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Abstract	The notion of a fuzzy set stems from considering sets where, in the words of Zadeh, the “transition from non-membership to membership is gradual rather than abrupt”. This paper introduces a new concept in fuzzy set theory, that of a gradual element. It embodies the idea of fuzziness only, thus contributing to the distinction between fuzziness and imprecision. A gradual element is to an element of a set what a fuzzy set is to a set. A gradual element is as precise as an element, but the former is flexible while the latter is fixed. The gradual nature of an element may express the idea that the choice of this element depends on a parameter expressing some relevance or describing some concept. Applications of this notion to fuzzy cardinality, fuzzy interval analysis, fuzzy optimization, and defuzzification principles are outlined.

Keywords (separated by '-')	Fuzzy set - Alpha-cuts - Fuzzy numbers
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DOI 10.1007/s00500-007-0187-6

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Gradual elements in a fuzzy set

Didier Dubois · Henri Prade

Abstract The notion of a fuzzy set stems from considering sets where, in the words of Zadeh, the "transition from non-membership to membership is gradual rather than abrupt". This paper introduces a new concept in fuzzy set theory, that of a gradual element. It embodies the idea of fuzziness only, thus contributing to the distinction between fuzziness and imprecision. A gradual element is to an element of a set what a fuzzy set is to a set. A gradual element is as precise as an element, but the former is flexible while the latter is fixed. The gradual nature of an element may express the idea that the choice of this element depends on a parameter expressing some relevance or describing some concept. Applications of this notion to fuzzy cardinality, fuzzy interval analysis, fuzzy optimization, and defuzzification principles are outlined.

Keywords Fuzzy set · Alpha-cuts · Fuzzy numbers

1 Introduction

Originally, Zadeh used the word "fuzzy" as referring specifically to the introduction of shades or grades in all-or-nothing concepts. A fuzzy set Zadeh (1965) is a generalization of subset (at least in the naive sense); it is a subset with boundaries that are "gradual rather than abrupt". It is defined by a membership function from a basic set to the unit interval (or a suitable lattice) and its cuts are sets. In fuzzy set theory, the term "fuzzy" explicitly refers to the idea of gradual transition, as opposed to the term "vague", which refers to lexical imprecision at large.

However, there is a recurrent confusion in the literature between the word "fuzzy" and other words or phrases like "imprecise", "inexact", "incompletely specified", "vague" that rather refer to a lack of sufficient information. For instance, what is often called a fuzzy number is understood as a generalized interval, not as a generalized number. The calculus of fuzzy numbers is basically an extension of interval arithmetics. Especially, fuzzy numbers in this sense, being fuzzy subsets of the real line, cannot be equipped with a group structure for the addition. Similarly, in engineering papers, defuzzification is generally understood as turning a fuzzy set of numbers (obtained from some fuzzy inference engine) into a number. Yet, since defuzzifying literally means sharpening, removing graduality, then defuzzifying a fuzzy set should yield a set, not a point. For instance, the notion of mean value of a fuzzy interval was proposed as a natural way of extracting an interval from a fuzzy interval (Dubois and Prade 1987, where the phrase "fuzzy number" was used in the sense of a unimodal fuzzy interval). On interval defuzzification, see also recent works by Roventa and Spircu (2003) and Ralescu (2002).

To be more credible, the defuzzification process as used in the engineering area should be split into two steps: removing fuzziness (thus getting an interval), and removing imprecision (by selecting a number in the interval). Suppose we perform defuzzification by swapping these two steps: given a fuzzy set of numbers, suppose we first remove imprecision. Then, we get what we could call a "fuzzy real number" or a "gradual number" that one may defuzzify eventually. Such a gradual number would then express fuzziness only, WITHOUT imprecision. To get a good intuition of a gradual number, one may view a fuzzy interval as a pair of such

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Journal: 500 MS: 187 Article No.: 0187 ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:2007/5/8 Pages: 11

gradual numbers, just as an interval is an ordered pair of numbers.

More generally, this discussion leads to introduce the notion of fuzzy or gradual element of a (fuzzy) set, a concept that was apparently missing in fuzzy set theory. Topologists tried to introduce ideas of fuzzy points in the past (attaching a membership value to a single element of a set), but this notion has often been controversial, and sterile in its applications. In fact they were meant to be fuzzy singletons, not really fuzzy elements. The aim of this paper is to introduce a natural notion of gradual element and gradual (integer or real) number, to outline elementary formal definitions related to this notion and discuss its potential at shedding light on some yet ill-understood aspects of fuzzy set theory and its applications. A full-fledged mathematical development is left for further research.

2 Basic definitions

Let S be a set. Consider a complete distributive lattice L with top 1 and bottom 0. In practice, L is often totally ordered (the unit interval, typically). Elements of L are often interpreted as degrees of membership (for fuzzy sets), degrees of possibility, of preference, of truth, etc. In this paper, they correspond to a set of more or less relevant contexts [an idea coined by Gebhardt and Kruse (1993) in a different scope]. As a consequence, L will be called a relevance scale. Let $L^+ = L \setminus \{0\}$ denote the set L where the bottom element has been removed.

Definition 1 A fuzzy (or gradual) element e in S is defined by its assignment function, a mapping $\mathcal{A}_e$ from L^+ to S.

Example 1 Assume $S = \{s_1, s_2, s_3\}, L = \{0, 0.25, 0.5, 0.75, 1\}$. Consider the assignment function defined by:

$$\mathcal{A}_e(1) = s_1; \mathcal{A}_e(0.25) = s_1; \mathcal{A}_e(0.5) = s_2; \mathcal{A}_e(0.25) = s_1.$$

In the following we shall use the term *gradual element* so as to avoid confusion, when such an element is a number, with what is usually meant by a fuzzy number in the literature. Several remarks are in order. First, the mapping we consider goes from the relevance scale to the referential (contrary to a fuzzy set). Given a degree of relevance $\lambda > 0$, $\mathcal{A}_e(\lambda) = s$ is the element of S representative of e at level λ. It is the element that is suitable in the corresponding context.

A crisp element $s \in S$ has its own assignment function $\mathcal{A}_s$, which is a constant mapping defined by $\forall \lambda > 0, \mathcal{A}_s(\lambda) = s$. On the other hand, $\mathcal{A}_s(0)$ is not defined because there is no counterpart to the empty set for elements of a set.

Remark The domain of $\mathcal{A}_e$ could be defined as any proper subset of L containing 1 and excluding 0. However, the fact that the domain of an assignment function may vary from one gradual element to the other would create difficulties when combining gradual elements. It is then always possible to augment the domain of $\mathcal{A}_e$ by building a mapping $\mathcal{A}^*_e$ from L^+ to S, such that $\forall \lambda \in \mathrm{Dom}(\mathcal{A}_e), \mathcal{A}^*_e(\lambda) = \mathcal{A}_e(\lambda)$, by letting:

$$\forall \lambda > 0, \mathcal{A}^*_e(\lambda) = \mathcal{A}_e(\lambda^*),$$

where $\lambda^* = \inf\{\alpha > \lambda, \alpha \in \mathrm{Dom}(\mathcal{A}_e)\}$. The idea is that if an element s is representative of the gradual element e to a certain extent, it is also representative to a lesser extent, unless otherwise specified.

One might also require an injective assignment function, in order to ensure that each $s \in S$ is attached a single degree of relevance, but there is no clear reason to do so, simply because as seen later, extending operations equipping S to gradual elements may fail to preserve this property. And note that it would exclude crisp elements, as their assignment function is not injective. But the lack of injectivity of assignment functions implies that they cannot always be reinterpreted as (inverses of) membership functions of fuzzy sets. For instance in Example 1, the membership grade of s_1 is not uniquely defined.

The idea of a gradual element, in contrast with the notion of a fuzzy set, can also be easily illustrated by the following important example of a gradual number. Consider a convex fuzzy set of the real line, i.e. a fuzzy interval M. Let m_α be the middle-point of the α-level cut of M. The set of pairs $m(M) = \{(\alpha, m_\alpha) \mid \alpha \in (0, 1]\}$ defines a gradual element of the real line, which can be called the fuzzy middle-point of M. If the membership function of M is symmetrical, then $m(M)$ reduces to an ordinary real number that is the common abscissa of the middle-points of all the α-cuts of M. For a non-symmetrical trapezoidal fuzzy interval, one obtains a straight-line segment from the mid-point of the core ($c(M) = \{r, \mu_M(r) = 1\}$) to the mid-point of the support ($supp(M) = \{r, \mu_M(r) > 0\}$). It is a particular gradual element, actually a gradual real number (or gradual number for short). In the general case, m_α is not a monotonic function of α, and the gradual number $m(M)$ may take shapes that can no longer be reinterpreted as a membership function, i.e., as a mapping from the real line into [0, 1]. Using the formal definition of a gradual element introduced in this paper, it is possible to extend, to fuzzy intervals, results like for instance: the middle-point of the sum of two intervals is equal to the sum of the middle-points of the intervals, as seen later. Note that there is no uncertainty in a gradual element since the assignment function reflects the idea of representativeness only. A gradual element is a "flexible" element, not so much an uncertain element: we still have the right of choice when picking a suitable representative for it. One can see a gradual element as an element depending on the relevance value λ, with the idea to postpone the choice of λ later in some decision process, for instance by selecting the proper context.

Journal: 500 MS: 187 Article No.: 0187 TYPESET DISK LE CP Disp.:2007/5/8 Pages: 11

2.1 Fuzzy sets as gradual elements of the power set

In order to check if the definition of a gradual element is meaningful, we must prove that a fuzzy subset of S can be defined as a gradual element of the power set 2^S. The natural assignment function of a fuzzy set F is its associated cut-mapping, which assigns to any $\lambda \in L^+$ its λ-cut $F_\lambda = \{s, \mu_F(s) \geq \lambda\}$. Define the assignment function of F as:

$$\mathcal{A}_F^{\geq}(\lambda) = F_\lambda, \quad \forall\lambda \in L^+.$$

This mapping has been especially studied by Negoita and Ralescu (1975). In the general case, this mapping may not be injective. Let $L_F = \mu_F(S) \setminus \{0\}$ be the set of non-zero membership grades of F. L_F can be a finite set and then the range of $\mathcal{A}_F$ consists of $|L_F|$ distinct nested subsets. Alternatively, one may try to represent F by an assignment function that map on disjoint subsets (λ-sections):

$$\mathcal{A}_F^{=}(\lambda) = \{s, \mu_F(s) = \lambda\}, \quad \forall\lambda \in L_F.$$

But the problem is that generally, $\mathcal{A}_F^{=}$ will not be a mapping since its domain L_F may be a proper subset of L^+.

However, Definition 1 is in agreement with a more general view of fuzzy sets whose crisp representatives are neither nested nor disjoint.

Definition 2 A *gradual subset* G of S is defined by an assignment function $\mathcal{A}_G$ from L^+ to 2^S.

Example 2 Consider the gradual subset G of the set S:

$$\mathcal{A}_G(1) = \{s_1, s_2\}; \mathcal{A}_G(0.75) = \{s_1\};$$
$$\mathcal{A}_G(0.5) = \{s_2, s_3\}; \mathcal{A}_G(0.25) = \{s_1, s_2\}.$$

The images of the assignment function are not nested.

A fuzzy set is then a special kind of gradual set. The assignment function of a crisp set is constant. A single gradual element e yields a gradual singleton E by letting $\mathcal{A}_E(\lambda) = \{\mathcal{A}_e(\lambda)\}, \quad \forall\lambda \in L^+$. More generally, a set of gradual elements forms a gradual set of S:

Definition 3 The gradual set G induced by the family of gradual elements $e_1, \ldots, e_k$ in S with assignment functions $\mathcal{A}_1, \ldots, \mathcal{A}_k$, has its assignment function $\mathcal{A}_G$ defined by

$$\mathcal{A}_G(\lambda) = \{\mathcal{A}_1(\lambda), \ldots, \mathcal{A}_k(\lambda)\}, \quad \forall\lambda \in L^+.$$

Example 3 The fuzzy element e in Example 1 induces a gradual singleton $\{e\} : \mathcal{A}_{\{e\}}(1) = \{s_1\}; \mathcal{A}_{\{e\}}(0.25) = \{s_1\}; \mathcal{A}_{\{e\}}(0.5) = \{s_2\}; \mathcal{A}_{\{e\}}(0.25) = \{s_1\}$.

It is possible to define a regular fuzzy set from a gradual subset defined via its assignment function $\mathcal{A}_G(\cdot)$. One natural approach is as follows:

Definition 4 The membership function of the fuzzy set $F(G)$ induced by the gradual set with assignment function $\mathcal{A}_G$ is

$$\mu_{F(G)}(s) = \sup\{\lambda, s \in \mathcal{A}_G(\lambda)\}$$
$$= 0 \quad \text{if} \quad \nexists\lambda, s \in \mathcal{A}_G(\lambda).$$

Example 4 The gradual set G in Example 2 induces the fuzzy set $F : \mu_{F(G)}(s_1) = \mu_{F(G)}(s_2) = 1; \mu_{F(G)}(s_3) = 0.5$.

The supremum is motivated by the idea that a set is the union of the singletons it contains. It is also in agreement with the representations of fuzzy sets in terms of its λ-cuts. A fuzzy set F clearly may correspond to several gradual sets, that form an equivalence class, even if we may consider the λ-cut representation as the canonical (nested) gradual set representative of F.

2.2 Fuzzy sets induced by gradual elements

Using Definitions 3 and 4, the fuzzy set induced from a set of gradual elements $e_1, \ldots, e_k$ is F such that

$$\mu_F(s) = \max_{i=1,\ldots,k} \sup\{\lambda_i, s = \mathcal{A}_i(\lambda_i)\}. \quad (1)$$

The set of gradual elements is said to generate the fuzzy set F. For instance, a single gradual element e having an injective assignment function yields a fuzzy set $F(e)$ by letting $\mu_{F(e)}(s) = \lambda$ if and only if $\mathcal{A}_e(\lambda) = s$, i.e., $\mu_{F(e)}$ is the inverse of the assignment function $\mathcal{A}_e$. In the non-injective case, we get $\mu_{F(e)}(s) = \sup\{\lambda, s = \mathcal{A}_e(\lambda)\}$, so as to account for the best representativeness level of s w.r.t e. If e is a crisp element s, then $F(e)$ is the singleton$\{s\}$. But it is not clear that, in general, a fuzzy set F induced by a single gradual element can be called a fuzzy singleton. Indeed, the cuts of $F(e)$ will generally not only contain a single element.

Example 5 The gradual element G in Example 1 induces the fuzzy set $F(e) : \mu_{F(e)}(s_1) = 1; \mu_{F(e)}(s_2) = 0.5; \mu_{F(e)}(s_3) = 0$.

Nevertheless, if F is the fuzzy set induced by the family of gradual elements $e_1, \ldots, e_k$ in S, then Eq. (1) reads $F = F(e_1) \cup F(e_2) \cup \cdots \cup F(e_n)$ (in the sense of the maximum).

So, in some sense, despite the dubious singleton status of fuzzy sets generated by single gradual elements, a fuzzy set is still viewed as the collection of gradual elements that generates it. But as seen later a fuzzy set may contain many more gradual elements than those that generated it. The notion of fuzzy singleton is difficult to define without a convenient notion of gradual element, when S is only equipped with the usual equality relation, as singletons are basically construed as the quotient set $S/=$ (see Hoehle 1998), hence are crisp. In the literature (especially in fuzzy topology) a fuzzy singleton is viewed as a fuzzy set whose support is a singleton and the member of this singleton has a membership grade that can be less than 1. Let alone the fact that it is most of the time an anomalous fuzzy set (since empty to some extent), it cannot be defined as some quotient set w.r.t an equality relation. In contrast, the notion of gradual singleton is clear (a mapping from L^+ to the set of singletons of S). Gradual singletons are singletons trivially induced by the set of gradual elements of S equipped by the equality relation defined by $e = e'$ if and only if $\mathcal{A}_e = \mathcal{A}'_e$.

2.3 Picking elements in a fuzzy set

An interesting question is how to pick an element in a fuzzy set. This is not so easy if the notion of gradual element is absent. Indeed, then only crisp elements can be picked. Either we pick an element in the core of F, but then one may argue that the fuzzy feature of the set is never accounted for; or we pick an element in the support. But, then one may argue that picking an element outside the core is not legitimate, since these elements are not totally representative of the fuzzy set. And picking an element with the associated membership grade is hard to interpret.

Now, looking at a gradual set G, it is clear that the idea of picking an element in a set is naturally generalized by picking an element in each set $\mathcal{A}_G(\lambda)$. So, gradual sets contain gradual elements. So, a gradual element in a fuzzy set F is naturally defined as one obtained by picking an element in each cut of F.

Definition 5 A gradual element e is said to belong to a fuzzy set F if and only if $\forall\lambda \in L^+, \mathcal{A}_e(\lambda) = s_\lambda \in F_\lambda$.

For instance, the fuzzy element e of Example 1 belongs to the fuzzy set $F(G)$ of Example 4. Mathematically, a gradual element in F is a *selection function* from the multivalued mapping $\alpha \longrightarrow F_\alpha$ defining the cuts of F. In other words, a gradual element e belongs to a fuzzy set F if and only if $\forall s \in S$, if $\exists\lambda \in L^+, \mathcal{A}_e(\lambda) = s$, then $\mu_F(s) \geq \lambda$. In some sense F collects all gradual elements e such that

$$\mu_F(\mathcal{A}_e(\lambda)) \geq \lambda, \quad \forall\lambda \in L^+. \tag{2}$$

Namely, $\mu_F(\mathcal{A}_e(\lambda)) \geq \lambda$ if and only if $\mathcal{A}_e(\lambda) \in F_\lambda$. The following simple condition is characteristic of the membership of a gradual element in a fuzzy set:

Proposition 1 *A gradual element e belongs to F if and only if $F(e) \subseteq F$ (that is, $\mu_{F(e)} \leq \mu_F$).*

Proof Since e belongs to F means Eq. (2) holds, it also reads $\forall s$, if $s = \mathcal{A}_e(\lambda)$, then $\mu_F(s) \geq \lambda$. So $\mu_F(s) \geq \sup\{\lambda, s = \mathcal{A}_e(\lambda)\}$. □

We can write, using the idempotent fuzzy set union, $F = \cup_{e\in F} F(e)$. In particular the only crisp elements in F are those in its core. Each gradual element e in a gradual set G belongs to the fuzzy set $F(G)$ induced by G. Namely : $\forall s$, if $\mathcal{A}_e(\lambda) = s$, then $s \in \mathcal{A}_G(\lambda)$ hence $\mu_{F(G)}(s) \geq \lambda$. So if F is generated by a collection of gradual elements, each of these gradual elements belongs to F. Of course, the family of generators of F is clearly not unique. The set of gradual elements of F is the maximal family generating F.

An interesting question is to find minimal families of gradual elements generating a fuzzy set F. One idea is to start from sections of F. A gradual element e of F can be built as follows: $\forall\lambda_i \in L_F$, pick an element $s_i \in \mathcal{A}_F^{=}(\lambda_i)$, the λ_i-section. Define $\mathcal{A}_e(\lambda_i) = s_i$. Then $\forall\lambda \notin L_F$, let $\mathcal{A}_e(\lambda) = \mathcal{A}_e(\lambda_i)$, where $\lambda_i = \inf\{\alpha > \lambda, \alpha \in L_F\}$. If the cardinality of $\mathcal{A}_F^{=}(\lambda_i)$ is n_i, the number of such generating gradual elements is $\prod_{\lambda_i\in L_F} n_i$. This set $\mathcal{G}$ of gradual elements clearly generates F.

A minimal set $\mathcal{G}^{\min}$ of gradual elements generating F can be built as follows: It contains $\max_i n_i$ elements $e_i \in \mathcal{G}^{\min}$ such that $\forall\lambda \neq 1 \in L_F, i \neq j, \mathcal{A}_{e_i}(\lambda) \neq \mathcal{A}_{e_j}(\lambda)$. It consists in emptying recursively the sections of F by picking an element in each λ_i-section of F, building the corresponding gradual element of $\mathcal{G}$ and doing it again until all sections are empty. When a section other than the core becomes empty, it is no longer picked from. When only one element remains in the core, it is kept and used repeatingly for building subsequent gradual elements until all other sections get empty.

Example 6 Consider a fuzzy set F such that $\mu_F(s_1) = \mu_F(s_2) = 1$, $\mu_F(s_3) = 0.7$, $\mu_F(s_4) = \mu_F(s_5) = \mu_F(s_6) = 0.3$. First build e_1, picking s_1, s_3, s_4. Then $\mathcal{A}_{e_1}(\lambda) = s_1$ if $\lambda \in (0.7, 1]$, s_3 if $\lambda \in (0.3, 0.7]$, s_4 if $\lambda \in (0, 0.3]$. Then the 0.7-section is empty. Next, build e_2, picking s_2, s_5. Then $\mathcal{A}_{e_2}(\lambda) = s_2$ if $\lambda \in (0.3, 1]$, s_5 if $\lambda \in (0, 0.3]$. Then the core is empty, so we keep s_2. Build e_3, picking s_2, s_6. Then $\mathcal{A}_{e_3}(\lambda) = s_2$ if $\lambda \in (0.3, 1]$, s_6 if $\lambda \in (0, 0.3]$.

By construction, if $\mu_F(s) = \lambda_i \neq 1$, there is a unique gradual element $e_j \in \mathcal{G}^{\min}$ such that $\mathcal{A}_{e_j}(\lambda_i) = s$. For all other $e_k \in \mathcal{G}^{\min}$, there is no $\lambda \in L_F$ such that $\mathcal{A}_{e_k}(\lambda) = s$. Hence, if F^{-j} denotes the fuzzy set generated by $\mathcal{G}^{\min} \setminus \{e_j\}$, it is clear that $\mu_{F^{-j}}(s) = 0$. If $\mu_F(s) = 1$, then either there is a singleton $e_j = \{s\} \in \mathcal{G}^{\min}$, so $\mu_{F^{-j}}(s) = 0$, or $\mathcal{A}_{e_j}(1) = s$ for a non-crisp gradual element e_j, and $\mu_{F^{-j}}(s') = 0$ for some other element s' of the support of F. Hence $\mathcal{G}^{\min}$ is a minimal set of gradual elements generating F.

2.4 The degree of membership of a gradual element in a fuzzy set

Let e be a gradual element of S and F be a fuzzy subset of S, such that $F(e) \not\subseteq F$. We may try to compute the degree of membership of this gradual element in the fuzzy set F. Naturally, this degree will be a gradual element of L.

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Definition 6 The degree of membership of a gradual element e in a fuzzy set F is a gradual element $\mu(e)$ of L defined by its assignment function $\mathcal{A}_{e\in F}$ such that

$$\mathcal{A}_{e\in F}(\lambda) = \mu_F(\mathcal{A}_e(\lambda)), \quad \forall\lambda \in L^+.$$

The lattice element $\mathcal{A}_{e\in F}(\lambda) \in L$ is a representative value of the membership grade of e in F to degree λ. Note that the obtained fuzzy degree of membership does not express imprecision. It just reflects the gradual nature of the fuzzy set and of the gradual element. Suppose $e = s$ is a regular element. Then, $\mathcal{A}_{e\in F}(\lambda) = \mu_F(s), \quad \forall\lambda \in L^+$.

Suppose e belongs to F in the sense of Definition 5. Then, by construction $\mathcal{A}_{e\in F} \geq Id_L$ the identity function of L. If F is not fuzzy, say a subset A, then $\mathcal{A}_{e\in A}(\lambda) = 1$ if $\mathcal{A}_e(\lambda) \in A$, and 0 otherwise. In other words, e belongs to A or does not belong to A according to its representative, indexed by λ. Then e fully belongs to A provided that $\mathcal{A}_{e\in F}(\lambda) = 1, \quad \forall\lambda > 0$. We can consider $\sup\{\lambda, \mathcal{A}_e(\lambda) \in A\}$ as the degree of membership of e in A, but this must be properly understood as the extent to which a gradual element belongs to a crisp set (the gradual nature of membership is not due to the set A). Alternatively, this degree could be defined as $\inf\{\lambda, \mathcal{A}_e(\lambda) \in A\}$. However the whole range $\{\lambda, \mathcal{A}_e(\lambda) \in A\}$ is more representative of this evaluation. More generally, one might compute a scalar degree of membership of e in F by considering the measure of the set $\{\lambda, \mathcal{A}_e(\lambda) \in F_\lambda\}$ (it may be the Lebesgue measure if $L = [0, 1]$, which is 1 when $e \in F$).

Note that defining a probability measure on L^+ changes a gradual set into a random set and a gradual element into a probability distribution. The latter fact emphasises the idea that a gradual element is not imprecise. If L is the unit interval, and $L_F = \{1 = \lambda_1 > \cdots > \lambda_k\}$, a canonical way of changing a fuzzy set into a random set is to assign probability mass $\lambda_i - \lambda_{i+1}$ to the cut $F_{\lambda_i}, \quad \forall i = 1\ldots k$. This idea goes back to Yager (1982), Goodman (1982), Dubois and Prade (1982). A similar procedure can be adopted for turning a gradual element e into a probability distribution, assigning probability $\lambda_i - \lambda_{i+1}$ to element $s_i = \mathcal{A}_e(\lambda_i)$. The meaning of this probability can be the degree of "stability" of representative element s_i for e. The gap between λ_i and λ_{i+1} then measures the reluctance to give up s_i for s_{i+1} as the proper representative of the gradual element e. For instance, this gap is of size 1 for a crisp element, indicating maximal stability. Then $\lambda_i - \lambda_{i+1}$ is the probability of picking s_i, in the sense that the more stable a crisp representative, the more likely it will be picked.

2.5 Fuzzy connectives and gradual sets

The next step is to show that connectives of fuzzy set theory are consistent with the notion of gradual element and gradual set. The union and intersection of gradual sets G_1 and G_2 can be defined by the classical union and intersection of representatives to the same degree[1]:

$$\mathcal{A}_{G_1\cup G_2}(\lambda) = \mathcal{A}_{G_1}(\lambda) \cup \mathcal{A}_{G_2}(\lambda);$$
$$\mathcal{A}_{G_1\cap G_2}(\lambda) = \mathcal{A}_{G_1}(\lambda) \cap \mathcal{A}_{G_2}(\lambda).$$

By construction, this definition is consistent with the usual idempotent fuzzy set connectives, namely if $F(G)$ is the fuzzy set induced by G, then:

$$\mu_{F(G_1\cup G_2)}(s) = \max(\mu_{F(G_1)}(s), \mu_{F(G_2)}(s));$$
$$\mu_{F(G_1\cap G_2)}(s) = \min(\mu_{F(G_1)}(s), \mu_{F(G_2)}(s)).$$

However one may also consider other connectives for gradual sets where unions

$$\mathcal{A}_{G_1\cup G_2}(\lambda, \nu) = \mathcal{A}_{G_1}(\lambda) \cup \mathcal{A}_{G2}(\nu),$$

for all $\lambda \in \mathrm{Dom}(\mathcal{A}_{G_1})$, $\nu \in \mathrm{Dom}(\mathcal{A}_{G_2})$, are computed, using $L \times L$ as a new relevance scale. Ultimately, the definition of a connective might depend on the design of a "correlation map" between the two gradual sets indicating which pairs of realizations of the gradual sets go together, the former definition pairing sets with equal representativeness, the latter accepting all pairs. This is the path to follow if other connectives than the idempotent ones are to be retrieved.

The complement G^c of a gradual set G can be defined levelwise as $\mathcal{A}_{G^c}(\lambda) = \mathcal{A}_G(\lambda)^c$, but this definition is not in agreement with fuzzy set complementation since the λ-cut of the fuzzy set F^c is not the complement of the λ-cut of F. In order to preserve consistency with the usual fuzzy complement, one must assume that the correspondence between representatives of G and of G^c is a negative correlation, and presuppose the existence of an order-reversing map neg on L exchanging 0 and 1. If $\mathcal{A}_{G^c}(\lambda)$ is defined as $\mathcal{A}_G(\nu)^c$, where $\nu = \inf\{\alpha > \mathrm{neg}(\lambda)\}$, then consistency with fuzzy set complementation can be restored. It comes down to a special case of permutation of cuts introduced by Ralescu (1992).

3 Examples and applications

There are many situations where gradual elements naturally appear. The first general situation is when extending scalar evaluation of sets by means of some index, like cardinality, measure, distance, and so on, to fuzzy sets. To this end, some try to evaluate an average over the cuts of the fuzzy set (using a Choquet integral for instance). Another path is to preserve a genuinely fuzzy index. More often than not, it has been assumed that if a set has a precise evaluation, a fuzzy set should have a fuzzy-valued evaluation interpreted as being

[1] However if assignment functions are partially defined and $\mathrm{Dom}(\mathcal{A}_{G_1}) \neq \mathrm{Dom}(\mathcal{A}_{G_2})$, we must consider the extensions $\mathcal{A}^*_{G_1}$ and $\mathcal{A}^*_{G_2}$ restricted to $\mathrm{Dom}(\mathcal{A}_{G_1}) \cup \mathrm{Dom}(\mathcal{A}_{G_2})$.

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imprecise. However, the above discussion does not suggest it: since a scalar evaluation of a set yields a precise number, the scalar evaluation of a fuzzy set should be a fuzzy (gradual but not imprecise) element in the range of the index. The same idea applies when extending the concept of optimum of a function on a domain to the optimum on a fuzzy domain. Finally, gradual numbers shed light on the nature of fuzzy intervals, and on the concept of defuzzification.

3.1 Fuzzy cardinality

Fuzzy-valued cardinality $\mathrm{CARD}(F)$ of a fuzzy set F on a finite set S was defined by Zadeh (1979) as a fuzzy subset of integers having membership function

$$\mu_{\mathrm{CARD}(F)}(n) = \sup\{\alpha, \mathrm{CARD}(F_\alpha) \geq n\}, \quad \forall n = 0, 1, 2, \ldots.$$

The fuzzy cardinality of fuzzy sets has been a topic of debate and many proposals appeared in the 1980s. See the monograph of Wygralak (1996) for a survey of various proposals. It is clear that the fuzzy-valued cardinality of a fuzzy set has been more often than not envisaged as another fuzzy set of integers representing various possible values of the actual cardinality of the fuzzy set (hence involving some imprecision, see for instance Dubois and Prade (1985) who interpret a fuzzy cardinality as a possibility distribution). However such a fuzzy set of integers has an extremely particular shape (strictly decreasing membership function on its support), and interpreting it as expressing a lack of knowledge about the cardinality of F is debatable, insofar as F is interpreted as a set having gradual boundaries (and not as an ill-known set).

On the contrary, $\mathrm{CARD}(F)$ is quite a refined description of the cardinality of F where the gradual nature of the set is reflected on the integer scale. In fact, fuzzy cardinality is naturally described by the following injective assignment function:

$$\mathcal{A}_{\mathrm{CARD}(F)}(\alpha) = \mathrm{CARD}(F_\alpha), \quad \forall \alpha \in L_F.$$

Integers are defined as cardinalities of (finite) sets. Hence we may claim that the fuzzy cardinality of a fuzzy set is precisely a gradual integer in the sense of a gradual element in the set of integers. For instance, the number of "young" employees in a firm is a gradual integer, if the fuzzy set "young" has a well-defined membership, which expresses a flexible (rather than ill-defined) query to a database. Delgado et al. (2002) consider the cardinality of a fuzzy set is a probability distribution on the set of natural integers where the probability of $\mathrm{CARD}(F) = i$ is of the form $\lambda_j - \lambda_{j+1}$ where the $\mathcal{A}_{\mathrm{CARD}(F)}(\lambda_j) = i$. This is also in the spirit of this paper, if we assume F is actually a consonant random set, so that its cardinality is a random integer.

Fuzzy relative cardinality $\mathrm{CARD}(F \mid C)$ can be treated likewise as a gradual number assigning to each membership value α the relative cardinality $\frac{\mathrm{CARD}(F_\alpha \cap C_\alpha)}{\mathrm{CARD}(C_\alpha)}$, the proposal by Delgado et al. (2002) being similar, but assuming a uniform probability distribution on [0, 1].

In the same vein, the fuzzy probability of a fuzzy event F is a gradual element $P(F)$ of the unit interval with assignment function:

$$\mathcal{A}_{P(F)}(\alpha) = P(F_\alpha), \quad \forall \alpha \in L_F,$$

rather than an imprecise (fuzzy interval-valued) probability.

The fuzzy Hausdorff distance between two fuzzy sets F and G generalises the Hausdorff distance d between sets: it can be viewed as the gradual number $d(F, G)$ with assignment function (Dubois and Prade 1984):

$$\mathcal{A}_{d(F,G)}(\lambda) = d(F_\lambda, G_\lambda),$$

rather than an imprecise distance. It is clear this assignment function has no special regularity. It is also true for the one of relative cardinality, contrary to fuzzy absolute cardinality. Such assignment functions can hardly be understood as fuzzy sets.

3.2 Gradual real numbers and fuzzy optimization

What is often called a fuzzy number is a fuzzy set of numbers whose cuts are intervals. Such fuzzy numbers account for both imprecision and fuzziness (regardless of whether their cores are reduced to a point or not). Their addition does not collapse to the regular addition on the real line when fuzziness is removed. It yields interval addition. Hence, the name "fuzzy number", used by many authors (including the authors of this paper) is debatable (even when its core reduces to a single number). This issue was a topic of (unresolved) debates in early Linz Seminars on Fuzzy sets between pure mathematicians and applied ones (see Klement 1979, pp. 139–140). In contrast, we here take it for granted that a "fuzzy real number" should be a gradual element of the real line, which should be a number for each relevance level. To avoid confusion we call it a *gradual* (*real*) *number*. Mathematically, a gradual real number $\tilde{r}$ can be modeled by a function $\mathcal{A}_{\tilde{r}}$ from the unit interval to the real line (and not the converse). Note that we do not require monotonicity of the function so that some gradual numbers cannot be interpreted as membership functions (a number would then sometimes have more than one membership degree…).

A monotonic and continuous gradual number is called a *fuzzy threshold* Dubois (1987). The idea is to model a fuzzy boundary between two regions of the real line. This fuzzy boundary is not an ill-known precise boundary, only a gradual one, i.e. it indicates only a gradual transition between two regions. Fuzzy thresholds are instrumental in fuzzy linear programming, for the definition of fuzzy linear constraints (Zimmermann 1976). Namely, linear constraints of the form $\sum_{i=1}^{n} a_i x_i \leq b$ become fuzzy by turning coefficient b into a

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fuzzy threshold. Note that this methodology does not introduce uncertainty in linear programming, only flexibility, by means of some gradual relaxation of the crisp constraint. The set of feasible solutions then becomes a fuzzy set, but the gradual boundary obtained by fuzzifying b is not a fuzzy set: it is a gradual number.

The notions of gradual number and gradual set actually shed some light on the ill-understood problem of maximizing a numerical function f over a fuzzy domain $\mathcal{D}$ of the real line, a topic at the heart of fuzzy mathematical programming. In their 1980 book (p. 102), Dubois and Prade (1980) notice that there are two approaches to this problem: the first approach stems from Bellman and Zadeh (1970); it consists in first turning the function f into a so-called maximizing set M, which is a fuzzy set obtained by rescaling the function f within the unit interval; then an optimal solution x^* of the fuzzy optimization problem is defined as one that maximizes $\min(\mu_{\mathcal{D}}(x), \mu_M(x))$. This approach may be questioned because the optimal solution crucially depends on the scaling method that enables the comparison of the degree of attainment of the goal, expressed by f, by a solution, and the degrees of feasibility of this solution. Yet, this is the methodology followed by the majority of researchers in fuzzy optimisation, starting with Tanaka et al. (1974), and Zimmermann (1976).

Quite in the same years, Orlovsky (1977) suggested quite a different approach. The idea was to define the optimum of f over $\mathcal{D}$ as a fuzzy set. Namely, for each $\lambda \in (0, 1]$, consider a supposedly unique solution x^*_λ maximizing f over the crisp domain $(\mathcal{D})_\lambda$. It is clear that, varying λ, a fuzzy optimal solution is obtained under the form of a gradual element x^* of $\mathcal{D}$ such that $\mathcal{A}_{x^*}(\lambda) = x_\lambda$. The set of fuzzy optimal solutions forms a gradual set, with assignment function $\mathcal{A}_{X^*}$ that assigns to each feasibility level λ the set X^*_λ of optimal solutions of f over the crisp domain $(\mathcal{D})_\lambda$. Note the sets X^*_λ are generally NOT nested. Orlovsky (1977) defines the fuzzy set of maximizing solutions precisely as the fuzzy set N induced by the gradual set X^* using Definition 4. The fuzzy optimum attained is then $f(N)$ defined using the extension principle. However it is clear that one could define this fuzzy maximum as the gradual set defined by $f(X^*_\lambda), \forall\lambda \in (0, 1]$. This view is also very close to the parametric programming approach to fuzzy linear programming first proposed by Chanas (1983).

3.3 The arithmetics of gradual numbers

Algebraic structures of numbers (like groups) should be preserved for the most part when moving from real numbers to gradual real numbers (while fuzzy intervals just preserve algebraic properties of intervals). Indeed, let $\tilde{r}_1$ and $\tilde{r}_2$ be two gradual numbers with assignment functions $\mathcal{A}_1$ and $\mathcal{A}_2$ (mappings from (0, 1] to the reals). Any operation $*$ between reals can be extended to gradual numbers as follows: $\tilde{r}_1 * \tilde{r}_2$ has assignment function $\mathcal{A}_{1*2}$ such that

$$\forall\lambda \in (0, 1], \mathcal{A}_{1*2}(\lambda) = \mathcal{A}_1(\lambda) * \mathcal{A}_2(\lambda).$$

Under this definition, it is obvious that, for instance, the set of gradual numbers forms an Abelian group for the addition, and that regular inverses exist $(\mathcal{A}_{-\tilde{r}}(\lambda) = -\mathcal{A}_{\tilde{r}}(\lambda))$. However it is clear that extended operations performed on (monotonic) fuzzy thresholds are not closed: if $\tilde{r}_1$ and $\tilde{r}_2$ are monotonically increasing, $\tilde{r}_1 - \tilde{r}_2$ may not be so (hence the necessity not to restrict to monotonic gradual numbers).

Note that arithmetic operations extended to gradual integers can solve the following paradox: the set difference $A \setminus A$ is the empty set, as is the fuzzy set difference $F \setminus F$ computed for instance as $F \cap F^c$ using Lukasiewicz conjunctions. The cardinality of $A \setminus A$ is $\mathrm{CARD}(A) - \mathrm{CARD}(A) = 0$. So should be $\mathrm{CARD}(F) - \mathrm{CARD}(F)$. It is clear that considering the fuzzy cardinality $\mathrm{CARD}(F)$ as a fuzzy set of integers and applying the extension principle to compute $\mathrm{CARD}(F) - \mathrm{CARD}(F)$ yields a symmetric fuzzy set of integers around 0, which is counterintuitive. But viewed as an operation between gradual integers, $\mathrm{CARD}(F) - \mathrm{CARD}(F) = 0$. In fact, we can retrieve the additivity property for fuzzy cardinality i.e.

$$\mathrm{CARD}(F_1) + \mathrm{CARD}(F_2) = \mathrm{CARD}(F_1 \cup F_2) + \mathrm{CARD}(F_1 \cap F_2),$$

and this is still equivalent to

$$\mathrm{CARD}(F_1) + \mathrm{CARD}(F_2) - \mathrm{CARD}(F_1 \cup F_2) - \mathrm{CARD}(F_1 \cap F_2) = 0.$$

Moreover, if we consider (in the spirit of Rocacher and Bosc 2003) the set of "natural gradual integers" as all gradual elements on the set of integers of the form $\mathrm{CARD}(F)$ for some finite fuzzy set F (hence assignment functions are decreasing), the set of gradual elements of the relative integers (gradual relative integers for short) can be obtained as $\mathrm{CARD}(F) - \mathrm{CARD}(G)$ for some finite fuzzy sets F and G. To see it, it is enough to consider a gradual relative integer z as a sequence of (usual) relative integers $k_1, \ldots, k_n$, such that $\mathcal{A}_z(k_i) = \lambda_i$ and to notice the following result:

Proposition 2 *Let $k_1, \ldots, k_n$ be any finite sequence of relative integers. There exist two increasing sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ of non-negative integers, such that $\forall i = 1 \ldots n, k_i = x_i - y_i$.*

Proof Define $x_1 = \max(k_1, 0)$ and $y_1 = \max(-k_1, 0)$ and, for $i > 1$,

$$y_i = y_{i-1} + 1 + \max(0, k_{i-1} - k_i),$$

and of course $x_i = y_i + k_i$. Clearly, $y_1 \geq 0, y_i > y_{i-1}, \forall i > 1$ as well. For x_i, a simple recursion will do. Clearly,

$x_2 > x_1 \geq 0$. Assume $x_k > 0$ for $k = 2, \ldots i-1$. Then

$x_i = x_{i-1} - k_{i-1} + k_i + 1 + \max(0, k_{i-1} - k_i)$.

If $k_{i-1} - k_i < 0$ then $x_i = x_{i-1} - k_{i-1} + k_i + 1 > x_{i-1}$.
If $k_{i-1} - k_i \geq 0$ then $x_i = x_{i-1} + 1 > x_{i-1}$. □

For any fuzzy relative integer $\tilde{z}$, the sequences $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ define gradual natural integers $\tilde{x}$ and $\tilde{y}$ such that $\tilde{z} = \tilde{x} - \tilde{y}$. In this sense, the set of gradual natural integers (understood as cardinalities of finite fuzzy sets) generates all gradual relative integers, via a canonical subtraction.

3.4 Fuzzy intervals as crisp intervals of gradual numbers

A fuzzy interval M is a normalized fuzzy set of reals with membership function μ_M such that

- its core (the 1-cut of M) is a closed interval $[m^-, m^+]$;
- its support $\{x | \mu_M(x) > 0\}$ is an open interval;
- it has a non-decreasing part, denoted μ_{M^-}, on $(-\infty, m^-]$;
- it has a non-increasing part, denoted μ_{M^+}, on $[m^+, +\infty)$.

The notion of gradual element enables a fuzzy interval to be defined as a pair of fuzzy thresholds having opposite monotonicities, and bounding gradual numbers, just as an interval is modeled by an ordered pair of numbers that are its bounds. In Dubois et al. (2004), such fuzzy thresholds are called profiles, which we consider here as genuine gradual numbers. Viewing a fuzzy interval as a fuzzy set of reals limited by two gradual numbers, enables classical interval analysis Moore (1979) to be directly applied to fuzzy intervals.

As $[a, b]$ stands for the set $\{r : a \leq r \leq b\}$, a fuzzy interval M can be defined by an ordered pair of (monotonic) gradual numbers $(\tilde{m}^-, \tilde{m}^+)$, with

$$\mathcal{A}_{\tilde{m}^-} = (\mu_{M^-})^{-1}, \quad \mathcal{A}_{\tilde{m}^+} = (\mu_{M^+})^{-1}$$

M is viewed as the crisp interval $[\tilde{m}^-, \tilde{m}^+]$ in a space of functions, and stands for a crisp interval of gradual numbers $\{\tilde{r} : (\mu_{M^-})^{-1} \leq \mathcal{A}_{\tilde{r}} \leq (\mu_{M^+})^{-1}\}$.

In standard interval analysis, the problem is to find the lower and the upper bounds of a function $\varphi(x, y, z, \ldots)$ when $x, y, z, \ldots$ range over intervals $I, J, K, \ldots$. The tuples of values called extreme configurations (the vertices of the hyper-rectangle $I \times J \times K \times \ldots$) play a decisive role as candidates for being tuples of values for which the optima of the function f are reached, when the function is locally monotonic (all functions obtained from φ by fixing all variables but one are monotonic). For instance consider product of (not necessarily positive) intervals

$$[a, b] \cdot [c, d] = \{(x \cdot y) : x \in [a, b], y \in [c, d]\}$$

The properties of product imply that we can restrict the computation of $x \cdot y$ to four extreme configurations (a, c), (b, c), (a, d), (b, d):

$$[a, b] \cdot [c, d] = [\min(a \cdot c, b \cdot c, a \cdot d, b \cdot d), \max(a \cdot c, b \cdot c, a \cdot d, b \cdot d)].$$

The product $A \cdot B$ of two fuzzy intervals A and B is defined by the extension principle of Zadeh (φ = product): $\mu_{A \cdot B}(z) =$

$$\left\{ \begin{array}{ll} \sup_{(x,y):z=x\cdot y} \min(\mu_A(x), \mu_B(y)) & \text{if } \exists (x, y) : z = x \cdot y \\ 0 & \text{otherwise} \end{array} \right\}$$

The usual method of fuzzy interval analysis is to perform regular interval analysis on α-cuts. Applying interval analysis to fuzzy intervals viewed as intervals of gradual numbers can now be envisaged: $[\tilde{a}^-, \tilde{a}^+] \cdot [\tilde{b}^-, \tilde{b}^+] =$

$$\{(\tilde{r} \cdot \tilde{s}) : (\mu_{A^-})^{-1} \leq \mathcal{A}_{\tilde{r}} \leq (\mu_{A^+})^{-1}, (\mu_{B^-})^{-1} \leq \mathcal{A}_{\tilde{s}} \leq (\mu_{B^+})^{-1}\}, \quad (3)$$

where $\tilde{r} \cdot \tilde{s}$ has an assignment function $\mathcal{A}_{\tilde{r}} \cdot \mathcal{A}_{\tilde{s}}$. It it is clear that the right hand side of Eq. (3) comes down to performing classical interval analysis at each level $\lambda \in (0, 1]$. From Nguyen (1978) theorem, under weak assumptions $(A \cdot B)_\lambda = A_\lambda \cdot B_\lambda$. Hence $A \cdot B$ exactly coincides with $[\tilde{a}^-, \tilde{a}^+] \cdot [\tilde{b}^-, \tilde{b}^+]$, in the sense of classical interval analysis. More generally, finding $f(A, B)$ requires optimization over a set of gradual numbers so as to find their fuzzy boundaries. For locally monotonic functions, it is possible to restrict computations to the fuzzy boundaries of fuzzy interval arguments (extreme fuzzy configurations). For product:

- Compute $\mathcal{A}_{\tilde{a}^-} \cdot \mathcal{A}_{\tilde{b}^-}$, $\mathcal{A}_{\tilde{a}^-} \cdot \mathcal{A}_{\tilde{b}^+}$, $\mathcal{A}_{\tilde{a}^+} \cdot \mathcal{A}_{\tilde{b}^-}$, $\mathcal{A}_{\tilde{a}^+} \cdot \mathcal{A}_{\tilde{b}^+}$.
- $A \cdot B$ is the fuzzy hull of these gradual numbers, obtained by applying the expression providing the optimal bounds in the interval case, replacing numbers by gradual numbers obtained above.

Some partial results may fail to be monotonic Dubois et al. (2004), even if when putting all partial results together, a genuine fuzzy interval is obtained in the end, due to the equivalence between the interval analysis approach and the extension principle.

Even if non-monotonic profiles appear as intermediary results in the computation, the above approach still avoids the pitfall of ending up with anomalous membership functions (like the anti-fuzzy numbers of Goetschel 1997) due to a definition of subtraction such that $M + N = Q$ if and only if $M = Q - N$, between fuzzy intervals. For instance, the fuzzy interval bounded by the pair $(a, \tilde{r})$ where a is a real number and $\mathcal{A}_{\tilde{r}}(\lambda) = a + (1 - \lambda)b$ (with $b > 0$), should not be confused with the gradual number $\tilde{r}$ itself, when it comes to performing subtraction. Indeed, while $\tilde{r} - \tilde{r} = 0$, interval analysis yields:

$$[a, \tilde{r}] - [a, \tilde{r}] = [a - \tilde{r}, \tilde{r} - a],$$

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which is an imprecise zero, but not zero. This is because $[a, \tilde{r}]$ is actually the fuzzy interval generated by the gradual real number $\tilde{r}$, as per Definition 4.

3.5 Defuzzification

A gradual number $\tilde{r}$ can be defuzzified and it yields a regular number that "summarizes" it. The most obvious candidate defuzzification method is the Riemann integral

$$r = \int_0^1 \mathcal{A}_{\tilde{r}}(\alpha) d\alpha.$$

It comes down to using the Lebesgue measure on the unit interval. For instance, defuzzifying the fuzzy cardinality CARD(F) of a finite fuzzy set F yields its usual scalar cardinality, namely:

$$\int_0^1 \mathcal{A}_{\mathrm{CARD(F)}}(\alpha) d\alpha = \sum_{s \in S} \mu_F(s).$$

Now, we can reconsider the problem of "defuzzifying" a fuzzy set of real numbers, understood as selecting a representative number for it. This vocabulary is not appropriate as explained earlier. The notion of mean interval of a fuzzy interval M was defined as follows (Dubois and Prade 1987): consider M as a pair of distribution functions (F_*, F^*) where

$$F^*(x) = \mu_M(x) \text{ for } \quad x \leq \inf c(M)$$

($c(M)$ is the core of M with membership value 1), and

$$F_*(x) = 1 - \mu_M(x) \text{ for } \quad x \geq \sup c(M).$$

The mean interval is $E(M) = [E_*(M), E^*(M)]$ where $E_*(M)$ (resp. $E^*(M)$) is the expectation of the probability function with cumulative distribution F^* (resp. F_*). This definition is justified from different points of view, as producing the upper and lower expectations of the set of probability functions dominated by the possibility measure induced by M Dubois and Prade (1987), but also as the mean α-cut obtained via an Aumann integral of the set-valued map associated to M (to each $\alpha \in (0, 1]$ assign the α-cut M_α, i.e., a gradual set; see Ralescu 2002). This set-valued average is linear with the fuzzy addition and scalar multiplication. It corresponds to stripping M from its fuzziness, not of its imprecision. It is, literally, a defuzzification. The next step is to select a number in $E(M)$ (for instance the mid-point, by symmetry; seeer Yager 1981). It provides a method for choosing a number representing a fuzzy set that is more natural than the center of area and the like. Using the notion of gradual number, one can exchange the steps of (genuine) defuzzification and selection. We can strip M from its imprecision, by selecting a gradual number $\tilde{r}(M)$ in M, and then we can defuzzify $\tilde{r}(M)$. A natural selection, in agreement with the symmetry argument is to pick the mid-point m_α of all α-cuts of M, and it defines a gradual number $\tilde{m}(M)$. Its (generally not monotonic) assignment function is $\mathcal{A}_{\tilde{m}(M)}(\alpha) = m_\alpha$. Now, we can defuzzify it, using the Riemann integral as suggested above. It is obvious that the obtained value $\underline{\mathrm{m}}(M)$ is also the mid-point of the mean interval (or average cut), i.e.

$$\underline{\mathrm{m}}(M) = (E^*(M) + E * (M))/2.$$

It follows that in terms of gradual numbers,

$$\tilde{m}(M + N) = \tilde{m}(M) + \tilde{m}(N).$$

Besides, the defuzzified $\underline{\mathrm{m}}(M)$ is also equal to the mean value of the probability distribution obtained by randomizing the fuzzy number (in the style of Chanas and Nowakowski 1988): picking an element α at random in (0, 1] and then a number at random in $[m_\alpha^-, m_\alpha^+]$: the obtained probability is the center of mass of the polyhedron restricting the set of probability functions induced by M. It is the random number obtained via the gradual number $\underline{\mathrm{m}}(M)$ when equipping the unit interval with a uniform probability distribution (also the Shapley value Shapley (1953) of the "unanimity game" generated by M).

4 Related works

Mathematicians of fuzzy sets in the past have introduced the notion of a "fuzzy real number", starting with Hutton (1975). Often, it takes the form of a decreasing mapping from the reals to the unit interval or a suitable lattice (Gantner et al. 1978), or a probability distribution function (Lowen 1996). Arithmetic operations on fuzzy reals were studied by Rodabaugh (1982), and contrast with fuzzy arithmetics based on the extension principle. Hoehle (1987) especially emphasized the role of fuzzy real numbers as modeling a fuzzy threshold softening the notion of Dedekind cut.

Recently, Rocacher and Bosc (2003) suggested to define what they call fuzzy natural integers as fuzzy cardinalities of fuzzy sets, defined as proposed by Zadeh, but interpreted differently. Their fuzzy natural integer is then modeled by a (monotonically decreasing) membership function from the natural integers to the unit interval. In order to make it clear that no uncertainty is involved, these authors suggest viewing such natural fuzzy integers as conjunctive sets (as opposed to possibility distributions, which are disjunctive sets whose elements are mutually exclusive, thus expressing a lack of knowledge). However fuzzy conjunctive sets are used to model fuzzy multivalued attributes Dubois and Prade (1988). We suggest that fuzzy cardinalities could as well be viewed as gradual integers, and are no sets at all.

Journal: 500 MS: 187 Article No.: 0187 ☐ TYPESET ☐ DISK ☐ LE ☐ CP Disp.:2007/5/8 Pages: 11

These authors then define fuzzy relative integers Rocacher and Bosc (2003) as equivalence classes of pairs of fuzzy integers $(\tilde{n}_1, \tilde{n}_2)$ such that $\tilde{n}_1 + \tilde{z} = \tilde{n}_2$. Fuzzy relative integers are no longer monotonic, nor membership functions, generally. In their paper, Rocacher and Bosc (2005) Rocacher and Bosc also build (what he calls) fuzzy rational numbers from fuzzy relative numbers in a similar way. This view is totally along the line discussed above. The idea of gradual set proposed here is just an extension of the definition of fuzzy sets in terms of α-cuts, dropping the nestedness condition. This view was first systematically explored by Negoita and Ralescu (1975) quite early (see Ralescu 1992 for more advanced considerations), and gradual sets were proposed by Goetschel (1997), under the name "fuzzy level sets".

The idea that a fuzzy interval is a pair of fuzzy thresholds or profiles is akin to the so-called graded numbers of Herencia and Lamata (1999) and the fuzzy darts of Goetschel (1997). These authors also consider mappings from the unit interval to the real line, instead of the usual mapping from the reals to the unit interval. A fuzzy interval is then viewed as a pair of such mappings. However, our gradual reals are more general because they are not necessarily monotonic. In fact, the very technique for deriving closed-form formulas for fuzzy arithmetic operations on L–R fuzzy intervals (see Dubois and Prade 1978, 1980) does rely on the separate treatment of left and right-hand sides of fuzzy intervals, applying the operations on the corresponding fuzzy thresholds. This technique is generalized to fuzzy interval analysis in Dubois et al. (2004).

This view of fuzzy numbers as pairs of functions from the unit intervals to the real line, is also the basics of the recent proposal by Kolesnik et al. (2004) and Kosinski et al. (2005). However, these authors propose a calculus that moves away from the extension principle and interval analysis. Especially these authors consider it natural that the Abelian group structure be preserved for addition and multiplication of fuzzy intervals. Demanding that $[a, b] - [a, b] = 0$ leads to define subtraction as $[a, b] - [c, d] = [a - c, b - d]$, with the anomaly that sometimes $a - c > b - d$. However, these authors argue that such entities make sense. Actually the extension of intervals to entities called generalized intervals $[a, b]$ where $a > b$ was proposed in the literature by Kaucher (1980), and they play some role in the solution of interval equations. In this vein, Kolesnik et al. (2004) and Kosinski et al. (2005) consider so-called "ordered fuzzy numbers" that are fuzzy intervals plus an orientation along the membership function from one end of the support to the other. They are viewed as pairs of oriented assignment functions. The calculations on ordered fuzzy numbers proposed by the Kosinski group combine assignment functions having the same orientation and provide results than can no longer be interpreted as fuzzy sets.

5 Conclusion

This paper introduces a new concept in fuzzy set theory, namely that of a gradual element. It seems that such a concept was missing in the theory. More exactly, this notion pervades parts of the fuzzy set literature without being given a proper name, as shown by the many examples provided above. Although of an abstract nature, we think that gradual elements are a crucial concept for understanding the nature of fuzziness introduced by Zadeh as concepts being a matter of degree, what can be named "graduality", (and called "membership gradience" by Lakoff (1987)), thus giving up Booleanity, as opposed to the idea of partial or incomplete information. Since sets are used for representing incomplete knowledge, fuzzy sets often capture both ideas of graduality and partial information at the same time (as in possibility theory). This has created confusion between fuzziness and uncertainty, sometimes leading to debatable developments in the theory or the applications of fuzzy sets. The merit of gradual elements is that they only embody the idea of being a matter of degree. Some applications of this concept have been surveyed, especially the notion of fuzzy real number that can be instrumental for developing a genuine fuzzy interval analysis as well as sound defuzzification procedures. Other applications of the new concept to fuzzy cardinality are relevant for a better handling of quantifiers in fuzzy queries to databases.

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