A Study on the Analytic Hierarchy Process Using a Fuzzy Reciprocal Matrix

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Abstract

Analytic Hierarchy Process (AHP) is one of the most popular method in the field of decision-making today. While there exist many kinds of extensions of AHP using fuzzy measures or fuzzy sets, one of the most natural uses of fuzzy sets is to employ a reciprocal matrix with fuzzy-valued entries. Indeed, using classical AHP, there is often not enough consistency among the data due to strong conditions bearing on the reciprocal data matrices. Fuzzy reciprocal matrices give us flexible specifications of pairwise preference. This paper presents an approach to using a fuzzy reciprocal matrix in AHP, as a way of specifying fuzzy restrictions on the possible values of the ratio judgments. Then, it can be computed to what extent there exists a consistent standard AHP matrix which is compatible with these restrictions. An optimal consistency index and optimal weights are derived using a fuzzy constraint satisfaction approach. Moreover we show an example of our approach in the later part of the paper.

Keywords: Decision Making; AHP; Fuzzy Sets.

1 Introduction

The AHP methodology was proposed by T.L. Saaty in 1977 [9] [10], and it has been widely used in the field of decision making. It elicits weights of criteria and alternatives through ratio judgments of relative importance. And finally the preference for each alternative can be derived. The classical method requires the decision-maker (DM) to express his or her preferences in the form of a precise ratio matrix encoding a valued preference relation. However, it can often be difficult for the DM to express exact estimates of the ratios of importance.

Therefore many kinds of methods employing intervals or fuzzy numbers as elements of a pairwise reciprocal data matrix have been proposed to cope with this problem. This allows for a more flexible specification of pairwise preference intensities accounting for the incomplete knowledge of the DM.

In practice, when interval-valued matrices are employed, the DM often gives ranges narrower than his or her actual perception would authorize, because he or she might be afraid of expressing information which is too imprecise. On the other hand, a fuzzy interval expresses rich information because the DM can provide i) the core of the fuzzy interval as a rough estimate of his perceived preference and ii) the support set of the fuzzy interval as the range that the DM believes to surely contain the unknown ratio of relative importance.

Usually, since components of the pairwise matrix are locally obtained from the DM by pairwise comparisons of activities or alternatives, its global consistency is not guaranteed. In classical AHP, consistency is usually measured by a consistency index (C.I.) based on the computation of an eigenvalue.

Very often, the transitivity of preferences between the elements to be compared is strongly related to the consistency of the matrix. Using intervals or fuzzy numbers as elements of the reciprocal matrices, strict transitivity is too hard to preserve in terms of equalities between intervals or fuzzy numbers. Therefore we only try to maintain consistency of precise matrices that fit the imprecise specifications provided by the DM. A new kind of consistency index for fuzzy-valued matrices is computed that corresponds to the degree of satisfaction of the fuzzy specifications by the best fitting consistent reciprocal preference matrices. Importance or priority weights are then derived based on these precise preference matrices.

2 Earlier work

The earliest work in AHP using fuzzy sets as data was published by van Laarhoven and Pedrycz [7]. They compared fuzzy ratios described by triangular membership functions. Lootsma's logarithmic least square was used to derive local fuzzy priorities. Later using a geometric mean, Buckley [3], [4] determined fuzzy priorities of comparison ratios whose membership functions were assumed trapezoidal. Modifying van Laarhoven and Pedrycz's method, Boender et al. [1] presented a more robust approach to the normalization of the local priorities.

The issue of consistency in AHP using fuzzy sets as elements of the matrix was first tackled by Salo [11]. Departing from the fuzzy arithmetic approach, he derived fuzzy weights using an auxiliary programming formulation, which described relative fuzzy ratios as constraints on the membership values of local priorities. Later Leung and Cao [8] proposed a notion of tolerance deviation of fuzzy relative importance that is strongly related to Saaty's consistency index C.I.

There also exist many other extensions of AHP using fuzzy measures and integrals [12]. These methods do not have to assume independence of activities and can avoid the rank reversal problems which the traditional AHP method may suffer from.

3 An approach using a fuzzy-valued reciprocal matrix

Since using fuzzy numbers as elements of a pairwise matrix is more expressive than using crisp values or intervals, we hope that the fuzzy approach allows a more accurate description of the decision making process. Rather than forcing the DM to provide precise representations of imprecise perceptions, we suggest using an imprecise representation instead. In the traditional method the obtained matrix does not exactly fit the AHP theory and thus must be modified so as to respect mathematical requirements. Here we let the DM be imprecise, and check if this imprecise data encompasses precise preference matrices obeying the AHP requirements.

3.1 Fuzzy reciprocal data matrix

In this paper, we employ a fuzzy pairwise comparison reciprocal $n{\times}n$ matrix $\tilde{R}=\{\tilde{r}_{ij}\}$ pertaining to n elements (criteria, alternatives). In the AHP model, entry r_{ij} of a preference matrix reflects the ratio of importance weights of element i over element j. In the fuzzy-valued matrix, diagonal elements are singletons (= 1) and the other entries $\tilde{r}_{ij}$ $(i \neq j)$ have membership function μ_{ij} whose support is positive:

$$\tilde{r}_{ii}=1, \quad \mathrm{supp}(\mu_{ij}) \subseteq (0, \ +\infty), \ i,j=1,\ldots,n$$

Moreover if element i is preferred to element j then supp(μ_{ij}) lies in $[1, +\infty)$, while supp(μ_{ij}) lies in $(0, 1]$ if the contrary holds. The DM is supposed to supply the core (modal value) r_{ij} of $\tilde{r}_{ij}$ and its support set $[l_{ij}, u_{ij}]$ for $i<j$. The support set is the range that the DM believes surely contains the unknown ratio of relative importance. The DM may only supply entries above the diagonal like in the classical AHP.

We assume reciprocity $\tilde{r}_{ij}=1/\tilde{r}_{ji}$ as follows [7]

(1) $$\mu_{ij}(r)=\mu_{ji}(1/r)\,.$$

Therefore

(2) $\quad \mathrm{core}\left(1/\tilde{r}_{ij}\right) = 1/r_{ij}$,

(3) $\quad \mathrm{supp}\left(1/\tilde{r}_{ij}\right) = [1/u_{ij}, 1/l_{ij}]$.

We may assume all entries whose core is larger than or equal to 1 form triangular fuzzy sets i.e., if $r_{ij} \geq 1$, we assume $\tilde{r}_{ij}$ is a triangular fuzzy number, denoted as

(4) $$\tilde{r}_{ij} = (l_{ij}, r_{ij}, u_{ij})_{\Delta},$$

but then the symmetric entry $\tilde{r}_{ji}$ is not triangular. Alternatively one may suppose that if $r_{ij} < 1$, $\tilde{r}_{ji}$ is $(1/u_{ij}, 1/r_{ij}, 1/l_{ij})_{\Delta}$. Therefore the following transitivity condition inherited from the AHP theory will not hold

(5) $$\tilde{r}_{ij} \otimes \tilde{r}_{jk} = \tilde{r}_{ik}$$

in particular because multiplication of fuzzy intervals does not preserve triangular membership functions. However, even with intervals, this equality is too demanding (since it corresponds to requesting two usual equalities instead of one) and impossible to satisfy. For instance take $i = k$ in the above equality. On the left hand side is an interval, on the right-hand side is a scalar value (=1). So it makes no sense to consider a fuzzy AHP theory where fuzzy intervals would simply replace scalar entries in the preference ratio matrix.

3.2 Consistency

In our approach, a fuzzy-valued ratio matrix is considered to be a fuzzy set of consistent non-fuzzy ratio matrices. Each fuzzy entry is viewed as a flexible constraint. A ratio matrix is consistent in the sense of AHP (or AHP-consistent) if and only if there exists a set of weights $w_1, w_2, \ldots, w_n$, summing to 1, such that for all i, j, $r_{ij} = w_i / w_j$.

Using a fuzzy reciprocal matrix, some kind of consistency index of the data matrix is necessary. This index will not measure the AHP-consistency of a non-fully consistent matrix, but instead will measure the degree to which an AHP-consistent matrix R exists, that satisfies the fuzzy constraints expressed in the fuzzy reciprocal matrix $\tilde{R}$. More precisely this degree of satisfaction can be attached to a n-tuple of weights $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$ since this n-tuple defines an AHP-consistent ratio matrix. This degree is defined as

(6) $$\alpha(\boldsymbol{w}) = \min_{i,j} \mu_{ij}(w_i / w_j).$$

It is the "degree of consistency" of the weight pattern $\boldsymbol{w}$ with the fuzzy ratio matrix $\tilde{R}$ in the sense of fuzzy constraint satisfaction problems (FCSPs) [5]. The coefficient $\alpha(\boldsymbol{w})$ is in some sense an empirical validity coefficient measuring to what extent a weight pattern is close to, or compatible with, the DM revealed preference.

The best fitting weight patterns can thus be found by solving the following FCSP:

$$\text{maximize } \alpha \equiv \min_{i,j}\left\{\mu_{ij}\left(\frac{w_i}{w_j}\right)\right\}$$

$$0 \leq w_i \leq 1,\ \sum_i^n w_i = 1,\ i=1,\ldots,n,$$

where w_i is the weight of alternative i, and n is the total number of alternatives. Maximizing α corresponds to getting as close as possible to the ideal preference patterns of the DM (in the sense of the Chebychev norm). Let

(7) $$\alpha^* \equiv \max_{w_1,\ldots,w_n} \min_{i,j}\left\{\mu_{ij}\left(\frac{w_i}{w_j}\right)\right\}.$$

α^* is a degree of consistency different from Saaty's index, but which can be used as a natural substitute to the AHP-consistency index for evaluating the DM's degree of rationality when expressing his or her preferences.

Solving this flexible constraint satisfaction problem in terms of α enables the fuzzy ratio matrix to be turned into an interval-valued matrix defining crisp constraints for the main problem of calculating local weights, as shown in the next subsection.

As usual, the FCSP problem can be re-stated as follows

$$\text{maximize } \alpha$$

$$\text{s.t.} \quad \mu_{ij}\left(\frac{w_i}{w_j}\right) \geq \alpha,$$

$$\sum_i^n w_i = 1,\ i,j = 1,\ldots,n,$$

and we can express the first constraint as follows

(8) $w_i / w_j \in [\underline{\mu_{ij}^{-1}(\alpha)}, \overline{\mu_{ij}^{-1}(\alpha)}]$,

where $\underline{\mu_{ij}^{-1}(\alpha)}$ and $\overline{\mu_{ij}^{-1}(\alpha)}$ are the lower and upper bound of the α-cut of $\mu_{ij}(w_i / w_j)$, respectively. This becomes

(9) $w_j \underline{\mu_{ij}^{-1}(\alpha)} \le w_i \le w_j \overline{\mu_{ij}^{-1}(\alpha)}$.

Here, if all μ_{ij} are triangular fuzzy numbers $(l_{ij}, r_{ij}, u_{ij})_\Delta$, the problem becomes a non-linear programming problem as follows,

[NLP]

maximize α

$$w_j \{l_{ij} + \alpha(r_{ij} - l_{ij})\} \le w_i \le w_j \{u_{ij} + \alpha(r_{ij} - u_{ij})\}$$

$$\sum_i^n w_i = 1, \ i,j=1,...,n.$$

The problem is one of finding a solution to a set of linear inequalities if we fix the value of α. Hence the following method:

[Solution] Dichotomy Method

I. Solve NLP for $\alpha = 0$ (support sets) using the simplex method,

if there is a solution, then let α^L be 0 and go to III

if there is no solution then stop and let α*=0 (the ratio matrix must be changed).

II. Solve NLP for $\alpha = 1$ (core),

if there is a solution, then stop and let α*=1,

if there is no solution then let α^U be 1 and go to III

III. Solve LP for $\alpha = (\alpha^L + \alpha^U) / 2$,

if there is a solution, then let α^L be α and go to IV,

if there is no solution then let α^U be α and go to IV.

IV. If $\alpha^U - \alpha^L \ge \varepsilon$, then go to III,

if $\alpha^U - \alpha^L < \varepsilon$, then let α be α* and stop.

3.3 Unicity of the optimal weight pattern

Results obtained by Dubois and Fortemps [6] on best solutions to maxmin optimization problems with convex domains can be applied here. Indeed, it is obvious that the set of weight patterns obeying (9), for all i, j is convex. Call this domain D_α. If $\boldsymbol{w}^1$ and $\boldsymbol{w}^2$ are in D_α, so is their convex combination $\boldsymbol{w} = \lambda \boldsymbol{w}^1 + (1 - \boldsymbol{w}^2)$. Note that the ratios w_i/w_j lie between w^1_i/w^1_j and w^2_i/w^2_j. Hence, if for all i, j, w^1_i/w^1_j differs from w^2_i/w^2_j, it is clear that

(10) $$\mu_{ij}\left(\frac{w_i}{w_j}\right) > \min\left\{\mu_{ij}\left(\frac{w_i^1}{w_j^1}\right), \mu_{ij}\left(\frac{w_i^2}{w_j^2}\right)\right\}$$

So in particular, suppose there are two optimal weight patterns $\boldsymbol{w}^1$ and $\boldsymbol{w}^2$ in the optimal α*-cuts, whose ratio matrices differ for all non-diagonal components. It implies that for $\boldsymbol{w} = (\boldsymbol{w}^1 + \boldsymbol{w}^2)/2$,

(11) $$\mu_{ij}\left(\frac{w_i}{w_j}\right) > \alpha^*, \ i,j = 1,\dots,n,$$

which is contradictory. In this case, there is only one weight pattern $\boldsymbol{w}$ coherent with the interval-valued matrix whose entries are intervals $(\tilde{r}_{ij})_{\alpha^*}$.

In the case when there are at least two optimal weight patterns $\boldsymbol{w}^1$ and $\boldsymbol{w}^2$ in the optimal α*-cuts, their ratio matrices coincide for at least one non-diagonal component ($w^1_i/w^1_j = w^2_i/w^2_j$). In [6], it is shown that in this case,

(12) $$\mu_{ij}\left(\frac{w_i^1}{w_j^1}\right) = \mu_{ij}\left(\frac{w_i^2}{w_j^2}\right) = \alpha^*.$$

So the procedure is then iterated: For such entries of the matrix, the fuzzy numbers in place (i, j) must be replaced by the $\alpha*$-cut of $\tilde{r}_{ij}$. It can be done using the best weight pattern $\boldsymbol{w}$* obtained from the dichotomy method, checking for (i, j) such that

(13) $$\mu_{ij}\left(\frac{w_i^*}{w_j^*}\right) = \alpha^*.$$

The problem [NLP] is solved again with the new fuzzy matrix. It yields an optimal consistency degree β*>α* (by construction). If the same lack of unicity phenomenon reappears, some fuzzy matrix entries are again turned into intervals, and so on, until all entries are interval. The obtained solution is called "discrimin"-optimal solution in [6] and is provably unique from theorem 5 in the paper.

4 An Example

In Table 1, a 4-dimensional fuzzy reciprocal matrix is presented; lower triangular components are omitted to write because they are reciprocal of their symmetric components and diagonals are singletons as in normal AHP.

Table 1: Fuzzy reciprocal matrix

1	$(1, 3, 5)_\triangle$	$(2, 5, 7)_\triangle$	$(6, 8, 9)_\triangle$
	1	$(1, 2, 4)_\triangle$	$(2, 4, 5)_\triangle$
		1	$(0.5, 2, 3)_\triangle$
			1

Table 2: Interval-valued matrix

1	[2.42,3.58]	[4.13,5.58]	[7.42,8.29]
	1	[1.71,2.58]	[3.42,4.29]
		1	[1.57,2.29]
			1

For the data matrix found in Table1, the degree of satisfaction can be calculated as $\alpha^* = 0.711$, and the fuzzy matrix can be turned into an interval-valued matrix presented in Table2. At last, unique weights $\boldsymbol{w} = (w_1, w_2, w_3, w_4)$ from the fuzzy data matrix are calculated as shown in Table 3. Also weights from the crisp data matrix (using the cores of Table 1) to compare with is written in this table.

Table 3: Weights from fuzzy and crisp matrix.

	fuzzy data	crisp data
w_1	0.581	0.590
w_2	0.240	0.228
w_3	0.110	0.119
w_4	0.070	0.063

The optimal maxmin solution found in this example is unique. This is not surprising, because the situation of non-unicity is not common. In case this solution is not unique, the discrimin solution is unique [6] as pointed out above.

5. Evaluating decisions

The global (aggregated) evaluation of a decision f is given by means of the unique (discrimin) optimal weight pattern $\boldsymbol{w}^*$ in D_α:

(14) $$V_f = \sum_i w_i^* u_i(f),$$

where $u_i(f)$ is the utility of decision f under criterion i.

In order to account for the imprecision of the DM's inputs, we should also use weight patterns consistent with all α-cuts of the fuzzy ratio matrix, for $\alpha \in (0, \alpha^*]$, and obtain intervals as global evaluations of decision f:

(15) $$[\underline{V}_f^\alpha, \overline{V}_f^\alpha],$$

by solving the following optimization problems,

(16) $$\underline{V}_f = \min \sum_i w_i u_i(f),$$

(17) $$\overline{V}_f = \max \sum_i w_i u_i(f),$$

$$\text{s.t.} \quad \frac{w_i}{w_j} \in (\tilde{r}_{ij})_\alpha \quad , \quad \sum_i^n w_i = 1.$$

In theory, this problem should be solved for all $\alpha \in (0, \alpha^*]$. Noticing that all intervals $[\underline{V}_f^\alpha, \overline{V}_f^\alpha]$ are nested, we can compute a fuzzy evaluation of decision f, which is a fuzzy interval with height α^*. The subnormalization (which is the usual situation here) indicates some inconsistency between the best weight pattern in agreement with the fuzzy preference ratio matrix and the best local ratio evaluations provided by the DM (the cores of the fuzzy intervals $\tilde{r}_{ij}$).

In practice, we can approximate the fuzzy global evaluations of decisions as a triangular fuzzy number

(18) $$\tilde{V} = (\underline{V}_f, V_f, \overline{V}_f)_\triangle,$$

with height α^* expressing the validity of the results, V_f being calculated from the optimal weight patterns, and $[\underline{V}_f, \overline{V}_f]$ being the widest interval calculated from the supports of the fuzzy entries $\tilde{r}_{ij}$. This representation is an approximation but it can be a useful compact way to show results. To refine it, it is enough to compute the interval-valued approximation for more cuts between 0 and α*.

6. Conclusion.

In classical AHP it is often difficult for the DM to provide an exact pairwise data matrix because it is hard to estimate ratios of importance in a precise way. Therefore we use fuzzy reciprocal matrices, and propose a new kind of consistency index. This index is considered as an empirical validity coefficient evaluating to what extent a weight pattern is close to the DM revealed preference.

In the next step, we will be able to show examples of AHP entirely. Moreover we plan to implement these results on actual data. We will also try to refine the search for appropriate weights that employs the DM's subjective distance.

Here we maintained the idea of using numerical pairwise preference degrees as importance weight ratios, only acknowledging the idea that a DM cannot provide precise data. As a consequence, even if our method is more faithful to the poor precision of the data, the obtained ranking of decisions will suffer from all limitations of the AHP method (described in [2] for instance).

Another research direction would be to reconstruct a counterpart of Saaty's method on an ordinal scale such as a finite chain of preference levels, in order to get more robust results. This would imply giving up the weighted sum as a basis for decision evaluation.

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