Refining Aggregation Operator-Based Orderings in Multifactorial Evaluation, Part I: Continuous Scales

D. Kyselová, D. Dubois, M. Komorníková, and R. Mesiar

Abstract—**Aggregation operators are often needed when building preference relations in multicriteria decision making problems. Most existing approaches have limitations due to incomparability between decisions or ties due to the use of some aggregation operations that produce a ranking. The natural way of overcoming the lack of discrimination power is to refine the obtained ranking. We bring an overview of methods that enable aggregation-based rankings to be refined, generalizing concepts like discrimin (max), leximin (max), and Lorentz orderings that refine such aggregation operations like the mininum (the maximum) and the sum.**

Index Terms—**Aggregation operator, multicriteria decision making, preference relation, preorder.**

I. Introduction

FINDING optimal alternatives in a decision-making problem strongly depends on the chosen approach to "optimality," especially when several criteria are involved. The two most acknowledged techniques for ranking decisions in the sense of several criteria are the Pareto ordering and the ranking induced by an aggregation operator. The Pareto ordering states that one decision is better than another if the former is as good as the latter with respect to all criteria and strictly better on one criteria. This principle is more a rationality postulate than an efficient decision method, as the set of Pareto-maximal elements is usually too large, due to incomparabilities.

Using an aggregation operation is more convenient, as it yields a ranking. However, it is the choice of the proper aggregation operation that is problematic. This choice can be done on the basis of specific requirements and by means of ranking samples obtained from the decision-maker on test cases. Even if the proper aggregation operation has been chosen, the set of optimal decisions may remain large enough while intuitively some optimal solutions may look better than others.

For instance, suppose decisions are ranked on the basis of their worst performance according to each criterion (min aggregation). Then the corresponding ranking is often coarse, and some min-optimal solutions may fail to be Pareto maximal. So it is important to do away with them. If the ranking of decisions is made according to the sum of satisfaction degrees according to several criteria, optimal solutions are Pareto optimal but still some additional discrimination may be needed. For instance, with two criteria ranging on the unit interval, suppose one decision gets (0, 1), and the other gets (0.5, 0.5). One may prefer the latter because it is more balanced, for instance. The usual temptation when some defect of this type is detected is to change the aggregation for another one, hopefully better. For instance, turning the minimum operation into a product, or the sum into minimum (for the latter example). However, if some ties due to the original aggregation operation are indeed broken by the new one, some previously discriminated decisions will end up being ties in turn, because changing the aggregation operation strongly modifies the ranking of decisions, especially inducing preference reversals. The only way to improve the result of an aggregation without creating any preference reversal is to refine the obtained ranking.

In the literature, some proposals were made to refine the minimum and maximum based rankings: the so-called LexiMin [2] and LexiMax consist in performing a lexicographic comparison of reordered performance rates vectors evaluating decisions according to several criteria. For the LexiMin, performance vector components are arranged in increasing order, and in decreasing order for the LexiMax. It yields a total ordering. A coarser refinement [3], [4] consists in canceling criteria where the performance of two decisions is equal and aggregating the remaining criteria for deciding which decision outperforms the other. It yields a partial ordering. Refinement of sum-based rankings was also proposed, under the name majorization (used by Hardy *et al.* [16]). The idea is to change each performance vector into a cumulative vector, called the Lorentz curve [19], whose ith component is the sum of the i worst evaluations in the original vector. Two vectors whose sums of components are equal can be discriminated by comparing their cumulative versions componentwise. For instance, the cumulative vectors of (0, 1) and (0.5, 0.5) are, respectively, (0, 1) and (0.5, 1), and the latter dominates the former.

Based on previous results in functional approximation, recent works (especially [6]) have shown that refined minimum and maximum aggregation rankings can be naturally generated by limit processes. Namely, under mild technical conditions, if an aggregation operation like a t-norm or a mean converges to the minimum, then optimal solutions in the sense of such an aggregation operation converge to an optimal solution in the sense of the LexiMin (actually the whole ranking converges to the LexiMin ranking). The aim of this paper is to check whether this limit process of refinement can be extended to other kinds of

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aggregation operations and propose general forms of refinement techniques that encompass the LexiMin and the Lorentz curve construction.

This paper is organized as follows. In the next section, basic notions and notations are introduced. Then we discuss preorders based on simple aggregation of scores from compared alternatives and limit refinements of such preorders. The third section extends the cancellation property of aggregation operators to Discri-methods and Lexi-methods. In the fourth section, orderings related to Lorentz approach are investigated.

II. Preliminaries

Let S be a fixed scale and denote $S* = \bigcup_{n\in\mathrm{N}} S^n$. Each alternative $\mathbf{x}$ is described as an n-tuple $\mathbf{x} = (x_1,\ldots,x_n) \in S^n, n \in \mathrm{N}$, and optimality refers to some preorder $\leqslant$ on S^* (sometimes, $\leqslant$ acts on S^n for a fixed n only).

Recall that the scale S is supposed to be a chain (linearly ordered set). The natural extension of the (total) order $\leqslant$ on S to the (partial) Cartesian product order $\leqslant$ on S^n is

$$\mathbf{x},\mathbf{y} \in S^n, \mathbf{x} \leqslant \mathbf{y} \text{ if and only if } x_i \leqslant y_i, i = 1,\ldots,n.$$

This relation $\leqslant$ is sometimes called also weak Pareto ordering. However, this partial order has many incomparable tuples (and, obviously, if $\mathbf{x} \in S^n$ and $\mathbf{y} \in S^m, n \neq m$, then $\mathbf{x}$ and $\mathbf{y}$ cannot be compared coordinatewise), though it has no nontrivial ties.

Therefore, it is desirable to introduce some other approaches to compare different alternatives. In all these approaches we expect compatibility with Cartesian partial order $\leqslant$ (Pareto property, i.e., $\mathbf{x} \leqslant \mathbf{y}$ implies $\mathbf{x} \leq \mathbf{y}$), i.e., we will look for refinements of the relation $\leqslant$. Our main aim is to review and link several approaches known from the literature, especially those leading to complete preorders (i.e., no incomparability but possibly several ties). If Cartesian (Pareto) ordering can be understood as the coarsest approach, any total order $\precsim$ is the other extreme (i.e., no refinement is possible). A typical example of such a total order on S^n is the lexicographic order $\leqslant_{\mathrm{Lex}}$,

$\mathbf{x} \leqslant_{\mathrm{Lex}} \mathbf{y}$ if and only if $\mathbf{x} = \mathbf{y}$ or there is $0 \leqslant m < n$ such that $x_1 = y_1,\ldots,x_m = y_m$ but $x_{m+1} < y_{m+1}$.

Observe that for any permutation $\sigma:\{1,\ldots,n\} \to \{1,\ldots,n\}$, σ-lexicographic total order $\leqslant_{\sigma-\mathrm{Lex}}$ is defined by $\mathbf{x} \leqslant_{\sigma-\mathrm{Lex}} \mathbf{y}$ if and only if $(x_{\sigma(1)},\ldots,x_{\sigma(n)}) \leqslant_{\mathrm{Lex}} (y_{\sigma(1)},\ldots,y_{\sigma(n)})$. There are several possible extensions of $\leqslant_{\mathrm{Lex}}$ to S^*. For example, for $\mathbf{x} \in S^n, \mathbf{y} \in S^m, n < m$, we can put $\mathbf{x} <_{\mathrm{Lex}} \mathbf{y}$ if $\mathbf{x} <_{\mathrm{Lex}} (y_1,\ldots,y_n)$ and $\mathbf{x} \geqslant_{\mathrm{Lex}} \mathbf{y}$ otherwise. Formally this approach corresponds to the extension of the scale S by an element ∞ such that $\mathrm{s} < \infty$ for all $\mathrm{s} \in S$. Then each $\mathbf{x} \in S*$ can be embedded into $(S \cup \{\infty\})^{\mathrm{N}}$ in the form $\mathbf{x}^* = (x_1,\ldots,x_n,\infty,\infty,\ldots)$ and $\leqslant_{\mathrm{Lex}}$ on S^* is a restriction of $\leqslant_{\mathrm{Lex}}$ on $(S\cup\{\infty\})^{\mathrm{N}}$ by $\mathbf{x} \leqslant_{\mathrm{Lex}} \mathbf{y}$ if and only if $\mathbf{x}^* \leqslant_{\mathrm{Lex}} \mathbf{y}^*$.

In applications, one mostly deals with two types of scales: either S is a continuous scale represented by an interval or S is a finite (ordinal) scale. In this paper, we will focus on the first case only, representing S by the standard unit interval [0, 1] and its corresponding standard order. Finite ordinal scales will be discussed in the second part of this paper.

III. Simple Aggregation Based Orderings and Limit Approach

Let $\boldsymbol{A} : \bigcup_{n\in\mathrm{N}} [0,1]^n \to [0,1]$ be an aggregation operator [1], [9], [18]. Observe that $\boldsymbol{A}$ is, in fact, a normed utility function, and the only substantial property of $\boldsymbol{A}$ is its nondecreasingness with respect to Cartesian partial ordering on S^*.

The standard $\boldsymbol{A}$-based partial ordering on evaluations from S^* is given simply by [8]

$$\mathbf{x} \leqslant_{\boldsymbol{A}} \mathbf{y} \text{ if and only if } \boldsymbol{A}(\mathbf{x}) \leqslant \boldsymbol{A}(\mathbf{y}).$$

Evidently, $\leqslant_{\boldsymbol{A}}$ is a preorder with no incomparability but many ties. So, for example, for any aggregation operator $\boldsymbol{A}, \underbrace{(0,\ldots,0)}_{k-\text{times}} = 0_k, 0_k \sim_{\boldsymbol{A}} 0_m$ for all $k,m \in \mathrm{N}$. Similarly $1_k \sim_{\boldsymbol{A}} 1_m$ (of course, there are many other ties, too).

Each kind of ordering is a relation R on the corresponding domain. Due to the natural duality $\eta(x) = 1 - x$ on the unit interval [0, 1], we can introduce the dual ordering described by the relation $R^{\mathrm{d}} = \{(\mathbf{x},\mathbf{y})|(\eta(\mathbf{y}),\eta(\mathbf{x})) \in R\}$, i.e., $\mathbf{x} \precsim^{\mathrm{d}} \mathbf{y}$ if and only if $\eta(\mathbf{y}) \precsim \eta(\mathbf{x})$.

It is not difficult to check that for an aggregation operator $\boldsymbol{A}$-based preorder $\leqslant_{\boldsymbol{A}}$, its dual $\leqslant_{\boldsymbol{A}}^{\mathrm{d}}$ can be described by the dual aggregation operator $\boldsymbol{A}^{\mathrm{d}}$

$$\boldsymbol{A}^{\mathrm{d}}(x_1,\ldots,x_n) = 1 - \boldsymbol{A}(1-x_1,\ldots,1-x_n)$$

i.e., $\leqslant_{\boldsymbol{A}}^{\mathrm{d}} \equiv \leqslant_{\boldsymbol{A}}^{\mathrm{d}}$. To improve the discrimination power of $\leqslant_{\boldsymbol{A}}$, we have several possibilities. We will first discuss the limit approach introduced in a specific form for ***Min*** operator in [6].

Let $(\boldsymbol{A}_k)_{k\in\mathrm{N}}$ be a sequence of aggregation operators such that the pointwise limit $\boldsymbol{A} = \lim_{k\to\infty}\boldsymbol{A}_k$ exists (then evidently $\boldsymbol{A}$ is also an aggregation operator).

For any fixed evaluations $\mathbf{x},\mathbf{y}$ such that $\boldsymbol{A}(\mathbf{x}) < \boldsymbol{A}(\mathbf{y})$, we have

$$\lim_{k\to+\infty}\boldsymbol{A}_k(\mathbf{x}) = \boldsymbol{A}(\mathbf{x}) < \boldsymbol{A}(\mathbf{y}) = \lim_{\mathbf{k}\to\infty}\boldsymbol{A}_{\mathbf{k}}(\mathbf{y})$$

and thus there exists a $k_0 \in \mathrm{N}$ such that also $\boldsymbol{A}_k(\mathbf{x}) < \boldsymbol{A}_k(\mathbf{y})$ for all $k \geqslant k_0$.

This observation allows us to refine the weak order $\leqslant_{\boldsymbol{A}}$ generalizing the approach introduced in [6].

Let $\mathcal{A} = (\boldsymbol{A}_k)_{k\in\mathrm{N}}$ be a sequence of aggregation operators such that the pointwise limit $\lim_{k\to+\infty}\boldsymbol{A}_k = \boldsymbol{A}$ exists.

Let $\boldsymbol{P}_k = \{(\mathbf{x},\mathbf{y})|\mathbf{x},\mathbf{y} \in \bigcup_{n\in\mathrm{N}}[0,1]^n, \boldsymbol{A}_k(\mathbf{x}) < \boldsymbol{A}_k(\mathbf{y})\}$ be the strict preference relation based on $\boldsymbol{A}_k$.

Proposition 1: The relation $\mathcal{P} = \liminf \boldsymbol{P}_k$ is a strict preference relation and the relation $\leqslant^{\mathcal{A}}$ given by $\mathbf{x} \leqslant^{\mathcal{A}} \mathbf{y}$ whenever $(\mathbf{y},\mathbf{x}) \notin \mathrm{P}$ is a complete preorder that refines the complete preorder $\leqslant_{\boldsymbol{A}}$.

Proof: Observe first that $(\mathbf{x},\mathbf{y}) \in \mathrm{P}$, i.e., $\mathbf{x} <^{\mathcal{A}} \mathbf{y}$ if and only if there is an integer $k_0 \in \mathrm{N}$ such that for all $k \geqslant k_0, (\mathbf{x},\mathbf{y}) \in \boldsymbol{P}_k$ (i.e., $\boldsymbol{A}_k(\mathbf{x}) < \boldsymbol{A}_k(\mathbf{y})$). As already mentioned, if $\boldsymbol{A}$ is the pointwise limit of $(\mathrm{A}_k)_{k\in\mathrm{N}}$ (on S^*) then $\boldsymbol{A}$ is an aggregation operator. Clearly, $\mathbf{x} <_{\boldsymbol{A}} \mathbf{y}$ implies $\mathbf{x} <^{\boldsymbol{A}} \mathbf{y}$ and $\mathbf{x} <^{\boldsymbol{A}} \mathbf{y}$ implies $\mathbf{x} \leqslant_{\boldsymbol{A}} \mathbf{y}$.

To complete the proof, we have to show the transitivity of relation P. Let $\mathbf{x},\mathbf{y},\mathbf{z} \in \bigcup_{n\in\mathrm{N}} S^n$ be such that $\mathbf{x} <^{\mathcal{A}} \mathbf{y}$ and $\mathbf{y} <^{\mathcal{A}} \mathbf{z}$. Then there are $k_0, k_1 \in \mathrm{N}$ such that for all $k \geqslant k_0$, $\boldsymbol{A}_k(\mathbf{x}) < \boldsymbol{A}_k(\mathbf{y})$ and for all $k \geqslant k_1, \boldsymbol{A}_k(\mathbf{y}) < \boldsymbol{A}_k(\mathbf{z})$. However, then for all $k \geqslant \max\,(k_0, k_1)$ it holds $\boldsymbol{A}_k(\mathbf{x}) < \boldsymbol{A}_k(\mathbf{z})$, i.e., $\mathbf{x} <^{\mathcal{A}} \mathbf{z}$.

The nonreflexivity of the relation P is obvious. Moreover, $(\mathbf{x},\mathbf{y})$ is in P only if there is $m \in \mathrm{N}$ such that, for each $k \geqslant m$, the couple $(\mathbf{x},\mathbf{y})$ is an element of $\boldsymbol{P}_k$. However, then a similar claim cannot hold for the couple $(\mathbf{y},\mathbf{x})$, excluding the couple $(\mathbf{x},\mathbf{y})$ from P. QED.

Proposition 1 gives a hint for possible refinements of preorders $\leqslant_{\boldsymbol{A}}$ breaking some ties.

Example 1:

i) Let $\mathcal{B} = (\boldsymbol{B}_k)_{k\in\mathrm{N}}$, where

$$\boldsymbol{B}_k(x_1,\ldots,x_n) = klog(\frac{\sum_{i=1}^{n} \exp\left(\frac{x_i}{k}\right)}{n}).$$

Then $\lim_{k\to\infty}\boldsymbol{B}_k = \boldsymbol{M}$ is the arithmetic mean. Note that for $\mathbf{x},\mathbf{y} \in [0,1]^*$, $\boldsymbol{B}_k(\mathbf{x}) < \boldsymbol{B}_k(\mathbf{y})$ if and only if

$$\frac{\sum_{i=1}^{n} \exp\left(\frac{x_i}{k}\right)}{n} < \frac{\sum_{i=1}^{\mathrm{p}} \exp\left(\frac{y_i}{k}\right)}{\mathrm{p}}$$

where $n = \dim\mathbf{x}$ and $\mathrm{p} = \dim\mathbf{y}$. By means of Taylor's series, we see that $\boldsymbol{B}_k(\mathbf{x}) < \boldsymbol{B}_k(\mathbf{y})$ if and only if

$$\frac{1}{k}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) + \frac{1}{2!k^2}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right) + \frac{1}{3!k^3}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^3\right) + \cdots$$
$$< \frac{1}{k}\left(\frac{1}{\mathrm{p}}\sum_{i=1}^{\mathrm{p}} y_i\right) + \frac{1}{2!k^2}\left(\frac{1}{\mathrm{p}}\sum_{i=1}^{\mathrm{p}} y_i^2\right) + \frac{1}{3!k^3}\left(\frac{1}{\mathrm{p}}\sum_{i=1}^{\mathrm{p}} y_i^3\right) + \cdots.$$

Then, for $\mathbf{x},\mathbf{y} \in [0,1]^*$, $\mathbf{x} \leqslant^{B} \mathbf{y}$ if and only if $\mathrm{MOM}_x \leqslant_{\mathrm{Lex}} \mathrm{MOM}_y$, where MOM_x:N $\to [0,1]$ is the moment function given by $\mathrm{MOM}_x(m) = (1/n)\sum_{i=1}^{n} x_i^m$, i.e., $\mathrm{MOM}_x(m)$ is the mth initial moment of a random variable described by the uniform sample $\mathbf{x} = (x_1,\ldots,x_n)$. Observe that $\mathbf{x} \sim^{\mathcal{B}} \mathbf{y}$ if and only if there exist $k,r,s \in \mathrm{N}$ and $\mathbf{z} \in [0,1]^k$ such that $\mathbf{x}$ is a permutation of $\mathbf{z}$ repeated r times and $\mathbf{y}$ is a permutation of $\mathbf{z}$ repeated s times.

We introduce also a "negative" example, showing that the complete preorder $\leqslant^{\mathcal{A}}$ need not be strictly finer than $\leqslant_{\boldsymbol{A}}$.

ii) For $k \in \mathrm{N}$, let $\boldsymbol{A}_k : \bigcup_{n\in\mathrm{N}} [0,1]^n \to [0,1]$ be given by

$$\boldsymbol{A}_{2k}(x_1,\ldots,x_n) = 2k\log\left(\frac{\sum_{i=1}^{n} \exp\left(\frac{x_i}{2k}\right)}{n}\right)$$

$$\boldsymbol{A}_{2k+1} = (2k+1)\log\left(\frac{n}{\sum_{i=1}^{n} \exp\left(\frac{-x_i}{2k+1}\right)}\right).$$

Then $\lim_{k\to+\infty}\boldsymbol{A}_k = \boldsymbol{M}$. Moreover, let $\mathcal{A} = (\boldsymbol{A}_k)_{k\in\mathrm{N}}$. Then, for $\mathbf{x},\mathbf{y} \in [0,1]^*$, it can be shown by similar methods as in part i) that $\mathbf{x} <^{\mathcal{A}} \mathbf{y}$ if and only if $\mathbf{x} <_M \mathbf{y}$, i.e., if and only if $(1/n)\sum_{i=1}^{n} x_i < (1/\mathrm{p})\sum_{i=1}^{\mathrm{p}} y_i, n = \dim\mathbf{x}$ and $\mathrm{p} = \dim\mathbf{y}$. Consequently, $\leqslant^{\mathcal{A}}$ and $\leqslant_M$ coincide.

In the next example, we introduce some refinements of $\leqslant_{\mathrm{Min}}$ ($\leqslant_{\mathrm{Max}}$) by means of $\leqslant^{\mathcal{A}}$, where $\mathcal{A} = (\boldsymbol{A}_k)_{k\in\mathrm{N}}$ and $\lim_{k\to+\infty}\boldsymbol{A}_k = \mathrm{Min}(\lim_{k\to+\infty}\boldsymbol{A}_k = \mathrm{Max})$.

Example 2:

i) Let $\mathcal{M}\dashv\S = (\boldsymbol{M}_k)_{k\in\mathrm{N}}$ be the system of root-power operators [10]

$$\boldsymbol{M}_k(x_1,\ldots,x_n) = \left(\frac{1}{n}\sum_{i=1}^{n} x_i^k\right)^{\frac{1}{k}}.$$

Then $\lim_{k\to+\infty}\boldsymbol{M}_k = \boldsymbol{M}ax$.

For $\mathbf{x} \in [0,1]^n$, define the occurrence function $\sigma_x : [0,1] \to \mathrm{N}_0$, $\mathrm{o}_x(t) = \mathrm{card}\{i \in \{1,\ldots,n\} | x_i = t\}$ and let $\bar{\sigma}_x : [0,1] \to [0,1]$ be the normalized occurrence function given by $\bar{\mathrm{o}}_x = (o_x/n)$.

Then $\mathbf{x} \leqslant^{\mathrm{Max}} \mathbf{y}$ if and only if $\bar{o}_x \circ \eta \leqslant_{\mathrm{Lex}} \bar{o}_y \circ \eta$.

Note that though [0, 1] is uncountable, the supports of both $\bar{o}_x \circ \eta$ and $\bar{o}_y \circ \eta$ are finite. Then the lexicographic relation $\bar{o}_x \circ \eta <_{\mathrm{Lex}} \bar{o}_y \circ \eta$ means that there is $t \in [0,1]$ such that $\bar{o}_x(1-\mathrm{t}) < \bar{o}_y(1-\mathrm{t})$ and for all $\mathrm{u} \in [0,t[$, it holds $\bar{o}_x(1-\mathrm{u}) = \bar{o}_y(1-\mathrm{u})$. Observe that for $\mathbf{x},\mathbf{y} \in [0,1]^n$, $\leqslant^{\mathrm{Max}} \equiv \leqslant_{\mathrm{LexiMax}}$ is just the LexiMax preorder [10], and thus $\leqslant$ can be viewed as an extension of LexiMax from $[0, 1]^n$, $n \in \mathrm{N}$ to $\bigcup_{n\in\mathrm{N}} [0,1]^n$.

By duality, we can introduce $\leqslant^{\mathrm{Min}}$, the extension of LexiMin [3], $\mathbf{x} \leqslant^{\mathrm{Min}} \mathbf{y}$, if and only if $\bar{o}_x \geqslant_{\mathrm{Lex}} \bar{o}_y$.

ii) For $\mathrm{Min}_1 = (\boldsymbol{M}_{-k})_{k\in\mathrm{N}}, \lim_{k\to+\infty}\mathrm{M}_{-k} = \mathrm{Min}$, we get $\mathbf{x} \leqslant^{\mathrm{Min}_1} \mathbf{y}$ if and only if $\boldsymbol{M}in(\mathbf{x}) = \boldsymbol{M}in(\mathbf{y}) = 0$ or $\mathbf{x} \leqslant^{\mathrm{Min}} \mathbf{y}$. Evidently, $\leqslant^{\mathcal{M}\rangle\backslash_1}$ is coarser than $\leqslant^{\mathcal{M}\rangle\backslash}$.

iii) For $\mathrm{Min}_2 = (\boldsymbol{T}^{\mathrm{DP}}_{1/k})_{k\in\mathrm{N}}$, the sequence of Dubois–Prade t-norms given (in binary form) by [7]

$$\boldsymbol{T}^{\mathrm{DP}}_{1/k}(\mathrm{u},\mathrm{v}) = \frac{\mathrm{uv}}{\min\left(\mathrm{u},\mathrm{v},\frac{1}{k}\right)}$$

$\leqslant^{\mathcal{M}\rangle\backslash_2} \equiv \leqslant_{\mathrm{Min}}$ i.e., we have no refinement of $\leqslant_{\mathrm{Min}}$ in this case.

iv) Starting from an arbitrary continuous Archimedean t-norm $\boldsymbol{T}$ with an additive generator t: $[0,1] \to [0,\infty]$—see [17]; also $t^k : [0,1] \to [0,\infty]$, $k \in \mathrm{N}$, is an additive generator and generates a continuous Archimedean t-norm $\boldsymbol{T}_k$. Then $\lim_{k\to\infty}\boldsymbol{T}_k = \boldsymbol{M}in$ and for $\mathcal{A} = (\boldsymbol{T}_k)_{k\in\mathrm{N}}$, $\mathbf{x} \lesssim^{\mathcal{A}} \mathbf{y}$ if and only if $\boldsymbol{M}in(\mathbf{x}) = \boldsymbol{M}in(\mathbf{y}) = 0$, or

$$\sigma_x|[0,1[\geqslant_{\mathrm{Lex}} o_y|[0,1].$$

Remark 1: Example 2 part iv) indicates another possible extension of the LexiMin to $\bigcup_{n\in \mathrm{N}}[0,1]^n$, namely, $\mathbf{x} \leqslant_{\mathrm{LexiMin}} \mathbf{y}$ if and only if $o_x|[0,1[\geqslant_{\mathrm{Lex}} o_y|[0,1[$.

Note that while $\mathbf{x} \sim^{\mathrm{Min}} \mathbf{y}$ if and only if there is $\mathbf{z} \in [0,1]^k$ and s, $\mathrm{r} \in \mathrm{N}$ such that $\mathbf{x}$ is a permutation of ks-tuple $(\mathbf{z},\ldots,\mathbf{z})$ (i.e., s repetitions of $\mathbf{z}$) and $\mathbf{y}$ is a permutation of kr-tuple $(\mathbf{z},\ldots,\mathbf{z})$ (i.e., r repetitions of $\mathbf{z}$), $\mathbf{x} \sim_{\mathrm{LexiMin}} \mathbf{y}$ if and only if $\mathbf{x}$ is a permutation of $\mathbf{y}$ or, for $\dim \mathbf{x} = m < k = \dim \mathbf{y}$, $\mathbf{y}$ is a permutation of k-dimensional vector $(x_1,\ldots,x_n,1,\ldots,1)$. Observe that this extension of LexiMin was proposed and discussed, e.g., in [5].

IV. Discri- and Lexi-Orderings

A widely applied approach to ordering alternatives (with score vectors of the same dimension) is based on extension of the standard algebraic cancellation in the sense that before comparing alternatives, equal scores in the same positions are simply omitted. The remaining coordinates (components) are then called discriminating, and the comparison is based only on scores in discriminating coordinates. This approach has been applied to refine the minimum and the maximum ordering thus recovering strict compatibility with the Pareto ordering [3]. The names DiscriMin and DiscriMax were suggested in [12]. An axiomatic approach to DiscriMin and LexiMin refinements of $\boldsymbol{M}in$-based ordering was proposed in [15].

For any ordering $\lesssim$ on [0, 1]*, the corresponding discri-ordering is denoted by $\lesssim_{\mathrm{Discri}}$, i.e., $\mathbf{x} \lesssim_{\mathrm{Discri}} \mathbf{y}$ if and only if $\mathbf{x}_y \lesssim \mathbf{y}_x$, where $\mathbf{x}_y$ is obtained from $\mathbf{x}$ omitting each coordinate i for which $x_i = y_i$ if $\dim \mathbf{x} = \dim \mathbf{y}$ (and else $\mathbf{x}_y = \mathbf{x}$). Note that $\emptyset \sim \emptyset$ by convention.

The main goal of discri-orderings is the refinement of the original orderings, and thus some properties of the later ones should be required [10].

Proposition 2: Let $\lesssim$ be a system of preorders on $[0,1]^n, n \in \mathrm{N}$. Then $\lesssim_{\mathrm{Discri}}$ is a refinement of $\lesssim$ if and only if $\lesssim$ is extensively preferentially consistent (EPC), i.e., for all $\mathrm{u} \in [0,1], \mathbf{x},\mathbf{y} \in [0,1]^n$ and $i \in \{1,2,\ldots,n+1\}$, $\mathbf{x} \lesssim \mathbf{y}$ implies $\mathbf{x}_{\mathrm{u},i} \lesssim \mathbf{y}_{\mathrm{u},i}$, where $\mathbf{x}_{\mathrm{u},i}$ is an $(n+1)$-dimensional vector $(x_1,\ldots,x_{i-1},\mathrm{u},x_i,\ldots,x_n)$.

Proof: The result follows from the additional fact that each extensively preferentially consistent preorder $\lesssim$ fulfils $\mathbf{x} \sim \mathbf{y} \Rightarrow \mathbf{x}_{\mathrm{u},i} \sim \mathbf{y}_{\mathrm{u},i}$. QED.

Typical examples of EPC preorders are those based on special aggregation operators, such as the quasi-arithmetic means, and all symmetric associative aggregation operators, such as t-norms, t-conorms, uninorms, and nullnorms. As an example of a nonsymmetric EPC aggregation operator, recall the first projection $\boldsymbol{P}_{\mathrm{F}}$, $\boldsymbol{P}_{\mathrm{F}}(x_1,\ldots,x_n) = x_1$.

Note that $\leqslant_{\mathrm{DiscriP_F}} \equiv \leqslant_{\mathrm{Lex}}$, i.e., applying the discri-approach to the first projection, we end up with the lexicographic total order.

For symmetric aggregation operators $\boldsymbol{A}$, the discri-$\boldsymbol{A}$ preorder was introduced in [10], covering DiscriMin, DiscriMax, among others (obviously, for cancellative aggregation operators like arithmetic mean we get no refinement in this way, $\leqslant_{\mathrm{DiscriM}} \equiv \leqslant_{\boldsymbol{M}}$). A further generalization to k-DiscriMin, whereby identical sets of k values are cancelled, as proposed in [10], can be easily adopted for any symmetric EPC aggregation operator $\boldsymbol{A}$, but the transitivity then may fail, in general. However, ∞ Discri-$\boldsymbol{A}$ is exactly the Lexi-$\boldsymbol{A}$ relation and does not suffer by this failure.

We generalize this approach to any symmetric EPC preorder $\lesssim$ (note that symmetry of a preorder $\lesssim$ on $[0, 1]^n$ means that $\mathbf{x} \sim \mathbf{y}$ whenever $\mathbf{x}$ is a permutation of $\mathbf{y}$).

Proposition 3: Let $\lesssim$ be a system of symmetric EPC preorders on $[0, 1]^n$, $n \in \mathrm{N}$. Then the relation $\lesssim_{\mathrm{Lex}}$ given by $\mathbf{x} \lesssim_{\mathrm{Lex}} \mathbf{y}$ if and only if $\mathbf{x}_{(y)} \lesssim \mathbf{y}_{(x)}$, where $\mathbf{x}_{(\mathrm{y})}$ is any score vector with occurrence function $o_{x_{(y)}} = \max(0, o_x - o_y)$, is a refinement of $\lesssim$.

Proof: It follows by induction from the symmetry and EPC property of $\lesssim$.

In the case when $\lesssim$ is induced by an aggregation operator $\boldsymbol{A}, \lesssim \equiv \leqslant_{\boldsymbol{A}}$, the preorder $\lesssim_{\mathrm{Lex}}$ is called Lexi-$\boldsymbol{A}$ and notation $\leqslant_{\mathrm{Lexi}\ \boldsymbol{A}}$ is used.

Note, however, that though LexiMin (LexiMax) are significant refinements of Min and DiscriMin (Max and DiscriMax) preorders, in most other cases the obtained refinements are negligible. Obviously, for symmetric cancellative aggregation operator $\boldsymbol{A}, \leqslant_{\mathrm{Lexi}-\boldsymbol{A}} \equiv \leqslant_{\boldsymbol{A}}$ on each $[0,1]^n$, $n \in \mathrm{N}$. Similarly, for the product Π (any strict t-norm $\boldsymbol{T}$), $\mathbf{x} \sim_{\mathrm{Lexi}\Pi} \mathbf{y}$ if and only if $\Pi x_i = \Pi y_{\mathrm{j}} \neq 0$ or $\Pi x_i = \Pi y_{\mathrm{j}} = 0, x_i > 0 x_i = \prod_{y_{\mathrm{j}}>0} y_{\mathrm{j}}$ and $\mathrm{card}\,\{i|x_i = 0\} = \mathrm{card}\{\mathrm{j}|y_{\mathrm{j}} = 0\}$, so the only ties of $\leqslant_{\Pi}$ violated by $\leqslant_{\mathrm{Lexi}\Pi}$ are special cases when both compared alternatives contain some zero score.

The approach described in Proposition 3 allows one to refine complete preorders based on order statistics to complete preorders with the only ties linked to permutations of score vectors (in the case of equal dimensions of the score vectors). LexiMin and LexiMax are typical examples of this type. However, we can consider LexiMed, linked to the median operator ***Med,*** as well. To ensure that ***Med*** is an order statistics (and thus it can be applied also on ordinal scales), we must let $(\boldsymbol{M}ed\mathbf{x}) = x'_{[(n+1/2)]}$, where $\mathbf{x} = (x_1,\ldots,x_n) \in [0,1]^n$ (or $\mathbf{x} \in S^n$), $[\cdot]$ is the integer part, and $(x'_1,\ldots,x'_n)$ is a nondecreasing permutation of $\mathbf{x}$.

V. Lorentz-Like Orderings

For $\mathbf{x},\mathbf{y} \in [0,1]^n$, Lorentz ordering $\leqslant_{\mathrm{Lorentz}}$ is related to the transformation of the information contained in $\mathbf{x}$ into an n-tuple $\tilde{x} = (\sum_{i=1}^n x'_i, \sum_{i=1}^{n-1} x'_i,\ldots,x'_1)$, where $(x'_1,\ldots,x'_n)$ is a nondecreasing permutation of $\mathbf{x} = (x_1,\ldots,x_n)$. Then $\mathbf{x} \leqslant_{\mathrm{Lorentz}} \mathbf{y}$ if and only if $\tilde{x} \leqslant \tilde{y}$. This is called majorization by Hardy *et al.* [16]. In the Introduction, we have mentioned an equivalent cumulative transformation $(x'_1, x'_1 + x'_2,\ldots,\sum_{i=1}^n x'_i)$. Because of the next consideration, we prefer now the transformation of $\mathbf{x}$ into $\tilde{x}$. Remember that the preference based on the Lorentz ordering fulfills the so-called Pigou–Dalton property, or, equivalently, the transfer principle [20]. It says that if we modify the vector $(x'_1,\ldots,x'_n)$ into

$$x^c = (x'_1,\cdots x'_i + c,\cdots x'_{\mathrm{j}} - c,\ldots,x'_n)$$

carrying the amount c over from component j to the smaller component i, thus making the vector $\mathbf{x}^c$ more balanced than $\mathbf{x}$, then $\mathbf{x} \leqslant_{\text{Lorentz}} \mathbf{x}^c$. Evidently, $\mathbf{x} \sim_{\text{Lorentz}} \mathbf{y}$ if and only if $\mathbf{x}$ is a permutation of $\mathbf{y}$. If $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, then $\mathbf{x} \leqslant_{\text{Lorentz}} \mathbf{y}$ if and only if $\mathbf{y}$ can be reached from $\mathbf{x}$ through a series of Pigou–Dalton transfers (see [16] and [11]). Hardy *et al.* [16] also show in this case the equivalence between Lorentz ordering $\mathbf{x} \leqslant_{\text{Lorentz}} \mathbf{y}$ and the existence of a bistochastic matrix W such that $\mathbf{y} = W\mathbf{x}$.

Note that the preorder $\leqslant_{\text{Lorentz}}$ is not complete. The lexicographic refinement of Lorentz preorder $\leqslant_{\text{Lexi-Lorentz}}$ given by $\mathbf{x} \leqslant_{\text{Lexi-Lorentz}} \mathbf{y}$ if and only if $\tilde{x} \leqslant_{\text{Lex}} \tilde{y}$ is a complete preorder with the same ties as by Lorentz ordering.

The two above approaches on how to build a preorder on score vectors, as well as some examples from Section II, have inspired us to introduce Lorentz-like orderings, generalizing the preorders $\leqslant_{\boldsymbol{A}}$ discussed in Section II.

Definition 1: Let $\boldsymbol{A} : \bigcup_{n\in\mathrm{N}} [0,1]^n \rightarrow [0,1]$ be an aggregation operator. For a fixed $n \in \mathrm{N}$, Lorentz-$\boldsymbol{A}$ ordering (denoted by $\leqslant_{\mathrm{LA}}$) is given by $\mathbf{x} \leqslant_{\mathrm{LA}} \mathbf{y}$ if and only if $(\boldsymbol{A}(x'_1,\ldots,x'_n), \boldsymbol{A}(x'_1,\ldots,x'_{n-1}),\ldots,\boldsymbol{A}(x'_1)) \leqslant (\boldsymbol{A}(y'_1,\ldots,y'_n), \boldsymbol{A}(y'_1,\ldots,y'_{n-1}),\ldots,\boldsymbol{A}(y'_1))$, where $(x'_1,\ldots,x'_n)$ is a nondecreasing permutation of $\mathbf{x} = (x_1,\ldots,x_n)$ (and similarly $(y'_1,\ldots,y'_n)$).

The Lexi-Lorentz $\boldsymbol{A}$ complete preorder $\leqslant_{\mathrm{LL}\boldsymbol{A}}$ is given by $\mathbf{x} \leqslant_{LL\boldsymbol{A}} \mathbf{y}$ if and only if $(\boldsymbol{A}(x'_1,\ldots,x'_n), \boldsymbol{A}(x'_1,\ldots,x'_{n-1}),\ldots,\boldsymbol{A}(x'_1)) \leqslant_{\text{Lex}} (\boldsymbol{A}(y'_1,\ldots,y'_n), \boldsymbol{A}(y'_1,\ldots,y'_{n-1}),\ldots,\boldsymbol{A}(y'_1))$.

Evidently, the usual Lorentz ordering is related to the arithmetic mean $\boldsymbol{M}$, $\leqslant_{\text{Lorentz}} \equiv \leqslant_{\mathrm{LM}}$. The Lorentz ordering induced by the minimum comes down to comparing $(x'_1,\ldots,x'_n)$ and $(y'_1,\ldots,y'_n)$ via the Pareto ordering. Namely, $\mathbf{x} \leqslant_{\text{Min-Lorentz}} \mathbf{y}$ if and only if $(x'_1,\ldots,x'_n) \leqslant (y'_1,\ldots,y'_n)$. This partial order refines the min-based ordering between vectors having the same minimal component. On the other hand, the LexiMin on $[0, 1]^n$ is the Lexi-Lorentz ordering related to the $\boldsymbol{Min}$ operator, $\leqslant_{\text{Leximin}} \equiv \leqslant_{\text{LLMin}}$. Definition 1 can be further generalized to allow to compare elements from $\bigcup_{n\in\mathrm{N}} [0,1]^n$.

Definition 2: Let $\mathrm{A} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots)$ be a sequence (finite or infinite) of aggregation operators. The relations $\leqslant_{\mathrm{L}\mathcal{A}}$ (Lorentz-A) and $\leqslant_{\mathrm{LL}\mathcal{A}}$ (Lexi-Lorentz-$\mathcal{A}$) on $\bigcup_{n\in\mathrm{N}} [0,1]^n$ are given by $\mathbf{x} \leqslant_{\mathrm{L}\mathcal{A}} \mathbf{y}$ if and only if $(\boldsymbol{A}_1(\mathbf{x}), \boldsymbol{A}_2(\mathbf{x}),\ldots) \leqslant (\boldsymbol{A}_1(\mathbf{y}), \boldsymbol{A}_2(\mathbf{y}),\ldots)$ and $\mathbf{x} \leqslant_{\mathrm{LL}\mathcal{A}} \mathbf{y}$ if and only if $(\boldsymbol{A}_1(\mathbf{x}), \boldsymbol{A}_2(\mathbf{x}),\ldots) \leqslant_{\text{Lex}} (\boldsymbol{A}_1(\mathbf{y}), \boldsymbol{A}_2(\mathbf{y}),\ldots)$.

It is not difficult to check that both relations $\leqslant_{\mathrm{L}\mathcal{A}}$ and $\leqslant_{\mathrm{LL}\mathcal{A}}$ are preorders and that $\leqslant_{\mathrm{L}\mathcal{A}} \equiv \leqslant_{\mathrm{LL}\mathcal{A}} \equiv \leqslant_{\boldsymbol{A}}$ whenever $\boldsymbol{A} = (\boldsymbol{A}, \boldsymbol{A}, \ldots)$. Observe that the usual Lorentz ordering $\leqslant_{\text{Lorentz}}$ corresponds to the infinite system $\Lambda = (\boldsymbol{L}_1, \boldsymbol{L}_2, \ldots)$ of aggregation operators given by $\boldsymbol{L}_k(\mathbf{x}) = 0$ if $n = \dim\mathbf{x} < k$, and otherwise $\boldsymbol{L}_k(\mathbf{x}) = (\sum_{i=1}^{n-k+1} x'_i / n - k + 1)$. Hence, on $[0,1]^n$, $\leqslant_{\text{Lorentz}} \equiv \leqslant_{\mathrm{LL}}$ and $\leqslant_{\text{Lexi Lorentz}} \equiv \leqslant_{\mathrm{LLL}}$. Moreover, $\leqslant_{\mathrm{LL}}$ as well as $\leqslant_{\mathrm{LLL}}$ act on $\bigcup_{n\in\mathrm{N}} [0,1]^n$. Observe that $\mathbf{x} \sim_{\mathrm{L}\Lambda} \mathbf{y}$ if and only if $\mathbf{x} \sim_{\mathrm{LL}\Lambda} \mathbf{y}$ if and only if either $\mathbf{x}$ is a permutation of $\mathbf{y}$ or, if dim $\mathbf{x} \neq$ dim $\mathbf{y}$ and all x_is and y_is are equal to zero.

Note that in the case of $\leqslant_{\mathrm{LL}\mathcal{A}}$, we can relax requirements on members of $\mathcal{A}$ not violating the Pareto property of $\leqslant_{\mathrm{LL}\mathcal{A}}$; see the next proposition.

Proposition 4: Let $\mathcal{A} = (\boldsymbol{A}_1, \boldsymbol{A}_2, \ldots)$ be a finite or infinite system such that $\boldsymbol{A}_1 : \bigcup_{n\in\mathrm{N}} [0,1]^n \rightarrow [0,1]$ is a strictly monotone aggregation operator (i.e., $\leqslant_{\boldsymbol{A}_1}$ satisfies the strong Pareto property, $\mathbf{x} \leqslant \mathbf{y}$ and $\mathbf{x} \neq \mathbf{y}$ implies $\mathbf{x} <_{\boldsymbol{A}_1} \mathbf{y}$), and for $k > 1$, $\boldsymbol{A}_k$, $\bigcup_{n\in\mathrm{N}} [0,1]^n \rightarrow R$ is a system of real functions. Then $\leqslant_{\mathrm{LL}\mathcal{A}}$ given by Definition 2 is a preorder satisfying the strong Pareto property.

Proof: The transitivity of $\leqslant_{\mathrm{LL}\mathcal{A}}$ follows from the transitivity of the lexicographic ordering. The strong Pareto property of $\leqslant_{\mathrm{LL}\mathcal{A}}$ follows from the strong Pareto property of $\boldsymbol{A}_1$. QED.

Observe that the (weak) Pareto property of $\leqslant_{\boldsymbol{A}_1}$ is not sufficient, in general, to ensure the (weak) Pareto property of $\leqslant_{\mathrm{LL}\mathcal{A}}$ if $\leqslant_{\boldsymbol{A}_2}$ does not fulfill this property. For example, for binary vectors let $\boldsymbol{A}_1(x_1, x_2) = \min(x_1 + x_2, 1)$ and $\boldsymbol{A}_2(x_1, x_2) = |x_1 - x_2|$. Then, for $\boldsymbol{A} = (\boldsymbol{A}_1\boldsymbol{A}_2), \mathbf{x}, \mathbf{y} \in [0,1]^2$, we have $\mathbf{x} \leqslant_{\mathrm{LL}\mathcal{A}} \mathbf{y}$ if and only if $x_1 + x_2 < \min(y_1 + y_2, 1)$ or if $x_1 + x_2 = y_1 + y_2 < 1$ and $|x_1 - x_2| \leqslant |y_1 - y_2|$ or if $x_1 + x_2 \geqslant 1$, $y_1 + y_2 \geqslant 1$ and $|x_1 - x_2| \leqslant |y_1 - y_2|$. However, then $(1, 1) <_{\mathrm{LL}\mathcal{A}} (0, 1)$ violating the Pareto property.

Example 3:

i) Let $\mathcal{A} = (\mathrm{M}_k^k)_{k\in\mathrm{N}}$ be a system of powers of root-power operators [see Example 2 i)], i.e., M_k^k is the kth initial moment

$$\boldsymbol{M}_k^k(x_1,\ldots,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i^k.$$

Then $\leqslant_{\mathrm{LL}\mathcal{A}} \equiv \leqslant^{\mathrm{B}}$ where $\leqslant^{\mathrm{B}}$ is the complete preorder described in Example 1 i).

Observe that if $\mathbf{x}, \mathbf{y} \in \bigcup_{n\in\mathrm{N}} [0,1]^n$ are treated as samples of uniform random variables, then we first compare the expected values. In the case of a tie, in the second step we compare the second initial moments. However, for equal expected values, the second step is equivalent to the comparison of dispersions (variances). Similarly, in the case of a tie also in the second step, in the third step we compare the coefficients of asymmetry, in a possible fourth step, the skewness coefficients, etc. Note also that $\leqslant_{\mathrm{LL}\mathcal{A}}$ is shift invariant, i.e., $\mathbf{x} \leqslant_{\mathrm{LL}\mathcal{A}} \mathbf{y}$ if and only if $\mathbf{x} + \mathrm{c} \leqslant_{\mathrm{LL}\mathcal{A}} \mathbf{y} + \mathrm{c}$ for any constant c such that both $\mathbf{x} + \mathrm{c}$ and $\mathbf{y} + \mathrm{c}$ are elements of $[0, 1]^n$. Concerning the preorder $\leqslant_{\mathrm{L}\mathcal{A}}$, a necessary condition for $\mathbf{x} \leqslant_{\mathrm{L}\mathcal{A}} \mathbf{y}$ is $\mathrm{Max}(\mathbf{x}) \leqslant (\mathrm{Max}\mathbf{y})$. For binary vectors $\mathbf{x}, \mathbf{y} \in [0,1]^2$, we have $\mathbf{x} \leqslant_{\mathrm{L}\mathcal{A}} \mathbf{y}$ if and only if $x_1 + x_2 \leqslant y_1 + y_2$ and $\boldsymbol{M}ax(\mathbf{x}) \leqslant \boldsymbol{M}ax(\mathbf{y})$. Moreover, $\mathbf{x} \sim_{\mathrm{L}\mathcal{A}} \mathbf{y}$ if and only if $(x_1, x_2) = (y_1, y_2)$ or $(x_1, x_2) = (y_2, y_1)$.

ii) Let $\mathcal{M} = (\mathrm{M}_{(k+1/k)})_{k\in\mathrm{N}}$ be a system of root-power operators. Then $lim_{k\rightarrow+\infty} \mathrm{M}_{(k+1/k)} = \mathrm{M}_1 = \mathrm{M}$ is the arithmetic mean and the complete preorder $\leqslant^{\mathrm{M}}$ (see Proposition 1) is identical with $\leqslant_{\mathrm{LL}\mathcal{A}}$, where $\mathcal{A} = (\boldsymbol{A}_k)_{k\in\mathrm{N}}$ is given by

$$\boldsymbol{A}_k(x_1,\ldots,x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i \log^{k-1} x_i$$

(with convention 0 log 0 = 0).

Observe that the system $\mathcal{A}$ fulfills the requirements of Proposition 4. Indeed, $\boldsymbol{A}_1 = \boldsymbol{M}$ is a strictly monotone aggregation operator, while for $k > 1$, $\boldsymbol{A}_k$ is a real function (observe that these functions are related to the entropy functions). Note also that the preorder $\leqslant_{\mathrm{LL}\mathcal{A}}$ is homogeneous, i.e., $\mathbf{x} \leqslant_{\mathrm{LL}\mathcal{A}} \mathbf{y}$ if and only if $c\mathbf{x} \leqslant_{\mathrm{LL}\mathcal{A}} c\mathbf{y}$ for any positive meaningful constant c.

Note that for $\mathbf{x}, \mathbf{y} \in [0,1]^2$, $\mathbf{x} \leqslant_{\mathrm{LL}\mathcal{A}} \mathbf{y}$ if and only if $x_1 + x_2 < y_1 + y_2$ or $x_1 + x_2 = y_1 + y_2$ and $(\boldsymbol{M}ax\mathbf{x}) \leqslant \boldsymbol{M}ax(\mathbf{y})$. However, $\leqslant_{\mathrm{L}\mathcal{A}}$ does not fulfill the Pareto property.

Indeed, (0.2, 0.2) and (0.3, 0.3) are incomparable in this case as far as $\boldsymbol{A}_2(0.2, 0.2) = 0.2 \log 0.2 < 0.3 \log 0.3 = \boldsymbol{A}_2(0.3, 0.3)$.

Generalizations of LexiMin ordering introduced in Section II can be described as follows: $\leqslant^{\mathcal{M}\rangle\backslash_3}$ [see Example 2 iv)] corresponds to $\leqslant_{\mathrm{LL}\mathcal{A}}$ with

$$\mathcal{A} = (\boldsymbol{A}_k)_{k \in \mathrm{N}}, \boldsymbol{A}_k(\mathbf{x}) = \begin{cases} x'_k, & \text{if } k \leqslant \dim x \\ 1, & \text{else} \end{cases}$$

where x'_k is the kth order statistics, i.e., $(x'_1, \ldots, x'_n)$ is a nondecreasing permutation of $\mathbf{x} = (x_1, \ldots, x_n)$; $\leqslant^{\mathrm{Min}}$ [see Example 2 i)] coincides with $\leqslant_{\mathrm{LLB}}$, where $\mathrm{B} = (\boldsymbol{B}_k)_{k \in \mathrm{N}}$, $\boldsymbol{B}_1(\mathbf{x}) = \boldsymbol{M}in(\mathbf{x})$ (and $\boldsymbol{B}_1(\emptyset) = 1$ by convention), $\boldsymbol{B}_2(\mathbf{x}) = -(\mathrm{card}(\{i | x_i = \mathrm{Min}(x)\})/\dim x)$ (and $\boldsymbol{B}_2(\emptyset) = 1$ by convention), and for $k > 2$, $\boldsymbol{B}_k(\mathbf{x}) = \boldsymbol{B}_{k-2}(\mathbf{x}^*)$, where the vector $\mathbf{x}^*$ is obtained from $\mathbf{x}$ omitting all minimal scores.

Note also that for t-norms (t-conorms), an alternative approach to lexi-refinement was proposed in [22]. This approach is equivalent to the one from Definition 2 in the case of weakly cancellative continuous t-norms, i.e., continuous t-norms for which the diagonal function $\delta : [0,1] \to [0,1], \delta(x) = \boldsymbol{T}(x,x)$, is strictly monotone. Indeed, it is enough to repeat the approach via Lexi-Lorentz ordering, applying the chosen weakly cancellative t-norm $\boldsymbol{T}$. We can extend the Lexi-$\boldsymbol{T}$ introduced in [22] for a weakly cancellative t-norm $\boldsymbol{T}$ to the domain [0, 1]* by means of a system $\mathrm{T} = (\boldsymbol{T}_k)_{k \in \mathrm{N}}$, putting (for $\mathbf{x} = (x_1, \ldots, x_n) \in [0,1]^n$)

$$\boldsymbol{T}_k(\mathbf{x}) = \begin{cases} T(x'_k, \ldots, x'_n), & \text{if } k \leqslant n \\ 1, & \text{else} \end{cases}$$

.

Now, $\leqslant_{\mathrm{LLT}}$ coincides on $[0,1]^n$ with $\leqslant_{\mathrm{Lexi}-\boldsymbol{T}}$ introduced in [22] [observe that for $\boldsymbol{T} = \boldsymbol{M}in$, $\mathcal{T} = \mathcal{A}$ from Example 3 iii)].

VI. Conclusion

In this paper, two types of results are proposed. First, we show how natural lexicographic refinements of rankings obtained via aggregation operations on the unit interval can be obtained as limit processes. It leads to extension of existing lexicographic ranking techniques like LexiMin to other aggregation operations. In addition, the notion of majorization using Lorentz curve has been generalized to any aggregation operation, thus casting LexiMin, LexiMax, and majorization within a unified setting.

Lexicographic refinements of aggregation-based ranking make full sense on finite ordinal evaluation scales S where no notion of arithmetic mean exists and where aggregation operations mapping S^n to S are poorly discriminant in essence. Our results open the door to rational refinements of such ordinal aggregation operations, including ordinal counterparts of Lorentz ordering or majorization. This is the topic of the next part of this work.

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