Post-processing the hybrid method for addressing uncertainty in risk assessments

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ABSTRACT: In this journal, a "hybrid method" was proposed for the joint propagation of probability distributions (expressing variability) and possibility distributions (i.e., fuzzy numbers, expressing imprecision or partial ignorance) in the computation of risk. In order to compare the results of the hybrid computation (a random fuzzy set) to a tolerance threshold (a tolerable level of risk), a post-processing method was proposed. Recent work has highlighted a shortcoming of this post-processing step which yields overly-conservative results. A post-processing method based on Shafer's theory of evidence provides a rigorous answer to the problem of comparing a random fuzzy set with a threshold. The principles behind the new post-processing scheme are presented and illustrated with a synthetic example.

INTRODUCTION

Variability and imprecision are two distinct facets of uncertainty in risk assessment. Variability, also referred to as "objective uncertainty", arises from heterogeneity or the random character of natural processes. Imprecision, also referred to as "subjective uncertainty", arises from the partial character of our knowledge of the natural world. As shown in particular by Ferson and Ginzburg (1996), distinct methods are needed to adequately represent and propagate variability and imprecision. One of the most common pitfalls in risk analysis is the confusion between these two types of uncertainty. When faced with partial information regarding a certain model parameter, for example the knowledge that the parameter value is located somewhere between a minimum value (*min*) and a maximum value (*max*), it is common to assume a uniform distribution of probability distribution between *min* and *max*. But this approach introduces information that is in fact not available and therefore biases the outcome of the analysis (see Ferson, 1996). While information regarding random variability is best conveyed using probability distributions, information regarding imprecision can be conveyed using families of probability distributions. This encompasses representation techniques ranging from simple intervals containing ill-known values, to more sophisticated tools. Namely, families of probabilities can be encoded by pairs of upper and lower cumulative probability functions (Ferson et al., 2004), by possibility distributions (also called fuzzy sets ; Dubois and Prade, 1988, Zadeh, 1978) or by belief functions of Dempster-Shafer (Shafer, 1976).

As risk assessments commonly face both types of uncertainty, it would seem beneficial to be able to distinguish between them when estimating risk, so that the way we represent uncertainty might be consistent with the information at hand. In Guyonnet et al. (2003) a method, dubbed "hybrid" method, was proposed for a joint handling of probability and possibility distributions in the computation of risk. Considering a "model", i.e. a function of several parameters, some of which are justifiably represented by probability distributions (expressing random variability), while others are better represented by possibility distributions (expressing imprecision), the hybrid method combines random sampling of the probability distributions (Monte Carlo method) with fuzzy interval analysis (Dubois and Prade, 1988). The result is a random fuzzy set (Gil, 2001). In order to compare this result to a tolerance threshold, Guyonnet et al. (2003) proposed a summarization of the obtained random fuzzy set in the form of a unique fuzzy interval, from which two cumulative (optimistic and pessimistic) probability distributions can be derived. But recent work (Baudrit et al., 2004) has laid bare a shortcoming of this post-processing method. This note describes a new method based on the theory of evidence that provides a rigorous answer to the problem of comparing a random fuzzy set with a threshold.

THEORY

For the sake of clarity, the "hybrid method" is briefly summarized below. Firstly, it is recalled that a possibility distribution (or fuzzy number) restricting the possible values of a real-valued parameter X takes the form of a fuzzy set membership function (denoted μ) such that $\mu(x^*) = 1$ for some value x^* of X (Dubois and Prade, 1988). This function, the values of which lie between zero and one, describes, for each value x, the degrees of possibility that X = x. The simplest possibility distribution is the well-known *min-max* interval. The membership function $\mu(x)$ for this distribution is equal to 1 for all values x within the interval, and zero elsewhere. But a possibility distribution enriches the simple *min-max* interval by allowing the expression of preferences within the interval. If available information is richer than that conveyed by a *min-max* interval (certain values are believed to be more likely than others) then a more elaborate possibility distribution can be used to convey such information. The interval containing all values of X that have a degree of possibility greater than or equal to a certain threshold α is called an α-cut and noted: $A_\alpha = \{x, \mu(x) \geq \alpha\}$.

Possibility theory provides two indicators of the likelihood of an event A, for example that the value of variable X should lie within a certain interval A; the possibility measure (Π) and the necessity measure (N). The possibility measure of the event is defined as:

$$\Pi(\mathrm{A}) = Sup_{\,x \in A}\ \mu(x) \tag{1}$$

where *Sup* denotes the largest value. Events are thus evaluated on the basis of their most plausible occurrences. In probabilistic terms, a degree of possibility can be viewed as an upper probability bound (Dubois and Prade, 1992). The dual function of a possibility measure Π is the necessity measure (N), defined as:

$$\mathrm{N(A)} = 1 - \Pi(\mathrm{A}^c) \tag{2}$$

where A^c denotes the complement of A. Therefore a degree of necessity can be viewed as a lower probability bound.

The hybrid method proposed in Guyonnet et al. (2003) combines the random sampling of probability distribution functions (PDFs) with fuzzy interval analysis on the α-cuts. Fuzzy interval analysis as performed herein can be thought of simply as interval calculus performed at different levels of possibility. For the purpose of describing the hybrid method we consider a "mathematical model", noted M (for example an equation for calculating a dose of exposure), that is a function of a certain number of parameters:

$$Dose = M(P_1, \ldots, P_n, F_1, \ldots, F_m), \tag{3}$$

where M = model; $P_1, \ldots, P_n = n$ independent model parameters represented by probability distribution functions (PDFs) ; $F_1, \ldots, F_m = m$ model parameters represented by fuzzy numbers. The hybrid procedure is summarized below (see Fig. 3 of Guyonnet et al., 2003):

1. Generate n random numbers ($\chi_1, \ldots, \chi_n$) from a uniform distribution and sample the n PDF's to obtain a realization of the n random variables: $p_1, \ldots p_n$
2. Select a value α of the membership function (a level of possibility).
3. Interval calculation: calculate the *Inf* (smallest) and *Sup* (largest) values of $M(p_1, \ldots, p_n, F_1, \ldots, F_m)$, considering all values located within the α-cuts of each fuzzy number.
4. Assign these *Inf* and *Sup* values to the lower and upper limits of the α-cut of $M(p_1, \ldots, p_n, F_1, \ldots, F_m)$.
5. Return to step 2 and repeat steps 3 and 4 for another α-cut. The fuzzy result of $M(p_1, \ldots, p_n, F_1, \ldots, F_m)$ (the fuzzy dose) is obtained from the *Inf* and *Sup* values of $M(p_1, \ldots, p_n, F_1, \ldots, F_m)$ for each α-cut.
6. Return to step 1 to generate a new realization of the random variables.

A family of ω fuzzy interval-valued doses (a random fuzzy set) is thus obtained (ω being the number of realizations of the random variables).

In order to compare the random fuzzy set to a tolerance threshold (for example a maximum tolerable dose) it is necessary to first perform a post-processing of this result. In Guyonnet et al. (2003) it was proposed to select the final *Inf* and *Sup* values of $M(P_1, \ldots, P_n, F_1, \ldots, F_m)$, for each level of possibility α, by building relative frequency distributions of the *Inf* and *Sup* values and extracting the final *Inf* and *Sup* values for a certain level of probability. The final *Inf* and *Sup* values could be selected such that there would be a 95% probability of respectively having values higher than *Inf* and lower than *Sup*. But as was shown by Baudrit et al. (2004), this method combines the *Inf* and *Sup* values of intervals that are in fact independent of one another and leads to an overestimation of the uncertainty range (see application section).

Baudrit et al. (2004) showed how the theory of evidence, also called theory of Dempster-Shafer (or theory of belief functions ; Shafer, 1976) could provide a simple and rigorous answer to the problem of summarizing the results of the hybrid computation for comparison with a tolerance threshold. The theory of evidence allows variability and imprecision to be treated within a single framework. While probability theory assigns probability weights (in the discrete case) or probability density weights (in the continuous case) to each possible value of an ill-known parameter, the theory of evidence may assign such weights to subsets (for example intervals) of values. A mass function m is thus defined which assigns a weight $m(A)$ to each subset A of values of the concerned parameter. Such a subset with positive mass is called a focal set (Shafer, 1976). The weight $m(A)$ is a probability mass that in the ideal case (rich information) would be shared among specific values within A, but which in the case of poorer information remains unassigned due to imprecision. If all focal sets contain single values, the mass function reduces to a standard probability assignment and expresses pure variability. Assigning all the mass to a single subset A corresponds to the case where all that is known is that the parameter value lies within A, which expresses pure imprecision (classical *min-max* interval). Finally, if all focal sets form a chain of n nested sets A_i, the mass function then defines a possibility distribution.

The theory of evidence provides two indicators to qualify the validity of a proposition B stating that the value of the parameter lies within a prescribed set B. These indicators, that generalize functions N and P to the non-nested case, are the degree of belief of B ; denoted *Bel*(B), and the degree of plausibility of B, denoted *Pl*(B), respectively defined by (Shafer, 1976):

$$Bel(\mathrm{B}) = \sum_{\mathrm{i}: A_\mathrm{i} \subseteq B} m(A_\mathrm{i}) \qquad \text{and} \qquad Pl(\mathrm{B}) = \sum_{\mathrm{i}: A_\mathrm{i} \cap B \neq 0} m(A_\mathrm{i}) \qquad (4)$$

Bel(B) is thus the sum of the weights of all subsets A_i (i = 1 to n where n is the number of subsets) such that A_i is completely included within prescribed set B, while *Pl*(B) is the sum of the weights of all subsets A_i such that the intersection of A_i and B is non empty. *Bel*(B) gathers the imprecise evidence that asserts B. *Pl*(B) gathers the imprecise evidence that does not contradict B ; the interval [*Bel*(B), *Pl*(B)] contains all potential probability values induced by the mass function m.

The theory of evidence naturally applies to the random fuzzy set resulting from the hybrid computation because each computed a_i-cut (interval of values) of each fuzzy dose M_j in the resulting sample can be viewed as a focal set of a single belief function representing this random fuzzy set. The mass of each such a_i-cut A_{ij} is a certain frequency related to the random swampling of the PDFs. Namely, to each focal set A_{ij} is associated a probability weight $m(A_i) = 1/(\omega \cdot \varepsilon)$, where ω is the number of realizations of the random variables while $\varepsilon+1$ is the number of α-cuts used to discretise the fuzzy doses. The construction of the degrees of belief (*Bel*) and of plausibility (Pl) of the proposition "the model result is lower than a certain threshold" is illustrated schematically in Fig. 1. Fig. 1a shows a few representatives of the random fuzzy set resulting from the hybrid computation of a dose of exposure. Fig.1b shows some focal sets corresponding to individual intervals extracted from the random fuzzy set. Considering an arbitrary dose threshold (D_o ; Fig.1b), the degrees of plausibility and belief for the proposition "the calculated dose is lower than the threshold D_o" (Fig. 1c) are calculated by summing the probability weights according to Eq. 4, for event $B=[-\infty, D_o]$. If the threshold D_o is lower than the lowest limit of all focal sets extracted from the random fuzzy set, then $Pl = Bel = 0$ (we are "sure" that the proposition is false). As soon as D_o exceeds this lowest limit, a probability weight of $1/(\omega*\varepsilon)$ is added to Pl (while Bel remains zero). Each time a new lower interval limit is exceeded by the threshold, that same probability weight is added to Pl. For Bel to become positive, the threshold must exceed at least one of the upper interval limits. Once the threshold exceeds the highest upper limit of all intervals, then $Pl = Bel = 1$ (we are "sure" that the proposition is true). The difference between Pl and Bel reflects the imprecision that has been propagated in the analysis. If the problem involves only variability, but no imprecision, then $Pl = Bel$.

From an operational viewpoint, let us assume that we have applied the hybrid method to calculate a dose of exposure and randomly sampled for example 400 realizations of the random variables and identified, for each realization, the *Inf* (minimum) and *Sup* (maximum) values of the dose for each value of possibility α (for example 11 values from 0 to 1 with step 0.1). We thus obtain the random fuzzy of the set D_i. The plausibility distribution will be obtained by sorting the 10 x 400 = 4 000 *Inf* values (for $\alpha = 0.1$ to 1 with step 0.1) in increasing order, and incrementally adding a weight of $1/(400\cdot 10) = 0.00025$ to each value. The belief distribution will be obtained likewise with the *Sup* values.

With respect to comparison with a threshold, if we consider the proposition A: "calculated dose is lower than threshold D_o", the probability P(A) that A is true is bounded by: $Bel(A)<P(A)<Pl(A)$. Bel(A) is therefore a conservative indicator of the probability that A is true (degree of belief that the threshold is not exceeded). Conversely, the probability of the inverse proposition $\overline{A}$: "calculated dose exceeds threshold D_o" is bounded by: $Bel(\overline{A}) < P(\overline{A}) < Pl(\overline{A})$ where: $Pl(\overline{A}) = 1 - Bel(A)$ and $Bel(\overline{A}) = 1 - Pl(A)$. In this case, $Pl(\overline{A})$ is the conservative indicator (degree of plausibility of exceeding the threshold).

APPLICATION

In order to clearly illustrate the drawback of the post-processing method proposed in Guyonnet et al. (2003) and the adequacy of the one proposed here, after Baudrit et al. (2004), we consider a generic "model" M that is a simple function of three parameters A, B and C:

$$M = \frac{AC}{B} \qquad (5)$$

Both A and B are represented by normal probability distributions (of averages respectively 15 and 50 and standard deviations respectively 2 and 5 ; Figs. 2a and 2b). If parameter C were a constant (for example C = 2), then M would be represented by a unique probability distribution. We will assume that C is tainted by a small degree of imprecision (Fig. 2c), encoded by a fuzzy number of core = 2 (value considered most likely) and support [1.9-2.1] (interval outside which values are considered not plausible). Given the limited extent of imprecision, one would expect the joint propagation of variability and imprecision for this problem, using the hybrid method, to yield a result very close to the case where C is not affected by any imprecision at all (C = 2).

Fig. 1d shows ten representatives of the random fuzzy set of AC/B resulting from the application of the hybrid method (500 realizations of the random parameters were in fact performed). We are now interested in the probability that AC/B is lower than some value *x* ; P(AC/B < x). The indicators Π_{95} and N_{95} (Fig. 1e) are, respectively, the degrees of possibility and necessity obtained from the post-processing method proposed in Guyonnet et al. (2003). They are the upper and lower probability bounds encoding the samples of fuzzy numbers of AC/B such that there is a 5% chance of having lower or higher values. The curve indicated "MC" in Fig. 1e is the result of the Monte Carlo calculation for C = 2. Despite the small imprecision on parameter C, there is seen to be a surprisingly large difference between the three curves.

Fig. 1f shows the degrees of plausibility and belief for the same proposition. As could be expected, there is very little difference between the different distributions. If the support of C were further reduced, to end up as a point value (C = 2), the distance between *Pl* and *Bel* would shrink until the curves eventually coincide (expressing pure variability with no imprecision). The example shows that the new proposed post-processing method yields results that are consistent with what we would expect for this problem. The post-processing method proposed in Guyonnet et al. (2003) yields a much broader range of uncertainty, but this range is unrealistic because the method combines upper and lower limits of intervals that are in fact independent of one another. It is worth noting that in 2005, a software program (dubbed HyRisk), implementing the hybrid method and the post-processing method described herein, will be available on the BRGM's web site.

CONCLUSIONS

This paper highlights a shortcoming of the post-processing method proposed in Guyonnet et al. (2003) for comparing the results of the hybrid computation to a tolerance threshold. This post-processing method yields overly-conservative estimates of the probability of respecting a tolerance threshold. A method that provides a rigorous post-processing of the hybrid computation is proposed based on the theory of evidence (Shafer, 1976). It should be noted, however, that a reassessment of the "possibility of exceeding a tolerable dose" in Fig. 11 of Guyonnet et al. (2003) shows values to be virtually unchanged by application of the new post-processing because the problem treated was mainly dominated by imprecision rather than by variability.

Further work is currently aiming at accounting for variability and imprecision directly within the common framework of the theory of evidence using a "homogeneous approach" (see Baudrit et al., 2003 for a preliminary account), without having to treat these two forms of uncertainty separately as in the hybrid approach. An advantage of such a homogeneous approach is that it enables all the possible dependencies between parameters to be accounted for (see also Ferson and Ginzburg, 1996). An underlying hypothesis of the hybrid method discussed here is that there is independence between the probabilistic variables and also independence between the group of probabilistic variables and the group of possibilistic variables. Note however that non-linear monotone dependency between the random variables could be accounted for in the hybrid approach using rank correlation methods (Connover and Iman, 1982).

While several options can be considered for a joint handling of variability and imprecision (partial ignorance) in risk assessments, it is the opinion of the authors that a clear distinction between these two types of uncertainty is necessary. In the face of partial ignorance, making the assumption of perfect random variability is one of the most common temptations in risk assessment. It presupposes a degree of confidence in the outcome of the analysis that is not consistent with the information that is truly available. It would seem important, with respect to communication with decision-makers or other stakeholders, that risk assessors be capable of distinguishing between what they know to be due to an underlying random process and what they simply do not know for lack of better information.

REFERENCES

Baudrit, C., Dubois, D. and Fargier, H. (2003). "Propagation of uncertainty involving imprecision and randomness." *In:* EUSFLAT, 3rd International conference in fuzzy logic and technology, Zittau, Germany, 10-12 September 2003, 653-658.

Baudrit, C., Dubois, D., Guyonnet, D., Fargier, H. (2004). "Joint treatment of imprecision and randomness in uncertainty propagation." *In:* Information Processing and Management of Uncertaity in Knowledge-Based Systems. IPMU-2004, July 4-9, Perugia, Italy, 873-880.

Connover, W. and Iman, R. (1982). "A distribution-free approach to inducing rank correlation among input variables." *Technometric*, 3, 311-334.

Dubois, D., and Prade, H. (1992). "When upper probabilities are possibility measures." *Fuzzy Sets and Systems*, 49, 95-74.

Dubois, D., and Prade, H. (1988). "Possibility theory." New York Plenum Press, 263 pp.

Ferson, S., Ginzburg, L. and Akcakaya, R. (2004) – "Whereof one cannot speak: when input distributions are unknown". Risk Analysis (in press).

Ferson, S. and Ginzburg, L. (1996). "Different methods are needed to propagate ignorance and variability." *Reliability Engineering and System Safety*, 54, 133-144.

Ferson, S. (1996). "What Monte Carlo methods cannot do." *Human and Environmental Risk Assessment*, 2, 990-1007.

Gil M.A. (2001). "Fuzzy Random Variables." Special issue of *Information Sciences*, 133, Nos. 1-2.

Guyonnet, D., Dubois, D., Bourgine, B., Fargier, H., Côme, B. and Chilès, J.-P. (2003). "Hybrid method for addressing uncertainty in risk assessments." *Journal of Environmental Engineering*, 129, 68-78.

Shafer, G., (1976). "A mathematical theory of evidence." Princeton University Press.

Zadeh, L.. (1978). "Fuzzy sets as a basis for a theory of possibility." *Fuzzy Sets and Systems,* 1, 3-28.

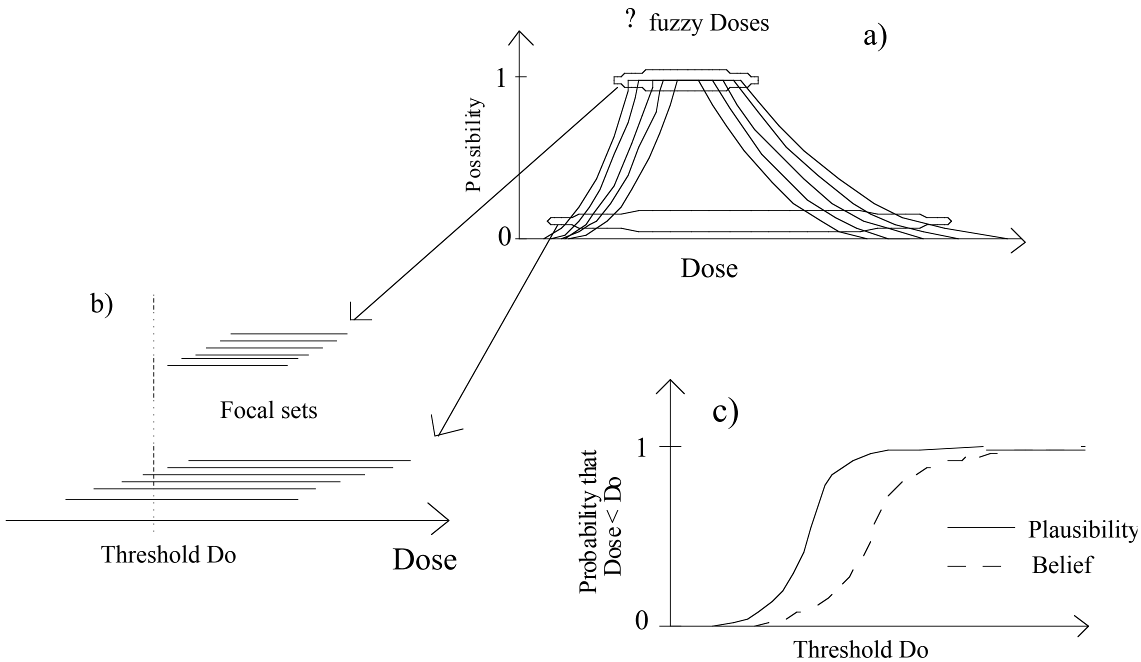

Fig. 1. Schematic illustration of the post-processing of the hybrid computation.

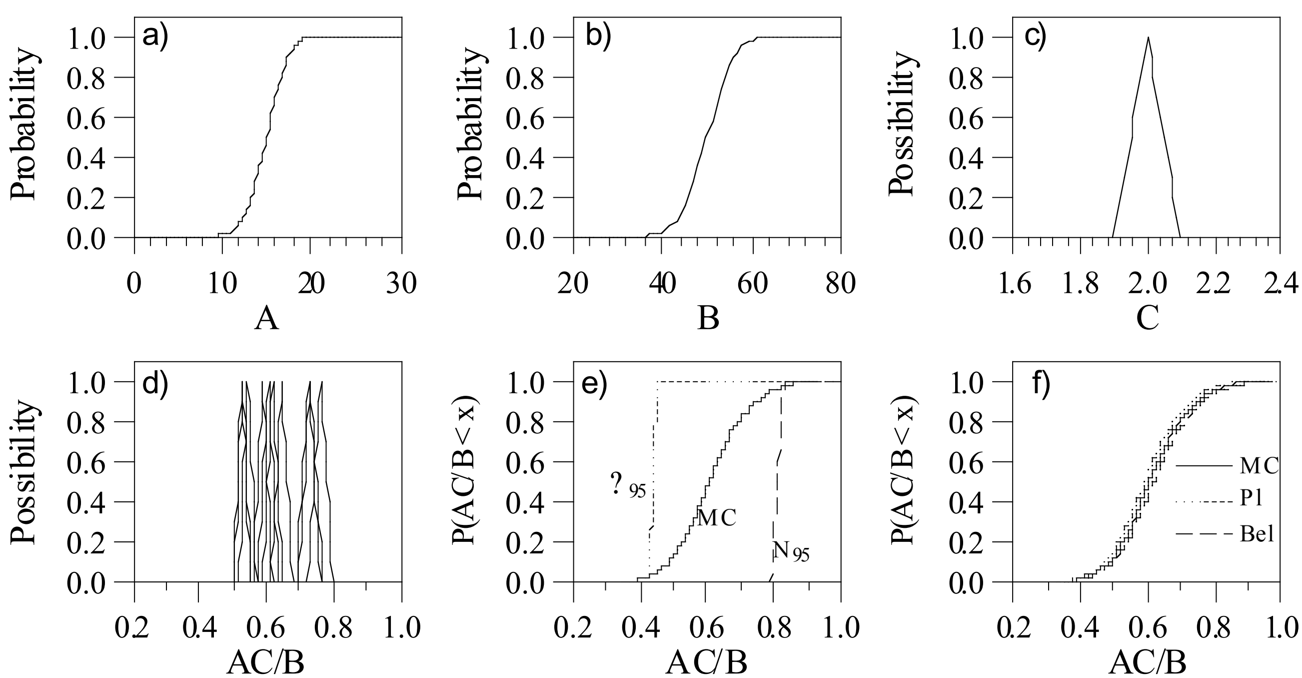

Fig. 2. Example calculation: probability distributions (a, b), possibility distribution (c), ten samples of the random fuzzy set of AC/B (d), indicators of the veracity of the proposal "AC/B < x" from Guyonnet et al. (2003) (e) and new indicators (f)