Aggregation Operators and Commuting

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Abstract— **Commuting is an important property in any two-step information merging procedure where the results should not depend on the order in which the single steps are preformed. We investigate the property of commuting for aggregation operators in connection with their relationship to bisymmetry. In case of bisymmetric aggregation operators we show a sufficient condition ensuring that two operators commute, while for bisymmetric aggregation operators with neutral element we even provide a full characterization of commuting n-ary operators by means of unary distributive functions. The case of associative operations, especially uninorms, is considered in detail.**

Index Terms— **Aggregation Operators, Bisymmetry, Commuting Operators, Consensus.**

I. Introduction

In various applications where information fusion or multi-factorial evaluation is needed, an aggregation process is carried out as a two-stepped procedure whereby several local fusion operations are performed in parallel and then the results are merged into a global result. It may happen that in practice the two steps can be exchanged because there is no reason to perform either of the steps first. For instance, in a multi-person multi-aspect decision problem, each alternative is evaluated by a matrix of ratings where the rows represent evaluations by persons and the columns represent evaluations by criteria. One may, for each row, merge the ratings according to each column with some aggregation operation $\mathbf{A}$ and form as such the global rating of each person, and then merge the persons opinions using another aggregation operation $\mathbf{B}$. On the other hand, one may decide first to merge the ratings in each column using the aggregation operation $\mathbf{B}$, thus forming the global ratings according to each criterion, and then merge these social evaluations across the criteria with aggregation operation $\mathbf{A}$. The problem is that it is not guaranteed that the results of the two procedures will be the same, while one would expect them to be so in any sensible approach. When the two procedures yield the same results operations $\mathbf{A}$ and $\mathbf{B}$ are said to commute.

This paper is devoted to a mathematical investigation of commuting aggregation operators which are used, e.g., in utility theory [15], but also in extension theorems for functional equations [33]. Very often, the commuting property is instrumental in the preservation of some property during an aggregation process, like transitivity when aggregating preference matrices or fuzzy relations (see, e.g., [13], [34]), or some form of additivity when aggregating set functions (see, e.g., [15]). In fact, early examples of commuting appear in probability theory for the merging of probability distributions. Suppose two joint probability distributions are merged by combining degrees of probability point-wisely. It is natural that the marginals of the resulting joint probability function are the aggregates of the marginals of the original joint probabilities. To fulfill this requirement the aggregation operation must commute with the addition operation involved in the derivation of the marginals. It enforces a weighted arithmetic mean as the only possible aggregation operation for probability functions [31]. This result is closely related to the theory of probabilistic mixtures that plays a key-role in the axiomatic derivation of expected utility theory [22]. In [15], the same question is solved for more general set functions, where the addition is replaced by a co-norm and the consequences for generalized utility theory are pointed out.

In this paper the problem of commuting operators is considered with more generality. After a section presenting necessary definitions and background, Section III considers the case of commuting unary operations, called distributive functions, that play a key role in the representation of commuting operators. Section IV provides characterization results concerning bisymmetric operations, i.e., aggregation operations that commute with themselves. Section V and VI focus on functions distributive over continuous t-(co)norms and particular uninorms respectively.

II. Preliminaries

A. Aggregation operators

Aggregation by itself is an important task in any discipline where the fusion of information is of vital interest. It comprehends the transformation of several items of input data into a single output value which is characteristic for the input data itself or some of its aspects. In case of aggregation operators it is assumed that a finite number of inputs from the same (numerical) scale, most often the unit interval, are being aggregated. Moreover, interpreting the inputs as evaluation results of objects according to some criterion, the monotonicity and boundary conditions of its formal definition look very natural:

Definition 1: A function $\mathbf{A}\colon \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is called an *aggregation operator* if it fulfills the following properties ([10])

(AO1) $\mathbf{A}(x_1,\dots,x_n) \leq \mathbf{A}(y_1,\dots,y_n)$ whenever $x_i \leq y_i$ for all $i \in \{1,\dots,n\}$,

(AO2) $\mathbf{A}(x) = x$ for all $x \in [0,1]$,

(AO3) $\mathbf{A}(0,\dots,0) = 0$ and $\mathbf{A}(1,\dots,1) = 1$.

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Each aggregation operator $\mathbf{A}$ can be represented by a family $(\mathbf{A}_{(n)})_{n\in\mathbb{N}}$ of n-ary operations, i.e., functions $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$ given by

$$\mathbf{A}_{(n)}(x_1,\ldots,x_n) = \mathbf{A}(x_1,\ldots,x_n).$$

In that case, $\mathbf{A}_{(1)} = \mathrm{id}_{[0,1]}$ and, for $n \geq 2$, each $\mathbf{A}_{(n)}$ is non-decreasing and satisfies $\mathbf{A}_{(n)}(0,\ldots,0) = 0$ and $\mathbf{A}_{(n)}(1,\ldots,1) = 1$. Usually, the aggregation operator $\mathbf{A}$ and the corresponding family $(\mathbf{A}_{(n)})_{n\in\mathbb{N}}$ of n-ary operations are identified with each other. Note that, n-ary operations $\mathbf{A}_{(n)} : [0,1]^n \to [0,1]$, $n \geq 2$, which fulfill properties **(AO1)** and **(AO3)** are referred to as *n-ary aggregation operators.*

Depending on the requirements applied to the aggregation process several properties for aggregation operators have been introduced. We only mention those few which are relevant for our further investigations. For more elaborated details on aggregation operators we refer to, e.g., [10].

Definition 2: Consider some aggregation operator $\mathbf{A}\colon \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$.

(i) $\mathbf{A}$ is called *symmetric* if for all $n \in \mathbb{N}$ and for all $x_i \in [0,1], i \in \{1,\ldots,n\}$,

$$\mathbf{A}(x_1,\ldots,x_n) = \mathbf{A}(x_{\alpha(1)},\ldots,x_{\alpha(n)})$$

for all permutations $\alpha = (\alpha(1),\ldots,\alpha(n))$ of $\{1,\ldots,n\}$.

(ii) $\mathbf{A}$ is called *bisymmetric* if for all $n,m \in \mathbb{N}$ and all $x_{i,j} \in [0,1]$ with $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$,

$$\mathbf{A}_{(m)}\big(\mathbf{A}_{(n)}(x_{1,1},\ldots,x_{1,n}),\ldots,\mathbf{A}_{(n)}(x_{m,1},\ldots,x_{m,n})\big) = \mathbf{A}_{(n)}\big(\mathbf{A}_{(m)}(x_{1,1},\ldots,x_{m,1}),\ldots,\mathbf{A}_{(m)}(x_{1,n},\ldots,x_{m,n})\big).$$

(iii) $\mathbf{A}$ is called *associative* if for all $n,m \in \mathbb{N}$ and all $x_i \in [0,1]$ and all $y_j \in [0,1]$ with $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$,

$$\mathbf{A}(x_1,\ldots,x_n,y_1,\ldots,y_m) = \mathbf{A}(\mathbf{A}(x_1,\ldots,x_n),\mathbf{A}(y_1,\ldots,y_m)).$$

(iv) An element $e \in [0,1]$ is called *neutral element* of $\mathbf{A}$ if for all $n \in \mathbb{N}$ and for all $x_i \in [0,1]$, $i \in \{1,\ldots,n\}$ it holds that if $x_i = e$ for some $i \in \{1,\ldots,n\}$ then

$$\mathbf{A}(x_1,\ldots,x_n) = \mathbf{A}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n).$$

(v) An element $d \in [0,1]$ is called an *idempotent element* of $\mathbf{A}$ if $\mathbf{A}(d,\ldots,d) = d$ for all $n \in \mathbb{N}$. We will abbreviate the set of idempotent elements by

$$\mathcal{I}(\mathbf{A}) = \{d \in [0,1] \mid \mathbf{A}(d,\ldots,d) = d\}.$$

In case that $\mathcal{I}(\mathbf{A}) = [0,1]$, the aggregation operator is called *idempotent*.

Associative aggregation operators $\mathbf{A}$, are completely characterized by their binary operators $\mathbf{A}_{(2)}$ since all n-ary, $n > 2$, aggregation operators $\mathbf{A}_{(n)}$ can be constructed by the recursive application of the binary operator $\mathbf{A}_{(2)}$.

Depending on the additional properties, several subclasses of aggregation operators can and have been distinguished, like, e.g., symmetric and associative operators with some neutral element e: For $e = 1$, they are referred to as *triangular norms* (*t-norm* for short), for $e = 0$, they are called *t-conorms*, for $e \in\,]0,1[$ we will refer to them as *uninorms* (see also [6], [17], [24]).

Note that associative and symmetric aggregation operators are also bisymmetric. On the other hand, bisymmetric aggregation operators with some neutral element are associative. Therefore, as just mentioned, the class of all associative and symmetric, and therefore bisymmetric, aggregation operators with neutral element e consists of all t-norms, t-conorms and uninorms.

Aggregation operators on other domains: Note that not all aggregation processes are carried out on input data from the unit interval, therefore, aggregation operators on other intervals as well as methods for transforming input data are needed to model the required aggregation process. Aggregation operators can be defined as acting on any closed interval $I = [a,b] \subseteq [-\infty,\infty]$. We will then speak of an *aggregation operator acting on I*. While **(AO1)** and **(AO2)** basically remain the same, only **(AO3)**, expressing the preservation of the boundaries, has to be modified accordingly

(AO3') $\mathbf{A}(a,\ldots,a) = a$ and $\mathbf{A}(b,\ldots,b) = b$.

Such aggregation operators can also be achieved from standard aggregation operators by means of isomorphic transformations. By such transformations many of the before mentioned properties are being preserved.

For an isomorphic transformation $\varphi\colon\ [a,b] \to [0,1]$, i.e., a monotone bijection, the *isomorphic transformation* $\mathbf{A}_\varphi$ of an aggregation operator $\mathbf{A}$ is given by

$$\mathbf{A}_\varphi(x_1,\ldots,x_n) = \varphi^{-1}(\mathbf{A}(\varphi(x_1),\ldots,\varphi(x_n)))$$

and is an aggregation operator on $[a,b]$. If for two aggregation operators $\mathbf{A}$, $\mathbf{B}$ on (possibly) different intervals, there exists a monotone bijection φ such that $\mathbf{A} = \mathbf{B}_\varphi$ or $\mathbf{A}_\varphi = \mathbf{B}$ we refer to $\mathbf{A}$ and $\mathbf{B}$ as *isomorphic aggregation operators*.

By means of increasing bijections, we can introduce t-norms T and t-conorms S on arbitrary interval $[a,b]$ preserving the boundary elements as the corresponding neutral elements. We will denote such t-norms, resp. t-conorms as t-(co)norms on the corresponding interval I.

B. Commuting and dominance

Definition 3: Consider two aggregation operators $\mathbf{A}$ and $\mathbf{B}$. We say that $\mathbf{A}$ *dominates* $\mathbf{B}$ ($\mathbf{A} \gg \mathbf{B}$) if for all $n,m \in \mathbb{N}$ and for all $x_{i,j} \in [0,1]$ with $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$, the following property holds:

$$\mathbf{B}_{(m)}\big(\mathbf{A}_{(n)}(x_{1,1},\ldots,x_{1,n}),\ldots,\mathbf{A}_{(n)}(x_{m,1},\ldots,x_{m,n})\big) \leq \mathbf{A}_{(n)}\big(\mathbf{B}_{(m)}(x_{1,1},\ldots,x_{m,1}),\ldots,\mathbf{B}_{(m)}(x_{1,n},\ldots,x_{m,n})\big). \quad (1)$$

Definition 4: Consider an n-ary aggregation operator $\mathbf{A}_{(n)}$ and an m-ary aggregation operator $\mathbf{B}_{(m)}$. Then we say that $\mathbf{A}_{(n)}$ *commutes* with $\mathbf{B}_{(m)}$ if for all $x_{i,j} \in [0,1]$ with $i \in \{1,\ldots,m\}$ and $j \in \{1,\ldots,n\}$, the following property holds:

$$\mathbf{B}_{(m)}\big(\mathbf{A}_{(n)}(x_{1,1},\ldots,x_{1,n}),\ldots,\mathbf{A}_{(n)}(x_{m,1},\ldots,x_{m,n})\big) = \mathbf{A}_{(n)}\big(\mathbf{B}_{(m)}(x_{1,1},\ldots,x_{m,1}),\ldots,\mathbf{B}_{(m)}(x_{1,n},\ldots,x_{m,n})\big). \quad (2)$$

Two aggregation operators $\mathbf{A}$ and $\mathbf{B}$ *commute* with each other if $\mathbf{A}_{(n)}$ commutes with $\mathbf{B}_{(m)}$ for all $n, m \in \mathbb{N}$. We will also refer to $\mathbf{A}$ and $\mathbf{B}$ as *commuting aggregation operators*.

Observe that the property of commuting as expressed by Eq. (2) is a special case of the so called *generalized bisymmetry* equation as introduced and discussed in [4], [5] and plays a key role in consistent aggregation.

It is an immediate consequence of the definition of commuting that two aggregation operators commute if and only if they dominate each other. Further that any aggregation operator commuting with itself is bisymmetric and vice versa. Note that in case of two associative aggregation operators commuting between the binary operators is a necessary and sufficient condition for their commuting in general.

Because of the preservation properties of dominance during isomorphic transformations (see also [34]) we immediately can state the following result:

Corollary 5: Let $\mathbf{A}$ and $\mathbf{B}$ be two aggregation operators. Then the following are equivalent:

(i) $\mathbf{A}$ commutes with $\mathbf{B}$.

(ii) $\mathbf{A}_\varphi$ commutes with $\mathbf{B}_\varphi$ for some isomorphic transformation φ.

(iii) $\mathbf{A}_\varphi$ commutes with $\mathbf{B}_\varphi$ for all isomorphic transformations φ.

Example 6: The projections to the first coordinate resp. to the last coordinate, i.e.,

$$\mathbf{P}_F(x_1, \ldots, x_n) = x_1$$
$$\mathbf{P}_L(x_1, \ldots, x_n) = x_n$$

commute with arbitrary aggregation operator $\mathbf{A}$.

III. Distributive Functions

A. Basic property

There is a close relationship between commuting aggregation operators and unary functions being distributive over one of the two aggregation operators involved. On the one hand, such functions can be constructed from commuting aggregation operators, on the other hand — as we will show in the next section — they can be used for constructing commuting operators. Note that such distributive functions are in fact commuting with the involved aggregation operator.

Proposition 7: For any n-ary aggregation operator $\mathbf{A}_{(n)}$ and any m-ary aggregation operator $\mathbf{B}_{(m)}$, $n, m \in \mathbb{N}$, it holds that if $\mathbf{A}_{(n)}$ commutes with $\mathbf{B}_{(m)}$, then the function $f_{d,i,\mathbf{A}_{(n)}}\colon [0,1] \to [0,1]$ defined by

$$f_{d,i,\mathbf{A}_{(n)}}(x) = \mathbf{B}_{(m)}(d, \ldots, d, \overset{\overset{i\text{-th position}}{\downarrow}}{x}, d, \ldots, d) \tag{3}$$

with $i \in \{1, \ldots, m\}$ and d some idempotent element of $\mathbf{A}_{(n)}$ is distributive over $\mathbf{A}_{(n)}$, i.e., it fulfills for all $i \in \{1, \ldots, m\}$ and all $x_j \in [0,1]$ with $j \in \{1, \ldots, n\}$

$$f_{d,i,\mathbf{A}_{(n)}}(\mathbf{A}_{(n)}(x_1, \ldots, x_n)) = \mathbf{A}_{(n)}(f_{d,i,\mathbf{A}_{(n)}}(x_1), \ldots, f_{d,i,\mathbf{A}_{(n)}}(x_n)).$$

Moreover, $f_{d,i,\mathbf{A}_{(n)}}$ is non-decreasing.

Proof: Consider some n-ary aggregation operator $\mathbf{A}_{(n)}$, one of its idempotent elements d, e.g., 0 or 1, and some m-ary aggregation operator $\mathbf{B}_{(m)}$ such that $\mathbf{A}_{(n)}$ commutes with $\mathbf{B}_{(m)}$. Then it holds for $f_{d,i,\mathbf{A}_{(n)}}\colon [0,1] \to [0,1]$ defined by Eq. (3) with arbitrary $i \in \{1, \ldots, n\}$ that

$$\begin{aligned}
&f_{d,i,\mathbf{A}_{(n)}}(\mathbf{A}_{(n)}(x_1, \ldots, x_n)) \\
&\quad= \mathbf{B}_{(m)}(d, \ldots, d, \mathbf{A}_{(n)}(x_1, \ldots, x_n), d \ldots, d) \\
&\quad= \mathbf{B}_{(m)}(\mathbf{A}_{(n)}(d, \ldots, d), \ldots, \mathbf{A}_{(n)}(d, \ldots, d), \\
&\qquad\quad \mathbf{A}_{(n)}(x_1, \ldots, x_n), \\
&\qquad\quad \mathbf{A}_{(n)}(d, \ldots, d) \ldots, \mathbf{A}_{(n)}(d, \ldots, d)) \\
&\quad= \mathbf{A}_{(n)}(\mathbf{B}_{(m)}(d, \ldots, d, x_1, d, \ldots, d), \ldots, \\
&\qquad\quad \mathbf{B}_{(m)}(d, \ldots, d, x_n, d, \ldots, d)) \\
&\quad= \mathbf{A}_{(n)}(f_{d,i,\mathbf{A}_{(n)}}(x_1), \ldots, f_{d,i,\mathbf{A}_{(n)}}(x_n)).
\end{aligned}$$

The non-decreasingness of $f_{d,i,\mathbf{A}_{(n)}}$ follows immediately from the monotonicity of $\mathbf{B}$. ∎

Analogously, we can define non-decreasing functions $f_{d',i,\mathbf{B}_{(m)}}$ which are distributive over $\mathbf{B}_{(m)}$ with d' some idempotent element of $\mathbf{B}_{(m)}$.

B. Distributive functions and lattice polynomials

We will denote by $\mathcal{F}_{\mathbf{A}_{(n)}}$ the set of all non-decreasing functions $f\colon [0,1] \to [0,1]$ that are distributive over the n-ary aggregation operator $\mathbf{A}_{(n)}$, i.e.,

$$\begin{aligned}
\mathcal{F}_{\mathbf{A}_{(n)}} = \{f\colon [0,1] \to [0,1] \mid\ & f \text{ is non-decreasing}, \\
& f(\mathbf{A}_{(n)}(x_1, \ldots, x_n)) = \mathbf{A}_{(n)}(f(x_1), \ldots, f(x_n))\}
\end{aligned}$$

Observe that $\mathbf{A}_{(1)}$ is the identity function and thus $\mathcal{F}_{\mathbf{A}_{(1)}}$ contains all non-decreasing functions $f\colon [0,1] \to [0,1]$. For the readers' convenience we will abbreviate this set simply by $\mathcal{F}$, i.e.,

$$\mathcal{F} = \{f\colon [0,1] \to [0,1] \mid f \text{ is non-decreasing}\} = \mathcal{F}_{\mathbf{A}_{(1)}}.$$

Evidently, $\mathcal{F}_{\mathbf{A}} = \cap_{n \in \mathbb{N}} \mathcal{F}_{\mathbf{A}_{(n)}}$ is the set of all functions $f \in \mathcal{F}$ that are distributive over the aggregation operator $\mathbf{A}$. Note that $\mathcal{F}_{\mathbf{A}_{(n)}}$ as well as $\mathcal{F}_{\mathbf{A}}$ contain at least the following functions

$$\begin{aligned}
\mathbf{0}\colon [0,1] &\to [0,1], x \mapsto 0, \\
\mathbf{1}\colon [0,1] &\to [0,1], x \mapsto 1, \\
\mathrm{id}\colon [0,1] &\to [0,1], x \mapsto x
\end{aligned}$$

and are therefore not empty for arbitrary aggregation operator $\mathbf{A}$. The following proposition shows that $\mathcal{F}_{\mathbf{A}}$ is maximal in case of lattice polynomials only, i.e., $\mathbf{A}$ can be expressed by $\wedge$, $\vee$ and its arguments only [8], compare also, e.g., [29], [30].

Proposition 8: Consider an aggregation operator $\mathbf{A}$. Then the following holds:

$$\forall n \in \mathbb{N}: \ \mathbf{A}_{(n)} \text{ is a lattice polynomial} \Leftrightarrow \mathcal{F}_{\mathbf{A}} = \mathcal{F}.$$

Proof: If all $\mathbf{A}_{(n)}$ with $n \in \mathbb{N}$ are lattice polynomials, it follows immediately from the non-decreasingness of all $f \in \mathcal{F}$ and the definition of $\mathcal{F}_{\mathbf{A}_{(n)}}$ that $\mathcal{F} \subseteq \mathcal{F}_{\mathbf{A}} \subseteq \mathcal{F}$.

Before showing the sufficiency, note that any n-variable lattice polynomial $L\colon [0,1]^n \to [0,1]$ can be put in the following *disjunctive normal form* [8]

$$L(x_1,\dots,x_n)=\bigvee_{\substack{I\subseteq N,\\ m(I)=1}}\bigwedge_{i\in I}x_i. \tag{4}$$

where $N=\{1,\dots,n\}$ and $m\colon 2^N\to\{0,1\}$ is a non-decreasing set function fulfilling $m(\emptyset)=0$ and $m(N)=1$. Therefore, in order to show that some n-ary aggregation operator $\mathbf{A}_{(n)}$ is a lattice polynomial, we have to show that a set function $m\colon 2^N\to\{0,1\}$ fulfilling the above conditions exists and that $\mathbf{A}_{(n)}$ can be written in the form of Eq. (4). For better readability, we will use in the sequel of this proof $\mathbf{A}$ instead of $\mathbf{A}_{(n)}$, as well as the additional notations $1_I=(x_1,\dots,x_n)$ where $x_i=1$ if $i\in I$ and $x_i=0$ otherwise, and $\mathbf{A}_I=\mathbf{A}(1_I)$. Now assume that $\mathcal{F}_{\mathbf{A}}=\mathcal{F}$.

- First, we show that $\mathbf{A}(x_1,\dots,x_n)\in\{x_1,\dots,x_n\}$ for all $x_i\in[0,1]$, $i\in\{1,\dots,n\}$. In case that there exist some $x_i\in[0,1]$, $i\in\{1,\dots,n\}$, such that $\mathbf{A}(x_1,\dots,x_n)=c\notin\{x_1,\dots,x_n\}$, depending on the value of c, one of the following functions $f_j\in\mathcal{F}$, $j\in\{1,2,3\}$,

$$f_1(a)=\begin{cases}x_* & \text{if } a\in[0,x_*]\,,\\ a & \text{otherwise,}\end{cases}$$

$$f_2(a)=\begin{cases}x_* & \text{if } a\in\,]x_*,x^*[\,,\\ a & \text{otherwise,}\end{cases}$$

$$f_3(a)=\begin{cases}x^* & \text{if } a\in[x^*,1]\,,\\ a & \text{otherwise,}\end{cases}$$

  contradicts $f_j(\mathbf{A}(x_1,\dots,x_n))=\mathbf{A}(f_j(x_1),\dots,f_j(x_n))$ with $x_*=\min(x_1,\dots,x_n)$ and $x^*=\max(x_1,\dots,x_n)$. Therefore, in particular $\mathbf{A}$ is idempotent, i.e., $\mathbf{A}(x,\dots,x)=x$ and $\mathbf{A}_I\in\{0,1\}$ for all $I\subseteq N$.
- Since for all $x\in[0,1]$ the functions $\varphi_x,\psi_x\colon[0,1]\to[0,1]$, $\varphi_x(a)=x\cdot a$ resp. $\psi_x(a)=a(1-x)+x$ fulfill $\varphi_x,\psi_x\in\mathcal{F}$ we can conclude the following for all $I\subseteq N$

$$\mathbf{A}(\varphi_x(1_I))=\varphi_x(\mathbf{A}_I)=\mathbf{A}_I\wedge x,$$
$$\mathbf{A}(\psi_x(1_I))=\psi_x(\mathbf{A}_I)=\mathbf{A}_I\vee x,$$

  since $\varphi_x(0)=0$, $\varphi_x(1)=x$, $\psi_x(0)=x$, $\psi_x(1)=1$, and $\mathbf{A}_I\in\{0,1\}$.
- Due to the monotonicity of $\mathbf{A}$ we can further conclude that for arbitrary $x_i\in[0,1]$, $i\in\{1,\dots n\}$,

$$\mathbf{A}(x_1,\dots,x_n)\ge\bigwedge_{i\in I}x_i\cdot\mathbf{A}_I=\mathbf{A}_I\wedge(\bigwedge_{i\in I}x_i),$$

  by replacing each x_i either by 0, if $i\notin I$, or by $\bigwedge_{i\in I}x_i$, if $i\in I$, for arbitrary choice of $I\subseteq N$. Therefore, also

$$\mathbf{A}(x_1,\dots,x_n)\ge\bigvee_{I\subseteq N}\mathbf{A}_I\wedge(\bigwedge_{i\in I}x_i)\ge\mathbf{A}_N\wedge(\bigwedge_{i\in N}x_i)$$
$$=1\wedge(\bigwedge_{i\in N}x_i)=x_*.$$

  We abbreviate by $y^*=\bigvee_{I\subseteq N}\mathbf{A}_I\wedge(\bigwedge_{i\in I}x_i)$ such that the previous inequality can be written as

$$\mathbf{A}(x_1,\dots,x_n)\ge y^*\ge x_*.$$

  As such it is immediately clear that the set $J=\{j\in N\mid x_j\le y^*\}$ is not empty. Moreover, the following holds for its complement $N\setminus J$

$$y^*=\bigvee_{I\subseteq N}\mathbf{A}_I\wedge(\bigwedge_{i\in I}x_i)\ge\mathbf{A}_{N\setminus J}\wedge(\bigwedge_{i\in N\setminus J}x_i)$$

  so that necessarily $\mathbf{A}_{N\setminus J}=0$.
  If we replace each x_i in $\mathbf{A}(x_1,\dots,x_n)$ either by y^* in case that $i\in J$ or by 1 in case that $i\notin J$, we can also conclude, due to the monotonicity of $\mathbf{A}$ and the properties shown before, that

$$\mathbf{A}(x_1,\dots,x_n)\le\mathbf{A}(\psi_{y^*}(1_{N\setminus J}))=\mathbf{A}_{N\setminus J}\vee y^*=y^*,$$

  showing that

$$\mathbf{A}(x_1,\dots,x_n)=\bigvee_{I\subseteq N}\mathbf{A}_I\wedge(\bigwedge_{i\in I}x_i).$$

  Finally, we define a set function $m\colon 2^N\to\{0,1\}$ by $m(I)=\mathbf{A}_I$, then it is immediate to show that it is non-decreasing and fulfills $m(\emptyset)=0$, $m(N)=1$, and that $\mathbf{A}$ is indeed a lattice polynomial.

■

Let us now focus on additional properties of $\mathcal{F}_{\mathbf{A}}$ in case of particular properties of the aggregation operator $\mathbf{A}$ involved.

C. Distributive functions for bisymmetric and associative aggregation operators

Proposition 9: Let $\mathbf{A}$ be a bisymmetric aggregation operator and fix some $n\in\mathbb{N}$. If we choose some $f_i\in\mathcal{F}_{\mathbf{A}_{(n)}}$,$i\in\{1,\dots,n\}$, not necessarily different, then also $g:[0,1]\to[0,1]$ defined by

$$g(x)=\mathbf{A}_{(n)}(f_1(x),\dots,f_n(x)) \tag{5}$$

belongs to $\mathcal{F}_{\mathbf{A}_{(n)}}$, i.e., $\mathcal{F}_{\mathbf{A}_{(n)}}$ is closed under $\mathbf{A}_{(n)}$.

Proof: Consider some bisymmetric aggregation operator $\mathbf{A}$ and fix some arbitrary $f_i\in\mathcal{F}_{\mathbf{A}_{(n)}}$, $i\in\{1,\dots,n\}$ for some $n\in\mathbb{N}$. Define a function $g\colon[0,1]\to[0,1]$ by Eq. (5) then the following holds for arbitrary $x_1,\dots,x_n\in[0,1]$ due to the bisymmetry of $\mathbf{A}$ and the distributivity of all f_i over $\mathbf{A}_{(n)}$

$$\begin{aligned}
&g(\mathbf{A}_{(n)}(x_1,\dots,x_n))\\
&\quad=\mathbf{A}_{(n)}\big(f_1(\mathbf{A}_{(n)}(x_1,\dots,x_n)),\dots,f_n(\mathbf{A}_{(n)}(x_1,\dots,x_n))\big)\\
&\quad=\mathbf{A}_{(n)}\big(\mathbf{A}_{(n)}(f_1(x_1),\dots,f_1(x_n))),\dots,\\
&\qquad\qquad\mathbf{A}_{(n)}(f_n(x_1),\dots,f_n(x_n))\big)\\
&\quad=\mathbf{A}_{(n)}\big(\mathbf{A}_{(n)}(f_1(x_1),\dots,f_n(x_1))),\dots,\\
&\qquad\qquad\mathbf{A}_{(n)}(f_1(x_n),\dots,f_n(x_n))\big)\\
&\quad=\mathbf{A}_{(n)}(g(x_1),\dots,g(x_n)).
\end{aligned}$$

■

Corollary 10: If $\mathbf{A}$ is a bisymmetric aggregation operator and additionally fulfills for all $n,m\in\mathbb{N}$ and all $x_{i,j}\in[0,1]$, $i\in\{1,\dots,m\}$, $j\in\{1,\dots,n\}$

$$\begin{aligned}
&\mathbf{A}_{(n\cdot m)}(x_{1,1},\dots,x_{1,n},\dots,x_{m,1},\dots,x_{m,n})\\
&\quad=\mathbf{A}_{(m)}\big(\mathbf{A}_{(n)}(x_{1,1},\dots,x_{1,n}),\dots,\mathbf{A}_{(n)}(x_{m,1},\dots,x_{m,n})\big)
\end{aligned}$$

then $g:[0,1]\to[0,1]$ defined by

$$g(x)=\mathbf{A}_{(m)}(f_1(x),\dots,f_m(x))$$

also belongs to $\mathcal{F}_{\mathbf{A}_{(n)}}$ for arbitrary $m\in\mathbb{N}$ and arbitrary $f_i\in\mathcal{F}_{\mathbf{A}_{(n)}}, i\in\{1,\dots,m\}$, i.e., $\mathcal{F}_{\mathbf{A}(n)}$ is closed under any $\mathbf{A}_{(m)}$, $m\in\mathbb{N}$.

Moreover, in case of an associative aggregation operator $\mathbf{A}$ the relationship can be generalized, expressing that it is sufficient (and necessary) to characterize all functions distributive over the binary aggregation operator $\mathbf{A}_{(2)}$ only in order to characterize the set $\mathcal{F}_{\mathbf{A}}$ of all unary mappings distributive over $\mathbf{A}$ with arbitrary arity.

Proposition 11: Let $\mathbf{A}$ be an associative aggregation operator, then the following holds:

$$\forall f\in\mathcal{F}:\quad f\in\mathcal{F}_{\mathbf{A}}\Leftrightarrow f\in\mathcal{F}_{\mathbf{A}_{(2)}}.$$

Proof: Consider an associative aggregation operator $\mathbf{A}$. If some non-decreasing function $f:[0,1]\to[0,1]$ fulfills $f\in\mathcal{F}_{\mathbf{A}}$, it is distributive over all n-ary aggregation operators $\mathbf{A}_{(n)}$, $n\in\mathbb{N}$, in particular over the binary aggregation operator $\mathbf{A}_{(2)}$. On the other hand if $f\in\mathcal{F}_{\mathbf{A}_{(2)}}$ the property follows directly from the associativity of $\mathbf{A}$, i.e., the fact that for all $n\in\mathbb{N}$ with $n\geq 2$ it holds that $\mathbf{A}_{(n)}(x_1,\dots,x_n)=\mathbf{A}_{(2)}(x_1,\mathbf{A}_{(n-1)}(x_2,\dots,x_n))$. ∎

Note that the associativity of an aggregation operator is a sufficient condition for $\mathcal{F}_{\mathbf{A}}=\mathcal{F}_{\mathbf{A}_{(2)}}$. But, as the following example will demonstrate, it is not necessary.

Example 12: Consider the arithmetic mean $\mathbf{M}\colon\bigcup_{n\in\mathbb{N}}[0,1]^n\to[0,1]$, $\mathbf{M}(x_1,\dots,x_n)=\frac{1}{n}\sum_{i=1}^n x_i$. Then (compare also [2], [3])

$$\mathcal{F}_{\mathbf{M}_{(2)}}=\mathcal{F}_{\mathbf{M}}=\{f\colon[0,1]\to[0,1]\mid f(x)=a+bx, a,b\in[0,1], a+b\in[0,1]\}$$

although clearly the arithmetic mean is not associative.

Example 13: Examples of associative and symmetric and therefore bisymmetric aggregation operators are a-medians med_a,

$$\mathrm{med}_a(x,y)=\mathrm{med}(x,y,a)$$

with $a\in[0,1]$ [18]. The set $\mathcal{F}_{\mathrm{med}_a}$ of distributive functions is characterized in the following way: Some non-decreasing function $f\colon[0,1]\to[0,1]$ is distributive over med_a, i.e., $f\in\mathcal{F}_{\mathrm{med}_a}$ if and only if either $f(a)=a$ or $f(a)=f(1)<a$ or $f(a)=f(0)>a$.

Besides associativity and bisymmetry, the possibility of building isomorphic aggregation operators leads to further insight to relationships between sets of distributive functions.

D. Distributive functions and isomorphisms

Proposition 14: Consider an aggregation operator $\mathbf{A}$ and some bijection $\varphi\colon[a,b]\to[0,1]$. Then for all $f\in\mathcal{F}_{\mathbf{A}}$ it holds that $f_\varphi:=\varphi^{-1}\circ f\circ\varphi\in\mathcal{F}_{\mathbf{A}_\varphi}$ where

$$\mathcal{F}_{\mathbf{A}_\varphi}=\{f\colon[a,b]\to[a,b]\mid f\text{ is non-decreasing and distributive over }\mathbf{A}_\varphi\}.$$

Proof: Consider the isomorphic aggregation operators $\mathbf{A}$ and $\mathbf{A}_\varphi$ with $\varphi\colon[a,b]\to[0,1]$ some bijection. Further assume $f\in\mathcal{F}_{\mathbf{A}}$, then the following are equivalent since for all $x_i\in[a,b]$, $i\in\{1,\dots,n\}$, $n\in\mathbb{N}$, there exists a unique $y_i\in[a,b]$ with $\varphi(y_i)=x_i$

$$\begin{aligned}
&f\circ\mathbf{A}(x_1,\dots,x_n)=\mathbf{A}(f(x_1),\dots,f(x_n)),\\
&\varphi^{-1}\circ f\circ\mathbf{A}(\varphi(y_1),\dots,\varphi(y_n))\\
&\qquad=\varphi^{-1}\circ\mathbf{A}(f\circ\varphi(y_1),\dots,f\circ\varphi(y_n)),\\
&\varphi^{-1}\circ f\circ\varphi\circ\varphi^{-1}\circ\mathbf{A}(\varphi(y_1),\dots,\varphi(y_n))\\
&\qquad=\varphi^{-1}\circ\mathbf{A}(\varphi\circ\varphi^{-1}\circ f\circ\varphi(y_1),\dots,\\
&\qquad\qquad\varphi\circ\varphi^{-1}\circ f\circ\varphi(y_n));\\
&f_\varphi\circ\varphi^{-1}\circ\mathbf{A}(\varphi(y_1),\dots,\varphi(y_n))\\
&\qquad=\varphi^{-1}\circ\mathbf{A}(\varphi\circ f_\varphi(y_1),\dots,\varphi\circ f_\varphi(y_n)),\\
&f_\varphi\circ\mathbf{A}_\varphi(y_1,\dots,y_n)=\mathbf{A}_\varphi(f_\varphi(y_1),\dots,\varphi(y_n)),
\end{aligned}$$

showing that $f_\varphi\in\mathcal{F}_{\mathbf{A}_\varphi}$. ∎

Example 15: Following Aczél [1], [3], the class of all continuous, strictly monotone, bisymmetric, and idempotent aggregation operators on the unit interval are just weighted quasi-arithmetic means

$$\mathbf{W}_\varphi(x_1,\dots,x_n)=\varphi^{-1}\left(\sum_{i=1}^n w_i\varphi(x_i)\right)$$

with $\varphi\colon[0,1]\to[0,1]$ some monotone non-decreasing bijection and weights w_i with $w_i>0$ for all $i\in\{1,\dots,n\}$ and $\sum_{i=1}^n w_i=1$. It is immediate that weighted quasi-arithmetic means are isomorphic transformations of weighted arithmetic means $\mathbf{W}$ with corresponding weights. Due to Proposition 14 the set of distributive functions $\mathcal{F}_{\mathbf{W}_\varphi}$ is therefore given by

$$\mathcal{F}_{\mathbf{W}_\varphi}=\{f\in\mathcal{F}\mid f(x)=\varphi^{-1}(a+b\varphi(x))\text{ and } a,b,a+b\in[0,1]\}$$

since

$$\mathcal{F}_{\mathbf{W}}=\{f\in\mathcal{F}\mid f(x)=a+bx\text{ and }a,b\in[0,1]\text{ such that }a+b\in[0,1]\}$$

in case that $w_i>0$ for all $i\in\{1,\dots,n\}$ and $\sum_{i=1}^n w_i=1$.

Example 16: For invariant aggregation operators $\mathbf{A}$, i.e., aggregation operators fulfilling $\mathbf{A}_\varphi=\mathbf{A}$ for all bijections $\varphi\colon[0,1]\to[0,1]$, it immediately holds that all non-decreasing bijections are included in $\mathcal{F}_{\mathbf{A}}$ (see also, e.g., [29], [30] for characterizations of aggregation operators invariant under non-decreasing bijections). This is, e.g., the case for the drastic product $T_{\mathbf{D}}$ and the weakest aggregation operator $\mathbf{A}_w$ being defined by

$$T_{\mathbf{D}}(x,y)=\begin{cases}\min(x,y) & \text{if }\max(x,y)=1,\\ 0 & \text{otherwise,}\end{cases}$$

$$\mathbf{A}_w(x_1,\dots,x_n)=\begin{cases}1 & \text{if }x_1=\ldots=x_n=1,\\ 0 & \text{otherwise.}\end{cases}$$

However, their set of distributive functions does not only contain all non-decreasing bijections, but is even much richer, namely

$$\begin{aligned}\mathcal{F}_{T_{\mathbf{D}}}&=\mathcal{F}_{\mathbf{A}_w}\\&=\{f\in\mathcal{F}\mid f(x)=1\Leftrightarrow x=1\text{ and }f(0)=0\}\cup\{\mathbf{0},\mathbf{1}\}.\end{aligned}$$

Similarly, lattice polynomials are invariant aggregation operators and we know already their sets of distributive functions equal the set of all non-decreasing functions.

However, for arbitrary aggregation operators $\mathbf{A}$ at least the following relationship between a bijective distributive function and its inverse can be stated.

Lemma 17: Consider an aggregation operator $\mathbf{A}$. If $f \in \mathcal{F}_{\mathbf{A}}$ is bijective then also $f^{-1} \in \mathcal{F}_{\mathbf{A}}$.

IV. Operators commuting with bisymmetric aggregation operators

After discussing unary operators being distributive over some aggregation operator and as such commuting, let us now turn to more general commuting operators.

Proposition 18: Let $\mathbf{A}$ be a bisymmetric aggregation operator. Then any n-ary operator B, $n \in \mathbb{N}$, on $[0,1]$ defined by

$$B(x_1,\ldots,x_n) = \mathbf{A}(f_1(x_1),\ldots,f_n(x_n)) \tag{6}$$

with $f_i \in \mathcal{F}_{\mathbf{A}}$ for $i \in \{1,\ldots,n\}$ commutes with $\mathbf{A}$.

Proof: Consider some bisymmetric aggregation operator A, choose some $m,n \in \mathbb{N}$ and arbitrary $f_i \in \mathcal{F}_{\mathbf{A}}$, $i \in \{1,\ldots,n\}$. Then the following holds for arbitrary $x_{i,j} \in [0,1]$ with $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,m\}$

$$\begin{aligned}
&B\big(\mathbf{A}(x_{1,1},\ldots,x_{1,m}),\ldots,\mathbf{A}(x_{n,1},\ldots,x_{n,m})\big)\\
&\quad= \mathbf{A}\big(f_1\circ\mathbf{A}(x_{1,1},\ldots,x_{1,m}),\ldots,f_n\circ\mathbf{A}(x_{n,1},\ldots,x_{n,m})\big)\\
&\quad= \mathbf{A}\big(\mathbf{A}(f_1(x_{1,1}),\ldots,f_1(x_{1,m})),\ldots\\
&\qquad\qquad\qquad \mathbf{A}(f_n(x_{n,1}),\ldots,f_n(x_{n,m}))\big)\\
&\quad= \mathbf{A}\big(\mathbf{A}(f_1(x_{1,1}),\ldots,f_n(x_{n,1})),\ldots,\\
&\qquad\qquad\qquad \mathbf{A}(f_1(x_{1,m}),\ldots,f_n(x_{n,m}))\big)\\
&\quad= \mathbf{A}\big(B(x_{1,1},\ldots,x_{n,1}),\ldots,B(x_{1,m},\ldots,x_{n,m})\big).
\end{aligned}$$

■

Note that the involved operator B need not be an aggregation operator, e.g., choose $f_i(x) = \mathbf{1}(x) = 1$ for all $i \in \{1,\ldots,n\}$, then

$$B(x_1,\ldots,x_n) = 1$$

for arbitrary $x_i \in [0,1]$, $i \in \{1,\ldots,n\}$, and therefore the boundary conditions **(A02)** and **(AO3)** are not fulfilled.

Remark 19: Note that the previous proposition provides a sufficient, but not a necessary condition for an operation B to commute with $\mathbf{A}$. As mentioned above, any aggregation operator $\mathbf{A}$ commutes with the projection to the first coordinate $\mathbf{P}_F$ which is a bisymmetric aggregation operator. However, using $\mathbf{P}_F(f_1(x_1),\ldots,f_n(x_n)) = f_1(x_1)$, only aggregation operators depending just on the first coordinate can be obtained although we have that $\mathcal{F}_{\mathbf{P}_F} = \mathcal{F}$, since $\mathbf{P}_F$ is a lattice polynomial.

A. Commuting aggregation operators

Let us briefly focus on the restrictions which additionally have to be applied to the selected functions $f_i \in \mathcal{F}_{\mathbf{A}}$ such that the constructed operator B also fulfills the requirements of an aggregation operator. If $n = 1$, the corresponding $f_1 \in \mathcal{F}_{\mathbf{A}}$ must be the identity function in order to guarantee $B(x) = x$. For $n > 1$, the functions $f_i \in \mathcal{F}_{\mathbf{A}}, i \in \{1,\ldots,n\}$ must be chosen accordingly to $\mathbf{A}$ such that

$$\begin{aligned}
B(0,\ldots,0) &= \mathbf{A}(f_1(0),\ldots,f_n(0)) = 0,\\
B(1,\ldots,1) &= \mathbf{A}(f_1(1),\ldots,f_n(1)) = 1
\end{aligned}$$

are both fulfilled at the same time. This is for sure guaranteed if for all f_i it holds that $f_i(0) = 0$ and $f_i(1) = 1$, but it need not be the case as the following example shows.

Example 20: The class of all aggregation operators commuting with the minimum

$$\begin{aligned}
\mathcal{D}^{(n)}_{\min} = \{\, &\min(f_1(x_1),\ldots,f_n(x_n)) \mid\\
&f_i \in \mathcal{F} \text{ with } f_i(1) = 1 \text{ for all } i \in \{1,\ldots,n\},\\
&f_i(0) = 0 \text{ for at least one } i \in \{1,\ldots,n\}\},
\end{aligned}$$

is also the class of all aggregation operators dominating the minimum (see also [34]).

B. The role of neutral elements

Let us now consider for which bisymmetric aggregation operators $\mathbf{A}$, operators B defined by Eq. (6) are the only commuting operators, i.e., if Eq. (6) does provide a sufficient as well as a necessary condition. For better readability, we will briefly restrict ourselves to binary operators only. Since the projections commute with any aggregation operator $\mathbf{A}$, they particularly commute also with such operators $\mathbf{A}$ for which Eq. (6) indeed is necessary and sufficient. In this case, there necessarily exist $f_i, g_i \in \mathcal{F}_{\mathbf{A}}$, $i = 1,2$, such that for all $x,y \in [0,1]$

$$\begin{aligned}
x &= \mathbf{P}_F(x,y) = \mathbf{A}(f_1(x),f_2(y)) = \mathbf{A}(f_1(x),f_2(0))\\
&= \mathbf{A}(f_1(x),f_2(1)),\\
y &= \mathbf{P}_F(x,y) = \mathbf{A}(g_1(x),g_2(y)) = \mathbf{A}(g_1(0),g_2(y))\\
&= \mathbf{A}(g_1(1),g_2(y)).
\end{aligned}$$

If there exists some $x_0, y_0 \in [0,1]$ such that $f_1(x_0) \in [g_1(0), g_1(1)]$ and $g_2(y_0) \in [f_2(0), f_2(1)]$ it follows from the monotonicity of $\mathbf{A}$ that

$$\begin{aligned}
x_0 &= \mathbf{A}(f_1(x_0),f_2(0)) \le \mathbf{A}(f_1(x_0),g_2(y_0))\\
&\le \mathbf{A}(g_1(0),g_2(y_0)) = y_0\\
y_0 &= \mathbf{A}(g_1(0),g_2(y_0)) \le \mathbf{A}(f_1(x_0),g_2(y_0))\\
&\le \mathbf{A}(f_1(x_0),f_2(1)) = x_0.
\end{aligned}$$

Therefore, independently of x_0, y_0, we have that

$$\mathbf{A}(f_1(x_0),g_2(y_0)) = x_0 = y_0,$$

i.e., such an element is unique. A typical candidate fulfilling the last property is a neutral element e. In such a case, it suffices to choose $f_1 = g_2 = \mathrm{id}$ and $f_2(x) = g_1(x) = e$ for all $x \in [0,1]$.

Indeed, we obtain a necessary and sufficient condition if the involved aggregation operator $\mathbf{A}$ is bisymmetric and possesses a neutral element e.

Proposition 21: Let $\mathbf{A}$ be a bisymmetric aggregation operator with neutral element e. An n-ary operator B, $n \in \mathbb{N}$, commutes with $\mathbf{A}$ if and only if for there exist $f_i \in \mathcal{F}_{\mathbf{A}}, i \in \{1, \dots, n\}$ such that

$$B(x_1, \dots, x_n) = \mathbf{A}(f_1(x_1), \dots, f_n(x_n)). \tag{7}$$

Proof: Consider some bisymmetric aggregation operator $\mathbf{A}$ with neutral element e. If B is defined by Eq. (7) for some $f_i \in \mathcal{F}_{\mathbf{A}}$ then it commutes with $\mathbf{A}$ due to Proposition 18. In order to show the necessity assume that B commutes with $\mathbf{A}$, i.e., especially for all $x_1, \dots, x_n \in [0,1]$ it holds that

$$\begin{aligned} B(x_1, \dots, x_n) &= B(\mathbf{A}(x_1, e, \dots, e), \dots, \mathbf{A}(e, \dots, e, x_n)) \\ &= \mathbf{A}(B(x_1, e, \dots, e), \dots, B(e, \dots, e, x_n)) \\ &= \mathbf{A}(f_{e,1,\mathbf{A}}(x_1), \dots, f_{e,n,\mathbf{A}}(x_n)) \end{aligned}$$

with $f_{e,i,\mathbf{A}}$ defined by Eq. (3), thus fulfilling $f_{e,i,\mathbf{A}} \in \mathcal{F}_{\mathbf{A}}$ and proving that B can be expressed as in Eq. (7). ■

Recall once again that any bisymmetric aggregation operator with neutral element is also associative and symmetric and therefore is either a t-norm, a t-conorm or a uninorm. But note that it is impossible that commuting operators having neutral elements are different operators.

Proposition 22: Consider two aggregation operators $\mathbf{A}$ and $\mathbf{B}$ with neutral elements e_a resp. e_b. If $\mathbf{A}$ commutes with $\mathbf{B}$ then $e_a = e_b$. Moreover, also $\mathbf{A} = \mathbf{B}$.

Proof: Assume that $\mathbf{A}$ and $\mathbf{B}$ are commuting aggregation operators with neutral elements e_a resp. e_b. Therefore,

$$\begin{aligned} e_a &= \mathbf{A}(e_a, e_a) = \mathbf{A}(\mathbf{B}(e_a, e_b), \mathbf{B}(e_b, e_a)) \\ &= \mathbf{B}(\mathbf{A}(e_a, e_b), \mathbf{A}(e_b, e_a)) = \mathbf{B}(e_b, e_b) = e_b \end{aligned}$$

and

$$\begin{aligned} \mathbf{A}(x_1, \dots, x_n) &= \mathbf{A}(\mathbf{B}(x_1, e, \dots, e), \dots, \mathbf{B}(e, \dots, e, x_n)) \\ &= \mathbf{B}(\mathbf{A}(x_1, e, \dots, e), \dots, \mathbf{A}(e, \dots, e, x_n)) \\ &= \mathbf{B}(x_1, \dots, x_n) \end{aligned}$$

for all $x_1, \dots, x_n \in [0,1]$ and arbitrary $n \in \mathbb{N}$. ■

As a consequence commuting does not appear among t-norms, t-conorms, or uninorms respectively. The only operators commuting with such bisymmetric operators with neutral element are, besides the operator itself, aggregation operators with no neutral element.

Example 23: As mentioned before the projection to the first coordinate $\mathbf{P}_F$ commutes with any aggregation operator and therefore also, e.g., with the product t-norm $T_{\mathbf{P}}$. Observe that $\mathbf{P}_F$ is bisymmetric but has no neutral element, while $T_{\mathbf{P}}$ is a bisymmetric aggregation operator with neutral element 1. According to Proposition 21, corresponding functions $f_i \in \mathcal{F}_{T_{\mathbf{P}}}$, $i \in \{1, \dots, n\}$, $n \in \mathbb{N}$, can be chosen such that

$$\mathbf{P}_F(x_1, \dots, x_n) = T_{\mathbf{P}}(f_1(x_1), \dots, f_n(x_n)),$$

namely $f_1 = \mathrm{id}$ and all other $f_j = \mathbf{1}$ for $j \in \{2, \dots, n\}$. However, for any $g_1, \dots, g_n \in \mathcal{F}_{\mathbf{P}_F} = \mathcal{F}$ the operator $\mathbf{P}_F(g_1(x_1), \dots, g_n(x_n)) = g_1(x_1)$ can never represent the product $T_{\mathbf{P}}$.

C. Consequences

Since Proposition 21 provides a full characterization of commuting operators in case that one of them is bisymmetric with some neutral element and further showed that these operators can be attained through functions distributive over the bisymmetric aggregation operator with neutral element involved, we will now focus on the set of such functions.

Note that a full characterization of all bisymmetric aggregation operators with neutral element, in particular if the neutral element is from the open interval, is still missing. Since the characterization of the set of unary functions distributing with such operators is heavily influenced by the structure of the underlying operator, we will later on focus on special subclasses of bisymmetric aggregation operators with neutral element only, namely on

- continuous t-norms,
- continuous t-conorms, and
- particular classes of uninorms.

Therefore, consider $*$ to be some continuous t-norm T, some continuous t-conorm S, or some uninorm U. Note that $f \in \mathcal{F}_*$ is equivalent to the fact that f fulfills a Cauchy like equation, i.e., for all $x, y \in [0,1]$

$$f(x * y) = f(x) * f(y). \tag{8}$$

Observe that besides $\mathbf{0}(x) = 0, \mathbf{1}(x) = 1$ and $\mathrm{id}(x) = x$ also the constant function $f_e(x) = e$ is included in $\mathcal{F}_*$.

Lemma 24: If $d \in \mathcal{I}(*)$, then $f_d \colon [0,1] \to [0,1]$, $f_d(x) = d$ for all $x \in [0,1]$ fulfills $f_d \in \mathcal{F}_*$.

V. Characterization of $\mathcal{F}_{\mathbf{A}}$ for continuous t-(co)norms

For the case of continuous t-conorms Eq. (8) has been solved by Benvenuti et al. in [7] and as such by duality also for continuous t-norms. Continuous t-(co)norms are particularly important subclasses of t-(co)norms. We briefly recall a few basic facts and properties, but refer the interested reader for more details to the monographs [6], [24] and the articles [25], [26], [27].

The class of continuous t-(co)norms consists exactly of all so called continuous Archimedean t-(co)norms and of ordinal sums of such continuous Archimedean t-(co)norms. Let us first turn to continuous Archimedean t-(co)norms T resp. S. They are in turn characterized as being generated by some continuous additive generator t resp. s, i.e., they can be written as

$$T(x, y) = t^{(-1)}(t(x) + t(y)), S(x, y) = s^{(-1)}(s(x) + s(y)).$$

In case of (continuous) t-norms, the additive generator $t \colon [0,1] \to \mathbb{R}$ is a strictly decreasing (continuous) function which fulfills $t(1) = 0$ and for which $t^{(-1)}(x) = t^{-1}(\min(t(0), x))$. In case of (continuous) t-conorms, the additive generator $s \colon [0,1] \to \mathbb{R}$ is a strictly increasing (continuous) function which fulfills $s(0) = 0$ and for which $s^{(-1)}(x) = s^{-1}(\min(s(1), x))$. Note that in both cases additive generators are unique up to a positive multiplicative constant. For continuous Archimedean t-(co)norms two subclasses can be further distinguished, namely *nilpotent* t-(co)norms for which $t(0) < \infty$ resp. $s(1) < \infty$, and *strict* t-(co)norms with $t(0) = \infty$ resp. $s(1) = \infty$.

Let us now turn to ordinal sum t-(co)norms, a concept applicable to all kinds of t-(co)norms. The main properties are based on results in the framework of semigroups, however, the basic idea of ordinal sums can be described the following way: Define a t-(co)norm T resp. S by t-(co)norms on pairwise non-overlapping subsquares along the diagonal of the unit square and choose for all other cases $\min$ in case of t-norms and $\max$ in case of t-conorms. Formally, consider a family $(]a_k, b_k[)_{k\in K}$ of pairwise disjoint open subintervals of the unitinterval and a corresponding family of t-(co)norms $(T_k)_{k\in K}$ resp. $(S_k)_{k\in K}$, then the *ordinal sums* $T = (\langle a_k, b_k, T_k\rangle)_{k\in k}\colon [0,1]^2 \to [0,1]$ resp. $S = (\langle a_k, b_k, S_k\rangle)_{k\in k}\colon [0,1]^2 \to [0,1]$ are given by

$$T(x,y) = \begin{cases} a_k + (b_k - a_k) T_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), \\ \qquad\qquad\qquad\qquad \text{if } (x,y) \in [a_k, b_k]^2, \\ \min(x,y), \qquad\qquad\qquad \text{otherwise,} \end{cases}$$

$$S(x,y) = \begin{cases} a_k + (b_k - a_k) S_k(\frac{x-a_k}{b_k-a_k}, \frac{y-a_k}{b_k-a_k}), \\ \qquad\qquad\qquad\qquad \text{if } (x,y) \in [a_k, b_k]^2, \\ \max(x,y), \qquad\qquad\qquad \text{otherwise,} \end{cases}$$

and are indeed a t-norm resp. a t-conorm. The ordinal sum t-norm T as well as the ordinal sum t-conorm S are continuous if and only if all T_k resp. S_k are continuous. Based on these facts let us now briefly recall the main results of [7] which will be further relevant for the investigation of particular classes of uninorms.

A. Continuous t-conorms

Theorem 25 ([7]): Consider a continuous t-conorm S. Then $[0,1] \setminus \mathcal{I}(S) = \bigcup_{k\in K}]a_k, b_k[$ for some index set K and there exists a family of continuous strictly increasing mappings $s_k\colon [a_k, b_k] \to [0,\infty]$ with $s(a_k) = 0$ such that

$$S(x,y) = \begin{cases} s_k^{-1}(\min(s_k(x) + s_k(y), s_k(b_k))), \\ \qquad\qquad\qquad\qquad \text{if } (x,y) \in [a_k, b_k]^2, \\ \max(x,y), \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Let $f \in \mathcal{F}_S$ and denote by f_k its restriction to the interval $]a_k, b_k[$.

(i) If $s_k(b_k) = \infty$, then one of the following holds:

- (ssi) $f_k(x) = i_k$ with $i_k \in \mathcal{I}(S)$ and $f(a_k) \le i_k \le f(b_k)$,
- (ssg) $f_k(x) = s_h^{-1}(\min(\lambda_k s_k(x), s_h(\beta_h)))$ for some $\lambda_k \in]0,\infty[$ and some $h \in K$ such that $f(a_k) \le a_h$ and $f(b_k) \ge b_h$.

(ii) If $s_k(b_k) < \infty$, then one of the following holds:

- (sni) $f_k(x) = f(b_k) \in \mathcal{I}(S)$,
- (sng) $f_k(x) = s_h^{-1}(\min(\lambda_k s_k(x), s_h(\beta_h)))$ for some $h \in K$ such that $f(a_k) \le a_h$, $f(b_k) = b_h$, $s_h(b_h)$ is finite and $\frac{s_h(b_h)}{s_k(b_k)} \le \lambda_k < \infty$.

Note that in case of (ssi) and (sni), f is constant on the whole corresponding interval $]a_k, b_k[$ resp. $]a_k, b_k]$ attaining its value at an idempotent element of S. In case of (ssg) and (sng) there exists at least one $x_0 \in]a_k, b_k[$ such that $f(x_0) \notin \mathcal{I}(S)$ so that necessarily there exists some $h \in K$ fulfilling $f(x_0) \in]a_h, b_h[$.

The previous theorem already indicates how all distributive functions $f \in \mathcal{F}_S$ for some continuous t-conorm S can be obtained:

Theorem 26 ([7]): Consider some continuous t-conorm S and use the notations as introduced in Theorem 25. Any $f \in \mathcal{F}_S$ is obtained from a generic function $f^+\colon \mathcal{I}(S) \to \mathcal{I}(S)$ which is monotone non-decreasing and from its restrictions f_k for every interval $]\alpha_k, \beta_k[$ whereas each restriction f_k is chosen either by expression (ssi) resp. (ssg) in case that $s(b_k) = \infty$ or by expression (sni) resp. (sng) in case that $s(b_k) < \infty$.

Example 27: Consider the basic t-conorm $S_{\mathbf{P}}(x,y) = x + y - xy$. It is continuous with $\mathcal{I}(S) = \{0,1\}$ and $s\colon [0,1] \to [0,\infty]$, $s(x) = -\ln(1-x)$. Its set of distributive functions is given by

$$\mathcal{F}_{S_{\mathbf{P}}} = \{\mathbf{0}, \mathbf{0}_{[0,1[}, \mathbf{0}_{\{0\}}, \mathbf{1}\} \cup \{f\colon [0,1] \to [0,1] \mid \\ f(x) = 1 - (1-x)^\lambda, \lambda \in]0,\infty[\}$$

where $\mathbf{0}_A\colon [0,1] \to [0,1]$ is defined by

$$\mathbf{0}_A(x) = \begin{cases} 0, & \text{if } x \in A, \\ 1, & \text{otherwise.} \end{cases}$$

Example 28: Consider the basic t-conorm $S_{\mathbf{L}}(x,y) = \min(x+y, 1)$. It is continuous with $\mathcal{I}(S) = \{0,1\}$, $s\colon [0,1] \to [0,\infty]$, $s(x) = x$, and

$$\mathcal{F}_{S_{\mathbf{L}}} = \{\mathbf{0}, \mathbf{0}_{\{0\}}, \mathbf{1}\} \cup \{f\colon [0,1] \to [0,1] \mid \\ f(x) = \min(\lambda x, 1), \lambda \in [1,\infty[\}.$$

B. Continuous t-norms

Since t-norms are dual to t-conorms we can get analogous results for functions f being distributive over some continuous t-norm T.

Corollary 29: Consider a continuous t-norm T. Then $[0,1] \setminus \mathcal{I}(T) = \bigcup_{k\in K}]a_k, b_k[$ for some index set K and there exists a family of continuous strictly decreasing mappings $t_k\colon [a_k, b_k] \to [0,\infty]$ with $t(b_k) = 0$ such that

$$T(x,y) = \begin{cases} t_k^{-1}(\min(t_k(x) + t_k(y), t_k(a_k))), \\ \qquad\qquad\qquad\qquad \text{if } (x,y) \in [a_k, b_k]^2, \\ \min(x,y), \qquad\qquad\qquad \text{otherwise.} \end{cases}$$

Let $f \in \mathcal{F}_T$ and denote by f_k its restriction to the interval $]a_k, b_k[$.

(i) If $t_k(a_k) = \infty$, then one of the following holds:

- (tsi) $f_k(x) = i_k$ with $i_k \in \mathcal{I}(T)$ and $f(a_k) \le i_k \le f(b_k)$,
- (tsg) $f_k(x) = t_h^{-1}(\min(\lambda_k t_k(x), t_h(a_h)))$ for some $\lambda_k \in]0,\infty[$ and some $h \in K$ such that $f(a_k) \le a_h$ and $f(b_k) \ge b_h$.

(ii) If $t_k(a_k) < \infty$, then one of the following holds:

- (tni) $f_k(x) = f(a_k) \in \mathcal{I}(S)$, (tng) $f_k(x) = t_h^{-1}(\min(\lambda_k t_k(x), t_h(a_h)))$ for some $h \in K$ such that $f(a_k) = a_h$, $f(b_k) \geq b_h$, $t_h(a_h)$ is finite and $\frac{t_h(a_h)}{t_k(a_k)} \leq \lambda_k < \infty$.

Analogous to the case of continuous t-conorms all functions f being distributive over some continuous t-norm T can be found.

Corollary 30: Consider some continuous t-norm t and use the notations as introduced in Corollary 29. Any $f \in \mathcal{F}_T$ is obtained from a generic function $f^+ \colon \mathcal{I}(T) \to \mathcal{I}(T)$ which is monotone non-decreasing and from its restrictions f_k for every interval $]\alpha_k, \beta_k[$ whereas each restriction f_k is chosen either by expression (tsi) resp. (tsg) in case that $t(a_k) = \infty$ or by expression (tni) resp. (tng) in case that $t(a_k) < \infty$.

Example 31: Consider the two basic t-norms $T_{\mathbf{P}}(x,y) = xy$ and $T_{\mathbf{L}}(x,y) = \max(x+y-1, 0)$. For both we have that $\mathcal{I}(T_{\mathbf{P}}) = \mathcal{I}(T_{\mathbf{L}}) = \{0,1\}$ and their additive generators are given by $t_{T_{\mathbf{P}}}(x) = -\ln(x)$ and $t_{T_{\mathbf{L}}} = 1 - x$, respectively. Further, we get that

$$\begin{aligned}\mathcal{F}_{T_{\mathbf{P}}} &= \{\mathbf{0}, \mathbf{0}_{[0,1[}, \mathbf{0}_{\{0\}}, \mathbf{1}\} \cup \{f \colon [0,1] \to [0,1] \mid \\ &\qquad f(x) = x^{\lambda} \text{ with } \lambda \in]0,\infty[\},\\ \mathcal{F}_{T_{\mathbf{L}}} &= \{\mathbf{0}, \mathbf{0}_{[0,1[}, \mathbf{1}\} \cup \{f \colon [0,1] \to [0,1] \mid \\ &\qquad f(x) = \max(\lambda x + 1 - \lambda, 0)\\ &\qquad\qquad \text{with } \lambda \in [1,\infty[\}.\end{aligned}$$

VI. Characterization of $\mathcal{F}_{\mathbf{A}}$ for (particular classes of) uninorms

Let us now turn to the last class of bisymmetric aggregation operators with some neutral element, namely uninorms whose neutral elements e fulfill $e \in]0,1[$ (see also [11], [17]). Note that uninorms U can be interpreted as combination of some t-norm and some t-conorm, i.e.,

$$\begin{aligned}U_{(n)}(x_1,\ldots,x_n) = U_{(2)}(&T(\min(x_1,e),\ldots,\min(x_n,e)),\\ &S(\max(x_1,e),\ldots,\max(x_n,e)))\end{aligned}$$

with T some t-norm acting on $[0,e]$ and S some t-conorm acting on $[e,1]$. To express explicitly that some uninorm U is related to some t-norm T and some t-conorm S, we will use the notation $U_{T,S}$.

Such created uninorms cover a quite large class of aggregation operators since on the remainder of their domains they can be chosen such that the monotonicity and associativity condition are not violated but otherwise arbitrarily. However, due to its properties any uninorm U fulfills

$$\min(x,y) \leq U(x,y) \leq \max(x,y)$$

whenever $\min(x,y) \leq e$ and $e \leq \max(x,y)$ for all $x, y \in [0,1]$, giving rise to the particular classes $U_{T,S,\min}$, $U_{T,S,\max}$ of uninorms. Note further, there exists no uninorm which is continuous on the whole domain [17]. Generated uninorms, which we will discuss later in more detail, therefore form another important subclass of uninorms, since they are continuous on the whole domain up to the case where $\{x,y\} = \{0,1\}$.

As the next section will show, functions f distributing with some uninorm U heavily depend on the structure of the uninorm. Therefore, since a full characterization of all uninorms is still missing, we restrict the discussion of $\mathcal{F}_U$ to two particular subclasses of uninorms — namely to uninorms which are either acting as the minimum or as the maximum on their remainders and to generated uninorms.

A. Distributive functions on uninorms

First of all let us investigate necessary and sufficient conditions for some non-decreasing function $f \colon [0,1] \to [0,1]$ being distributive over some uninorm U, i.e., for all $x, y \in [0,1]$

$$U(f(x), f(y)) = f(U(x,y)).$$

If we choose $x = e$ we see that $U(f(e), f(y)) = f(y)$ for all $y \in [0,1]$, expressing that $f(e)$ acts as a neutral element of U on the range of f. Moreover, $U(f(e), f(e)) = f(e)$ so that necessarily $f(e) \in \mathcal{I}(U)$.

From this, we see already, that the set of idempotent elements as well as the range of $f \in \mathcal{F}_U$ will play a crucial role in characterizing $\mathcal{F}_U$.

Lemma 32: Consider some $f \in \mathcal{F}_U$. Then the following holds:

(i) If $e \in \mathrm{Ran}_f$, then $f(e) = e$.
(ii) If $d \in \mathcal{I}(U)$, then also $f(d) \in \mathcal{I}(U)$.

Proof: Consider some $f \in \mathcal{F}_U$. If $e \in \mathrm{Ran}_f$ then there exists some $x_0 \in [0,1]$, such that $f(x_0) = e$ and

$$\begin{aligned}f(e) = U(e, f(e)) &= U(f(x_0), f(e))\\ &= f(U(x_0, e)) = f(x_0) = e.\end{aligned}$$

Moreover, if $d \in \mathcal{I}(U)$ then also

$$f(d) = f(U(d,d)) = U(f(d), f(d)),$$

i.e., $f(d) \in \mathcal{I}(U)$. ■

Let us now briefly focus on particular cases where $e \notin \mathrm{Ran}_f$:

Proposition 33: Consider some uninorm $U_{T,S}$ with neutral element e and some $f \in \mathcal{F}$ with either $\mathrm{Ran}_f \subseteq [0,e[$ or $\mathrm{Ran}_f \subseteq]e,1]$. Then the following holds:

$\mathrm{Ran}_f \subseteq [0,e[\,:\, f \in \mathcal{F}_{U_{T,S}}$ if and only if

(i) $f(e) \in \mathcal{I}(U) \cap [0,e[$,
(ii) $\forall x \in [e,1] : f(x) = f(e)$,
(iii) $f|_{[0,e]}$ is distributive over T,
(iv) $\forall x \in [0,1] : f(U(x,1)) = f(x)$.

$\mathrm{Ran}_f \subseteq]e,1] \,:\, f \in \mathcal{F}_{U_{T,S}}$ if and only if

(i) $f(e) \in \mathcal{I}(U) \cap]e,1]$,
(ii) $\forall x \in [0,e] : f(x) = f(e)$,
(iii) $f|_{[e,1]}$ is distributive over S,
(iv) $\forall x \in [0,1] : f(U(x,0)) = f(x)$.

Proof: Consider some uninorm $U = U_{T,S}$ with neutral element e, some $f \in \mathcal{F}$ with $\mathrm{Ran}_f \subseteq [0,e[$. Assume that $f \in \mathcal{F}_U$.

Since e is an idempotent element of U and $\mathrm{Ran}_f \subseteq [0,e[$, it immediately follows that $f(e) \in \mathcal{I}(U) \cap [0,e[$, i.e., $f(e)$ is an idempotent element of t-norm T involved.

Further, since $f(e)$ acts as a neutral element on Ran_f we know that for all $x \in [e,1]$ it holds that

$$\begin{aligned}f(x) = U(f(x), f(e)) &= T(f(x), f(e))\\ &\leq \min(f(x), f(e)) \leq f(e).\end{aligned}$$

Moreover, due to the non-decreasingness of f, $f(e) \leq f(x)$ for all $x \in [e,1]$ such that indeed $f(x) = f(e)$ for all $x \in [e,1]$.

The fact that $f|_{[0,e]}$ is distributive over T follows immediately from $f \in \mathcal{F}_{U_{T,S}}$. Finally, choose arbitrary $x \in [0,1]$, then due to property (ii)

$$\begin{aligned} f(U(x,1)) &= U(f(x), f(1)) = U(f(x), f(e)) \\ &= f(U(x,e)) = f(x). \end{aligned}$$

To prove the sufficiency, assume that $\mathrm{Ran}_f \subseteq [0,e[$ and that conditions *(i)–(iv)* are fulfilled. If both $x, y \leq e$ then also $U(x,y) \leq e$, such that f distributes over U due to condition *(iii)*. In case that both $x, y \geq e$, also $U(x,y) \geq e$ such that

$$f(U(x,y)) = f(e) = U(f(e), f(e)) = U(f(x), f(y))$$

due to condition *(ii)* and the fact that $f(e)$ is an idempotent element of U. Finally, let us consider w.l.o.g. some $x \leq e \leq y$. Due to condition *(iv)* and the non-decreasingness of f and U we can conclude that

$$f(U(x,1)) = f(x) = f(U(x,e)) \quad \Rightarrow \quad f(x) = f(U(x,y)).$$

Moreover, since $f|_{[0,e]}$ commutes with T resp. U and condition *(ii)*, we also know that

$$f(x) = f(U(x,e)) = U(f(x), f(e)) = U(f(x), f(y)),$$

such that $f \in \mathcal{F}_U$. Analogously, the remaining case and the characterization of $f \in \mathcal{F}_U$ in case of $\mathrm{Ran}_f \subseteq]e,1]$ can be shown. ■

Let us illustrate the previous results by some examples.

Example 34: Consider the following uninorm $U\colon [0,1]^2 \to [0,1]$ with neutral element $e = \frac{1}{2}$

$$U(x,y) = \begin{cases} 2xy\,, & \text{if } (x,y) \in \left[0, \frac{1}{2}\right]^2, \\ \max(x,y)\,, & \text{otherwise.} \end{cases}$$

Note that $U = U_{T,S}$ with $T\colon [0,e]^2 \to [0,e]$, $T(x,y) = 2xy$ is an isomorphic transformation of the product and $S\colon [e,1]^2 \to [e,1]$, $S(x,y) = \max(x,y)$ (see also Figure 1). Its set of idempotent elements $\mathcal{I}(U)$ is given by $\{0\} \cup \left[\frac{1}{2}, 1\right]$ since the continuous t-norm T has its boundaries as its only trivial idempotent elements.

- Therefore, there is only one function $f \in \mathcal{F}_U$ with $\mathrm{Ran}_f \subseteq [0,e[$, namely the constant function $\mathbf{0}$, since $\mathcal{I}(U) \cap [0,e[= \{0\}$ and f has to be non-decreasing.
- On the other hand, there exist several functions $f \in \mathcal{F}_U$ with $\mathrm{Ran}_f \subseteq]e,1]$: We can choose $f(e) \in]e,1] \subseteq \mathcal{I}(U) \cap]e,1]$ arbitrarily and fix as such $f(x)$ for all $x \in [0,e]$. Because $S = \max$ is a lattice polynomial, f has just to be non-decreasing on $[e,1]$ to distribute over S such that condition *(iii)* of Proposition 33 is fulfilled. And finally, condition *(iv)* trivially holds since $f(U(x,0)) = f(0) = f(x)$ in case of $x \in [0,e]$ and $f(U(x,0)) = f(\max(x,0)) = f(x)$ for all $x \in]e,1]$.
  Therefore, e.g., all functions $f_\lambda\colon [0,1] \to [0,1]$ with $\lambda \in]0.5,1]$ given by

  $$f_\lambda(x) = \begin{cases} \lambda & \text{if } x \in [0, 0.5]\,, \\ 2(1-\lambda)x + 2\lambda - 1 & \text{otherwise} \end{cases}$$

  distribute over U (see also Figure 1).

Example 35: Consider the following uninorm $U\colon [0,1]^2 \to [0,1]$ with neutral element $e = \frac{1}{2}$

$$U(x,y) = \begin{cases} \max(x,y) & \text{if } (x,y) \in \left[\frac{1}{2}, 1\right]^2, \\ 4xy & \text{if } (x,y) \in \left[0, \frac{1}{4}\right]^2, \\ \frac{1}{4}\,(4x-1)(4y-1) + \frac{1}{4} & \text{if } (x,y) \in \left[\frac{1}{4}, \frac{1}{2}\right]^2, \\ \min(x,y) & \text{otherwise.} \end{cases}$$

Again $U = U_{T,S}$ with T on ordinal sum t-norm on $\left[0, \frac{1}{2}\right]$ with twice the product as its summands and $S = \max$ a basic t-conorm on $\left[\frac{1}{2}, 1\right]$ (see also Figure 2). The set of idempotent elements $\mathcal{I}(U)$ is given by $\{0, \frac{1}{4}\} \cup \left[\frac{1}{2}, 1\right]$.

Let us now focus just on those $f \in \mathcal{F}_U$ with $\mathrm{Ran}_f \subseteq \left[0, \frac{1}{2}\right[$, i.e., $f(e) \in \{0, \frac{1}{4}\}$:

- $f(e) = 0$: Necessarily $f = \mathbf{0}$ due to the non-decreasingness of f and the necessary properties given in Proposition 33. Therefore, $\mathbf{0}$ is the only element of $\mathcal{F}_U$ for which $\mathrm{Ran}_f \subseteq [0,e[$ and $f(e) = 0$.
- $f(e) = \frac{1}{4}$: Necessarily, we fix $f(x) = \frac{1}{4}$ for all $x \in \left[\frac{1}{2}, 1\right]$ and as such fulfill conditions *(i)*, *(ii)*, *(iv)* of Proposition 33 immediately, i.e.,

  $$f\colon [0,1] \to [0,1], \quad f(x) = \begin{cases} \frac{1}{4} & \text{if } x \in \left[\frac{1}{2}, 1\right], \\ g(x) & \text{otherwise.} \end{cases}$$

  The function $g\colon \left[0, \frac{1}{2}\right[\to \left[0, \frac{1}{2}\right[$ and as such also $f|_{[0,e]}$ distributes over the ordinal sum t-norm T if it is one of the following functions (see also Figure 2):

  $$g_1(x) = 0, \quad \forall x \in \left[0, \tfrac{1}{2}\right[,$$

  $$g_{2,\lambda}(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ \frac{1}{4}(4x-1)^\lambda, & \text{otherwise,} \end{cases} \qquad \text{with } \lambda \in [1, \infty[,$$

  $$g_3(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{4}\right], \\ \frac{1}{4}, & \text{otherwise,} \end{cases}$$

  $$g_4(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{4}\right[, \\ \frac{1}{4}, & \text{otherwise,} \end{cases}$$

  $$g_{5,\lambda}(x) = \begin{cases} 4^{\lambda-1}x^\lambda, & \text{if } x \in \left[0, \frac{1}{4}\right[, \\ \frac{1}{4}, & \text{otherwise,} \end{cases} \qquad \text{with } \lambda \in [1, \infty[,$$

  $$g_6(x) = \begin{cases} 0, & \text{if } x = 0, \\ \frac{1}{4}, & \text{otherwise,} \end{cases}$$

  $$g_7(x) = \tfrac{1}{4}, \quad \forall x \in \left[0, \tfrac{1}{2}\right[.$$

So far, we have investigated non-decreasing functions f with particular domains being distributive over some uninorm U. However, in case that $e \in \mathrm{Ran}_f$ the characterization of those $f \in \mathcal{F}_U$ heavily depends on the structure of the uninorm U involved. Therefore, we will now turn to special subclasses of uninorms.

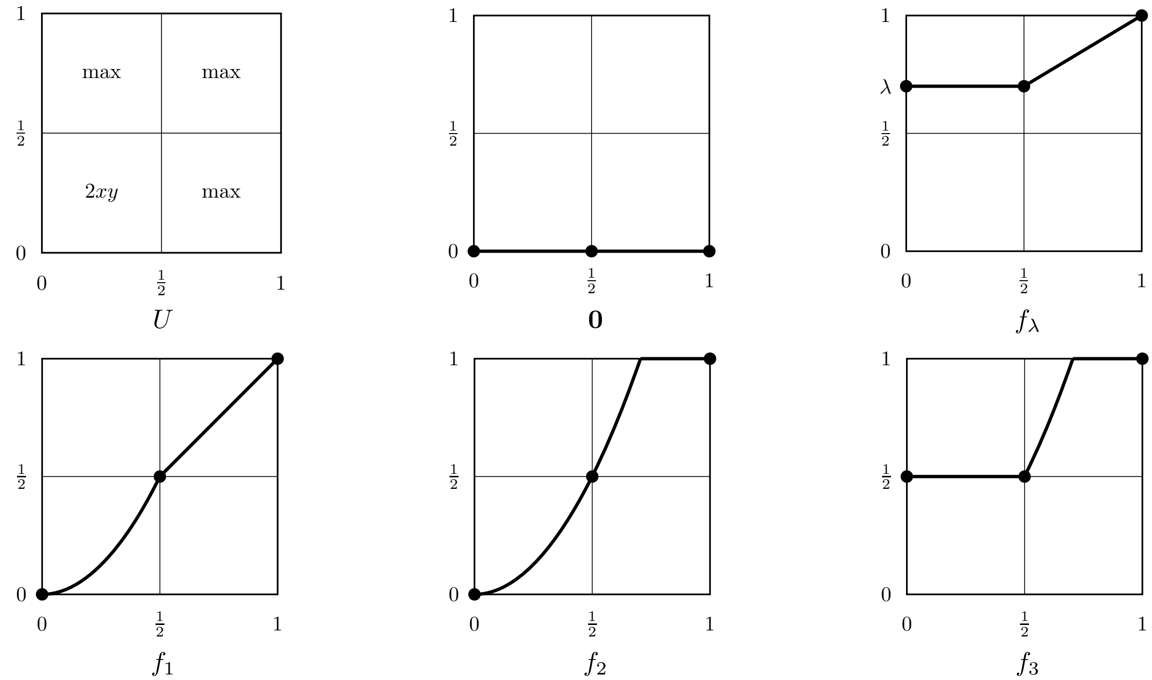

Fig. 1. Uninorm U and some $f \in \mathcal{F}_U$ as discussed in Examples 34 and 40.

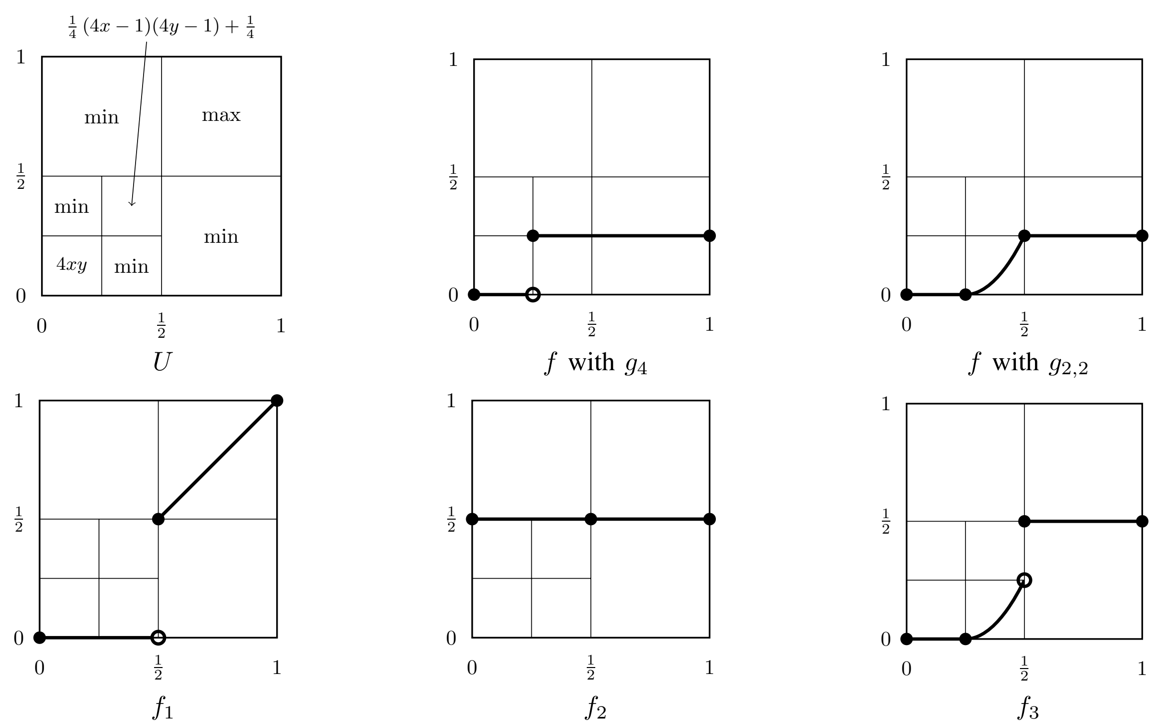

Fig. 2. Uninorm U and some $f \in \mathcal{F}_U$ as discussed in Examples 35 and 41.

B. Special case: Uninorms $U_{T,S,\min}$, $U_{T,S,\max}$

We now assume that the uninorm U is such that $U|_{[0,e]^2}$ is some t-norm T on $[0,e]$, $U|_{[e,1]^2}$ some t-conorm S on $[e,1]$ and on the remainder U acts either as the minimum or as the maximum. We will denote such uninorms by $U_{T,S,\min}$ resp. $U_{T,S,\max}$. In case that the t-norm T as well as the t-conorm S involved are continuous, we refer to the uninorm $U_{T,S}$ as *weakly continuous* t-norm.

We will focus on functions f based on a composition of functions distributive over T resp. S, i.e., on functions $f : [0,1] \to [0,1]$ defined by

$$f(x) = \begin{cases} f_T(x) & \text{if } x \in [0,e[\,, \\ f_S(x) & \text{if } x \in \,]e,1]\,, \\ e & \text{if } x = e, \end{cases} \tag{9}$$

with some $f_T \in \mathcal{F}_T$ and $f_S \in \mathcal{F}_S$. We will use the abbreviation $f = f_T \boxplus_e f_S$.

Note that not all $f \in \mathcal{F}_U$ are of the type $f = f_T \boxplus_e f_S$ as the following example shows.

Example 36: Consider some weakly continuous uninorm $U_{T,S}$. Then $f:[0,1]\to[0,1]$ defined by

$$f(x)=\begin{cases}0 & \text{if } x\in[0,e]\,,\\ 1 & \text{if } x\in\,]e,1]\,.\end{cases}$$

fulfills $f\in\mathcal{F}_{U_{T,S,\min}}$ and $f\in\mathcal{F}_{U_{T,S,\max}}$, but $f\neq f_T\boxplus_e f_S$.

However, since uninorms can be interpreted as operators acting on a bipolar scale with neutral element e, it is natural to investigate distributive functions f preserving that neutrality level, i.e., fulfilling $f(e)=e$.

By the construction $f=f_T\boxplus_e f_S$ provided by Eq. (9), it is guaranteed that the restrictions of some $f\in\mathcal{F}_{U_{T,S}}$ to $[0,e]$ resp. $[e,1]$ are distributive over the corresponding T resp. S. Note that this construction also forces that, due to the non-decreasingness of f, that $f(x)\le e$ for all $x\in[0,e]$ and $f(x)\ge e$ for all $x\in[e,1]$. Depending on whether $U=U_{T,S,\min}$ or $U=U_{T,S,\max}$, f has to fulfill additional properties for $f\in\mathcal{F}_U$.

Proposition 37: Consider some weakly continuous uninorm $U_{T,S}$, further some $f_T\in\mathcal{F}_T$ and $f_S\in\mathcal{F}_S$ and define $f:[0,1]\to[0,1]$ by $f=f_T\boxplus_e f_S$.

(i) $f\in\mathcal{F}_{U_{T,S,\min}}$ if and only if $\forall x\in[0,e[\,:f(x)<e$ or $\forall y\in[e,1]:f(y)=e$.

(ii) $f\in\mathcal{F}_{U_{T,S,\max}}$ if and only if $\forall x\in[0,e]:f(x)=e$ or $\forall y\in\,]e,1]:f(y)>e$.

Proof: Consider some weakly continuous uninorm $U_{T,S}$, further some $f_T\in\mathcal{F}_T$ and $f_S\in\mathcal{F}_S$ and define $f:[0,1]\to[0,1]$ as $f=f_T\boxplus_e f_S$ by Eq. (9).

Assume that $f\in\mathcal{F}_{U_{T,S,\min}}$. Further assume that there exists some $x_0\in[0,e[$ with $f(x_0)=e$ and some $y_0\in[e,1]$ with $f(y_0)>e$, then the following holds

$$\begin{aligned}f(y_0)&=U(e,f(y_0))=U(f(x_0),f(y_0))=f(U(x_0,y_0))\\&=f(\min(x_0,y_0))=f(x_0)=e\end{aligned}$$

leading to a contradiction. Vice versa, since $f=f_T\boxplus_e f_S$ it distributes over $U_{T,S,\min}$ for all $(x,y)\in[0,e]^2$ and for all $(x,y)\in[e,1]^2$ due to its construction. Therefore, it suffices to prove that f distributes over U for all $(x,y)\in[0,e]\times[e,1]\cup[e,1]\times[0,e]$.

Assume that f additionally fulfills either for all $x\in[0,e[\,:f(x)<e$ or for all $y\in[e,1]:f(y)=e$ and choose an arbitrary $x\in[0,e[$ and an arbitrary $y\in[e,1]$. Therefore, either $f(x)<e$ or $f(y)=e$, in any case $f(x)\le f(y)$, such that

$$\begin{aligned}f(U(x,y))&=f(\min(x,y))=f(x)=\min(f(x),f(y))\\&=U(f(x),f(y)).\end{aligned}$$

In case that $x=e$ and $y\in[e,1]$, it immediately holds that $f(U(x,y))=f(U(e,y))=f(y)=U(e,f(y))=U(f(e),f(y))=U(f(x),f(y))$. Analogously, the distributivity of f over $U_{T,S,\min}$ for some $(x,y)\in[e,1]\times[0,e]$ can be shown as well as the characterization of all $f\in U_{T,S,\max}$. ■

Based on this result, we can immediately state which functions $f=f_T\boxplus_e f_S$ are distributive over both $U_{T,S,\min}$ as well $U_{T,S,\max}$.

Lemma 38: Consider some weakly continuous uninorm $U_{T,S}$, further some $f_T\in\mathcal{F}_T$ and $f_S\in\mathcal{F}_S$ and define $f:[0,1]\to[0,1]$ by $f=f_T\boxplus_e f_S$.
$f\in\mathcal{F}_{U_{T,S,\min}}\cap\mathcal{F}_{U_{T,S,\max}}$ if and only if either

- $\forall x\in[0,e[\,:f(x)<e$ and $\forall x\in\,]e,1]:f(x)>e$, or
- $\forall x\in[0,1]:f(x)=e$.

Moreover, due to Proposition 37 and the non-decreasingness of f we can further draw the following conclusions.

Corollary 39: Consider some weakly continuous uninorm $U_{T,S}$, further some $f_T\in\mathcal{F}_T$ and $f_S\in\mathcal{F}_S$ and define $f:[0,1]\to[0,1]$ by $f=f_T\boxplus_e f_S$.

(i) If $f\in\mathcal{F}_{U_{T,S,\min}}$ and there exists some $x_0\in[0,e[$ such that $f(x_0)=e$, then $f(x)=e$ for all $x\in[x_0,1]$.

(ii) If $f\in\mathcal{F}_{U_{T,S,\max}}$ and there exists some $y_0\in\,]e,1]$ such that $f(y_0)=e$, then $f(x)=e$ for all $x\in[0,y_0]$.

Example 40 (Continuation of Example 34): Let us once again consider the uninorm U as introduced in Example 34, i.e.,

$$U(x,y)=\begin{cases}2xy\,, & \text{if } (x,y)\in\left[0,\frac{1}{2}\right]^2,\\ \max(x,y)\,, & \text{otherwise.}\end{cases}$$

It is of the type $U_{T,S,\max}$ with $T\colon[0,e]^2\to[0,e]$, $T(x,y)=2xy$ an isomorphic transformation of the product and $S\colon[e,1]^2\to[e,1]$ the maximum. Now we are looking for those $f\in\mathcal{F}_U$ which are constructed by $f=f_T\boxplus_e f_S$. The sets $\mathcal{F}_T$ and $\mathcal{F}_S$ of non-decreasing functions distributing with T resp. S are given by

$$\begin{aligned}\mathcal{F}_T=\{f\colon\left[0,\tfrac{1}{2}\right]\to\left[0,\tfrac{1}{2}\right]\mid\forall x\in\left[0,\tfrac{1}{2}\right]:&\\ f(x)=2^{(\lambda-1)}x^\lambda \text{ with } \lambda\in\,]0,\infty[\text{ or}&\\ f(x)=0\text{ or } f(x)=\tfrac{1}{2}\},&\\ \mathcal{F}_S=\{f\colon\left[\tfrac{1}{2},1\right]\to\left[\tfrac{1}{2},1\right]\mid f\text{ is non-decreasing}\}.&\end{aligned}$$

In accordance with Proposition 37, we now have to choose either $f_T(x)=\frac{1}{2}$ for all $x\in\left[0,\frac{1}{2}\right[$ or $f_S(y)>\frac{1}{2}$ for all $y\in\left]\frac{1}{2},1\right]$ such that $f=f_T\boxplus_e f_S$ fulfills $f\in\mathcal{F}_U$, so, e.g., $f_i\colon[0,1]\to[0,1]$, $i\in\{1,2,3\}$, (see also Figure 1)

$$f_1(x)=\begin{cases}2x^2, & \text{if } x\in\left[0,\frac{1}{2}\right],\\ x, & \text{otherwise}\,,\end{cases}\qquad f_2(x)=\min(2x^2,1),$$

$$f_3(x)=\max(\tfrac{1}{2},f_2(x)).$$

Example 41 (Continuation of Example 35): Note that the uninorm U defined by

$$U(x,y)=\begin{cases}\max(x,y) & \text{if } (x,y)\in\left[\frac{1}{2},1\right]^2,\\ 4xy & \text{if } (x,y)\in\left[0,\frac{1}{4}\right]^2,\\ \frac{1}{4}\,(4x-1)(4y-1)+\frac{1}{4} & \text{if } (x,y)\in\left[\frac{1}{4},\frac{1}{2}\right]^2,\\ \min(x,y) & \text{otherwise.}\end{cases}$$

is a uninorm of the type $U_{T,S,\min}$. Since T is an ordinal sum t-norm on $\left[0,\frac{1}{2}\right]$, its set of distributive functions $\mathcal{F}_T$ is rather large. Some of its elements are already listed in Example 35. Similarly, also $\mathcal{F}_S$ contains many, namely all non-decreasing functions on $\left[\frac{1}{2},1\right]$. In accordance with Proposition 37 we have to choose functions f_T, f_S such that either $f_T(x)<\frac{1}{2}$ for all $x\in\left[0,\frac{1}{2}\right[$ or that $f_S(y)=\frac{1}{2}$ for all $x\in\left]\frac{1}{2},1\right]$, such that $f = f_T \boxplus_e f_S \in \mathcal{F}_U$, e.g., $f_i\colon [0,1] \to [0,1]$, $i \in \{1,2,3\}$, (see also Figure 2),

$$f_1(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2}\right[\,, \\ x, & \text{otherwise,} \end{cases} \qquad f_2(x) = \frac{1}{2},$$

$$f_3(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{4}\right[\,, \\ \frac{1}{4}(4x-1)^2, & \text{if } x \in \left[\frac{1}{4}, \frac{1}{2}\right[\,, \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

C. *Special case: Generated uninorms*

An important subclass of uninorms are those generated by some additive generator. They are continuous on the whole domain up to the case where $\{x,y\} = \{0,1\}$.

Definition 42: An operator $\mathbf{U} : \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ is an Archimedean uninorm continuous in all points up to $(x_1,\ldots,x_n)$, $\{0,1\} \in \{x_1,\ldots,x_n\}$, if and only if there exists a monotone bijection $h : [0,1] \to [-\infty,\infty]$ such that

$$\mathbf{U}(x_1,\ldots,x_n) = h^{-1}(\sum_{i=1}^{n} h(x_i)),$$

with convention $+\infty + (-\infty) = -\infty$. The uninorm $\mathbf{U}$ is then called a *generated* uninorm with *additive generator* h.

Note that the neutral element e of such a generated uninorm is given by $h^{-1}(0) = e$. The increasingness of the additive generator is equivalent to its conjunctive form. Moreover, generated uninorms are related to strict t-norms and strict t-conorms, since $t(x) = -h(ex)$ and $s(x) = h(e + (1-e)x)$ are additive generators of a strict t-norm, resp. t-conorm, associated with $\mathbf{U}$.

In case of some $f \in \mathcal{F}_U$ with U generated by the additive generator h, we get

$$\begin{aligned} f(U(x,y)) &= f \circ h^{-1}(h(x) + h(y)) \\ &= h^{-1}(h \circ f(x) + h \circ f(y)) = U(f(x), f(y)). \end{aligned}$$

Since h is a bijection this is equivalent to

$$h \circ f \circ h^{-1}(u+v) = h \circ f \circ h^{-1}(u) + h \circ f \circ h^{-1}(v)$$

with $h(x) = u$ and $h(y) = v$ both elements from $[-\infty,\infty]$ such that for $h^* = h \circ f \circ h^{-1}$ and arbitrary $u, v \in [-\infty,\infty]$ it holds that

$$h^*(u+v) = h^*(u) + h^*(v).$$

In case that h^* is continuous the solutions of this equation (see also [3]) are given by

$$h^*(u) = c \cdot u$$

with $c > 0$. As a consequence

$$f(x) = h^{-1}(c \cdot h(x))$$

leading to the following lemma.

Lemma 43: Consider some uninorm U generated by some additive generator h. If $f \in \mathcal{F}_U$ and f continuous, but not constant, then there exists some $c > 0$ such that

$$f(x) = h^{-1}(c \cdot h(x))$$

for all $x \in [0,1]$.

Example 44: Consider some uninorm U generated by some additive generator h and choose $c_i \geq 0$ for all $i \in \{1,\ldots,n\}$ and $c_i > 0$ for at least one $i \in \{1,\ldots,n\}$. Then the operator A defined by

$$A(x_1,\ldots,x_n) = h^{-1}(\sum_{i=1}^{n} c_i h(x_i))$$

commutes with U.

Example 45: Consider the additive generator $h\colon [0,1] \to [-\infty,\infty]$, $h(x) = \ln \frac{x}{1-x}$. The generated uninorm $U^*\colon [0,1]^2 \to [0,1]$ is then given by

$$U^*(x,y) = \frac{xy}{xy + (1-x)(1-y)}$$

with neutral element $e = 0.5$. Note that U^* is also known as 3-Π-operator and has already been discussed by several authors [14], [17], [23], [35], [37]. It is worth to remark that it plays an important role as combining functions of uncertainty factors in expert systems like MYCIN and PROSPECTOR (see also [9], [12], [21]).

In accordance with the previous example, aggregation operators $\mathbf{A}\colon \bigcup_{n\in\mathbb{N}}[0,1]^n \to [0,1]$ defined by

$$\mathbf{A}(x_1,\ldots,x_n) = \frac{\prod_{i=1}^{n} x_i^{c_i}}{\prod_{i=1}^{n}(1-x_i)^{c_i} + \prod_{i=1}^{n} x_i^{c_i}}$$

with $c_i \geq 0$ for all $i \in \{1,\ldots,n\}$ and $c_i > 0$ for at least one $i \in \{1,\ldots,n\}$ commute with U^*.

VII. Final Remarks

The issue of commuting aggregation has been considered in the general case and in some important particular cases especially the one of uninorms, where new non-trivial results are obtained. Finding commuting operations can be a difficult exercise sometimes leading to impossibility results. So, e.g., in the class of OWA operators [36], the set of all aggregation operators commuting with an n-ary OWA operator different from $\min$, $\max$, or the arithmetic mean respectively, is trivial, namely, consisting just of the projections [32]. However, for bisymmetric operations such that as weighted arithmetic mean, results on commuting exist for some 25 years in connection with the problem of consensus functions for probabilities [28], more recently for t-norms and conorms in connection with generalized utility theory [15] or transitivity preservation in the aggregation of fuzzy relations [34]. Commuting operators for uninorms can be relevant in multi-criteria decision-making with bipolar scales where bipolar set-functions are used to evaluate the importance of criteria [19], [20]. Indeed the neutral element of uninorm separates a bipolar evaluation scale in its positive and negative parts [16]. Our results can be instrumental in laying bare consensus functions for multi-person multi-criteria decision-making problems on bipolar scales, a topic to be investigated at a further stage.

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