FUZZY ELEMENTS IN A FUZZY SET

Didier Dubois Henri Prade
IRIT, Université Paul Sabatier, 31062 Toulouse, France
Email: dubois@irit.fr, prade@irit.fr

ABSTRACT: This paper introduces a new concept in fuzzy set theory, that of a fuzzy element. It embodies the idea of fuzziness only, thus contributing to the distinction between fuzziness and ideas of imprecision. A fuzzy element is to a fuzzy set what an element is to a set. A fuzzy element is as precise as an element, just more gradual that the latter. Applications of this notion to fuzzy cardinality, fuzzy interval analysis and defuzzification principles are outlined.

Keywords: Fuzzy set, alpha-cuts, fuzzy numbers

1 INTRODUCTION

Originally, Zadeh used the word "fuzzy" as referring specifically to the introduction of shades or grades in all-or-nothing concepts. A fuzzy set [25] is a generalization of subset (at least in the naive sense); it is a subset with boundaries that are "gradual rather than abrupt". It is defined by a membership function from a basic set to the unit interval (or a suitable lattice) and its cuts are sets.

However, there is a recurrent confusion in the literature between the words "fuzzy" and other words or phrases like "imprecise", "inexact", "incompletely specified", "vague" that rather refer to a lack of sufficient information. For instance, what is often called a fuzzy number is understood as a generalized interval, not as a generalized number. The calculus of fuzzy numbers is an extension of interval analysis. Especially, fuzzy numbers under the sum operation do not form a group.

Similarly, in engineering papers, there is a misunderstanding about the notion of defuzzification whereby a fuzzy set of numbers obtained from some fuzzy inference engine is changed into a number. Yet, defuzzifying means removing gradedness, strictly speaking, so that defuzzifying a fuzzy set should yield a set, not a point. And indeed, in the past, the notion of mean value of a fuzzy interval was proposed as a natural way of extracting an interval from a fuzzy interval (Dubois & Prade [5], where the phrase "fuzzy number" was used in the sense of a fuzzy interval). See also recent works by Roventa and Spircu [12] and Ralescu [17]. To be more credible, the defuzzification process as used in the engineering area should be split into two steps: removing fuzziness (thus getting an interval), and removing imprecision (by selecting a number in the interval).

Suppose we perform defuzzification by swapping these two steps: given a fuzzy set of numbers, first remove imprecision, get what we could call a "fuzzy real number", and then defuzzify this fuzzy real number. Such a fuzzy real number would then express fuzziness only, WITHOUT imprecision. To get a good intuition of a fuzzy real number, one may view a fuzzy interval as a pair of such fuzzy numbers, just as an interval is an ordered pair of numbers. More generally, this discussion leads to introduce the notion of fuzzy element of a (fuzzy) set, a concept that was apparently missing in fuzzy set theory. Topologists tried to introduce ideas of fuzzy points in the past (attaching a membership value to a single element of a set), but this notion has often been controversial, and sterile in its applications. In fact they were fuzzy singletons, not really fuzzy elements. The aim of this paper is to informally introduce a natural notion of fuzzy element and fuzzy (integer or real) number, to outline elementary formal definitions related to this notion and discuss its potential at shedding light on some yet ill-understood aspects of fuzzy set theory and its applications. A full-fledged mathematical development is left for further research.

2 BASIC DEFINITIONS

Let S be a set. Consider a complete lattice L (of membership grades) with top 1 and bottom 0.

Definition 1 *A fuzzy (or gradual) element e in S is defined by its assignment function a_e from $L \setminus \{0\}$ to S.*

Several remarks are in order. First the function we consider goes from the membership set to the referential (contrary to a fuzzy set). Given a membership grade λ, $a_e(\lambda) = s$ is the element of S representative of e at level λ. Note that the domain of a_e can be a proper subset of L. The element s itself has assignment function a_s defined only for $\lambda = 1 : a_s(1) = s$. Generally we assume that $a_s(1)$ always exists. On the other hand, $a_s(0)$ is not defined because there is no counterpart to the empty set for elements of a set. The fact that the domain of an assignment function varies from one fuzzy element to the other may create difficulties when combining fuzzy elements. It is always possible to augment the domain by building a mapping a_e^* from $L \setminus \{0\}$ to S, such that

$$\forall \lambda \notin Dom(a_e), a_e^*(\lambda) = a_e(\lambda^*),$$

where $\lambda^* = \inf\{\alpha > \lambda, \alpha \in Dom(a_e)\}$. The idea is that if an element s is representative of the fuzzy element e to a certain extent, it is also representative to a lesser extent, unless otherwise specified.

One might also require an injective assignment function, in order to ensure that each $s \in S$ is attached a single membership grade, but there is no clear reason to do so, simply because as seen later, extending operations equipping S to fuzzy elements may fail to preserve this property.

The idea of a fuzzy element, in contrast with the notion of a fuzzy set, can be easily illustrated by the following example. Consider a convex fuzzy set of the real line, i.e. a fuzzy interval M. Let m_α be the middle-point of the α-level cut of M. The set of pairs $m(M) = \{(\alpha, m_\alpha) \mid \alpha \in (0,1]\}$ defines a fuzzy element of the real line, which can be called the fuzzy middle-point of M. If the membership function of M is symmetrical, then $m(M)$ reduces to an ordinary real number that is the common abscissa of the middle-points of all the α-cuts of M. For a non-symmetrical trapezoidal fuzzy interval, one obtains a straight-line segment from the mid-point of the core ($c(M) = \{r, \mu_M(r) = 1\}$) to the mid-point of the support ($supp(M) = \{r, \mu_M(r) > 0\}$). It is a particular fuzzy element, a fuzzy real number. In the general case, m_α is not a monotonic function of α, and the fuzzy real number $m(M)$ may take shapes that can no longer be reinterpreted as a membership function, i.e., a mapping from the real line into [0, 1]. Using the formal definition of a fuzzy element introduced in this paper, it is possible to extend to fuzzy intervals, results such as the middle-point of the sum of two intervals is equal to the sum of the middle-points of the intervals.

Note that there is no uncertainty in a fuzzy element since the assignment function reflects the idea of representativeness only. A fuzzy element is a "flexible" element not so much an uncertain element: we have some choice when picking a suitable representative for it.

2.1 Fuzzy sets generated by fuzzy elements

In order to check if the definition is meaningful, we must prove that a fuzzy subset F of S can be defined as a fuzzy element of the power set 2^S. We can define its assignment function via λ-cuts $F_\lambda = \{s, \mu_F(s) \geq \lambda\}$. Let $L_F = \mu_F(S) \setminus \{0\}$ be the set of non-zero membership grades of F. Now define the assignment function of F as:

$$a_F^{\geq}(\lambda) = \{s, \mu_F(s) \geq \lambda\}, \forall \lambda \in L_F.$$

Alternatively, another assignment function may map on disjoint subsets (λ-sections):

$$a_F^{=}(\lambda) = \{s, \mu_F(s) = \lambda\}, \forall \lambda \in L_F.$$

However, Definition 1 is in agreement with a more general view of fuzzy sets whose crisp representatives are neither nested not disjoint.

Definition 2 *A gradual subset G in S is defined by its assignment function a_G from $L \setminus \{0\}$ to 2^S.*

A single fuzzy element e yields a gradual singleton E by letting $a_E(\lambda) = \{a_e(\lambda)\} \forall \lambda \in L \setminus \{0\}$. More generally, a set of fuzzy elements forms a gradual set of S:

Definition 3 *The gradual set G induced by the family of fuzzy elements $e_1, \ldots, e_k$ in S with assignment functions $a_1, \ldots a_k$, has its assignment function a_G defined by*

$$a_G(\lambda) = \{a_1^*(\lambda), \ldots, a_k^*(\lambda)\}, \forall \lambda \in \cup_{i=1,\ldots k} Dom(a_i).$$

It is possible to define a regular fuzzy set from a gradual subset defined via its assignment function $a_G(\lambda)$.

Definition 4 *The membership function of the fuzzy set induced by the gradual set with assignment function a_G is*

$$\mu_G(s) = \sup\{\lambda, s \in a_G(\lambda)\}.$$

We can check that the fuzzy set thus induced using Definition 4 is F such that

$$\mu_F(s) = \max_{i=1,\ldots k} \sup\{\lambda_i, s = a_i(\lambda_i)\}.$$

The set of fuzzy elements $e_1, \ldots, e_k$ is said to generate the fuzzy set F. For instance, a single fuzzy element e having an injective assignment function yields a fuzzy set F by letting $\mu_F(s) = \lambda$ if and only if $a_e(\lambda) = s$, i.e., μ_F is the inverse of the assignment function a_e. If e is a crisp element, then F is a singleton. In the non-injective case, we get $\mu_E(s) = \sup\{\lambda, s = a_e(\lambda)\}$, to account for the best representativeness level of s w.r.t e. But it is not clear that F can be called a fuzzy singleton if induced by a single fuzzy element. This is due to the fact that there are several possible representations of F as a gradual set : its cuts are not singletons, even if its sections are singletons. In contrast, the notion of gradual singleton is clear (a mapping from from $L \setminus \{0\}$ to the set of singletons of S). The notion of fuzzy singleton is more difficult to define in the absence of a notion of fuzzy element, when S is only equipped with the usual equality relation, as singletons are basically construed as the quotient set $S/=$ (see Hoehle[11]), hence are crisp. However, gradual singletons are singletons trivially induced by the set of fuzzy elements of S equipped by the equality relation defined by $e = e'$ if and only if $a_e = a'_e$.

Intuitively, if an element s is representative of the fuzzy elements generating F only to degree at most λ, then its degree of membership in F cannot exceed λ.

Definition 5 *A fuzzy element e is said to belong to a fuzzy set F if and only if $\forall s \in S$, if $\exists \lambda \in L \setminus \{0\}, a_i(\lambda) = s$, then $\mu_G(s) \geq \lambda$.*

The only crisp elements in F are those in its core. Each fuzzy element in the family generating a gradual set G belongs to the fuzzy set F induced by G. More precisely: $\forall i = 1, \ldots k, \forall s$, if $a_i(\lambda) = s$, then $\mu_F(s) \geq \lambda$. The family of generators of F is clearly not unique. The set of fuzzy elements of (= belonging to) F is the maximal family generating F. It collects all fuzzy elements e such that

$$\mu_F(a_e(\lambda)) \geq \lambda, \forall \lambda \in Dom(a_e).$$

To build such fuzzy elements, it is enough to select one element s_λ in each λ-cut F_λ of F, letting $a_e(\lambda) = s_\lambda, \forall \alpha \in L_F$. Namely, $\mu_F(a_e(\lambda)) \geq \lambda$ if and only if $a_e(\lambda) \in F_\lambda$. Besides, we can restrict to fuzzy elements of F with an injective assignment function. It consists of all e such that $\forall \lambda_i \in L_F$, an element $s_i \in a_F^{=}(\lambda_i)$, the λ_i-section, exists, such that $a_e(\lambda_i) = s_i$. If the cardinality of $a_F^{=}(\lambda_i)$ is n_i, the number of such generating fuzzy elements is $\prod_{\lambda_i \in L_F} n_i$. An interesting question is to find minimal families of fuzzy elements generating a fuzzy set F.

Now, we may try to compute the degree of membership of a fuzzy element in a fuzzy set. Naturally, this degree will be a fuzzy element of L. Let e be a fuzzy element of S and F a fuzzy subset of S.

Definition 6 *The degree of membership of a fuzzy element e in a fuzzy set F is a fuzzy element of L defined by its assignment function $a_{e \in F}$ such that*

$$a_{e \in F}(\lambda) = \mu_F(a_e(\lambda)), \forall \lambda \in Dom(f)$$

The fuzzy element $a_{e \in F}(\lambda) \in L$ is a representative value of the membership grade of e in F to degree λ. Note that the obtained fuzzy degree of membership does not express imprecision. It just reflects the gradual nature of the fuzzy set and of the fuzzy element. Suppose $e = s$ is a regular element. Then, $a_{e \in F}(1) = \mu_F(s)$. If F is not fuzzy, say a subset A, then $a_{e \in A}(\lambda) = 1$ if $a_e(\lambda) \in A$, and 0 otherwise. Suppose e belongs to F in the sense of Definition 5. Then, by construction $a_{e \in F} \geq Id_L$ the identity function of L. If $F = A$ is not fuzzy, then e belongs to A provided that $a_{e \in F}(\lambda) = 1, \forall \lambda > 0$. We can consider $\inf\{\lambda, a_e(\lambda) \in A\}$ as the degree of membership of e in A, but this must be properly understood as the extent to which a crisp set contains a fuzzy element (the gradual nature of membership is not due to the set A). However the whole range $\{\lambda, a_e(\lambda) \in A\}$ is more representative of this evaluation.

Note that defining a probability measure on $Dom(a_G)$ changes a gradual set into a random set and a fuzzy element into a probability distribution. It emphasises the idea that a fuzzy element is not imprecise. If L is the unit interval, and $L_F = \{1 = \lambda_1 > \ldots > \lambda_k\}$, a canonical way of changing a fuzzy set into a random set is to assign probability mass $\lambda_i - \lambda_{i+1}$ to the cut $F_{\lambda_i}, \forall i = 1 \ldots k$. A similar procedure can be adopted for turning a fuzzy element e into a probability distribution, assigning probability $\lambda_i - \lambda_{i+1}$ to element $s_i = a_e(\lambda_i)$. The meaning of this probability can be the degree of "stability" of representative element s_i for e, the gap between λ_i and λ_{i+1} measuring the reluctance to give up s_i for s_{i+1} as the proper representative of the fuzzy element e. For instance, this gap is of size 1 for a crisp element, indicating maximal stability. Then $\lambda_i - \lambda_{i+1}$ is the probability of picking s_i, in the sense that the more stable a crisp representative, the more likely it will be picked.

2.2 Fuzzy connectives and gradual sets

The next step is to show that connectives of fuzzy set theory are consistent with the notion of fuzzy element and gradual set. The union and intersection of gradual sets G_1 and G_2 can be defined by the classical union and intersection of representatives to the same degree:

$$a_{G_1 \cup G_2}(\lambda) = a_{G_1}(\lambda) \cup a_{G_2}(\lambda);$$

$$a_{G_1 \cap G_2}(\lambda) = a_{G_1}(\lambda) \cap a_{G_2}(\lambda).$$

However if $Dom(a_{G_1}) \neq Dom(a_{G_2})$, we must consider the extensions $a^*_{G_1}$ and $a^*_{G_2}$ restricted to $Dom(a_{G_1}) \cup Dom(a_{G_2})$. By construction, this definition is consistent with the usual idempotent fuzzy set connectives, namely if $F(G)$ is the fuzzy set induced by G, then:

$$\mu_{F(G_1 \cup G_2)}(s) = \max(\mu_{F(G_1)}(s), \mu_{F(G_2)}(s));$$

$$\mu_{F(G_1 \cap G_2)}(s) = \min(\mu_{F(G_1)}(s), \mu_{F(G_2)}(s)).$$

However one may also consider other connectives for gradual sets where unions

$$a_{G_1 \cup G_2}(\lambda, \nu) = a_{G_1}(\lambda) \cup a_{G2}(\nu),$$

for all $\lambda \in Dom(a_{G_1}), \nu \in Dom(a_{G_2})$, are computed, using $L \times L$ as a new membership set. Ultimately, the definition of a connective might depend on the design of a "correlation map" between the two gradual sets indicating which pairs of realizations of the gradual sets go together, the former definition pairing sets with equal representativeness, the latter accepting all pairs.

The complement G^c of a gradual set G can be defined levelwise as $a_{G^c}(\lambda) = a_G(\lambda)^c$, but this definition is not in agreement with fuzzy set complementation since the λ-cut of the fuzzy set F^c is not the complement of the λ-cut of F. In order to preserve consistency with the usual fuzzy complement, one must assume that the correspondence between representatives of G and of G^c is a negative correlation, and presuppose the existence of an order reversing map neg on L exchanging 0 and 1. If $a_{G^c}(\lambda)$ is defined as $a_G(\nu)^c$, where $\nu = \inf\{\alpha > neg(\lambda)\}$, then consistency with fuzzy set complementation can be restored. It comes down to a special case of permutation of cuts introduced by Ralescu [16].

3 EXAMPLES AND APPLICATIONS

There are many cases where fuzzy elements naturally appear. The first general situation is when evaluating sets by means of some index, like cardinality, measure, distance, and so on. When extending these indices to fuzzy sets, some try to evaluate an average over the cuts of the fuzzy set (using a Choquet integral for instance). Another path is to preserve a genuinely fuzzy index. More often than not, it has been assumed that while a set has a precise evaluation, a fuzzy set should have a fuzzy-valued evaluation interpreted as being imprecise. However, the above discussion does not suggest it: since a scalar evaluation of a set yields a precise number, the scalar evaluation of a fuzzy set should be a fuzzy (gradual but not imprecise) element in the range of the index.

3.1 Fuzzy cardinality

Consider cardinality, for instance. Fuzzy-valued cardinality $CARD(F)$ of a fuzzy set F on a finite set S was defined by Zadeh [26] as a fuzzy subset of integers having membership function

$$\mu_{CARD(F)}(n) = \sup\{\alpha, CARD(F_\alpha) = n\} \forall n = 0, 1, 2, \ldots.$$

The fuzzy cardinality of fuzzy sets has been a topic of debate and many proposals appeared in the 1980s. See the monograph of Wygralak [23] for a survey of various proposals. It is clear that the fuzzy-valued cardinality of a fuzzy set has been more often than not envisaged as another fuzzy set of integers representing various possible values of the actual cardinality of the fuzzy set (hence involving some imprecision). However this fuzzy set of integers has an extremely particular shape (strictly decreasing membership function on its support), and interpreting it as expressing a lack of knowledge about the cardinality of F is extremely dubious, insofar as F is interpreted as a set having gradual boundaries (and not an ill-known set). On the contrary, $CARD(F)$ is quite a refined description of the cardinality of F where the gradual nature of the set is reflected on the integer scale. In fact, this membership function is equivalently described by the following injective assignment function:

$$a_{CARD(F)}(\alpha) = CARD(F_\alpha) \forall \alpha \in L_F.$$

Integers are defined as cardinalities of (finite) sets. Hence we claim that the fuzzy cardinality of a fuzzy set is precisely a fuzzy integer in the sense of a fuzzy element in the set of integers. For instance, the number of "young" employees in a firm is a fuzzy integer, if the fuzzy set "young" has a well-defined membership, which expresses a flexible (rather than ill-defined) query to a database. The same rationale can be put forward to justify the idea that the fuzzy probability of a fuzzy event, or the Hausdorff distance between fuzzy sets are fuzzy real numbers, rather than an imprecise probability or imprecise distance, respectively. The fuzzy Hausdorff distance between two fuzzy sets F and G generalises the Hausdorff distance d between sets: it can be viewed as the fuzzy real number $d(F,G)$ with assignment function $a_d(F,G)(\lambda) = d(a_F^{\geq}(\lambda), a_G^{\geq}(\lambda))$ (Dubois and Prade [4]). It is clear this assignment function has no special regularity, and can hardly be understood as a fuzzy set.

3.2 Fuzzy real numbers vs. fuzzy intervals

Another interesting case is the notion of *fuzzy interval*, a fuzzy set of numbers whose cuts are intervals. Fuzzy intervals account for both imprecision and fuzziness (regardless of whether their cores are reduced to a point or not). The addition of fuzzy intervals does not collapse to the regular addition when fuzziness is removed. It yields interval addition. Hence, calling a fuzzy interval a "fuzzy number", as many authors (including us) often do is debatable (even when its core reduces to a single number). This issue was a topic of (unresolved) debates in early Linz Seminars on Fuzzy sets between pure mathematicians and applied ones (see Proc. of the 1st Linz Seminar, pp. 139-140, 1979).

In contrast, we here take it for granted that a *fuzzy real number* should be a fuzzy *element* of the real line, each cut of which should be a number. Mathematically, a fuzzy real number r can be modeled by a function a_r from the unit interval to the real line (and not the converse). Note that we do not require monotonicity of the function so that some fuzzy numbers cannot be interpreted as membership functions (a number would then sometimes have more than one membership degree...). A monotonic and continuous fuzzy real number is called a *fuzzy threshold*. The idea is to model a fuzzy boundary between two regions of the real line. This fuzzy boundary is not an ill-known precise boundary, only a gradual one.

Algebraic structures of numbers (like groups) should be preserved for the most part when moving from real numbers to fuzzy real numbers (while fuzzy intervals just preserve algebraic properties of intervals). Indeed, let r_1 and r_2 be two fuzzy real numbers with assignment functions a_1 and a_2 (assumed to be mappings from (0, 1] to the reals) any operation $*$ between reals can be extended to fuzzy real numbers as follows: $r_1 * r_2$ has assignment function a_{1*2} such that

$$\forall \lambda \in (0,1], a_{1*2}(\lambda) = a_1(\lambda) * a_2(\lambda).$$

Under this definition, it is obvious that, for instance, the set of fuzzy real numbers forms an Abelian group for the addition, and that regular inverses exist ($a_{-r}(\lambda) = -a_r(\lambda)$). However it is clear that extended operations performed on (monotonic) fuzzy thresholds are not closed : if r_1 and r_2 are monotonically increasing, $r_1 - r_2$ may not be so (hence the necessity not to restrict to monotonic fuzzy real numbers).

Note that arithmetic operations extended to fuzzy elements can solve the following paradox: The set difference $A \setminus A$ is the empty set, as is the fuzzy set difference $F \setminus F$ computed for instance using Lukasiewicz conjunctions. The cardinality of $A \setminus A$ is $CARD(A) - CARD(A) = 0$. So should be $CARD(F) - CARD(F)$. It is clear that considering the fuzzy cardinality $CARD(F)$ as a fuzzy set of integers and applying the extension principle to compute $CARD(F) - CARD(F)$ yields a fuzzy set which is symmetric around 0, which is counterintuitive. As an operation between fuzzy integers, $CARD(F) - CARD(F) = 0$.

Moreover, if we consider (in the spirit of Rocacher and Bosc [18]) the set of "natural" fuzzy integers as all fuzzy elements on the set of integers of the form $CARD(F)$ for some finite fuzzy set F, the set of fuzzy elements of the relative integers (fuzzy relative integers for short) can be obtained as $CARD(F) - CARD(G)$ for some finite fuzzy sets F and G. To see it, it is enough to consider a fuzzy relative integer z as a sequence or relative integers $k_1, \ldots, k_n$, such that $a_z(k_i) = \lambda_i$ and to notice the following result:

Theorem 1 *Let $k_1, \ldots, k_n$ be any finite sequence of relative integers. There exist two increasing sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of non-negative integers, such that $\forall i = 1 \ldots n, k_i = x_i - y_i$.*

To see it define $x_1 = \max(k_1, 0)$ and $y_1 = \max(-k_1, 0)$ and, for $i > 1$,

$$y_i = y_{i-1} + 1 + \max(0, k_{i-1} - k_i),$$

and of course $x_i = y_i + k_i$. In some sense, the set of fuzzy natural integers (understood as cardinalities of finite fuzzy sets) generates all fuzzy relative integers, via a canonical subtraction. The sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ define fuzzy natural integers x and y such that $z = x - y$.

The notion of fuzzy element enables a fuzzy interval to be defined as a pair of fuzzy thresholds having opposite monotonicities, just as an interval is modeled by an ordered pair of numbers. In [2], such fuzzy thresholds are called profiles, which we consider here as genuine fuzzy real numbers. In fuzzy interval analysis, the problem is to find the lower and the upper bounds of a function $\varphi(x,y,z,...)$ when $x,y,z,...$ range over intervals $I, J, K, \ldots$. The tuples of values called extreme configurations (the vertices of the hyper-rectangle $I \times J \times K \times ...$) play a decisive role as candidates for being tuples of values for which the optima of the function f are reached, when the function is locally monotonic. The usual method of fuzzy interval analysis is to perform regular interval analysis on α-cuts. Viewing a fuzzy interval as a fuzzy set of reals limited by two fuzzy real numbers, enables interval analysis to be directly applied to fuzzy extreme configurations (viewed as tuples of fuzzy boundaries).

Performing fuzzy interval analysis in the style of interval calculations, for a locally monotonic function, comes down to applying the function to (at worst) all extreme fuzzy configurations. Some partial results may fail to be monotonic [2], even if when putting all partial results together, a genuine fuzzy interval is obtained in the end. This use of non-monotonic profiles avoids the pitfall of ending up with anomalous membership functions (like the anti-fuzzy numbers of Goetschel [8]) due to a definition of subtraction such that $M + N = Q$ if and only if $M = Q - N$, between fuzzy intervals. For instance, the fuzzy interval bounded by the pair of fuzzy reals (r^-, r^+) where $r^-(\lambda) = a, \forall \lambda \in (0,1]$ and $r^+(\lambda) = a + (1-\lambda)b$ (with $b > 0$), should not be confused with the fuzzy real number r^+ itself, when it comes to performing subtraction. Indeed, while $r^+ - r^+ = 0$, the extension principle of fuzzy arithmetics yields:

$$(r^-, r^+) - (r^-, r^+) = (r^- - r^+, r^+ - r^-),$$

which is an imprecise zero, but not zero. This is because the pair of fuzzy reals (r^-, r^+) actually represents the fuzzy interval generated by the fuzzy real number r^+, as per Definition 4.

3.3 Defuzzification

Finally we can reconsider the problem of "defuzzifying" a fuzzy set of real numbers, understood as selecting a representative number for it. This vocabulary is not appropriate as explained earlier. The notion of mean interval of a fuzzy interval M was defined as follows (Dubois and Prade [5]): consider M as a pair of distribution functions (F_*, F^*) where

$$F^*(x) = \mu_M(x) \text{ for } x \leq \inf c(M)$$

($c(M)$ is the core of M with membership value 1), and

$$F_*(x) = 1 - \mu_M(x) \text{ for } x \geq \sup c(M).$$

The mean interval is $E(M) = [E_*(M), E^*(M)]$ where $E_*(M)$ (resp. $E^*(M)$) is the expectation of the probability function with cumulative distribution F^* (resp. F_*). This definition is justified from different points of view, as producing the upper and lower expectations of the set of probability functions dominated by the possibility measure induced by M [5], but also as the mean α-cut obtained via an Aumann integral of the set-valued map associated to M (to each $\alpha \in (0,1]$ assign the α-cut M_α, i.e., a gradual set; see Ralescu [17]). This set-valued average is linear with the fuzzy addition and scalar multiplication. It corresponds to stripping M from its fuzziness, not of its imprecision. It is, literally, a defuzzification. The next step is to select a number in $E(M)$ (for instance the mid-point, by symmetry; see Yager [24]). It provides a method for choosing a number representing a fuzzy set that is more natural than the center of area and the like. Using the notion of fuzzy real number, and a fuzzy element in the real line, one can exchange the steps of (genuine) defuzzification and selection. We can strip M from its imprecision, by selecting a fuzzy real number $r(M)$ in M, and then we can defuzzify $r(M)$. A natural selection, in agreement with the symmetry argument is to pick the mid-point m_α of all α-cuts of M, and it defines a fuzzy real number $m(M)$. Its (generally not monotonic) assignment function is $a_{m(M)}(\alpha) = m_\alpha$. Now, we can defuzzify it, using the Riemann integral:

$$\underline{\text{m}}(M) = \int_0^1 m_\alpha d\alpha.$$

It is obvious that the obtained value is also the mid-point of the mean interval (or average cut), i.e.

$$\underline{\text{m}}(M) = (E^*(M) + E*(M))/2.$$

It follows that in terms of fuzzy real numbers,

$$m(M+N) = m(M) + m(N).$$

Besides, the defuzzified $\underline{\text{m}}(M)$ is also equal to the mean value of the probability distribution obtained by randomizing the fuzzy number (in the style of Chanas and Nowakowski [1]): picking an element α at random in $(0,1]$ and then a number at random in $[m_\alpha^-, m_\alpha^+]$: the obtained probability is the center of mass of the polyhedron restricting the set of probability functions induced by M. It is the random number obtained via the fuzzy real number $\underline{\text{m}}(M)$ when equipping the unit interval with a uniform probability distribution (also the Shapley value [22] of the "unanimity game" generated by M).

4 RELATED WORKS

Mathematicians of fuzzy sets in the past have introduced the notion of a fuzzy real number, starting with Hutton [12]. Often, it takes the form of a decreasing mapping from the reals to the unit interval or a suitable lattice (Grantner et al. [7]), or a probability distribution function (Lowen [13]). Arithmetic operations on fuzzy reals were studied by Rodabaugh [20], and contrast with fuzzy arithmetics based on the extension principle. Hoehle [10] especially emphasized the role of fuzzy real numbers as modeling a fuzzy threshold softening the notion of Dedekind cut.

The idea of gradual set proposed here is just an extension of the definition of fuzzy sets in terms of α-cuts, dropping the nestedness condition. This view was first systematically explored by Negoita and Ralescu [15] quite early (see Ralescu [16] for more advanced considerations), and gradual sets were proposed by Goetschel [8], under the name fuzzy level sets.

Recently, Rocacher and Bosc [18] suggested to define fuzzy integers as (precise, but gradual) cardinalities of fuzzy sets, defined using an inequality : $\mu_{CARD(F)}(n) = \sup\{\alpha, CARD(F_\alpha) \geq n\} \forall n = 0, 1, 2, ...$. A fuzzy integer is then a (monotonic) mapping from the unit interval to the natural integers. They then define fuzzy negative integers [18] and fuzzy rationals [19] as equivalence classes of pairs of fuzzy integers (r_1, r_2) such that $r_1 + r = r_2$ and $r_1 \cdot r = r_2$, as in the classical setting. Fuzzy negative integers are no longer monotonic, generally. This view is totally along the line discussed above.

The idea that a fuzzy interval is a pair of fuzzy thresholds or profiles is akin to the so-called graded numbers of Herencia and Lamata [9] and the fuzzy darts of Goetschel [8]. These authors also consider mappings from the unit interval to the real line, instead of the usual mapping from the reals to the unit interval. A fuzzy interval is then viewed as a pair of such mappings. However, our fuzzy reals are more general because they are not necessarily monotonic. In fact, the very technique for deriving closed-form formulas for fuzzy arithmetic operations on L-R fuzzy intervals (see Dubois and Prade [3]) does rely on the separate treatment of left and right-hand sides of fuzzy intervals, applying the operations on the corresponding fuzzy thresholds. This technique is generalized to fuzzy interval analysis in [2].

5 CONCLUSION

This paper introduces a new concept in fuzzy set theory, namely that of a fuzzy element. It seems that such a concept was missing in the theory. Although of an abstract nature, we think it is a crucial concept for understanding the nature of fuzziness introduced by Zadeh as gradedness and flexibility in concepts, thus giving up Booleanity, as opposed to the idea of partial or incomplete information. Since sets are used for representing incomplete knowledge, fuzzy sets often capture both ideas of gradedness and partial information at the same time (as in possibility theory). This has created confusion between fuzziness and uncertainty, sometimes leading to debatable developments in the theory or the applications of fuzzy sets. The merit of fuzzy elements is that they only embody the idea of gradedness. Some applications of this concept have been surveyed, especially the notion of fuzzy real number that can be instrumental for developing a genuine fuzzy interval analysis as well as sound defuzzification procedures. Other applications of the new concept to fuzzy cardinality are relevant for a better handling of quantifiers in fuzzy queries to databases.

REFERENCES

[1] S. Chanas and M. Nowakowski. Single value simulation of fuzzy variable, Fuzzy Sets and Systems, vol. 25, pp. 43–57, 1988.

[2] D. Dubois, H. Fargier, J. Fortin A generalized vertex method for computing with fuzzy intervals. Proc. of the Int. Conf. on Fuzzy Systems, Budapest, Hungary, 2004, pp. 541–546.

[3] D. Dubois and H. Prade Operations on fuzzy numbers, Int. J. Systems Science, vol. 9, pp. 613–626, 1978.

[4] D. Dubois and H. Prade On distances between fuzzy points and their use for plausible reasoning, Proc. IEEE Inter. Conf. on Cybernetics & Society, Bombay - New Dehli, Dec. 30, 1983-Jan. 7, 1984, pp. 300–303.

[5] Dubois D. and Prade H. The mean value of a fuzzy number, Fuzzy Sets and Systems, vol.24, pp. 279–300, 1987.

[6] D. Dubois, E. Kerre, R. Mesiar, H. Prade . Fuzzy interval analysis. In: Fundamentals of Fuzzy Sets, Dubois,D. Prade,H., Eds, The Handbooks of Fuzzy Sets Series, Kluwer, Boston, Mass., 2000, pp. 483–581.

[7] T. Gantner, R. Steinlage, R. Warren Compactness in fuzzy topological spaces, J. Math. Anal. Appl. vol.62, pp. 547–562, 1978.

[8] R.H. Goetschel, Jr, Representations with fuzzy darts, Fuzzy Sets and Systems, vol.89, pp. 77–105, 1997.

[9] J. Herencia and M. Lamata, A total order for the graded numbers used in decision problems, Int. J. of Uncertainty, Fuzziness and Knowledge-Based Systems, vol. 7, pp. 267–276, 1999.

[10] U. Hoehle Fuzzy real numbers as Dedekind cuts with respect to a multiple-valued logic, Fuzzy Sets and Systems, vol.24, pp. 263–278,1987.

[11] U. Hoehle Many-valued equalities, singletons and fuzzy partitions, Soft Computing, 2, pp. 134-140,1998.

[12] B Hutton Normality in fuzzy topological spaces. J. Math. Anal. Appl. vol.50, pp. 74–79, 1975.

[13] R. Lowen Fuzzy Set Theory, (Ch. 5 : Fuzzy real numbers), Kluwer, Dordrecht, 1996.

[14] R. Moore, Methods and applications of interval analysis. SIAM Studies in Applied Mathematics, 1979.

[15] C.V. Negoita and D.A. Ralescu. Representation theorems for fuzzy concepts, Kybernetes, vol.4, pp. 169–174, 1975.

[16] D.A. Ralescu A generalization of the representation theorem, Fuzzy Sets and Systems, vol.51, pp. 309-311, 1992.

[17] D.A. Ralescu Average level of a fuzzy set. In Statistical Modeling, Analysis and Management of Fuzzy Data (C. Bertoluzza et al. Eds,) Springer, Heidelberg, 2002, pp.119-126.

[18] D. Rocacher, P. Bosc (2003) About Zf, the set of relative integers, and the definition of fuzzy bags on Zf, IFSA03 Istanbul, LNCS 2715, Springer Verlag, 2003, pp. 95-102.

[19] D. Rocacher, P. Bosc The set of fuzzy rational numbers and flexible querying, To appear in Fuzzy Sets and Systems, 2005

[20] S. E. Rodabaugh, Fuzzy addition in the L-fuzzy real line, Fuzzy Sets and Systems, vol.8, pp. 39–51, 1982.

[21] E. Roventa and T. Spircu Averaging procedures in defuzzification processes, Fuzzy Sets and Systems, vol.136, pp. 375–385, 2003.

[22] L.S. Shapley A value for n-person games. In Kuhn and Tucker, eds., In: Contributions to the Theory of Games, II, Princeton Univ. Press, 1953, pp. 307–317.

[23] M. Wygralak Vaguely Defined Objects Representations, Fuzzy Sets and Nonclassical Cardinality Theory, Kluwer Academic Publ., Dordrecht, 1996.

[24] R. R. Yager A procedure for ordering fuzzy subsets of the unit interval, Information Sciences, vol.24, pp. 143–161, 1981.

[25] L.A. Zadeh Fuzzy sets, Information & Control, vol.8, pp. 338-353, 1965.

[26] L.A. Zadeh A computational approach to fuzzy quantifiers in natural languages, Computers & Mathematics with Applications, vol.9(1), pp. 149–184, 1983.