, [2004b]).

A recent extension [Dubois and Prade, 2006] of possibilistic logic allows a calculus where formulas, which can be nested, encode the beliefs of different agents and their mutual beliefs. One can for instance express that all the agents in a group have some beliefs, or that there is at least one agent in a group that has a particular belief, where beliefs may be more or less entrenched.

Symbolic weights

Rather than dealing with weights in a partially ordered structure, one may consider weights belonging to a linearly ordered structure, but handled in a symbolic manner in such a way that the information that some formulas are known to be more certain than others (or equally certain as others) can be represented by constraints on the weights. This may be useful in particular in case of multiple source knowledge. This idea already present in Benferhat *et al.* [1998a] (where constraints encodes a partial order on the set of sources), has been more recently reconsidered by encoding the constraints as propositional formulas and rewriting the propositional possibilitic logic knowledge base in a two-sorted propositional logic [Benferhat *et al.*, 2004a]. The principle is to translate (p, α) into $p \vee A$ (understood as "p is true or situation is A-abnormal") and $\alpha \leq \beta$ into $\neg B \vee A$ (a statement is all the more certain, as it is more abnormal to have it false, and strong abnormality implies weaker abnormality). This view appears to be fruitful by leading to efficient compilation techniques both when the constraints partially order the weights [Benferhat and Prade, 2005], or linearly order them as in standard possibilistic logic [Benferhat and Prade, 2006].

Variable weights and fuzzy constants

It has been noticed that subparts of classical logic formulas may be 'moved' to the weight part of a possibilistic logic formula. For instance, the possibilistic formula $(\neg p(x) \vee q(x), \alpha)$ is semantically equivalent to $(q(x), \min(\mu_P(x), \alpha))$, where $\mu_P(x) = 1$ if $p(x)$ is true and $\mu_P(x) = 0$ if $p(x)$ is false. It expresses that $q(x)$ is α-certainly true given the proviso that $p(x)$ is true. This is the basis of the use of possibilistic logic in hypothetical reasoning [Dubois *et al.*, 1991a] and case by case reasoning [Dubois and Prade, 1996b], which enables us to compute under what conditions a conclusion could be at least somewhat certain, when information is missing for establishing it unconditionally.

Such *variable weights* can be also useful for fuzzifying the scope of a universal quantifier. Namely, an expression such that $(\neg p(x) \vee q(x), \alpha)$ can be read "$\forall x \in P, (q(x), \alpha)$" where the set $P = \{x \mid p(x) \text{ is true}\}$. Making one step further, P can be allowed to be fuzzy [Dubois *et al.*, 1994c]. The formula $(q(x), \mu_P(x))$ then expresses a piece of information of the form "the more x is P, the more certain $q(x)$ is true". A fuzzy restriction on the scope of an existential quantifier can be also introduced in the following way [Dubois *et al.*, 1998]. From the two classical first order logic premises "$\forall x \in A, \neg p(x, y) \vee q(x, y)$", and "$\exists x \in B, p(x, c)$", where c is a constant, we can conclude that "$\exists x \in B, q(x, c)$" provided that $B \subseteq A$. Let $p(B, c)$ stand for that $\exists x \in B, q(x, c)$". Then B can be called *imprecise constant*. Letting A and B be fuzzy sets, the following pattern can be established:

$$(\neg p(x, y) \vee q(x, y), \min(\mu_A(x), \alpha)); (p(B, c), \beta) \vdash (q(B, c), \min(N_B(A), \alpha, \beta).$$

where $N_B(A) = \inf_t \max(\mu_A(t), 1 - \mu_B(t))$ is the necessity measure of the fuzzy event A based on fuzzy information B and it can be seen as a (partial) degree of unification of A given B. See [Alsinet *et al.*, 1999 ; Alsinet, 2001; Alsinet *et al.*, 2002] for a further development and logical formalization of these ideas in a logic programming framework. In particular, in that context the above pattern can be turned into a sound rule by replacing B by the cut B_β in $N_B(A)$. A complete proof procedure based on a similar resolution rule dealing only with fuzzy constants has been defined [Alsinet and Godo, 2000; Alsinet and Godo, 2001]. This framework has been recently extended in order to incorporate elements of argumentation theory in order to deal with conflicting information [Chesñevar *et al.*, 2004; Alsinet *et al.*, 2006].

Embedding possibilistic logic in a non-classical logic

Another type of extension consists in embedding possibilistic logic in a wider object language adding new connectives between possibilistic formulas. In particular, it is possible to cast possibilistic logic inside a (regular) many-valued logic such as Gödel or Łukasiewicz logic. The idea is to consider many-valued atomic sentences φ of the form (p, α) where p is a formula in classical logic. Then, one can define well-formed formulas of the form $\varphi \vee \psi, \varphi \wedge \psi, \varphi \rightarrow \psi$, etc. where the "external" connectives linking φ and ψ are those of the chosen many-valued logic. From this point of view, possibilistic logic can be viewed as a fragment of Gödel or Łukasiewicz logic that uses only one external connective: conjunction $\wedge$ interpreted as minimum. This approach involving a Boolean algebra embedded in a non-classical one has been proposed by Boldrin and Sossai [1997; 1999] with a view to augment possibilistic logic with fusion modes cast at the object level. Hájek *et al.* [1995] use this method for both probability and possibility theories, thus understanding the probability or the necessity of a classical formula as the truth degree of another formula. This kind of embedding inside a fuzzy logic works for other uncertainty logics as well as explained in section 4.5.

Lastly, possibilistic logic can be cast in the framework of modal logic. Modal accounts of qualitative possibility theory involving conditional statements were already proposed by Lewis [1973a] (this is called the VN conditional logic, see [Dubois and Prade, 1998a; Fariñas and Herzig, 1991]). Other embeddings of possibilistic logic in modal logic are described in [Boutilier, 1994; Hájek, 1994; Hájek *et al.*, 1994].

Possibilistic extensions of non-classical logics

One may consider counterparts to possibilistic logic for non-classical logics, such as many-valued logics. A many-valued logic is cast in the setting of possibility theory by changing the classical logic formula p present in the possibilistic logic formula (p, α) into a many-valued formula, in Gödel or Łukasiewicz logic, for instance. Now (p, α) is interpreted as $C(p) \geq \alpha$, where $C(p)$ is the degree of necessity of a fuzzy event as proposed by Dubois and Prade [Dubois and Prade, 1990] (see section 2.3). Alsinet and Godo [Alsinet, 2001; Alsinet and Godo, 2000] cast possibilistic logic in the framework of Gödel many-valued logic. A possibilistic many-valued formula can also be obtained in first-order logic by making a fuzzy restriction of the scope of an existential quantifier pertaining to a standard first order possibilistic formula, as seen above.

Besnard and Lang [1994] have proposed a possibilistic extension of paraconsistent logic in the same spirit. Quasi-possibilistic logic (Dubois, Konieczny, and Prade [2003]) encompasses both possibilistic logic and quasi-classical logic (a paraconsistent logic due to Besnard and Hunter [1995]; see also [Hunter, 2002]). These two logics cope with inconsistency in different ways, yet preserving the main features of classical logic. Thus, quasi-possibilistic logic preserves their respective merits, and can handle plain conflicts taking place at the same level of certainty (as in quasi-classical logic), while it takes advantage of the stratification of the knowledge base into certainty layers for introducing gradedness in conflict analysis (as in possibilistic logic).

Lehmke [2001a; 2001b] has tried to cast Pavelka-style fuzzy logics and possibilistic logic inside the same framework, considering weighted many-valued formulas of the form (p, τ), where p is a many-valued formula with truth set T, and τ is a "label" defined as a monotone mapping from the truth-set T to a validity set L. T and L are supposed to be complete lattices, and the set of labels has properties that make it a fuzzy extension of a filter in L^T. Labels encompass what Zadeh [1975a] called "fuzzy truth-values" of the form "very true", "more or less true". They are continuous increasing mappings from $T = [0, 1]$ to $L = [0, 1]$ such that $\tau(1) = 1$. A (many-valued) interpretation Val, associating a truth-value $\theta \in T$ to a formula p, satisfies (p, τ), to degree $\lambda \in L$, whenever $\tau(\theta) = \lambda$. When $T = [0, 1], L = \{0, 1\}$, $\tau(\theta) = 1$ for $\theta \geq t$, and 0 otherwise, then (p, τ) can be viewed as a weighted formula in some Pavelka-style logic. When $T = \{0, 1\}$, $L = [0, 1]$, $\tau(\theta) = 1 - \alpha$ for $\theta = 0$, and 1 for $\theta = 1$, then (p, τ) can be viewed as a weighted formula in possibilistic logic. Lehmke [2001a] has laid the foundations for developing such labelled fuzzy logics, which can express uncertainty about (many-valued) truth in a graded way. It encompasses proposals of Esteva *et al.* [1994] who suggested that attaching a certainty weight α to a fuzzy proposition p can be modeled by means of a labeled formula (p, τ), where $\tau(\theta) = \max(1 - \alpha, \theta)$, in agreement with semantic intuitions formalized in [Dubois and Prade, 1990]. This type of generalization highights the difference between many-valued and possibilistic logics.

Refining possibilistic inference

A last kind of extension consists in keeping the language and the semantics of possibilistic logics, while altering the inference relation with a view to make it more productive. Such inference relations that tolerate inconsistency can be defined at the syntactic level [Benferhat *et al.*, 1999a]. Besides, proof-paths leading to conclusions can be evaluated by more refined strategies than just their weakest links [Dubois and Prade, 2004].

4.3 Possibilistic nonmonotonic inference

A nonmonotonic inference notion can be defined in possibilistic logic as $K \vdash_{pref} p$ if and only if $K \vdash (p, \alpha)$ with $\alpha > Inc(K)$. It can be rewritten as $K^{cons} \vdash (p, \alpha)$, where $K^{cons} = K \setminus \{(p_i, \alpha_i) \mid \alpha_i \leq Inc(K)\}$ is the set of weighted formulas whose weights are above the level of inconsistency (they are thus not involved in the inconsistency). Indeed, $Inc(K^{cons}) = 0$. This inference is nonmonotonic because due to the non-decreasingness of the inconsistency level when K is augmented, $K \vdash_{pref} p$ may not imply $K \cup \{(q, 1)\} \vdash_{pref} p$.

The semantic counterpart to the preferential nonmonotonic inference $K \vdash_{pref} p$ (that is, $K \vdash (p, \alpha)$ with $\alpha > Inc(K)$) is defined as $K \models_{pref} p$ if and only if $N_K(p) > Inc(K)$, where N_K derives from the possibility distribution π_K that describes the fuzzy set of models of K. The set $\{\omega \mid \pi_K(\omega) \text{ is maximal}\}$ forms the set of best models $B(K)$ of K. It turns out that $K \models_{pref} p$ if and only if $B(K) \subseteq [p]$ if and only if $K \vdash_{pref} p$. It can be shown that $B(K) \subseteq [p]$ is equivalent to $\Pi_K(p) > \Pi_K(\neg p)$ where Π_K is the possibility measure defined from π_K [Dubois and Prade, 1991c]. Similarly $K \cup \{(p,1)\} \models_{pref} q$ is equivalent to $\Pi_K(p \wedge q) > \Pi_K(p \wedge \neg q)$. The latter corresponds to the idea of inferring a belief q from a contingent proposition p in the context of some background knowledge described by π_K (encoded in K), which we denote $p \models_{\pi_K} q$.

Conversely, a constraint of the form $\Pi(p \wedge q) > \Pi(p \wedge \neg q)$ is a proper encoding of a default rule expressing that in context p, having q true is the normal course of things. Then a knowledge base made of a set of default rules is associated with a set of such constraints that induces a family (possibly empty in case of inconsistency) of possibility measures. Two types of nonmonotonic entailments can be then defined (see [Benferhat *et al.*, 1992; Benferhat *et al.*, 1997a; Dubois and Prade, 1995] for details):

1. the above preferential entailment $\models_\pi$ based on the unique possibility distribution π obeying the above constraints (it leads to an easy encoding of default rules as possibilistic logic formulas);
2. a more cautious entailment, if we restrict to beliefs inferred from all possibility measures obeying the above constraints.

Clearly $p \models_\pi q$ means that when only p is known to be true, q is an expected, normal conclusion since q is true in all the most plausible situations where p is true. This type of inference contrasts with the similarity-based inference of Section 4.4 since in the latter the sets of models of q is enlarged so as to encompass the models of p, while in possibilistic entailment, the set of models of p is restricted to the best ones. Preferential possibilistic entailment $\models_\pi$ satisfies the following properties that characterize nonmonotonic consequence relations $\mid\!\sim$:

Restricted Reflexivity:	$p \mid\!\sim p$, if $\not\models p \equiv \bot$
Consistency Preservation:	$p \not\mid\!\sim \bot$
Left logical equivalence:	if $\models p \equiv p'$, from $p \mid\!\sim q$ deduce $p' \mid\!\sim q$
Right weakening:	from $q \models q'$ and $p \mid\!\sim q$ deduce $p \mid\!\sim q'$
Closure under conjunction:	$p \mid\!\sim q$ and $p \mid\!\sim r$ deduce $p \mid\!\sim q \wedge r$
OR:	from $p \mid\!\sim r$ and $q \mid\!\sim r$ deduce $p \vee q \mid\!\sim r$
Rational monotony:	from $p \mid\!\sim r$ and $p \not\mid\!\sim \neg q$ deduce $p \wedge q \mid\!\sim r$
Cut:	from $p \wedge q \mid\!\sim r$ and $p \mid\!\sim q$ deduce $p \mid\!\sim r$.

But for the two first properties (replaced by a mere reflexivity axiom), these are the properties of the so-called rational inference of Lehmann and Magidor [1992]. Let us explain some of these axioms. Restricted reflexivity just excludes the assumption that everything follows by default from a contradiction. Consistency preservation ensures the consistency of lines of reasoning from consistent arguments. Right weakening and closure for conjunction ensures that the set of plausible consequences of p is a deductively closed set. The OR rule copes with reasoning by cases. Rational monotony controls the amount of monotonicity of the possibilistic inference: from $p \models_\pi r$ we can continue concluding r if q is also true, provided that it does not hold that, in the context p, $\neg q$ is expected. The cut rule is a weak form of transitivity.

Liau and Lin [1996] have augmented possibilistic logic with weighted conditionals of the form

$$p \xrightarrow{\langle c \rangle} q \text{ and } p \xrightarrow{\langle c \rangle^+} q$$

that encode Dempster rule of conditioning ($\Pi(q \mid p) = \Pi(p \wedge q)/\Pi(p)$), and correspond to constraints $\Pi(p \wedge q) \geq c \cdot \Pi(p)$ and $\Pi(p \wedge q) > c \cdot \Pi(p)$ respectively with c being a coefficient in the unit interval. Liau [1998] considers more general conditionals where a t-norm is used instead of the product. Note that if $p = \top$ (tautology), then

$$\top \xrightarrow{\langle c \rangle} q \text{ and } \neg(\top \xrightarrow{\langle 1-c \rangle^+} \neg q)$$

stands for $\Pi(q) \geq c$ for $N(q) \geq c$ respectively. This augmented possibilistic logic enables various forms of reasoning to be captured such as similarity-based and default reasoning as surveyed in [Liau and Lin, 1996].

4.4 Deductive Similarity Reasoning

The question raised by interpolative reasoning is how to devise a logic of similarity, where inference rules can account for the proximity between interpretations of the language. This kind of investigation has been started by Ruspini [1991] with a view to cast fuzzy patterns of inference such as the generalized modus ponens of Zadeh into a logical setting, and pursued by Esteva *et al.* [1994]. Indeed in the scope of similarity modeling, a form of generalized modus ponens can be expressed informally as follows,

$$\frac{\begin{array}{l} p \text{ is } \textit{close} \text{ to being true} \\ p \ \textit{approximately} \text{ implies } q \end{array}}{q \text{ is } \textit{not far} \text{ from being true}}$$

where “close”, “approximately”, and “not far” refer to a similarity relation S, while p and q are classical propositions. The universe of discourse Ω serves as a framework for modeling the meaning of classical propositions p_1, p_2, $\ldots$, p_n in a formal language L, by means of constraints on a set of interpretations Ω. Interpretations are complete descriptions of the world in terms of this language, and assign a truth-value to each propositional variable. Let $[p]$ denote the set of models of proposition p, i.e., the set of interpretations which make p true. If ω is a model of p, this is denoted denoted $w \models p$. The set of interpretations Ω is thus equipped with a similarity relation S, that is a reflexive, symmetric and t-norm-transitive fuzzy relation. The latter property means that there is a triangular norm T such that $\forall \omega, , \omega', \omega'', T(S(\omega, \omega'), S(\omega', \omega'')) \leq S(\omega, \omega'')$. For any subset A of Ω, a fuzzy set A^* can be defined by

(24) $A^*(\omega) = \sup_{\omega' \in A} S(\omega, \omega')$

where $S(\omega, \omega')$ is the degree of similarity between ω and ω'. A^* is the fuzzy set of elements close to A. Then proposition p can be fuzzified into another proposition p^* which means "*approximately* p" and whose fuzzy set of models is $[p^*] = [p]^*$ as defined by (24). Clearly, a logic dealing with propositions of the form p^* is a fuzzy logic in the sense of a many-valued logic, whose truth-value set is the range of $S(\omega, \omega')$, for instance $[0, 1]$. The satisfaction relation is graded and denoted $\models^\alpha$ namely,

$$\omega \models^\alpha p \quad \text{iff} \quad \text{there exists a model } \omega' \text{ of } p \text{ which is } \alpha\text{-similar to } \omega,$$

in other words, iff $[p^*](\omega) \geq \alpha$, i.e., ω belongs to the α-cut of $[p^*]$, that will be denoted by $[p^*]_\alpha$.

One might be tempted by defining a multiple-valued logic of similarity. Unfortunately it cannot be truth-functional. Namely given S, truth evaluations v_ω defined as $v(p) = [p^*](\omega)$, associated to the interpretation ω, are truth-functional neither for the negation not for the conjunction. Indeed, in general, $[p \wedge q]^*(\omega)$ is not a function of $[p^*](\omega)$ and $[q^*](\omega)$ only. This feature can be observed even if S is a standard equivalence relation. Indeed, for $A \subseteq \Omega$, $A^* = S \circ A$ is the union of equivalence classes of elements belonging to A, i.e., it is the upper approximation of A in the sense of rough set theory [Pawlak, 1991], and it is well known that $[A \cap B]^* \subseteq [A]^* \cap [B]^*$ and no equality is obtained (e.g., when $A \cap B = \emptyset$, but $[A]^* \cap [B]^* \neq \emptyset$). This fact stresses the difference between similarity logic and other truth-functional fuzzy logics. The reason is that here all fuzzy propositions are interpreted in the light of a single similarity relation, so that there are in some sense less fuzzy propositions here than in more standard many-valued calculi. Similarity logic is more constrained, since the set of fuzzy subsets $\{[p]^* : p \in L\}$ of Ω induced by classical propositions of the language L, is in a one-to-one correspondence to a Boolean algebra (associated with L), and is only a proper subset of the set $[0, 1]^\Omega$ of all fuzzy subsets of Ω. However it holds that $[A \cup B]^* = [A]^* \cup [B]^*$.

The graded satisfaction relation can be extended over to a graded semantic entailment relation: a proposition p entails a proposition q at degree α, written $p \models^\alpha q$, if each model of p makes q^* at least α-true, where q^* is obtained by means of a T-transitive fuzzy relation S [Dubois *et al.*, 1997]. That is,

$$p \models^\alpha q \text{ holds iff } [p] \subseteq [q^*]_\alpha.$$

$p \models^\alpha q$ means "p entails q, approximately" and α is a level of strength. The properties of this entailment relation are:

Nestedness:	if $p \models^{\alpha} q$ and $\beta \leq \alpha$ then $p \models^{\beta} q$;
T-Transitivity:	if $p \models^{\alpha} r$ and $r \models^{\beta} q$ then $p \models^{T(\alpha,\beta)} q$;
Reflexivity:	$p \models^{1} p$;
Right weakening:	if $p \models^{\alpha} q$ and $q \models r$ then $p \models^{\alpha} r$;
Left strengthening:	if $p \models r$ and $r \models^{\alpha} q$ then $p \models^{\alpha} q$;
Left OR:	$p \vee r \models^{\alpha} q$ iff $p \models^{\alpha} q$ and $r \models^{\alpha} q$;
Right OR:	if r has a single model, $r \models^{\alpha} p \vee q$ iff $r \models^{\alpha} p$ or $r \models^{\alpha} q$.

The fourth and fifth properties are consequences of the transitivity property (since $q \models r$ entails $q \models^{1} r$ due to $[q] \subseteq [r] \subseteq [r^*]_1$). They express a form of monotonicity. The transitivity property is weaker than usual and the graceful degradation of the strength of entailment it expresses, when $T \neq \min$, is rather natural. It must be noticed that $\models^{\alpha}$ does not satisfy the Right And property, i.e., from $p \models^{\alpha} q$ and $p \models^{\alpha} r$ it does not follow in general that $p \models^{\alpha} q \wedge r$. Hence the set of approximate consequences of p in the sense of $\models^{\alpha}$ will not be deductively closed. The left OR is necessary to handle disjunctive information, and the right OR is a consequence of the decomposability w.r.t. the $\vee$ connective in similarity logic. Characterization of the similarity-based graded entailment in terms of the above properties as well as for two other related entailments are given in [Dubois *et al.*, 1997].

The idea of approximate entailment can also incorporate "background knowledge" in the form of some proposition K. Namely, [Dubois *et al.*, 1997] propose another entailment relation defined as $p \models^{\alpha}_{K} q$ iff $[K] \subseteq ([p^*] \to [q^*])_{\alpha}$, where $\to$ is the R-implication associated with the triangular norm T and $[p^*] \to [q^*]$ expresses a form of gradual rule "the closer to the truth of p, the closer to the truth of q". Then, using both $\models^{\alpha}$ and $\models^{\alpha}_{K}$, a deductive notion of interpolation based on gradual rules, as described in Section 2.5, can be captured inside a logical setting. The relations and the differences between similarity-based logics and possibilistic logic are discussed in [Esteva *et al.*, 1994] and in [Dubois and Prade, 1998b]. The presence of a similarity relation on the set of interpretation suggests a modal logic setting for similarity-based reasoning where each level cut S_{α} of S is an accessibility relation. Especially $p \models^{\alpha} q$ can be encoded as $p \models \Diamond_{\alpha} q$, where $\Diamond_{\alpha}$ is the possibility modality induced by S_{α}. Such a multimodal logic setting is systematically developed by Esteva *et al.* [1997b].

Finally, let us mention that a different approach to similarity-based reasoning, with application to the framework of logic programming, has been formally developed in [Ying, 1994; Gerla and Sessa, 1999; Biacino *et al.*, 2000; Formato *et al.*, 2000]. The idea is to extend the classical unification procedure in classical first order logic by allowing partial degrees of matching between predicate and constants that are declared a priori to be similar to some extent. A comparison between both approaches can be found in [Esteva *et al.*, 2001].

4.5 Fuzzy logic theories to reason under uncertainty

Although fuzzy logic is not a logic of uncertainty per se, as it has been stressed in Sections 1 and 2, a fuzzy logic apparatus can indeed be used in a non standard (i.e. non truth-functional) way to represent and reasoning with probability or other uncertainty measures. This is the case for instance of the approach developed by Gerla [1994b]. Roughly speaking, Gerla devises a probability logic by defining a suitable fuzzy consequence operator C, in the sense of Pavelka (see Section 3.8), on fuzzy sets v of the set B of classical formulas (modulo classical equivalence) in a given language, where the membership degree $v(p)$ of a proposition p is understood as lower bound on its probability. A (finitely additive) probability w on B is a fuzzy set (or theory) that is complete, i.e. fulfilling $w(p) + w(\neg p) = 1$ for each $p \in B$. Models of fuzzy set v are probabilities w such that $v \leq w$ (i.e. $v(p) \leq w(p)$ for each p). The probabilistic theory $C(v)$ generated by v is the greatest lower bound of the probabilities greater than or equal to v. Then Gerla defines a fuzzy deduction operator D based on some inference rules to deal with probability envelopes (called the h-m-k-rules and the h-m-collapsing rules) and shows that C and D coincide, this gives the probabilistic completeness of the system.

In a series of works starting in [Hájek *et al.*, 1995], a different logical approach to reason about uncertainty has been developed that is able to combine notions of different classical uncertainty measures (probability, necessity/possibility and belief functions) with elements of t-norm based fuzzy logics: the basic observation is that "uncertainty" or belief is itself a gradual notion, e.g. a proposition may be totally, quite, more or less, or slightly certain (in the sense of probable, possible, believable, plausible, etc.).

For instance in the case of probability, one just starts with Boolean formulas φ and a probability on them; then there is nothing wrong in taking as truth-degree of the fuzzy proposition $P\varphi :=$ "φ is probable" just the probability degree of the crisp proposition φ. Technically speaking, the approach boils down to considering the following identity

$$\text{probability degree of } \varphi = \text{truth degree of } P\varphi,$$

where P is a (fuzzy) modality with the intended reading: $P\varphi$ stands for the fuzzy proposition "φ is probable". Notice that such an approach clearly distinguishes between assertions like "(φ is probable) and (ψ is probable)" on the one hand and "($\varphi \wedge \psi$) is probable" in the other. This is the basic idea exposed in [Hájek *et al.*, 1995] and then later refined by Hájek in [Hájek, 1998a]. Taking Łukasiewicz logic Ł as base logic, this is done by first enlarging the language of Ł by means of a unary (fuzzy) modality P for *probably*, defining two kinds of formulas:

- classical Boolean formulas: $\varphi, \psi, \ldots$ (which are definable in Ł), and
- modal formulas: for each Boolean formula φ, $P(\varphi)$ is an atomic modal formula and, moreover, such a class of modal formulas, MF, is taken closed under the connectives of Łukasiewicz logic $\to_L$ and $\neg_L$,

and then by defining a set of axioms and an inference rule reflecting those of a probability measure, namely:

(FP1) $P(\neg\varphi \vee \psi) \rightarrow_L (P\varphi \rightarrow_L P\psi)$,
(FP2) $P(\neg\varphi) \equiv_L \neg_L P\varphi$,
(FP3) $P(\varphi \vee \psi) \equiv_L ((P\varphi \rightarrow_L P(\varphi \wedge \psi)) \rightarrow_L P\psi)$,

and the necessitation rule for P: *from* φ *infer* $P(\varphi)$, for any Boolean formula φ.

The resulting fuzzy probability logic, $FP(\text{Ł})$, is sound and (finite) strong complete [Hájek, 1998a] with respect to the intended probabilistic semantics given by the class of *probabilistic Kripke models*. These are structures $\mathcal{M} = \langle W, \mu, e\rangle$ where W is a non-empty set, $e : W \times BF \rightarrow \{0, 1\}$ (where BF denotes the set of Boolean formulas) is such that, for all $w \in W$, $e(w, \cdot)$ is a Boolean evaluation of non-modal formulas, and μ is a finitely additive probability measure on a Boolean subalgebra $\Omega \subseteq 2^W$ such that, for every Boolean formula φ, the set $[\varphi]_W = \{w \in W : e(w, \varphi) = 1\}$ is μ-measurable, i.e. $[\varphi]_W \in \Omega$ and hence $\mu([\varphi]_W)$ is defined. Then, the truth-evaluation of a formula $P\varphi$ in a model $\mathcal{M}$ is given by

$$\| P(\varphi) \|_{\mathcal{M}} = \mu([\varphi]_W)$$

and it is extended to compound (modal) formulas using Łukasiewicz logic connectives. The completeness result for $FP(\text{Ł})$ states that a (modal) formula Φ follows (using the axioms and rules of $FP(\text{Ł})$) from a finite set of (modal) formulas Γ iff $\| \Phi \|_{\mathcal{M}} = 1$ in any probabilistic Kripke model $\mathcal{M}$ that evaluates all formulas in Γ with value 1. The same results holds for $FP(RPL)$, that is, if the expansion of Ł with rational truth-constants RPL is used instead of Ł as base logic. Thus both $FP(\text{Ł})$ and $FP(RPL)$ are adequate for a treatment of simple probability.

Let us comment that the issue of devising fuzzy theories for reasoning with conditional probability has also been developed for instance in [Godo *et al.*, 2000; Flaminio and Montagna, 2005; Flaminio, 2005; Godo and Marchioni, 2006] taking $\text{Ł}\Pi\frac{1}{2}$ as base logic instead of Łukasiewicz logic in order to express axioms of conditional probability involving product and division.

The same easy approach can be used to devise a fuzzy modal theory to reason with necessity measures, hence very close to possibilistic logic. In fact, buiding the modal formulas MF as above, just replacing the modality P by another modality N, the logic $FN(\text{Ł})$ is defined as $FP(\text{Ł})$ by replacing the axioms (FP1), (FP2) and (FP3) by the following ones:

(FN1) $N(\neg\varphi \vee \psi) \rightarrow_L (N\varphi \rightarrow_L N\psi)$,
(FN2) $\neg N\bot$
(FN3) $N(\varphi \wedge \psi) \equiv_L (N\varphi \wedge N\psi)$

and keeping the necessitation rule for N: *from* φ *infer* $N(\varphi)$, for any Boolean formula φ. This axiomatization gives completeness with respect to the intended semantics, i.e. w. r. t. the class of *necessity Kripke models* $\mathcal{M} = \langle W, \mu, e\rangle$, where now μ is a necessity measure on a suitable Boolean subalgebra $\Omega \subseteq 2^W$. Note that one can define the dual modality Π, $\Pi\varphi$ as $\neg N \neg \varphi$, and then the truth-value of $\Pi\varphi$ in a necessity Kripke models is just the corresponding possibility degree. If we consider the theory $FN(RPL)$, the necessity modal theory over RPL (thus introducing rational truth-constants), then one faithfully cast possibilistic logic into $FN(RPL)$ by transforming possibilistic logic expressions (p, α) (with α rational) into the modal formulas $\overline{\alpha} \rightarrow_L p$. See [Marchioni, 2006] for an extension to deal with generalized conditional necessities and possibilities.

This kind of approach has been generalized to deal with Dempster-Shafer belief functions[19]. The idea exploited there is that belief functions on propositions can be understood as probabilities of necessities (in the sense of S5 modal formulas). So, roughly speaking, what one needs to do is to define the above $FP(Ł)$ over S5 formulas rather than over propositional calculus formulas. Then the belief modal formula $B\varphi$, where φ is a classical (modality free) formula, is defined as $P\Box\varphi$. The details are fully elaborated in [Godo *et al.*, 2003], including completeness results of the defined fuzzy belief function logic $FB(Ł)$ w. r. t. the intended semantics based on belief functions.

5 CONCLUSION

The idea of developing something like fuzzy logic was already part of Zadeh's concerns in the early fifties. Indeed, one can read in an early position paper of his, entitled *Thinking machines: a new field in Electrical Engineering* the following premonitory statement[20]:

> "Through their association with mathematicians, the electrical engineers working on thinking machines have become familiar with such hitherto remote subjects as Boolean algebra, multivalued logic, and so forth. And it seems that the time is not so far distant when taking a course in mathematical logic will be just as essential to a graduate student in electrical engineering as taking a course in complex variables is at the present time."

It seems that Zadeh's prediction was correct to a large extent.

The historical development of fuzzy logic may look somewhat erratic. The concept of approximate reasoning developed by Zadeh in the seventies in considerable details did not receive great attention at the time, neither from the logical community, nor from the engineering community, let alone the artificial intelligence community, despite isolated related works in the eighties. Many logicians did not like it by lack of a syntax. Engineers exploited very successful, sometimes ad hoc, numerical techniques borrowing only a small part of fuzzy set concepts. They did

[19] A belief function [Shafer, 1975] on a set W is a mapping $bel : 2^W \rightarrow [0,1]$ satisfying the following conditions: $bel(W) = 1$, $bel(\emptyset) = 0$ and $bel(A_1 \cup \ldots \cup A_n) \geq \sum_{\emptyset \neq I \subseteq \{1,\ldots,n\}} (-1)^{|I|+1} bel(\cap_{i \in I} A_i)$, for each n.

[20] appearing in the Columbia Engineering Quarterly, Vol. 3, January 1950, p. 31

not implement the combination projection principle which is the backbone of approximate reasoning (see [Dubois *et al.*, 1999] on this point). Finally there is a long tradition of mutual distrust between artificial intelligence and fuzzy logic, due to the numerical flavor of the latter. Only later on, in the late nineties, approximate reasoning would be at work in possibilistic counterparts to Bayesian networks.

The nineties witnessed the birth of new important research trend on the logical side, which is no less than a strong revival of the multiple-valued logic tradition, essentially prompted by later theoretical developments of fuzzy set theory (especially the axiomatization of connectives). However multiple-valued logic had been seriously criticized at the philosophical level (see the survey paper by Urquart[1986], for instance) because of the confusion between truth-values on the one hand and degrees of belief, or various forms of incomplete information, on the other hand, a confusion that even goes back to pioneers including Łukasiewicz (e.g., the idea of "possible" as a third truth-value). Attempts to encapsulate ideas of non-termination and error values (suggested by Kleene) in many-valued logics in formal specification of software systems also seem to fail (see Hähnle [2005]). In some sense, fuzzy set theory had the merit of giving multiple-valued logic a more natural interpretation, in terms of gradual properties. The point is to bridge the gap between logical notions and non-Boolean (even continuous) representation frameworks. This has nothing to do with the representation of belief. It is interesting to see that the current trend towards applying Łukasiewicz infinite-valued logic and other multiple valued logics is not focused on the handling of uncertainty, but on the approximation of real functions via normal forms (see the works of Mundici [1994], Perfilieva [2004], Aguzzoli and Gerla [2005], etc.). Another emerging topic is the reconsideration of mathematical foundations of set theory in the setting of the general multiple-valued logic setting recently put together [Behounek and Cintula, 2006b], sometimes in a category-theoretical framework [Höhle, 2007].

However, in this new trend, the fundamental thesis of Zadeh, namely that "fuzzy logic is a logic of approximate reasoning" is again left on the side of the road. Yet our contention is that a good approach to ensuring a full revival of fuzzy logic is to demonstrate its capability to reasoning about knowledge and uncertainty. To this end, many-valued logics must be augmented with some kind of modalities, and the natural path to this end is the framework of possibility theory. The case of possibilistic logic is typical of this trend, as witnessed by its connections to modal logic, nonmonotonic logics and non-standard probabilities, along the lines independently initiated by Lewis [1973b] and by Kraus, Lehmann and Magidor [1990]. However, possibilistic logic handles sharp propositions. Recent works pointed out in the last part of this survey make first steps towards a reconciliation between possibility theory, other theories of belief as well, and many-valued logic. Fuzzy logic in the narrow sense being essentially a rigorous symbolic setting to reason about gradual notions (including belief), we argue that this is the way to follow.

BIBLIOGRAPHY

[Adillon and Verdú, 2000] R. J. Adillon and V. Verdú. On a Contraction-Less Intuitionistic Propositional Logic with Conjunction and Fusion. *Studia Logica* 65(1):11-30 (2000)

[Aglianó *et al.*, to appear] P. Aglianó, I. M. A. Ferreirim and F. Montagna. Basic hoops: an algebraic study of continuous t-norms. *Studia Logica*, to appear.

[Aguzzoli, 2004] S. Aguzzoli. Uniform description of calculi for all t-norm logics. In L. Henkin *et al.*(eds.), Proceedings of 34th IEEE International Symposium on Multiple-Valued Logic (ISMVL 2004),pages 38-43, 2004.

[Aguzzoli and Gerla, 2005] S. Aguzzoli and B. Gerla. Normal Forms for the One-Variable Fragment of Hájek's Basic Logic. Proceedings of 35th IEEE International Symposium on Multiple-Valued Logic (ISMVL 2005), pp. 284-289.

[Aguzzoli *et al.*, 2005] S. Aguzzoli, B. Gerla and Z. Haniková. Complexity issues in basic logic. *Soft Computing* 9: 919–934, 2005.

[Aguzzoli *et al.*, 2005] S. Aguzzoli, B. Gerla and C. Manara. Poset Representation for Gdel and Nilpotent Minimum Logics. Proceedings of The 8th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty, ECSQARU 2005, Barcelona, Spain. Lecture Notes in Artificial Intelligence, 3571 Springer, pp. 662-674, 2005.

[Aguzzoli *et al.*, 2006] S. Aguzzoli, B. Gerla and C. Manara. Structure of the algebras of NMG-formulas. Proceedings of the 11th Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems, IPMU 2006, Paris, France, pp. 1620-1627, 2006.

[Alsina *et al.*, 2006] C. Alsina, M. Frank and B. Schweizer. *Associative Functions: Triangular Norms and Copulas*, World Scientific, 2006.

[Alsina *et al.*, 1980] C. Alsina, E. Trillas and L. Valverde (1980) On Some Logical Connectives for Fuzzy Set Theory. Busefal 3, Summer 1980, Université Paul Sabatier. pp. 18-29. Long version in *Journal of Mathematical Analysis and Applications*.

[Alsina *et al.*, 1983] C. Alsina, E. Trillas and L. Valverde. On Some Logical Connectives for Fuzzy Set Theory. *Journal of Math. Analysis and Applications* 93 (1), 1983, pp. 15-26.

[Alsinet, 2001] T. Alsinet. *Logic Programming with Fuzzy Unification and Imprecise Constants: Possibilistic Semantics and Automated Deduction*, Ph. D. Thesis, Technical University of Catalunya, Barcelona, 2001.

[Alsinet *et al.*, 2006] T. Alsinet, C. Chesñevar, L. Godo, S. Sandri and G. Simari (2006) Modeling Defeasible Argumentation within a Possibilistic Logic Framework with Fuzzy Unification. Proc. of the 11th International Conference IPMU 2006 (Information Processing and Mangament of Uncertainty), 1228-1235.

[Alsinet and Godo, 2000] T. Alsinet and L. Godo, A complete calculus for possibilistic logic programming with fuzzy propositional variables. Proc. of the 16th Conference on Uncertainty in Artificial Intelligence (UAI'00), Stanford, Ca., (Morgan Kaufmann, San Francisco, Ca.), 2000, pp.1-10.

[Alsinet and Godo, 2001] T. Alsinet and L. Godo. A proof procedure for possibilistic logic programming with fuzzy constants. In *Proc. of the ECSQARU-2001 Conference*, LNAI 2143, Springer, pages 760–771, 2001.

[Alsinet *et al.*, 1999] T. Alsinet, L. Godo and S. Sandri. On the semantics and automated deduction for PLFC, a logic of possibilistic uncertainty and fuzziness. Proc. of the 15th Conference on Uncertainty in Artificial Intelligence, (UAI'99), Stockholm, Sweden, (Morgan Kaufmann, San Francisco, Ca.), 1999, pp. 3-20.

[Alsinet *et al.*, 2002] T. Alsinet, L. Godo, S. Sandri. Two formalisms of extended possibilistic logic programming with context-dependent fuzzy unification: a comparative description. *Elec. Notes in Theor. Computer Sci.* 66 (5), 2002.

[Atanassov, 1986] K. T. Atanassov. Intuitionistic fuzzy sets. *Fuzzy Sets and Systems*, 20, 87-96,1986.

[Atanassov, 1999] K. T. Atanassov. *Intuitionistic Fuzzy Sets: Theory And Applications*, Physica Verlag, 1999.

[Avron, 1991] A. Avron. Hypersequents, Logical Consequence and Intermediate Logics for Concurrency. *Annals of Mathematics and Artificial Intelligence*,**4**: 225–248, 1991.

[Avron and Konikowska, 2001] A. Avron and B. Konikowska. Decomposition Proof Systems for Gödel-Dummett Logics.*Studia Logica*, 69(2):197?219, 2001.

[Baader *et al.*, 2003] F. Baader, D. Calvanese, D. McGuinness, D. Nardi and P. Patel-Schneider (eds.). *The Description Logic Handbook: Theory, Implementation, and Applications*, Cambridge University Press, 2003.

[Baaz, 1996] M. Baaz. Infinite-valued Gödel logic with 0-1-projections and relativisations. In Petr Hájek, editor, *Gödel'96: Logical Foundations of Mathematics, Computer Science, and Physics*, volume 6 of *Lecture Notes in Logic*, pages 23–33. Springer-Verlag, Brno, 1996.

[Baaz *et al.*, 2002] M. Baaz, P. Hájek, F. Montagna and H. Veith. Complexity of t-tautologies. *Ann Pure Appl Logic* 113: 3–11, 2002.

[Baaz *et al.*, 2004] M. Baaz, A. Ciabattoni and F. Montagna. A proof-Theoreticalinvestigation of Monoidal T-norm Based Logic. *Fundamenta Informaticae* 59, 315-322, 2004.

[Baldwin, 1979] J.F. Baldwin. A new approach to approximate reasoning using a fuzzy logic, *Fuzzy Sets and Systems*, 2, 309-325, 1979.

[Baldwin and Pilsworth, 1979] J.F. Baldwin and B. W. Pilsworth. Fuzzy truth definition of possibility measure for decision classification, *Int. J. of Man-Machine Studies*, 11, 447-463, 1979..

[Baldwin and Pilsworth, 1980] J.F. Baldwin and B. W. Pilsworth. Axiomatic approach to implication for approximate reasoning with fuzzy logic, *Fuzzy Sets and Systems*, 3, 193-219, 1980.

[Behounek and Cintula, 2006a] Libor Behounek and Petr Cintula. Fuzzy logics as the logics of chains. *Fuzzy Sets and Systems* 157(5): 604-610 (2006).

[Behounek and Cintula, 2006b] Libor Behounek and Petr Cintula. From fuzzy logic to fuzzy mathematics: A methodological manifesto. *Fuzzy Sets and Systems* 157(5): 642-646 (2006).

[Bellman and Giertz, 1973] R. Bellman and M. Giertz. On the analytical formalism of the theory of fuzzy sets. Information Sciences 5, 149-156, 1973

[Bellman and Zadeh, 1977] R.E. Bellman and L.A. Zadeh. Local and fuzzy logics, Modern Uses of Multiple-Valued Logic (Epstein G., ed.), D. Reidel, Dordrecht, 103-165, 1977.

[Bělohlávek, 2001] R. Bělohlávek Fuzzy Closure Operators.*Journal of Mathematical Analysis and Applications* 262, 473–489, 2001.

[Bělohlávek, 2002a] R. Bělohlávek Fuzzy closure operators II:induced relations, representation, and examples. *Soft Computing* 7, 53-64, 2002.

[Bělohlávek, 2002b] R. Bělohlávek *Fuzzy Relational Systems: Foundations and Principles.* Kluwer Academic/Plenum Press (Vol. 20 of IFSR Int. Series on Systems Science and Engineering), New York, 2002.

[Bělohlávek, 2002c] R. Bělohlávek. Fuzzy equational logic. *Archive for Math. Logic* 41(2002), 83–90.

[Bělohlávek and Vychodil, 2005] R. Bělohlávek and V. Vychodil. *Fuzzy Equational Logic*. Springer (series: Studies in Fuzziness and Soft Computing, vol. 186), Berlin, 2005.

[Benferhat *et al.*, 2002] S. Benferhat, D. Dubois, S. Kaci and H. Prade. Possibilistic merging and distance-based fusion of propositional information. *Annals of Mathematics and Artificial Intelligence*, 34, 217-252, 2002.

[Benferhat *et al.*, 2002c] S. Benferhat, D. Dubois, S. Kaci and H. Prade, Bipolar representation and fusion of preferences in the possibilistic logic framework. Proc. of the 8th Int. Conf. on Principles of Knowledge Representation and Reasoning, KR?02, Toulouse, France. (Morgan Kaufmann, San Francisco, Ca.), 2002, pp. 421-432.

[Benferhat *et al.*, 1998a] S. Benferhat, D. Dubois, J. Lang, H. Prade, A. Saffiotti and P. Smets . A general approach for inconsistency handling and merging information in prioritized knowledge bases. Proc. of 6th Int. Conf. Principles of Knowledge Representation and Reasoning, Trento, Italy. Morgan Kaufmann, San Francisco, CA, pp. 466-477, 1998.

[Benferhat *et al.*, 1992] S. Benferhat, D. Dubois and H. Prade . Representing default rules in possibilistic logic. Proc. 3rd Inter. Conf. Principles of Knowledge Representation and Reasoning (KR'92), Cambridge, MA,(Morgan Kaufmann, San Francisco, Ca.), 1992, pp. 673-684.

[Benferhat *et al.*, 1997a] S. Benferhat, D. Dubois and H. Prade. Nonmonotonic reasoning, conditional objects and possibility theory. *Artificial Intellig. J.*, 92 (1997), 259-276.

[Benferhat *et al.*, 1998c] S. Benferhat, D. Dubois and H. Prade, From semantic to syntactic approaches to information combination in possibilistic Logic. In: Aggregation and Fusion of Imperfect Information, (B. Bouchon-Meunier, ed.), Physica Verlag, 1998, pp. 141-161.

[Benferhat *et al.*, 1999a] S. Benferhat, D. Dubois and H. Prade Some syntactic approaches to the handling of inconsistent knowledge bases: A comparative study. Part 2: The prioritized case. In: Logic at Work, (E. Orlowska, Ed.), Physica-Verlag, Heidelberg, 1999, pp. 473-511.

[Benferhat *et al.*, 2001d] S. Benferhat, D. Dubois, and H. Prade Towards a possibilistic logic handling of preferences. Applied Intelligence, 14 (2001), 303-317.

[Benferhat *et al.*, 2004a] S. Benferhat, D. Dubois and H. Prade. Logique possibiliste avec calcul symbolique sur des poids partiellement constraints. In Actes des Rencontres Francophones sur la Logique Floue et ses Applications (LFA'04), Toulouse, France, Nov. 18-19, Cpadus, Toulouse, 67-74, 2004.

[Benferhat *et al.*, 2004b] S. Benferhat, S. Lagrue and O. Papini. Reasoning with partially ordered information in a possibilistic framework, *Fuzzy Sets and Systems*, 144, 25-41, 2004.

[Benferhat and Prade, 2005] S. Benferhat and H. Prade. Encoding formulas with partially constrained weights in a possibilistic-like many-sorted propositional logic. Proc. 19th International Joint Conference on Artificial Intelligence IJCAI'05, Edinburgh, Scotland, UK, July 30-August 5, 1281-1286, 2005.

[Benferhat and Prade, 2006] S. Benferhat and H. Prade. Compiling Possibilistic Knowledge Bases. Proc. 17th European Conference on Artificial Intelligence, (ECAI 2006), August 29 - September 1, Riva del Garda, Italy, 337-341, 2006.

[Besnard and Lang, 1994] P. Besnard and J. Lang. Possibility and necessity functions over non-classical logics. Proc of the 10th Conf. Uncertainty in Artificial Intelligence (R. Lopez de Mantaras, D. Poole, eds.) (Morgan Kaufmann, San Francisco, Ca.), 1994, pp.69-76.

[Besnard and Hunter, 1995] P. Besnard and A. Hunter. Quasi-classical logic: Non-trivializable classical reasoning from inconsistent information, Proc. of the European Conference on Symbolic and Quantitative Approaches to Reasoning and Uncertainty (ECSQARU), Fribourg, July 3-5, (Ch. Froidevaux, J. Kohlas, eds.), Lecture Notes in Computer Science 946, Springer, 44-51, 1995.

[Biacino and Gerla , 1992] L. Biacino and G. Gerla. Generated necessities and possibilities, *Int. J. Intelligent Systems*, 7 (1992), 445-454.

[Biacino *et al.*, 2000] L. Biacino, G. Gerla and M. Ying Approximate Reasoning Based on Similarity. *Mathematical Logic Quarterly*, Vol. 46, N. 1, pp. 77-86, 2000.

[Blok and Pigozzi, 1989] J. Willem Blok and Don Pigozzi. *Algebraizable Logics*, volume 396 of *Memoirs of the American Mathematical Society.* American Mathematical Society, Providence, 1989.

[Boixader and Jacas, 1998] D. Boixader, J. Jacas Extensionality Based Approximate Reasoning. *International Journal of Approximate Reasoning* 19, 221-230, 1998.

[Boldrin, 1995] L. Boldrin. A substructural connective for possibilistic logic. In: Symbolic and Quantitative Approaches to Reasoning and Uncertainty (Proc. of Europ. Conf. ECSQARU'95) C. Froidevaux, J. Kohlas, eds.), Springer Verlag, Fribourg, pp. 60-68, 2005.

[Boldrin and Sossai, 1995] L. Boldrin and C. Sossai, An algebraic semantics for possibilistic logic. Proc of the 11th Conf. Uncertainty in Artificial Intelligence (P. Besnard, S. Hank, eds.) Morgan Kaufmann, San Francisco, CA, 1995, pp. 27-35.

[Boldrin and Sossai, 1997] L. Boldrin and C. Sossai, Local possibilistic logic. *J. Applied Non-Classical Logics*, 7 (1997), 309-333

[Boldrin and Sossai, 1999] L. Boldrin and C. Sossai. Truth-functionality and measure-based logics. In: Fuzzy Sets, Logic and Reasoning about Knowledge, (Dubois, D., Prade, H. and Klement, E.P., Eds.), Kluwer, Dordrecht, vol. 15 in Applied Logic Series, 1999, pp.351-380.

[Bouchon-Meunier *et al.*, 1999] B. Bouchon-Meunier, D. Dubois, L. Godo, H. Prade . Fuzzy sets and possibility theory in approximate and plausible reasoning. Fuzzy Sets in Approximate reasoning and Information Systems (Bezdek, J. Dubois, D. Prade, H., Eds): Kluwer, Boston, Mass., The Handbooks of Fuzzy Sets, 15-190, 1999.

[Boutilier, 1994] C. Boutilier. Modal logics for qualitative possibility theory, *Int. J. Approximate Reasoning*, 10, 173-201, 1994.

[Bova and Montagna, 2007] S. Bova and F. Montagna. Proof Search in Hajek's Basic Logic. *ACM Transactions on Computational Logic*, to appear.

[Butnariu and Klement , 1995] D. Butnariu, E. P. Klement, and S. Zafrany. On triangular norm-based propositional fuzzylogics. *Fuzzy Sets Systems* 69: 241-255, 1995.

[Carnap, 1949] Carnap R. (1949). The two concepts of probability, *Philosophy and Phenomenological Research*, 513-532, 1949.

[Castro and Klawonn, 1994] F. Klawonn and J.L. Castro. Similarity in Fuzzy Reasoning. *Mathware and SoftComputing* 2, 197-228, 1994.

[Castro and Trillas, 1991] J.L. Castro and E. Trillas. Tarski's Fuzzy Consequences. Proc. of IFES'91. Yokohama (Japan). Vol. 1, 70-81, 1991.

[Castro *et al.*, 1994] J.L. Castro, E. Trillas and S. Cubillo (1994). On consequence in approximate reasoning. *Journal ofApplied Non-Classical Logics*, vol. 4, n. 1, 91-103.

[Chakraborty, 1988] M.K. Chakraborty. Use of fuzzy set theory in introducing graded consequence in multiple valued logic. In *Fuzzy Logic in KnowledgeSystems, Decision and Control*, M.M. Gupta and T.Yamakawa (eds).North Holland, Amsterdam, pp. 247-257, 1998.

[Chakraborty, 1995] M.K. Chakraborty. Graded consequence: further studies. *Journal of Applied Non-Classical Logics* 5 (2), 227-237, 1995.

[Chang, 1958] Chen Chung Chang. Algebraic analysis of many-valued logics. *Trans. Amer. Math. Soc.*, 88:456–490, 1958.

[Chang, 1959] C.C. Chang. A new proof of the completeness of the Łukasiewicz axioms. *Transactions of the American Mathematical Society* 93 (1959), pp. 74–90.

[Chesñevar *et al.*, 2004] C. Chesñevar, G.Simari, T. Alsinet and L.Godo (2004) A Logic Programming Framework for Possibilistic Argumentation with Vague Knowledge. Procs. of the Uncertainty in Artificial Intelligence Conference (UAI-2004), Banff (Canada), July 7-11, (M. Chickering and J. Halpern Eds), AUAI Press, pages 76–84.

[Chung and Schwartz, 1995] H.-T. Chung and D.G. Schwartz. A resolution-based system for symbolic approximate reasoning, *International Journal of Approximate Reasoning*, 13, 3 (1995) 201-246.

[Ciabattoni *et al.*, 2002] A. Ciabattoni, F. Esteva and L. Godo. T-norm based logics with n-contraction. *Special Issue on SOFSEM2002 of Neural Network World*, 12(5):453–460, 2002.

[Ciabattoni *et al.*, 2005] A. Ciabattoni, C. Fermüller and G. Metcalfe. Uniform Rules and Dialogue Games for Fuzzy Logics. In Proceedings of LPAR 2004, volume 3452 of LNAI, pages 496-510, 2005

[Cignoli *et al.*, 1999] R. Cignoli, I.M.L. D'Ottaviano and D. Mundici. *Algebraic Foundations of many-valued reasoning*. Kluwer Academic Press, Dordrecht-Boston-London, 1999.

[Cignoli *et al.*, 2000] R. Cignoli, F. Esteva, L. Godo, and A. Torrens. Basic fuzzy logic is the logic of continuous t-norms and their residua. *Soft Computing*, 4(2):106–112, 2000.

[Cintula, 2001a] P. Cintula. The ŁΠ. and ŁΠ1/2 propositional and predicate logics. *Fuzzy Sets and Systems* 124(3): 289-302 (2001)

[Cintula, 2001c] P. Cintula. An alternative approach to the ŁΠ logic. *Neural Network World* 124: 561-575 (2001).

[Cintula, 2003] P. Cintula: Advances in the ŁΠ and ŁΠ$\frac{1}{2}$ logics. *Arch. Math. Log.* 42(5): 449-468 (2003).

[Cintula, 2005a] P. Cintula. Short note: on the redundancy of axiom (A3) in BL and MTL. *Soft Comput.* 9(12): 942-942 (2005)

[Cintula, 2005b] P. Cintula. A note to the definition of the ŁΠ-algebras. *Soft Comput.* 9(8): 575-578 (2005)

[Cintula, 2005d] P. Cintula. *From Fuzzy Logic to Fuzzy Mathematics*. Ph.D. dissertation, Czech Technical University, Prague (Czech Republic), 2005.

[Cintula, 2006] P. Cintula. Weakly implicative (fuzzy) logics I: Basic properties. *Archive for Mathematical Logic*, 45(6):673–704, 2006.

[Cintula and Gerla, 2004] Petr Cintula and Brunella Gerla. Semi-normal forms and functional representation of product fuzzy logic. *Fuzzy Sets and Systems* 143, 89–110, 2004.

[Cintula *et al.*, 2006] P. Cintula, E.P. Klement, R. Mesiar, and M. Navara. Residuated logics based on strict triangular norms with an involutive negation *Mathematical Logical Quarterly* 52(3): 269-282 (2006).

[Cross and Sudkamp, 1994] V. Cross and T. Sudkamp. Patterns of fuzzy rule-based inference, *Int. J. of Approximate Reasoning*, 11, 235-255, 1994.

[de Finetti, 1936] B. de Finetti. La logique de la probabilité, Actes Congrès Int. de Philos. Scient., Paris 1935. Hermann et Cie Editions, Paris, IV1- IV9.

[Di Nola *et al.*, 2002] A. di Nola, G. Georgescu, and A. Iorgulescu. Pseudo-BL algebras I and II. *J. Multiple-Valued Logic* 8: 671–750, 2002.

[Di Nola *et al.*, 1985] A. di Nola, W. Pedrycz and S. Sessa. Fuzzy relation equations and algorithms of inference mechanism in expert systems, Approximate Reasoning in Expert Systems (Gupta M. M., Kandel A., Bandler W. and Kiszka J. B., eds.), North-Holland, Amsterdam, 355-367, 1985.

[Di Nola *et al.*, 1989] A. di Nola, W. Pedrycz and S. Sessa. An aspect of discrepancy in the implementation of modus ponens in the presence of fuzzy quantities, *Int. J. of Approximate Reasoning*, 3, 259-265, 1999.

[Domingo *et al.*, 1981] X. Domingo, E. Trillas and L. Valverde. Pushing Łukasiewicz-Tarski implication a little farther. Proc. IEEE Int. Symposium on Multiple-valued Logic (ISMVL'81), 232-234, 1981..

[Dubois, 1980] D. Dubois. Triangular norms for fuzzy sets. Proc. of 2nd International Linz Seminar on Fuzzy Set Theory, E.P. Klement (ed.), 39-68, 1980.

[Dubois *et al.*, 1997] D. Dubois, F. Esteva, P. Garcia, L. Godo and H. Prade. A logical approach to interpolation based on similarity relations. *International Journal of Approximate Reasoning*, 17(7), 1997, 1-36.

[Dubois *et al.*, 2005] D. Dubois, F. Esteva, L. Godo and H. Prade. An information-based discussion of vagueness. Handbook of Categorization in Cognitive Science, (Henri Cohen, Claire Lefebvre, Eds.) Chap. 40, Elsevier, 2005 pp. 892-913.

[Dubois *et al.*, 1994] D. Dubois, M. Grabisch and H. Prade. Gradual rules and the approximation of control laws, Theoretical Aspects of Fuzzy Control (Nguyen H. T., Sugeno M., Tong R. and Yager R. R., eds.), Wiley, New York, 147-181, 1984.

[Dubois *et al.*, 2005] D. Dubois, S. Gottwald, P. Hájek, J. Kacprzyk and H. Prade. Terminological difficulties in fuzzy set theory - The case of Intuitionistic Fuzzy Sets (with a reply by K. T. Atanassov, 496-499), *Fuzzy Sets and Systems* 156, 485-491, 2005.

[Dubois *et al.*, 2000] D. Dubois, P. Hájek and H. Prade. Knowledge-Driven versus data-driven logics. *Journal of Logic, Language and Information* 9, 65–89, 2000.

[Dubois *et al.*, 2003] D. Dubois, S. Konieczny and H. Prade. Quasi-possibilistic logic and its measures of information and conflict. *Fundamenta Informaticae* 57, 101-125, 2003.

[Dubois *et al.*, 1987] D. Dubois, J. Lang and H. Prade. Theorem proving under uncertainty - A possibility theory-based approach. Proc. of the 10th Inter. Joint Conf. on Artificial Intelligence IJCAI'87, Milano, Italy, 1987, pp. 984-986.

[Dubois *et al.*, 1991a] D. Dubois, J. Lang and H. Prade. A possibilistic assumption-based truth maintenance system with uncertain justifications and its application to belief revision. In: Truth Maintenance Systems, (J.P. Martins, M. Reinfrank, eds.), LNAI 515, Springer Verlag, 1991, pp. 87-106.

[Dubois *et al.*, 1991b] D. Dubois, J. Lang and H. Prade. Timed possibilistic logic. *Fundamenta Informaticae*, XV (1991), Numbers 3-4, 211-237.

[Dubois *et al.*, 1991c] D. Dubois, J. Lang and H. Prade. Fuzzy sets in approximate reasoning - Part 2: Logical approaches, *Fuzzy Sets and Systems*, 40, 203-244, 1991.

[Dubois *et al.*, 1992b] D. Dubois, J. Lang and H. Prade Dealing with multi-source information in possibilitic logic. Proc. of the 10th Eur. Conf. on Artificial Intelligence (ECAI'92), Vienna, Austria. Wiley, New-York, 1992, pp. 38-42.

[Dubois *et al.*, 1994a] D. Dubois, J. Lang and H. Prade. Possibilistic logic. In: Handbook of Logic in Artificial Intelligence and Logic Programming, (D.M. Gabbay *et al.*, eds.), Vol. 3, Oxford Univ. Press, Oxford, UK, 1994, pp. 439-513.

[Dubois *et al.*, 1994b] D. Dubois, J. Lang and H. Prade. Automated reasoning using possibilistic logic: Semantics, belief revision and variable certainty weights, *IEEE Trans. Data & Knowledge Engineering* 6 (1994), 64-71.

[Dubois *et al.*, 1994c] D. Dubois, J. Lang and H. Prade. Handling uncertainty, context, vague predicates, and partial inconsistency in possibilistic logic. In: Fuzzy Logic and Fuzzy Control (Proc. of the IJCAI'91 Workshop) (D. Driankov, P.W. Eklund, A.L. Ralescu, eds.), LNAI 833, Springer-Verlag, Berlin, 1994, pp.45-55.

[Dubois *et al.*, 1999] D. Dubois, D. Le Berre and H. Prade, R. Sabbadin. Using possibilistic logic for modeling qualitative decision: ATMS-based algorithms. *Fundamenta Informaticae* 37 (1999), 1-30.

[Dubois *et al.*, 1997] D. Dubois, S. Lehmke and H. Prade. A comparative study of logics of graded uncertainty and logics of graded truth. Proc. 18th International on Fuzzy Set Theory (Enriched Lattice Structures for Many-Valued and Fuzzy Logics), Linz, Austria, 1997.

[Dubois *et al.*, 2006] D. Dubois, J. Mengin and H. Prade. Possibilistic uncertainty and fuzzy features in description logic. A preliminary discussion. In: Fuzzy logic and the semantic web (E. Sanchez, ed.), Elsevier, 101-113, 2006.

[Dubois and Prade, 1979a] D. Dubois and H. Prade. New Results about Properties and Semantics of Fuzzy Set-theoretic Operators. First Symposium on Policy Analysis and Information Systems. Durham, North Caroline, USA, 167-174, 1979.

[Dubois and Prade, 1979b] D. Dubois and H. Prade. Operations in a fuzzy-valued logic, *Information and Control* 43(2), 224-240, 1979.

[Dubois and Prade, 1980] D. Dubois and H. Prade, *Fuzzy Sets and Systems - Theory and Applications*, New York: Academic Press, 1980.

[Dubois and Prade, 1984a] D. Dubois and H. Prade. A theorem on implication functions defined from triangular norms, *Stochastica* 8(3), 267-279, 1984.

[Dubois and Prade, 1984b] D. Dubois and H. Prade. Fuzzy logics and the generalized modus ponens revisited, *Cybernetics and Systems* 15, 293-331, 1984.

[Dubois and Prade, 1985a] D. Dubois and H. Prade. Evidence measures based on fuzzy information, *Automatica* 21(5), 547-562, 1985.

[Dubois and Prade, 1985b] D. Dubois and H. Prade. The generalized modus ponens under sup-min composition -A theoretical study-, Approximate Reasoning in Expert Systems (Gupta M. M., Kandel A., Bandler W. and Kiszka J. B., eds.), North-Holland, Amsterdam, 217-232, 1985.

[Dubois and Prade, 1987] D. Dubois and H. Prade. Necessity measures and the resolution principle. *IEEE Trans. Systems, Man and Cybernetics*, 17 (1987), 474-478.

[Dubois and Prade, 1988a] D. Dubois and H. Prade. *Possibility Theory*, Plenum Press, New-York, 1988.

[Dubois and Prade, 1988b] D. Dubois and H. Prade. An introduction to possibilistic and fuzzy logics (with discussions and a reply), Non Standard Logics for Automated Reasoning (Smets P., Mamdani A., D. Dubois and H. Prade., eds.), Academic Press, 287-315 and 321-326, 1988.

[Dubois and Prade, 1989] D. Dubois and H. Prade. A typology of fuzzy "if... then..." rules, Proc. of the 3rd Inter. Fuzzy Systems Association (IFSA'89), Congress, Seattle, WA, Aug. 6-11, 782-785, 1989.

[Dubois and Prade, 1990] D. Dubois and H. Prade. Resolution principles in possibilistic logic. *Int. J. Approx. Reasoning*, 4 (1990), pp. 1-21.

[Dubois and Prade, 1991a] D. Dubois and H. Prade. Fuzzy sets in approximate reasoning - Part 1: Inference with possibility distributions, *Fuzzy Sets and Systems*, 40, 143-202, 1991.

[Dubois and Prade, 1991b] D. Dubois and H. Prade. Epistemic entrenchment and possibilistic logic, *Artificial Intelligence*, 50 (1991), 223-239.

[Dubois and Prade, 1991c] D. Dubois and H. Prade. Possibilistic logic, preferential models, non-monotonicity and related issues. In Proc. of the Inter. Joint Conf. on Artificial Intelligence (IJCAI'91), Sydney, Australia, Aug. 24-30, 419-424, 1991.

[Dubois and Prade, 1992a] D. Dubois and H. Prade. Gradual inference rules in approximate reasoning, *Information Sciences*, 61, 1992, 103-122, 1992.

[Dubois and Prade, 1992b] D. Dubois and H. Prade. (1992b). Fuzzy rules in knowledge-based systems. Modelling gradedness, uncertainty and preference, An Introduction to Fuzzy Logic Applications in Intelligent Systems (Yager R. R. and Zadeh L. A., eds.), Kluwer, Dordrecht, 45-68, 1992.

[Dubois and Prade, 1992c] D. Dubois and H. Prade. Possibility theory as a basis for preference propagation in automated reasoning. Proc. 1st IEEE Inter. Conf. on Fuzzy Systems (FUZZ-IEEE'92), San Diego, Ca., 1992, pp. 821-832.

[Dubois and Prade, 1994] D. Dubois and H. Prade. Can we enforce full compositionality in uncertainty calculi?, Proc. of the 12th National Conf. on Artificial Intelligence (AAAI'94), Seattle, WA, 149-154, 1994.

[Dubois and Prade, 1995] D. Dubois and H. Prade. Conditional objects, possibility theory and default rules. In: Conditionals: From Philosophy to Computer Science (G. Crocco, L. Fariñas del Cerro, A. Herzig, eds.), Oxford University Press, Oxford, UK, 1995, pp. 311-346.

[Dubois and Prade, 1996a] D. Dubois and H. Prade. What are fuzzy rules and how to use them, *Fuzzy Sets and Systems*, 84, 169-185, 1996.

[Dubois and Prade, 1996b] D. Dubois and H. Prade, Combining hypothetical reasoning and plausible inference in possibilistic logic. *J. of Multiple Valued Logic*, 1 (1996), 219-239 .

[Dubois and Prade, 1998a] D. Dubois and H. Prade, Possibility theory: Qualitative and quantitative aspects, In: Handbook of Defeasible Reasoning and Uncertainty Management Systems, Vol. 1. (D. M. Gabbay and P. Smets P., Eds.), Dordrecht: Kluwer Academic, 1998, pp. 169-226.

[Dubois and Prade, 1998b] D. Dubois and H. Prade. Similarity vs. preference in fuzzy set-based logics. In *Incomplete information: Rough Set Analysis* (Orlowska E. ed.) Physica-Verlag, Heidelberg, 1998.

[Dubois and Prade, 2001] D. Dubois and H. Prade. Possibility theory, probability theory and multiple-valued logics: A clarification. *Annals of Mathematics and Artificial Intelligence* 32, 35-66, 2001.

[Dubois and Prade, 2004] D. Dubois and H. Prade. Possibilistic logic: a retrospective and prospective view. *Fuzzy Sets and Systems*, 144, 3-23, 2004.

[Dubois and Prade, 2006] D. Dubois and H. Prade. Extensions multi-agents de la logique possibiliste. In Actes des Rencontres Francophones sur la Logique Floue et ses Applications (LFA'06), Toulouse, France, Oct. 19-20, 137-144, 2006.

[Dubois *et al.*, 1993] D. Dubois, H. Prade and R. R. Yager (eds) *Readings in Fuzzy Sets for Intelligent Systems*, Morgan Kaufmann, 1993.

[Dubois *et al.*, 1998] D. Dubois, H. Prade and S. Sandri. A possibilistic logic with fuzzy constants and fuzzily restricted quantifiers. In: Logic Programming and Soft Computing (T.P. Martin and F. Arcelli-Fontana, Eds.), Research Studies Press, Ltd, Baldock, England, 1998, pp. 69-90.

[Dubois *et al.*, 2001] D. Dubois, H. Prade and P. Smets, "Not impossible" vs. "guaranteed possible" in fusion and revision. Proc. 6th Europ. Conf. on Symbolic and Quantitative Approaches to reasoning with Uncertainty ECSQARU-01, Toulouse, LNAI 2143, Springer Verlag, Berlin, 2001, pp. 522-531.

[Dubois *et al.*, 1999] D. Dubois, H. Prade and L. Ughetto. Fuzzy logic, control engineering and artificial intelligence. In: Fuzzy Algorithms for Control, (H.B. Verbruggen, H.-J. Zimmermann, R. Babuska, eds.), Kluwer Academic, 17-57, 1999.

[Dubois *et al.*, 2003] D. Dubois and H. Prade and L. Ughetto. New perspective on reasoning with fuzzy rules. *International Journal of Intelligent Systems*, 18,541-567, 2003.

[Dummett, 1959] Michael Dummett. A propositional calculus with denumerable matrix. *Journal of Symbolic Logic*, 27:97–106, 1959.

[Dvořák and Novák, 2004] Antonín Dvořák and Vilém Novák. Formal theories and linguistic descriptions. *Fuzzy Sets and Systems* 143(1), 169–188, 2004.

[Elkan, 1994] C. Elkan. The paradoxical success of fuzzy logic. (with discussions by many scientists and a reply by the author), *IEEE Expert*, August, 3-46, 1994.

[Elorza and Burillo, 1999] J. Elorza, P. Burillo. On the relation of Fuzzy Preorders and Fuzzy-consequence Operators. *Int. J. of Uncertainty, Fuzziness andKnowledge-based Systems*, Vol 7, (3), 219-234, 1999.

[Esteva *et al.*, 1994] F. Esteva, P. Garcia and L. Godo. Relating and extending semantical approaches to possibilistic reasoning. *International Journal of Approximate Reasoning*, 10(4), 311-344, 1994.

[Esteva *et al.*, 1997a] F. Esteva. P. Garcia and L. Godo. On the Semantics of Fuzzy Statements Based on Possibilistic Constraints, Proc. VII Congreso Español sobre Tecnologías y Lógica Fuzzy, ESTYLF'97, Tarragona. Universitat Rovira i Virgili, 21-27, 1997.

[Esteva *et al.*, 2001] F. Esteva, P. Garcia and L. Godo. On syntactical and semantical approaches to similarity-based approximate reasoning, Proceedings of Joint 9th IFSA World Congress and 20th NAFIPS International Conference, July 25-28, 2001,Vancouver (BC), Canada, pp. 1598-1603.

[Esteva *et al.*, 1997b] F. Esteva, P. Garcia, L. Godo, and R. Rodríguez. A modal account of similarity-based reasoning. *International Journal of Approximate Reasoning*, 16(3-4):235–260, 1997.

[Esteva *et al.*, 1998] F. Esteva, P. Garcia, L. Godo, R.O.Rodríguez. Fuzzy Approximation Relations, Modal Structures and Possibilistic Logic. *Mathware and SoftComputing*, 5, n. 2-3, 151-166, 1998.

[Esteva *et al.*, 2002] F. Esteva, J. Gispert, L. Godo and F. Montagna. On the standard and rational completeness of some axiomatic extensions of the monoidal t-norm logic. *Studia Logica*, 71(2):199-226, 2002

[Esteva *et al.*, 2007a] F. Esteva, J. Gispert, L. Godo, and C. Noguera. Adding truth-constants to continuous t-norm based logics: Axiomatization and completeness results. *Fuzzy Sets and Systems* 158: 597–618, 2007.

[Esteva and Godo, 1999] F. Esteva and L. Godo. Putting together Łukasiewicz and product logics. *Mathware and Soft Computing*, Vol. VI, n.2-3, pp. 219-234, 1999.

[Esteva and Godo, 2001] Francesc Esteva and Lluís Godo. Monoidal t-norm based logic: Towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems*, 124(3):271–288, 2001.

[Esteva and Godo (eds.), 2005] F. Esteva and L. Godo (eds.), Special issue on BL-algebras *Soft Computing* 9(2), 2005.

[Esteva *et al.*, 2003] F. Esteva, L. Godo and A.García-Cerdaña. On the hierarchy of t-norm based residuated fuzzy logics. In *Beyond Two: Theory andApplications of Multiple-Valued Logic*, Ed. M.Fitting and E.Orlowska, Springer-Verlag (2003) 251–272.

[Esteva *et al.*, 2000] F. Esteva, L. Godo, P. Hájek and M. Navara. Residuated fuzzy logics with an involutive negation. *Archive for Mathematical Logic*, 39(2):103–124, 2000.

[Esteva *et al.*, 2003] F. Esteva, L. Godo, P. Hájek, and F. Montagna. Hoops and fuzzy logic. *Journal of Logic and Computation*, 13:531?555, 2003.

[Esteva *et al.*, 2001] F. Esteva, L. Godo and F. Montagna. The ŁΠ and ŁΠ$\frac{1}{2}$ logics: Two complete fuzzy systems joining Łukasiewicz and product logics. *Archive for Mathematical Logic*, 40(1):39–67, 2001.

[Esteva *et al.*, 2004] F. Esteva, L. Godo and F. Montagna. Equational characterization of the subvarieties of BL generated by t-norm algebras. *Studia Logica* 76, 161-200, 2004.

[Esteva *et al.*, 2006] Francesc Esteva, Lluís Godo, and Carles Noguera. On rational Weak Nilpotent Minimum logics. *Journal of Multiple-Valued Logic and Soft Computing*, Vol. 12, Number 1-2, pp. 9-32, 2006.

[Esteva *et al.*, 2007b] F. Esteva, L. Godo, and C. Noguera. On expansions of t-norm based logics with truth-constants. To appear in the book *Fuzzy Logics and Related Structures* (S. Gottwald, P. Hájek, U. Höhle and E.P. Klement eds.), Elsevier, 2007.

[Fariñas and Herzig, 1991] L. Fariñas del Cerro and A. Herzig. A modal analysis of possibility theory, Fundamentals of Artificial Intelligence Research (FAIR'91) (Jorrand Ph. and Kelemen J., Eds.), Lecture Notes in Computer Sciences, Vol. 535, Springer Verlag, Berlin, 1991, pp. 11-18.

[Fariñas *et al.*, 1994] L. Fariñas del Cerro, A. Herzig and J. Lang, From ordering-based nonmonotonic reasoning to conditional logics, *Artificial Intelligence*, 66, 375-393, 1994.

[Fine, 1975] K. Fine (1975). Vagueness, truth and logic, *Synthese*, 30: 265-300.

[Flaminio, 2005] T. Flaminio. *A Zero-Layer Based Fuzzy Probabilistic Logic for Conditional Probability.* Lecture Notes in Artificial Intelligence, **3571**: 8th European Conference on Symbolic and Quantitaive Approaches on Reasoning under Uncertainty ECSQARU'05, Barcelona, Spain, July 2005. 714–725.

[Flaminio and Marchioni, 2006] Tommaso Flaminio and Enrico Marchioni. T-norm based logics with an independent involutive negation. *Fuzzy Sets and Systems*, Vol. 157, Issue 24, 3125-3144, 2006.

[Flaminio and Montagna, 2005] T. Flaminio and F. Montagna. A Logical and Algebraic Treatment of Conditional Probability. *Archive for Mathematical Logic*, **44**, 245–262 (2005).

[Fodor, 1989] J. Fodor. Some remarks on fuzzy implication operations, BUSEFAL (IRIT, Univ. P. Sabatier, Toulouse, France), 38, 42-46, 1989.

[Fodor, 1995] J. Fodor. Nilpotent minimum and related connectives for fuzzy logic. Proc. of FUZZ–IEEE'95, 1995, pp. 2077–2082.

[Fodor and Yager, 2000] J.C. Fodor and R.R. Yager Fuzzy Set-theoretic Operators and Quantifiers. Chapter 1.2 in: (D. Dubois and H. Prade, Eds.) Handbook of Fuzzy Sets and Possibility Theory, Vol. 1: Basic Notions, Kluwer,Boston, MA, 2000, pp. 125-193.

[Formato *et al.*, 2000] F. Formato, G. Gerla, and M.I. Sessa. Similarity-based unification. *Fundamenta Informaticae*, 40:1–22, 2000.

[Fukami *et al.*, 1980] S. Fukami, M. Mizumoto and K. Tanaka K. (1980). Some considerations on fuzzy conditional inference, *Fuzzy Sets and Systems*, 4, 243-273.

[Fung and Fu, 1975] L.W. Fung and K.S. Fu. An axiomatic approach to rational decision making in a fuzzy environment. In "Fuzzy sets and their applications to cognitive and decision processes", (L.A. Zadeh, K.S. Fu, K. Tanaka, M. Shimura eds.), Academic Press, 227-256, 1975.

[Gabbay, 1996] D.M. Gabay. How to make your logic fuzzy (fibred semantic and weaving of logics, part 3. In D. Dubois, E.P. Klement, and H. Prade, editors, *Fuzzy Set, Logics, and Artificial Intelligence*, pages 69–89, 1996.

[Gabbay, 1997] D.M. Gabay. Fibring and labelling: Two methods for making modal logic fuzzy. In M. Mares, et al. eds. *Proc. of Seventh International Fuzzy Systems Association World Congress IFSA'97*, Vol. 1, Prague, Czech Republic, June 1997.

[Gabbay *et al.*, 2004] D.M. Gabbay, G. Metcalfe and N. Olivetti. Hypersequents and Fuzzy Logic. *Revista de la Real Academia de Ciencias* 98(1), pages 113-126, 2004.

[Gaines, 1976] B.R. Gaines. Foundations of fuzzy reasoning, *Int. J. of Man-Machine Studies*, 6, 623-668, 1976.

[Gaines, 1978] B.R. Gaines. Fuzzy and probability uncertainty logics, *Information and Control*, 38, 154-169, 1978.

[Galatos *et al.*, 2007] N. Galatos, P. Jipsen, T. Kowalski and H. Ono. *Residuated Lattices: an algebraic glimpse at substructural logics*, Studies in Logics and the Foundations of Mathematics 151, Elsevier 2007.

[Gentilhomme, 1968] Gentilhomme Y. (1968). Les ensembles flous en linguistique, *Cahiers de Linguistique Théorique et Appliquée* (Bucarest), 5, 47-63.

[Gerla, 2000] B. Gerla. A Note on Functions Associated with Gödel Formulas, *Soft Computing*, vol 4 (2000), 206-209.

[Gerla, 2001a] B. Gerla. *Many-Valued Logics Based on Continuous t-Norms and Their Functional Representation*, Ph.D. Università di Milano, 2001.

[Gerla, 2001b] B. Gerla. Rational Łukasiewicz logic and DMV-algebras. *Neural Networks World*, vol 11 (2001), 579-584.

[Gerla, 1994a] G. Gerla (1994). An Extension Principle for Fuzzy Logic. *Mathematical Logic Quarterly*, n. 40, 357-380.

[Gerla, 1994b] G. Gerla. Inferences in probability logic. *Artificial Intelligence* 70, 33–52, 1994.

[Gerla, 1996] G. Gerla. Graded Consequence Relations and Fuzzy Closure Operators. *Journal of Applied Non-Clasical Logics*, vol. 6 num. 4, 369-379, 1996.

[Gerla, 2001] G. Gerla (2001) *Fuzzy Logic: Mathematical Tols for Approximate Reasoning.* Trends in Logic, vol. 11. Kluwer Academic Publishers.

[Gerla and Sessa, 1999] G. Gerla and M.I. Sessa. Similarity in logic programming. In G. Chen, M. Ying, and K. Cai, editors, *Fuzzy Logic and Soft Computing*, chapter 2, pages 19–31. Kluwer, 1999.

[Giles, 1988a] R. Giles. The concept of grade of membership, *Fuzzy Sets and Systems*, 25, 297-323, 1988.

[Giles, 1988b] R. Giles. A utility-valued logic for decision-making, *Int. J. of Approximate Reasoning*, 2, 113-141, 1988.

[Gödel, 1932] Kurt Gödel. Zum intuitionistischen Aussagenkalkül. *Anzieger Akademie der Wissenschaften Wien*, 69:65–66, 1932.

[Godo, 1990] L. Godo. *Contribució a l'Estudi de Models d'inferència en els Sistemes Possibilístics.* PhD Thesis, Universitat Politcnica de Catalunya, Barcelona, Spain, 1990.

[Godo *et al.*, 2000] L. Godo, F. Esteva and P. Hájek, Reasoning about probability using fuzzy logic. *Neural Network World*, Vol. 10, Number 5, 811-824 (2000).

[Godo and Hájek, 1999] L. Godo and P. Hájek. Fuzzy Inference as Deduction. *Journal of Applied non-Classical Logics* 9(1), 37-60, 1999.

[Godo *et al.*, 2003] L. Godo, P. Hájek and F. Esteva. A fuzzy modal logic for belief functions. *Fundamenta Informaticae* 57(2-4), 127-146, 2003.

[Godo and Marchioni, 2006] L. Godo and E. Marchioni. Reasoning about coherent conditional probability in a fuzzy logic setting. *Logic Journal of the IGPL*, Vol. 14, Number 3, 457-481, 2006.

[Goguen, 1967] J.A. Goguen. L-fuzzy sets, *J. Math. Anal. Appl.* 8:145-174, 1967.

[Goguen, 1969] J.A. Goguen. The logic of inexact concepts, *Synthese*, 19, 325-37, 1969.

[Gottwald, 1993] Siegfried Gottwald. *Fuzzy Sets and Fuzzy Logic: Foundations of Application–from a Mathematical Point of View.* Vieweg, Wiesbaden, 1993.

[Gottwald, 2001] S. Gottwald. *A Treatise on Many-valued Logics*, Studies in Logic and Computation 9, Research Studies Press Ltd., Baldock, UK, 2001.

[Gottwald and Hájek, 2005] S. Gottwald and P. Hájek. Triangular norm based mathematical fuzzy logic. In Erich Petr Klement and Radko Mesiar, editors, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms, pages 275?300. Elsevier, Amsterdam, 2005.

[Haack, 1979] Susan Haack. Do We Need "Fuzzy Logic"? *International Journal of Man-Machine Studies*, 11 (4), 437–445, 1979.

[Hájek, 1994] P. Hájek . A qualitative fuzzy possibilistic logic, *Int. J. of Approximate Reasoning*, 12 (1994), 1-19.

[Hájek, 1998a] P. Hájek. *Metamathematics of Fuzzy Logic*, Trends in Logic, vol. 4, Kluwer, Dordercht, 1998.

[Hájek, 1998b] P. Hájek. Basic fuzzy logic and BL-algebras, *Soft Computing* **2** (1998), 124 – 128.

[Hájek, 2002] Petr Hájek. Observations on the monoidal t-norm logic. *Fuzzy Sets and Systems*, 132(1):107–112, 2002.

[Hájek, 2003a] P. Hájek. Fuzzy logics with non-commutative conjunctions. *Journal of Logic and Computation*, 13: 469-479, 2003.

[Hájek, 2003b] P. Hájek. Observations on non-commutative fuzzy logics. *Soft Computing*, 8:28?43, 2003.

[Hájek, 2005a] P. Hájek. Making fuzzy description logic more general. *Fuzzy Sets and Systems*, 154, 1-15, 2005.

[Hájek, 2005b] P. Hájek. Arithmetical complexity of fuzzy predicate logics – a survey. *Soft Computing* 9: 935–941, 2005.

[Hájek, 2005c] P. Hájek. Fleas and fuzzy logic. *Journal of Multiple-Valued Logic and Soft Computing*, Volume 11, Number 1-2, 137-152, 2005.

[Hájek, 2006a] P. Hájek What does mathematical fuzzy logic offer to description logic? In Capturing Intelligence: Fuzzy Logic and the Semantic Web, Elie Sanchez, ed., Elsevier, 91-100, 2006.

[Hájek, 2006b] P. Hájek. Computational complexity of t-norm based propositional fuzzy logics with rational truth-constants. *Fuzzy Sets and Systems* 157 (2006) 677–682.

[Hájek and Cintula, 2006] Petr Hájek and Petr Cintula. On theories and models in fuzzy predicate logics. *Journal of Symbolic Logic*, 71(3):863–880, 2006.

[Hájek and Cintula, 2007] Petr Hájek and Petr Cintula. Triangular norm predicate fuzzy logics. To appear in the book *Fuzzy Logics and Related Structures* (S. Gottwald, P. Hájek, U. Höhle and E.P. Klement eds.), Elsevier, 2007.

[Hájek *et al.*, 1995] P. Hájek, L. Godo and F. Esteva. Fuzzy logic and probability, Proc. of the 12th Conf. on Uncertainty in Artificial Intelligence (Besnard P. Hanks S., eds.), Morgan Kaufmann, San Francisco, CA, 1995, pp. 237-244.

[Hájek *et al.*, 1996] Petr Hájek, Lluís Godo, and Francesc Esteva. A complete many-valued logic with product conjunction. *Archive for Mathematical Logic*, 35(3):191–208, 1996.

[Hájek *et al.*, 1994] P. Hájek, D. Harmancová, F. Esteva, P. Garcia and L. Godo. On modal logics for qualitative possibility in a fuzzy setting, Proc. of the 11th Conf. on Uncertainty in Artificial Intelligence (López de Mántaras R. and Poole D., eds.), Morgan Kaufmann, San Francisco, CA, 1994, pp. 278-285.

[Hähnle, 1994] R. Hähnle. *Automated Deduction in Multiple-Valued Logics*, volume 10 of *International Series of Monographs in Computer Science.* Oxford University Press, 1994.

[Hähnle, 2005] R. Hähnle. Many-valued logic, partiality, and abstraction in formal specification languages. *Logic Journal of IGPL* 2005 13(4):415-433.

[Höhle, 1979] U. Höhle. Minkowski functionals of L-fuzzy sets. First Symposium on Policy Analysis and Information Systems. Durham, North Caroline, USA, 178- 186, 1979.

[Höhle, 1995] U. Höhle. Commutative, residuated l-monoids. In Höhle, U. and Klement. E.P. eds., Non-Classical Logics andTheir Applications to Fuzzy Subsets, Kluwer Acad. Publ., Dordrecht (1995) 55–106.

[Höhle, 2007] U. Höhle. Fuzzy Sets and Sheaves. Part I: Basic Concepts. Part II: Sheaf-theoretic Foundations of Fuzzy Set Theory with Applications to Algebra and Topology, *Fuzzy Sets and Systems* to appear.

[Hollunder, 1994] B. Hollunder. An alternative proof method for possibilistic logic and its application to terminological logics. Proc. of the 10th Conference on Uncertainty in Artificial Intelligence (UAI'94), (R.L. de Mántaras, D. Poole, eds.), San Francisco, CA, USA, 1994, 327-335.

[Hollunder , 1995] B. Hollunder. An alternative proof method for possibilistic logic and its application to terminological logics, *Int. J. of Approximate Reasoning*, 12 (1995), 85-109.

[Horčík, 2005b] Rostislav Horčík. Standard completeness theorem for ΠMTL. *Archive for Mathematical Logic*, 44(4): 413–424, 2005.

[Horčík, 2007] Rostislav Horčík. On the Failure of Standard Completeness in ΠMTL for Infinite Theories, *Fuzzy Sets and Systems*, 158(6): 619-624, March 2007.

[Horčík and Cintula, 2004] Rostislav Horčík and Petr Cintula. Product Łukasiewicz logic. *Archive for Mathematical Logic*, 43(4): 477–503, 2004.

[Hunter, 2002] A. Hunter. Measuring inconsistency in knowledge via quasi-classical models. Proc. 18th National Conference on Artificial Intelligence (AAAI 2002), Edmonton, Canada, pp. 68 - 73, 2002.

[Jenei and Montagna, 2002] Sándor Jenei and Franco Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica*, 70(2):183–192, 2002.

[Jenei and Montagna, 2003] S. Jenei and F. Montagna. A proof of standard completeness for non-commutative monoidal t-norm logic. *Neural Network World*, 13: 481–488, 2003.

[Klawonn, 1995] F. Klawonn. Prolog extensions to many-valued logics. In Höhle, U. and Klement, E.P., editor, *Non-Classical Logics and Their Applications to Fuzzy Subsets. A Handbook of the Mathematical Foundations of Fussy Sets Theory*, pages 271–289. Kluwer, 1995.

[Klawonn and Kruse, 1994] F. Klawonn and R. Kruse. A Łukasiewicz logic based Prolog. *Mathware and Soft Computing*, 1:5–29, 1994.

[Klawonn and Novák, 1996] F. Klawonn and V. Novák V. The relation between inference and interpolation in the framework of fuzzy systems, *Fuzzy Sets and Systems*, 81, 331-354, 1996.

[Kleene, 1952] S.C. Kleene. *Introduction to Metamathematics* North Holland, Amsterdam, 1952.

[Klement , 1980] E.P. Klement (1980) Some remarks on t-norms, fuzzy σ-algebras and fuzzy measures. Proc. of 2nd International Linz Seminar on Fuzzy Set Theory, E.P. Klement (ed.), 125-142.

[Klement and Navara, 1999] E.P. Klement and M. Navara. A survey of different triangular norm-based fuzzy logic, *Fuzzy Sets and Systems* 101, 1999, 241–251.

[Klement *et al.*, 2000] E.P. Klement, R. Mesiar and E. Pap. *Triangular Norms.* Kluwer Academic Publisher, Dordrecht. 2000.

[Konieczny *et al.*, 2002] S. Konieczny, J. Lang and P. Marquis . Distance-based merging: A general framework and some complexity results . Proc. of the 8th International Conference, Principles of Knowledge Representation and Reasoning (KR2002), Toulouse. Morgan Kaufmann, San Francisco, Ca., 2002, pp. 97-108.

[Konieczny and Pino-Pérez, 1998] S. Konieczny and R. Pino Pérez. On the logic of merging. Proc. of the 1998 Conf. on Knowledge Representation and Reasoning Principles (KR-98), Trento. Morgan Kaufmann, San Francisco, Ca., 1998, 488-498.

[Kowalski and Ono, 2001] T. Kowalski and H. Ono. Residuated lattices: An algebraic glimpse at logics withoutcontraction, JAIST Report, March 2001, 1-67.

[Kraus *et al.*, 1990] S. Kraus, D. J. Lehmann and M. Magidor. Nonmonotonic Reasoning, Preferential Models and Cumulative Logics. *Artificial Intelligence* 44(1-2): 167-207 (1990).

[Lafage *et al.*, 2000] C . Lafage, J. Lang and R. Sabbadin. A logic of supporters. In: Information, Uncertainty and Fusion, (B. Bouchon-Meunier, R. R. Yager and L. A. Zadeh, Eds.), Kluwer Acad. Publ.,Dordrecht, 2000, pp.381-392.

[Lang, 1991] J. Lang. *Logique Possibiliste: Aspects Formels, Déduction Automatique et Applications.* Thèse de Doctorat, Université Paul Sabatier, Toulouse, 1991.

[Lang, 2001] J. Lang. Possibilistic logic: complexity and algorithms. In: Algorithms for Uncertainty and Defeasible Reasoning . (J. Kohlas,S. Moral, Eds.), Vol. 5 of the Handbook of Defeasible Reasoning and Uncertainty Management Systems, Kluwer Acad. Publ., Dordrecht, 179-220, 2001, pp. 179-220.

[Lang *et al.*, 1991] J. Lang, D. Dubois and H. Prade. A logic of graded possibility and certainty coping with partial inconsistency, Proc. of the 7th Conf. on Uncertainty in Artificial Intelligence, UCLA, Los Angeles, July 13-15, 1991, (Morgan Kaufmann, San Francisco, Ca.), 1991, 188-196

[Lee, 1972] R.C.T. Lee. Fuzzy logic and the resolution principle. *Journal of the ACM*, 19(1):109–119, 1972.

[Lee and Chang, 1971] R.C.T. Lee and C.L. Chang. Some properties of fuzzy logic. *Information and Control*, 19(5):417–431, 1971.

[Lehmann and Magidor, 1992] D. Lehmann and M. Magidor. What does a conditional knowledge base entail? *Artificial Intelligence* 55 (1992), pp. 1-60.

[Lehmke, 1995] S. Lehmke. *On resolution-based theorem proving in propositional fuzzy logic with 'bold' connectives.* Universität Dortmund, Fachbereich Informatik, 1995. Master's Thesis.

[Lehmke, 2001a] S. Lehmke. *Logics which Allow Degrees of Truth and Degrees of Validity.* PhD dissertation, Universität Dortmund, Germany, 2001.

[Lehmke, 2001b] S. Lehmke, Degrees of Truth and Degrees of Validity. In Discovering the World with Fuzzy Logic (V. Novák, I. Perfilieva, Eds) Physica Verlag, Heidelberg, 2001, pp. 192-23791.

[Lewis, 1973a] D. L. Lewis. *Counterfactuals*. Oxford: Basil Blackwell, 1973.

[Lewis, 1973b] D.L. Lewis. Counterfactuals and comparative possibility, *J. Philosophical Logic* 2 (1973), pp 418-46

[Liau, 1998] C.J. Liau, Possibilistic residuated implications logics with applications. *Int. J. Uncertainty, Fuzziness, and Knowledge-based Systems*, 6 (1998): 365-385.

[Liau, 1999] C.J. Liau, On the possibility theory-based semantics for logics of preference. *International J. Approximate Reasoning*, 20 (1999), 173-190.

[Liau and Lin, 1988] C.J. Liau and B.I.P Lin. Fuzzy logic with equality. *International Journal Pattern Recognition and Artificial Intelligence*, 2(2):351–365, 1988.

[Liau and Lin, 1993] C. J. Liau and B. I-P. Lin. Proof methods for reasoning about possibility and necessity, *Int. J. of Approximate Reasoning*, 9, 327-364, 1993.

[Liau and Lin, 1996] C.J. Liau and I.P. Lin. Possibilistic reasoning: a mini-survey and uniform semantics. *Artificial Intelligence*, 88 (1996), 163-193.

[Łukasiewicz, 1920] Jan Łukasiewicz. O logice trojwartosciowej (On three-valued logic). *Ruch filozoficzny*, 5:170–171, 1920.

[Łukasiewicz, 1930] J. Łukasiewicz. Philosophical remarks on many-valued systems of propositional logic, 1930. Reprinted in Selected Works (Borkowski, ed.), Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1970, 153-179.

[Łukasiewicz, 1970] J. Łukasiewicz. Selected Works, Borkowski, ed., Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1970..

[Lukasiewicz, 2006] T. Lukasiewicz. Fuzzy description logic programs under the answer set semantics for the semantic web. Proc. of the 2nd International Conference on Rules and Rule Markup Languages for the Semantic Web (RuleML 2006), (T. Eiter, E. Franconi, R. Hodgson, and S. Stephens, eds.) pp. 89-96, Athens, Georgia, IEEE Computer Society, 2006.

[Mamdani, 1977] E.H. Mamdani. Application of fuzzy logic to approximate reasoning using linguistic systems, *IEEE Trans. on Comput.*, 26, 1182-1191, 1977.

[Marchioni, 2006] E. Marchioni. Possibilistic conditioning framed in fuzzy logics. *International Journal of Approximate Reasoning*, Vol. 43, Issue 2, 133-165, 2006.

[Marchioni and Montagna, 2006] E. Marchioni and F. Montagna. A note on definability in ŁΠ1/2. In Proc. of the 11th IPMU International Conference, 1588–1595, 2006.

[Marchioni and Montagna, to appear] E. Marchioni and F. Montagna. Complexity and definability issues in Ł$\Pi\frac{1}{2}$.*Journal of Logic and Computation*, doi:10.1093/logcom/exl044, to appear.

[McNaughton, 1951] R. McNaughton. A theorem about infinite-valued sentencial logic. *Journal of Symbolic Logic*, 16:1–13, 1951.

[Medina *et al.*, 2001] J. Medina, M. Ojeda-Aciego and P. Vojtátš, Multi-adjoint logic programming with continuous semantics, Proc of Logic Programming and Non-Monotonic Reasoning, LPNMR'01, Springer-Verlag, Lecture Notes in Artificial Intelligence 2173 (2001), 351-364.

[Mendel, 2000] J. Mendel (2000) *Uncertain Rule-Based Fuzzy Logic Systems: Introduction and New Directions*, Prentice-Hall, 2000.

[Metcalfe *et al.*, 2003] G. Metcalfe, N. Olivetti, and D. Gabbay. Goal-directed calculi for Gödel-Dummett logics.In M. Baaz and J. A. Makowsky, editors, Proceedings of CSL 2003, volume 2803 of LNCS,pages 413-426. Springer, 2003.

[Metcalfe *et al.*, 2004a] 14. G. Metcalfe, N. Olivetti, and D. Gabbay. Analytic proof calculi for product logics. *Archive for Mathematical Logic*, 43(7): 859-889, 2004.

[Metcalfe *et al.*, 2004b] G. Metcalfe, N. Olivetti, and D. Gabbay. Goal-directed methods for Łukasiewicz logics. InJ. Marcinkowski and A. Tarlecki, editors, Proceedings of CSL 2004, volume 3210 of LNCS,pages 85?99. Springer, 2004.

[Metcalfe *et al.*, 2005] 16. G. Metcalfe, N. Olivetti, and D. Gabbay. Sequent and Hypersequent Calculi for Łukasiewicz and Abelian Logics. *ACM Transactions on Computational Logic* 6(3): 578-613, 2005.

[Metcalfe *et al.*, to appear] G. Metcalfe, N. Olivetti, and D. Gabbay. *Proof Theory for Fuzzy Logics.* Book in preparation for Research Studies Press.

[Michálek, 1975] J. Michálek (1975). Fuzzy Topologies. *Kibernetika*, vol. II, n. 5, 345-354.

[Mizumoto and Tanaka, 1976] M. Mizumoto and K. Tanaka. Some properties of fuzzy sets of type 2, *Information Control*, 31, 312-340, 1976.

[Mizumoto and Zimmermann, 1982] M. Mizumoto and H. J. Zimmermann. Comparison of fuzzy reasoning methods, *Fuzzy Sets and Systems*, 8, 253-283, 1982.

[Moisil, 1972] G. Moisil. La logique des concepts nuancs, Essais sur les Logiques Non Chrysippiennes, Editions Acad. Repub. Soc. Roum, Bucharest, 157-163, 1972.

[Montagna, 2000] Franco Montagna. An algebraic approach to propositional fuzzy logic. *Journal of Language, Logic and Information* 9, 91-124, 2000.

[Montagna, 2001] Franco Montagna. Functorial representation of MV_Δ algebras with additional operations. *Journal of Algebra* 238, 99-125, 2001.

[Montagna, 2005] F. Montagna. Subreducts of MV-algebras with product and product residuation. *Algebra Universalis* 53, 109-137, 2005.

[Montagna *et al.*, 2006] Franco Montagna, Carles Noguera, and Rostislav Horčík. On weakly cancellative fuzzy logics. *Journal of Logic and Computation*, 16(4): 423–450, 2006.

[Montagna and Ono, 2002] F. Montagna and H. Ono. Kripke semantics, undecidability and standard completeness for Esteva and Godo's logic MTL∀. *Studia Logica*, 71(2): 227-245, 2002.

[Montagna and Panti, 2001] F. Montagna and G. Panti. Adding structure to MV-algebras. *Journal of Pure and Applied Algebra* 164, 365–387, 2001.

[Morsi and Fahmy, 2002] N. N. Morsi and A. A. Fahmy. On generalized modus ponens with multiple rules and a residuated implication, *Fuzzy Sets and Systems*, Volume 129, Issue 2, 16 July 2002, Pages 267-274.

[Mostert and Shields, 1957] P.S. Mostert and A.L. Shields. On the structure of semigroups on a compact manifold with boundary. *Ann. Math.*, 65:117–143, 1957.

[Mukaidono *et al.*, 1989] M. Mukaidono, Z.L. Shen, and L. Ding. Fundamentals of fuzzy Prolog. *International Journal of Approximate Reasoning*, 3:179–193, 1989.

[Mukaidono and Kikuchi, 1993] M. Mukaidono and H. Kikuchi, Foundations of fuzzy logic programming, in: P.-Z. Wang, K.-F. Loe (Eds.), Between Mind And Computer - Fuzzy Science and Engineering, World Scientic Publ, pp. 225-244, 1993.

[Mundici, 1994] D. Mundici. A constructive proof of McNaughton's theorem in infinite-valued logic. *Journal of Symbolic Logic*, 59(2):596–602, 1994.

[Negoita and Ralescu, 1975] C.V. Negoita and D.A. Ralescu. Representation theorems for fuzzy concepts, *Kybernetes*, 4, 169-174, 1975.

[Nilsson, 1974] N.J. Nilsson Probabilistic Logic. *Artificial Intelligence*, 28, 71-87, 1974.

[Niskanen, 1988] V. A. Niskanen. An alternative approach for specifying fuzzy linguistic truth values: truth as a distance, Cybernetics and Systems'88 (Trappl R., ed.), Kluwer Academic Publishers, 627-634, 1988.

[Noguera, 2006] Carles Noguera. *Algebraic study of axiomatic study of t-norm based fuzzy logics.* PhD thesis, University of Barcelona, Barcelona, 2006.

[Novák, 1990a] V. Novák. On the syntactico-semantical completeness of first-order fuzzy logic. Part I: Syntax and Semantics. *Kybernetika*, 26:47–66, 1990.

[Novák, 1990b] V. Novák. On the syntactico-semantical completeness of first-order fuzzy logic. Part II: Main results. *Kybernetika*, 26:134–154, 1990.

[Novák, 1999] V. Novák. Weighted inference systems. In J. Bezdek, D. Dubois, and H. Prade, editors, *Fuzzy Sets in Approximate Reasoning and Information Systems*, Fuzzy Sets Series, pages 191–241. Kluwer, 1999.

[Novák, 2004] V. Novák. On fuzzy equality and approximation in fuzzy logic. *Soft Computing* 8 (2004) 668–675.

[Novák, 2005] Vilém Novák. On fuzzy type theory. *Fuzzy Sets and Systems* 149(2),235–273, 2005.

[Novák, 2006] Vilém Novák editor. Special section "What is Fuzzy Logic", *Fuzzy Sets and Systems* 157(5), 595–718, 2006.

[Novák and Lehmke, 2006] Vilém Novák and Stephan Lehmke. Logical structure of fuzzy IF-THEN rules. *Fuzzy Sets and Systems* 157(15) 2003–2029, 2006.

[Novák and Perfilieva, 2000] V. Novák and I. Perfilieva. Some consequences of Herbrand and McNaughton theorems in fuzzy logic. In V. Novák and I. Perfilieva, editors, *Discovering the World with Fuzzy Logic*, Studies in Fuzziness and Soft Computing, pages 271–295. Physica-Verlag, 2000.

[Novák *et al.*, 1999] Vilém Novák, Irina Perfilieva, and Jiří Močkoř. *Mathematical Principles of Fuzzy Logic.* Kluwer, Dordrecht, 1999.

[Ono and Komori, 1985] H. Ono and Y. Komori. Logics without the contraction rule. Journal of Symbolic Logic 50, 169-201, 1985.

[Parikh, 1983] R. Parikh. The problem of vague predicates, Language, Logic and Method, (Cohen R.S. and Wartopsky M.W., eds.), D. Reidel, Dordrecht, 241-261, 1983.

[Pavelka, 1979] J. Pavelka. On Fuzzy Logic I, II, III. *Zeitschrift fur Math. Logik und Grundlagen der Math.* 25 (1979) 45-52, 119-134, 447-464.

[Pawlak, 1991] Z. Pawlak.*Rough Sets: Theoretical Aspects of Reasoning About Data*. Dordrecht: Kluwer Academic Publishing, 1991.

[Pei, 2003] D. Pei. On equivalent forms of fuzzy logic systems NM and IMTL. *Fuzzy Sets and Systems* 138 (2003) 187 - 195.

[Perfilieva, 2004] Irina Perfilieva. Normal forms in BL-algebra and their contribution to universal approximation of functions. *Fuzzy Sets and Systems* 143(1): 111-127, 2004.

[Polya, 1954] G. Polya (1954). *Patterns of Plausible Inference*, Princeton University Press.

[Prade, 1980] H. Prade. Unions et intersections d'ensembles flous. *Busefal* 3, 58-62, 1980.

[Prade, 1982] H. Prade. *Modèles Mathématiques de l'Imprécis et de l'Incertain en vue d'Applications au Raisonnement Naturel*, Thèse de Doctorat d'État, Université Paul Sabatier, 1982.

[Prade, 1985] H. Prade. A computational approach to approximate and plausible reasoning with applications to expert systems, *IEEE Trans. on Pattern Analysis and Machine Intelligence*, 7(3), 260-283. Corrections in 7(6), 747-748, 1985.

[Prade, 1988] H. Prade. Raisonner avec des règles d'inférence graduelle - Une approche basée sur les ensembles flous, *Revue d'Intelligence Artificielle* (Hermès, Paris), 2(2), 29-44, 1988.

[Rasiowa, 1974] Helena Rasiowa. *An Algebraic Approach to Non-Classical Logics.* North-Holland, Amsterdam, 1974.

[Reichenbach, 1949] H. Reichenbach. *The theory of probability*, University of California Press, 1949.

[Rescher, 1976] N. Rescher. *Plausible Reasoning.* Van Gorcum, Amsterdam, 1976.

[Rescher and Manor, 1970] N. Rescher, R. Manor, On inference from inconsistent premises. *Theory and Decision*, 1(1970), 179-219.

[Rodríguez *et al.*, 2003] R. Rodríguez, F. Esteva, P. Garca and L. Godo. On Implicative Closure Operators in Approximate Reasoning. *International Journal of Approximate Reasoning* 33 (2003) 159–184. Preliminary version in Proc. of 1999 Eusflat-Estylf Joint Conference, Palma de Mallorca, Sep. 99, pp. 35-37.

[Rose and Rosser, 1958] A. Rose and J.B. Rosser. Fragments of many-valued statement calculi. *Transactions of the American Mathematical Society* 87, 1–53, 1958.

[Ruspini, 1991] E.H. Ruspini. On the semantics of fuzzy logic, *Int. J. of Approximate Reasoning*, 5, 45-88, 1991.

[Russell, 1923] B. Russell. Vagueness, *Austr. J. of Philosophy*, 1, 84-92, 1923.

[Sanchez, 1978] E. Sanchez. On possibility-qualification in natural languages, *Information Sciences*, 15, 45-76, 1978.

[Savický *et al.*, 2006] P. Savický, R. Cignoli, F. Esteva, L. Godo, and C. Noguera. On product logic with truth constants. *Journal of Logic and Computation*, 16(2):205–225, 2006.

[Schotch, 1975] P.K. Schotch. Fuzzy modal logic. In Proc. of the 5th Intl. Symposium on Multiple-valued Logic (ISMVL-75), IEEE press, pp. 176-183, 1975.

[Schweizer and Sklar , 1963] B. Schweizer and A. Sklar. Associative functions and abstract semigroups. *Publ. Math. Debrecen* 10, pp. 69-180, 1963.

[Schweizer and Sklar , 1983] B. Schweizer and A. Sklar. *Probabilistic metric spaces*, North-Holland, 1983.

[Shackle, 1961] G. L.S. Shackle. *Decision, Order and Time in Human Affairs*, (2nd edition), Cambridge University Press, UK, 1961.

[Shafer, 1975] G. Shafer *A mathematical theory of evidence.* Princeton Univ. Press 1975.

[Shen *et al.*, 1988] Z.L. Shen, L. Ding, and M. Mukaidono. Fuzzy resolution principle. In *Proceedings of the Eighteenth International Symposium on Multiple-Valued Logic, ISMVL-88*, pages 210–214, Palma de Mallorca, Spain, 1988. IEEE Press.

[Smets and Magrez, 1987] P. Smets and P. Magrez. Implication in fuzzy logic, *Int. J. of Approximate Reasoning*, 1, 327-347, 1987.

[Smets and Magrez, 1988] P. Smets and P. Magrez. The measure of the degree of truth and the grade of membership, *Fuzzy Sets and Systems*, 25, 297-323, 1988.

[Spohn, 1990] W. Spohn. A general non-probabilistic theory of inductive reasoning, Uncertainty in Artificial Intelligence 4 (Shachter R.D., Levitt T.S., Kanal L.N. and Lemmer J.F., Eds.), North-Holland, Amsterdam, 149-158, 1980.

[Straccia, 1998] U. Straccia. A fuzzy description logic. Proc. 15th National Conf. on Artificial Intelligence (AAAI'98) and 10th Conf. on Innovative Applications of Artificial Intelligence (IAAI'98), AAAI Press, 594-599, 1998.

[Straccia, 2001] U. Straccia. Reasoning within fuzzy description logics. *J. of Artif. Intellig. Research*, 14:137-166, 2001.

[Straccia, 2006a] U. Straccia. A fuzzy description logic for the semantic web. In Capturing Intelligence: Fuzzy Logic and the Semantic Web, Elie Sanchez, ed., Elsevier, 2006.

[Straccia, 2006b] U. Straccia. Uncertainty and description logic programs over lattices. In Capturing Intelligence: Fuzzy Logic and the Semantic Web, Elie Sanchez, ed., Elsevier, 2006.

[Straccia, 2006c] U. Straccia. Description logics over lattices. *International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems* 14(1): 1-16, 2006.

[Sudkamp, 1993] T. Sudkamp. Similarity, interpolation, and fuzzy rule construction, *Fuzzy Sets and Systems*, 58, 73-86, 1993.

[Sugeno, 1985] M. Sugeno. An introductory survey of fuzzy control, *Information Sciences*, 36, 59-83, 1985.

[Sugeno and Takagi, 1983] M. Sugeno and T. Takagi. Multi-dimensional fuzzy reasoning, *Fuzzy Sets and Systems*, 9, 313-325, 1983.

[Takeuti and Titani, 1992] G. Takeuti and S. Titani. Fuzzy Logic and Fuzzy Set Theory. *Archive for Mathematical Logic* 32, 1-32, 1992.

[Tarski, 1930] A. Tarski. Fundamentale Begriffe der Methodologie der deduktiven Wissenschaften. *Monat-shefte Mathematik Physik,* 37: 361-404, 1930

[Thiele and Lehmke, 1994] H. Thiele and S. Lehmke. On 'bold' resolution theory. In *Proceedings of the Third IEEE International Conference on Fuzzy Systems, Fuzz-IEEE-94*, pages 1945–1950, Orlando, Florida, 1994. IEEE Press.

[Tresp and Molitor, 1998] C.B. Tresp and R. Molitor. A description logic for vague knowledge. Proc. of the 13th European Conference on Artificial Intelligence (ECAI'98), J. Wiley and Sons, 1998, 361-365.

[Trillas, 1979] E. Trillas. Sobre funciones de negacin en la teora de conjuntos difusos (in spanish), *Stochastica*, III, 1, 47-60, 1979. English version: On negation functions in fuzzy set theory, Advances of Fuzzy Logic (Barro S., Bugarn A. and Sobrino A., eds.), Publicacin de la Universidade de Santiago de Compostela, 1998, 31-40.

[Trillas and Valverde, 1981] E. Trillas and L. Valverde. On some functionally expressable implications for fuzzy set theory, Proc. of the 3rd Inter. Seminar on Fuzzy Set Theory, Linz, Austria, 173-190, 1981.

[Trillas and Valverde, 1985a] E. Trillas and L. Valverde. On implication and indistinguishability in the setting of fuzzy logic, Management Decision Support Systems using Fuzzy Sets and Possibility Theory (Kacprzyk J. and Yager R. R., eds.), Verlag TV Rheinland, Kln, 198-212, 1985.

[Ughetto *et al.*, 1997] L. Ughetto, D. Dubois and H. Prade. Efficient inference procedures with fuzzy inputs, Proc. of the 6th IEEE Inter. Conf. on Fuzzy Systems (FUZZ-IEEE'97), Barcelona, Spain, 567-572, 1997.

[Urquhart, 1986] A. Urquhart. Many-Valued Logic. In Dov M. Gabbay and Franz Guenthner, eds. Handbook of Philosophical Logic: Volume III, Alternatives to Classical Logic, pp.71-116. Dordrecht: Reidel, 1986.

[Valverde and Trillas, 1985] L. Valverde and E. Trillas. On modus ponens in fuzzy logic. In *Proceedings of the Fifteenth International Symposium on Multiple-Valued Logic, ISMVL-85*, pages 294–301. IEEE Press, 1985.

[Vojtáš, 1998] P. Vojtáš. Fuzzy reasoning with tunable *t*-operators. *Journal for Advanced Computer Intelligence*, 2:121–127, 1998.

[Vojtáš, 2001] P. Vojtáš. Fuzzy logic programming. *Fuzzy Sets and Systems*, 124(3):361–370, 2001.

[Wang, 1999] G.J. Wang. On the logic foundation of fuzzy reasoning, *Information Sciences* 117 (1999) 47–88.

[Wang, 2000] G.J. Wang. *Non-classical Mathematical Logic and Approximate Reasoning*, Science Press, Beijing, 2000 (in Chinese).

[Wang *et al.*, 2005a] San-Min Wang, Bao-Shu Wang, and Dao-Wu Pei. A fuzzy logic for an ordinal sum t-norm. *Fuzzy Sets and Systems*, 149(2):297–307, 2005.
[Wang *et al.*, 2005b] S. Wang, B. Wang, Ren-Fang. NML, a schematic extension of F. Esteva and L. Godo's logic MTL. Fuzzy Sets Syst 149, 285-295, 2005.
[Wang *et al.*, 2004] S.-M.San-Min Wang, Bao-Shu Wang and Xiang-Yun Wang. A characterization of truth-functions in the nilpotent minimum logic. *Fuzzy Sets and Systems*, Volume 145, 253-266, 2004.
[Weber, 1983] S. Weber. A general concept of fuzzy connectives, negations and implications based on t-norms and t-co-norms, *Fuzzy Sets and Systems*, 11, 115-134, 1983.
[Weston, 1987] T. Weston. Approximate truth, *J. Philos. Logic*, 16, 203-227, 1987.
[Weyl, 1946] H. Weyl. Mathematic and logic, *Amer. Math. Month.*, 53, 2-13, 1946.
[Whalen, 2003] T. Whalen. Parameterized R-implications Fuzzy Sets and Systems, 134, 2003, 231-281, 2003.
[Whalen and Schott, 1983] T. Whalen and B. Schott. Issues in fuzzy production systems, *Int. J. of Man-Machine Studies*, 19, 57-71, 1983.
[Whalen and Schott, 1985] T. Whalen and B. Schott. Alternative logics for approximate reasoning in expert systems: A comparative study, *Int. J. of Man-Machine Studies*, 22, 327-346.
[Wójcicki, 1988] R. Wójcicki. *Theory of Logical Calculi: Basic Theory of Consequence Operations.* Kluwer Academic Publishers, Dordrecht, 1988.
[Yager, 1983a] R.R. Yager. An introduction to applications of possibility theory, *Human Systems Management*, 3, 246-269, 1983.
[Yager, 1983b] R.R. Yager. Some relationships between possibility, truth and certainty, *Fuzzy Sets and Systems*, 11, 151-156, 1983.
[Yager, 1985] R.R. Yager. Inference in a multivalued logic system. *International Journal Man-Machine Studies*, 23:27–44, 1985.
[Yager, 1985b] R.R. Yager. Strong truth and rules of inference in fuzzy logic and approximate reasoning, *Cybernetics and Systems*, 16, 23-63, 1985.
[Yen, 1991] J. Yen. Generalizing term subsumption languages to fuzzy logic. Proc. of the 12th International Joint Conference on Artificial Intelligence (IJCAI'91), Sidney, August 1991, 472-477.
[Ying, 1994] M. Ying (1994). A Logic for Approximate Reasoning. *Journal of Symbolic Logic*,vol. 59, n. 3, 830-837.
[Zadeh, 1965] L.A. Zadeh. Fuzzy sets, *Information and Control*, 8, 338-353, 1965.
[Zadeh, 1972] L.A. Zadeh. A fuzzy-set-theoretic interpretation of linguistic hedges, *J. of Cybernetics*, 2, 4-34, 1972.
[Zadeh, 1973] L.A. Zadeh. Outline of a new approach to the analysis of complex systems and decision processes, *IEEE Trans. on Systems, Man and Cybernetics*, 3, 28-44, 1973.
[Zadeh, 1975a] L.A. Zadeh. Fuzzy Logic and approximate reasoning (In memory of Grigore Moisil), *Synthese*, 30, 407-428, 1975.
[Zadeh, 1975b] L.A. Zadeh. Calculus of fuzzy restrictions, Fuzzy Sets and their Applications to Cognitive and Decision Processes (Zadeh L. A., Fu K. S., Tanaka K. and Shimura M., eds.), Academic Press, New York, 1-39, 1975.
[Zadeh, 1975c] L.A. Zadeh. The concept of a linguistic variable and its application to approximate reasoning, *Information Sciences*, Part 1: 8, 199-249; Part 2: 8, 301-357; Part 3: 9, 43-80, 1975.
[Zadeh, 1976] L.A. Zadeh. A fuzzy-algorithmic approach to the definition of complex or imprecise concepts, *Int. J. of Man-Machine Studies*, 8, 249-291, 1976.
[Zadeh, 1978a] L.A. Zadeh. Fuzzy sets as a basis for a theory of possibility, *Fuzzy Sets and Systems*, 1, 3-28, 1978.
[Zadeh, 1978b] L.A. Zadeh. PRUF - A meaning representation language for natural languages, *Int. J. of Man-Machine Studies*, 10, 395-460, 1978.
[Zadeh, 1979a] L.A. Zadeh. A theory of approximate reasoning. In J.E: Hayes, D. Michie, and L.I. Mikulich, editors, *Machine Intelligence*, volume 9, pages 149–194. Elsevier, 1979.
[Zadeh, 1979b] L.A. Zadeh, Fuzzy sets and information granularity. In M.M. Gupta, R.K. Ragade and R.R. Yager (eds.), Advances in Fuzzy Set Theory and Applications, North-Holland, Amsterdam, pp. 3-18,1979.
[Zadeh, 1981] L.A. Zadeh. Test score semantics for natural languages and meaning representation via PRUF, Empirical Semantics, Vol. 1 (Rieger B. B., ed.), Brockmeyer, Bochum, 281-349, 1981.

[Zadeh, 1987] L.A. Zadeh. A computational theory of dispositions, *Int. J. of Intelligent Systems*, 2, 39-63, 1987.

[Zadeh, 1988] L.A. Zadeh.Fuzzy Logic. *IEEE Computer* 21(4): 83-93 (1988)

[Zadeh, 1989] L.A. Zadeh. Knowledge Representation in Fuzzy Logic. *IEEE Trans. Knowl. Data Eng.* 1(1): 89-100 (1989)

[Zadeh, 1992] L.A. Zadeh.The calculus of fuzzy if/then rules, *AI Expert*, 7(3), 27-27, 1992.

[Zadeh, 1994a] L.A. Zadeh. Preface in *Fuzzy Logic technology and Applications*, (R. J. Marks-II Ed.), IEEE Technical Activities Board (1994).

[Zadeh, 1994b] L.A. Zadeh. Soft computing and fuzzy logic, *IEEE Software*, November issue, 48-56, 1994.

[Zadeh, 1995] L.A. Zadeh. Fuzzy logic = Computing with words, *IEEE Trans. on Fuzzy Systems*, 4, 103-111, 1995.

[Zadeh, 1997] L.A. Zadeh. Toward a theory of fuzzy information granulation and its centrality in human reasoning and fuzzy logic, *Fuzzy Sets and Systems*, 90, 1997, 111-127.

[Zadeh, 1999] L.A. Zadeh. A New Direction in System Analysis: From Computation with Measurements to Computation with Perceptions (Abstract). RSFDGrC 1999: 10-11.

[Zadeh, 2001] L.A. Zadeh. A New Direction in AI: Toward a Computational Theory of Perceptions. *AI Magazine* 22(1): 73-84 (2001).

[Zadeh, 2005] L.A. Zadeh. Toward a generalized theory of uncertainty (GTU)–an outline, *Information Sciences*, 172, 2005, 1-40.

[Zhang and Zhang, 2004] Wen-Ran Zhang, Lulu Zhang. YinYang bipolar logic and bipolar fuzzy logic. *Information Sciences*, 165, 265-287, 2004.

INDEX

ŁΠ$\frac{1}{2}$ logic, 69
Łukasiewicz logic, 51
Łukasiewicz, J., 1

$\mathbf{L}_K$-closure operator, 86

abstract fuzzy logic, 67
algebraic semantics, 53
approximate reasoning, 6
axiomatic extensions of MTL, 56

Bellman, R., 21
bipolar possibilistic logic, 93
BL logic, 51

canonical chain, 66
canonical completeness, 66
canonical extension, 83
Carnap, R., 3
comparative truth, 63
compositionality, 29
computational complexity, 73
core fuzzy logic, 58

De Finetti B., 3
degree of inclusion, 84
Delta Deduction Theorem, 68
DMV-algebra, 72

Frank t-norms, 78
fuzzy belief function logic, 105
fuzzy closure operator, 83
fuzzy consequence relation, 84
fuzzy constants, 96
fuzzy description logic, 88
fuzzy if-then rules, 32
fuzzy implication function, 12
fuzzy logic, 6, 48
fuzzy logic programming system, 80
fuzzy logic with equality, 73
fuzzy logic with evaluated syntax, 67
fuzzy probability logic, 103
fuzzy resolution, 79
fuzzy set, 7
fuzzy truth-value, 21
fuzzy type therory, 75

Gödel logic, 51
generalized modus ponens, 40
graded consequence relation, 83
guaranteed possibility, 36

Hájek, P., 51
Hoop fuzzy logics, 72
hypersequent, 61

implicative closure operators, 85
interpolation, 46
intuitionistic fuzzy sets, 94

Lee, R.T.C., 27
level of inconsistency, 91, 98
local deduction theorem, 56
logic of supporters, 95
logic with an involutive negation, 68
logics with Δ, 67

McNaughton theorem, 74
Moisil, G., 2
Monoidal logic, 57
MTL logic, 54
multi-adjoint residuated lattice, 81
multiple-source possibilistic logic, 95
Mundici, D., 73
MV-algebras, 53

necessity measure, 18
nilpotent minimum, 12

Nilpotent minimum logic, 57
non-commutative fuzzy logic, 73
nonmonotonic possibilistic entailment, 99

Pavelka-style completeness, 64
possibilistic entailment, 91
possibilistic logic, 89
possibility distribution, 15
possibility measure, 17
predicate fuzzy logic, 59
principle of minimal specificity, 16
probability logic, 102
Product logic, 51
PŁ logic, 71

Rational Pavelka logic, 64
Reichenbach, H., 2
relational hypersequent, 62

S-fuzzy logic, 78
safe structures, 60
Shackle, G. , 18
similarity logic, 100
Spohn, W., 18
standard completeness, 57
standard semantics, 53
strong negation, 10
substructural logic, 57

t-norm based fuzzy logics, 49
Takeuti and Titani's logic, 72
timed possibilistic logic, 95
triangular conorm, 11
triangular norm, 11
truth-constant, 64

uncertainty logic, 102

variable weight, 96

weakly implicative fuzzy logic, 74
weighted clauses, 90

Zadeh, L.A., 3, 6