Terminological Difficulties in Fuzzy Set Theory – The Case of “Intuitionistic Fuzzy Sets.”

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Abstract: *This note points out a terminological clash between Atanassov's “intuitionistic fuzzy sets” and what is currently understood as intuitionistic logic. They differ both by their motivations and their underlying mathematical structure. Furthermore, Atanassov's construct is isomorphic to interval-valued fuzzy sets and other similar notions, even if their interpretive settings and motivation are quite different, the latter capturing the idea of ill-known membership grade, while the former starts from the idea of evaluating degrees of membership and non-membership independently. This paper is a plea for a clarification of terminology, based on mathematical resemblances and the comparison of motivations between “intuitionistic fuzzy sets” and other theories .*

1. Introduction

Mathematical objects introduced by Atanassov [1, 2, 3] and studied under the name “intuitionistic fuzzy sets” (IFS) have become a popular topic of investigation in the fuzzy set community. The first public statement of this notion was made in 1983 [1], and the first widely accessible reference was published in 1986 [2]. An intuitionistic fuzzy set in the sense of Atanassov is defined by a pair of membership functions (F^+, F^-) denoted by IF, where $F^+(u)$ is the degree of membership of u in IF and $F^-(u)$ is its degree of non-membership. It is worth pointing out that this may be seen as a fuzzification of the idea of sub-definite set, introduced some years before by Narin′yani [30] who separately handles the (ordinary) set F^+ of elements known as belonging to the sub-definite set and the (ordinary) set F^- of elements known as not belonging to it, with the condition $F^+ \cap F^- = \varnothing$ (together with some bounds on the cardinalities of F^+ and F^-). Such a condition is extended to the two membership functions F^+ and F^-, which for IFS are supposed to verify the constraint

$$F^+(u) + F^-(u) \leq 1. \qquad (1)$$

Mind that what become gradual here are the membership degrees, not the uncertainty about membership. In other words, $F^+(u)$ is viewed as a lower bound on membership, and

$F^{-}(u)$ is a lower bound on non-membership. The basic intuitionistic fuzzy set-theoretic (or logical) operations for such IFSs are proposed as follows

Conjunction : $IF \cap IG(u) = (\min(F^{+}(u),G^{+}(u)), \max(F^{-}(u),G^{-}(u)))$;

Disjunction : $IF \cup IG(u) = (\max(F^{+}(u),G^{+}(u)), \min(F^{-}(u),G^{-}(u)))$;

Negation : $IF^{c}(u) = (F^{-}(u), F^{+}(u))$.

On this basis many subsequent papers have been written, developing set-theoretic as well as logical aspects of the theory.

2. Are (Atanassov′s) intuitionistic fuzzy sets intuitionistic?

A so-called "Intuitionistic fuzzy set theory" was independently introduced by Takeuti and Titani [16] as a set theory developed in (a kind of) intuitionistic logic. Takeuti-Titani′s intuitionitic fuzzy logic is simply a extension of intuitionistic logic[36], i.e. all formulas provable in the intuitionistic logic are provable in their logic. They give a sequent calculus which extends Heyting intuitionistic logic LJ, an extension that does not collapse to classical logic and keeps the flavour of intuitionism.

The name "intuitionistic" in Atanassov′s theory of IFSs was most probably motivated by the inequality (1) which is supposed to express the rejection of the excluded middle law, like in intuitionistic logic. Such mathematical objects look reasonable and interesting from the point of view of the theory of fuzzy sets as well as from application viewpoints, but it can be argued that the name "intuitionistic fuzzy sets" (IFSs) for Atanassov theory is unsuitable and misleading, at least for the following three reasons:

1. Intuitionistic fuzzy set theory by Takeuti and Titani [35] is an absolutely legitimate approach, in the scope of intuitionistic logic, but it has nothing to do with Atanassov′s intuitionistic fuzzy sets.
2. As well known, the law of excluded middle is not valid in fuzzy logic in general; so, the fact that IFSs do not satisfy it is no sufficient reason to use the name "intuitionistic". Much worse, as analyzed in the paper by Cattaneo and Ciucci [9], the connectives of IFS theory violate properties of intuitionistic logic by validating the double negation (involution) axiom $IF^{cc}=IF$ which is not valid in intuitionistic logic. (Recall that axioms of intuitionistic logic extended by the axiom of double negation imply classical logic, and thus imply excluded middle; see e.g. Kleene [28]). On the other hand, the axiom of non-contradiction $A \cap A^{c}=0$ is valid in intuitionistic logic but it is not satisfied in IFS theory by the commonly used involutive negation.
3. Finally, the philosophical ideas behind intuitionism in general, and intuitionistic mathematics and intuitionistic logic in particular have a strong tendency toward constructivist points of view. There are no relationship between these ideas and the basic intuitive ideas of IFS theory.

The main objection to the terminology used by Atanassov is the fact that he calls "intuitionistic fuzzy set theory" something which accepts rules and principles (as double negation) that, added to the intuitionistic logic, make it classical, i.e. nothing from intuitionism remains. Calling the Atanassov theory intuitionistic leads to a misunderstanding.

3. Mathematical similarities between interval-valued fuzzy sets and intuitionistic fuzzy sets

The older notion of interval-valued fuzzy sets was introduced independently by Zadeh [41], Grattan-Guiness [23], Jahn [26], Sambuc [33], in the seventies, in the same year. An interval-valued fuzzy set (IVF) is defined by an interval–valued membership function: a mapping F from the universe U to the set of closed intervals in [0, 1]. Let $F(u) = [F_*(u), F^*(u)]$. The union, intersection and complementation of IVFs is obtained by canonically extending fuzzy set-theoretic operations to intervals. As such operations are monotonic, this step is mathematically obvious. For instance, the most elementary fuzzy set operations are extended as follows:

Conjunction : $F \cap G(u) = [\min(F_*(u), G_*(u)), \min(F^*(u), G^*(u))]$;

Disjunction : $F \cup G(u) = [\max(F_*(u), G_*(u)), \max(F^*(u), G^*(u))]$;

Negation : $F^c(u) = [1 - F^*(u), 1 - F_*(u)]$.

IVFs are a special case of L-fuzzy sets in the sense of Goguen [21] and a special case of type 2 fuzzy set (also introduced by Zadeh [41]). See Walker and Walker [37] for a careful study of connectives for type 2 fuzzy sets; their results apply to the special case of IVFs.

The use of an involutive negation acting on the pair of membership and non-membership functions make IFS theory formally collapse to IVF theory. Indeed, constraint (1) always guarantees the existence of the membership degree interval $[F^+(u), 1- F^-(u)]$, which can thus be identified with $[F_*(u), F^*(u)]$, and the set-theoretic operations defined for IFS agree with the standard extension of basic fuzzy set connectives to interval-valued membership recalled above. For instance, negation in IFS theory becomes the above complementation to 1 extended to intervals, i.e., $1- [F^+(u), 1- F^-(u)] = [F^-(u), 1- F^+(u)]$. The same holds for min and max connectives. This collapse was already noticed by Atanassov and Gargov [5] in the eighties, and later emphasized by several scholars (e.g. Dubois et al. [16], G. Deschrijver, E. Kerre [12], Cornelis et al. [10]) even if the initial intuitions behind IFS theory and interval-valued fuzzy sets differ. The formal study of IFS can thus also naturally be cast in the wider setting of lattice-valued fuzzy sets (Wang and He [38], Gutierrez-Garcia and Rodabaugh [24]).

The above mathematical equivalence of IFSs with interval-valued fuzzy sets confirm that F^+ and F^- can be respectively viewed as lower bounds on the membership function of a regular fuzzy set F and its regular complement $1 - F$ (a subdefinite fuzzy set, in the terminology of Narin′yani [30]). Then, F^+ and F^- represent incomplete knowledge about F, which explains that one may have two IFSs (F^+, F^-) and (G^+, G^-) with for instance $F^+ = G^+$ but $F^- \neq G^-$ (as two different pieces of knowledge about F). So, the temptation to see the pair (F^+, F^-) as made of a well-defined fuzzy set and its (independently, but uniquely defined) non-truth-functional complement is hard to sustain. This formal equivalence between IFSs and IVFs confirms that IFSs have little to do with the algebraic structure of intuitionistic logic.

Remark : Interestingly, interval-valued fuzzy sets were several times reintroduced later on, under yet other names, like vague sets [20, 7], and grey fuzzy sets (used in modelling and decision [25]). In fact, grey sets [13] are another name for many-valued quantities introduced by Young [40] and grey set theory often reduces to interval analysis pioneered by Ramon Moore [29]. So grey sets are not interval-valued fuzzy sets (Yang and John [39]) but grey membership functions come down to interval-valued fuzzy sets most of the time [18]. Adopting a unique terminology for the interval-valued fuzzy sets would also avoid the case of scholars proving the same results in different settings.

4. On Specific Practical Motivations for Atanassov Theory

One should of course bear in mind that a mathematical equivalence is one thing, and what a particular mathematical concept or property may model (its semantics) is another thing, and clearly the latter is what matters more for applications. For instance, in the case of interval-valued fuzzy sets, the idea is that membership grades can hardly be precise. As fuzzy sets are supposed to model ill-defined concepts, some scientists have argued that requiring precision in membership grades may sound paradoxical. Although this view could be challenged, it naturally leads to interval-valued fuzzy sets in a first step of departure away from standard fuzzy sets. Indeed it is a long tradition in economics, engineering, etc., that intervals were used to represent values of quantities in case of uncertainty. For instance, dealing with uncertain possibilistic information about the potential elements of an ill-known set, give birth to a special kind of interval-valued fuzzy set named twofold fuzzy set [17]. A rough set [31] is a kind of interval-valued set induced by indiscernibility of elements.

However, the membership and non-membership degrees in the IFSs may represent something else, namely the idea that concepts are more naturally approached by separately envisaging positive and negative instances. It leads to the idea of loosely related membership and nonmembership functions. This idea of positive vs. negative information is actually confirmed by psychological investigations [8], and is currently studied in various domains of information engineering including preference modelling, learning, and reasoning (see for instance Grabish[22] Dubois et al.[14], and papers from two special sessions at the IPMU 2004 conference in Perugia [15]) under the term “bipolarity”.

In Szmidt and Kacprzyk [34] the IFSs were illustrated on a voting example when “yes”, “no” and “abstain” votes are possible. Abstention votes may be accounted for by means of the IFSs if they are viewed to represent votes that are “unclassifiable”. This is in line with, for instance, Kang [27] who presents, and experimentally verifies, a novel model of abstention as an expression of a voter′s discomfort with particular political options, not as an expression of a voter′s uncertainty about what to do (how to vote). So, we can say that an abstention vote is an “unclassifiable” vote, i.e. which may be represented by an IFS, not the one that expresses “uncertainty”, i.e. which may be represented by an interval-valued fuzzy set. Of course, this only means that a “pure” IFS may be a more intuitive model in a particular situation than its equivalent interval-valued formulation. But, again, such intuitions are at odds with the intuitionistic tradition even if one feels they can be perfectly legitimate.

5. Towards a Modified Terminology

For the above reasons, the term “intuitionistic fuzzy sets”, and still worse “intuitionistic fuzzy logic” in the sense of [1, 2], *turns out to be unjustified, misleading, and possibly offensive to people in intuitionistic mathematics and logic*, let alone its clashing with the correct usage of “intuitionistic fuzzy logic” by Takeuti and Titani [35] and those building on their work (Baaz & Zach [6], also Fermueller and Preining [19]).

The clarification of formal matters and nomenclature related to Atanassov theory, namely the representation of imprecise concepts via membership and non-membership degrees that do not sum up to 1, is useful for a proper development of applications. Applications of IFS appear more and more frequently in the literature, and some of them seem to be serious and claim good results. Of course, authors of papers coming from engineering, economics, etc., may be unaware of, or even may not care about the fact that the name of the formal apparatus employed is inappropriate. But they should be offered a "clean" name anyway.

In a paper [4] presented at the EUSFLAT′03 conference, it is argued that it is now too late to change the name, be it good or not, in view of a growing number of theoretical and applied papers. Unfortunately, we have to disagree with this argumentation. On the contrary, we believe that it would be crucial and helpful for a further development of Atanassov′s IFS theory and its applications to refrain from using the name "intuitionistic fuzzy sets" (and "intuitionistic fuzzy logic") when speaking about the IFSs.

To do it smoothly and painlessly, a good solution might be, for instance, to use the term I-fuzzy sets with "I" resembling the old inappropriate name but allowing also other interpretation like "interval", "imprecise", etc. However, in view of the above discussion on motivations, these names do not seem to faithfully capture the motivations of IFSs. In fact, since the term "bipolarity" seems to be agreed upon in some communities, from preference modelling to cognitive psychology, as capturing the separate handling of positive and negative aspects of information, one may suggest a more radical change and call IFSs "bipolar fuzzy sets". Besides there also exists the proposal of "neurostrophic logic" [33] which generalizes Atanassov idea by dropping the condition $F^{+}(u) + F^{-}(u) \leq 1$, so as to capture a form of contradiction. Ultimately of course, only a consensus among scientists will legitimate the final choice of a proper name.

A change of name has occurred quite frequently in mathematics, science and technology, and in this case the entire community, theoreticians and practitioners, would only benefit by having a clean point of departure to further works, free from accusations that terms, concepts and properties from already existing and established areas have not been taken into account. Simply speaking, this small change would make a big difference.

6. Conclusion

This short note is an attempt at finding a constructive solution to a dilemma concerning the appropriateness of the name "intuitionistic fuzzy set" in Atanassov theory. We think that, since this apparatus belongs to a broadly perceived area of tools and techniques for the representation of imperfect (uncertain, imprecise, vague, …) information, then its name, foundations, formal tools, etc., should be in line with the terminology in this area, and not borrow its name from another field (intuitionistic logic), which is also well-established, but currently deals with other matters. We feel that since Atanassov′s IFSs have without doubt an application potential, as shown by growing interest in diverse fields, such a terminological clarification would lead to a common and clean ground for both theoreticians and practitioners, who should use the same terms to denote the same things. It might avoid clashes between communities dealing with different matters under the same names, and promote interaction between communities dealing with similar topics but working separately, each with its own terminology, on topics where the main issue is the handling of imprecision, irrelevance or bipolarity notions for membership grades.

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