Qualitative decision making with bipolar information

Didier Dubois, Hélène Fargier
IRIT
118 route de Narbonne
31062 Toulouse Cedex 9, France
{dubois,fargier}@irit.fr

Abstract

Decisions can be evaluated by sets of positive and negative arguments — the problem is then to compare these sets. Studies in psychology have shown that in this case the scale of evaluation of decisions is generally bipolar. Moreover decisions are often made on the basis of an ordinal ranking of the arguments rather than on a genuine numerical evaluation of their degrees of attractiveness or rejection, hence the qualitative nature of the decision process in practice. In this paper, assuming bipolarity of evaluations and qualitative ratings, we present and axiomatically characterise two decision rules based on possibilistic order of magnitude reasoning that are capable of handling positive and negative affects. They are extensions of the maximin and maximax criteria to the bipolar case. A bipolar extension of possibility theory is thus obtained. In order to overcome the lack of discrimination power of the decision rules, refinements are also proposed, capturing both the efficiency principle and the idea of order of magnitude reasoning.

Introduction

It is shown from many experiements in psychology that our way of evaluating alternatives , objects and making decision is guided by bipolar scales of evaluation (**?**; **?**; **?**; **?**), i.e. considers simultaneously the positive and negative character of the alternatives. Decisions are moreover often made on the basis of an ordinal ranking of the strength of the arguments rather than on a numerical evaluation, hence the qualitative nature of the decision process (**?**).

The present work is a first temptative of formalization and axiomatic characterization of bipolar qualitative decision rules. We consider the simple situation where each possible decision d is assessed by a finite subset of arguments $\mathcal{C}(d) \subseteq X$. X is the set of all possible arguments: an argument is typically a criterion satisfied by d, a risk run by choosing d, a good, or a bad, consequence of d. Under this view, comparing decisions amounts to comparing sets of arguments, i.e. subsets A, B of 2^X. The point is that some arguments are positive, and thus attractive for the decision maker, while others are negative and should be avoided. For instance, when choosing a house, having a garden is a positive argument g^+, missing a house is a negative argument g^-. Being close to an airport is a negative argument a^-– but being far is not necessarily positive, and being to far is definitly negative – hence a second argument a'^-. So, comparing a house with a garden but close to the airport with one without garden and very far from the airport amount at comparing $\{g^+, a^-\}$ and $\{g^-, a'^-\}$. The final decision obviously depend on the strenght of the different elements of decision.

As said previously we focus in this paper on *qualitative* bipolar decision making, i.e. on models that rank decisions on the basis of an ordinal rather than on a numerical evaluation of their pros and cons. Among other motivations is the fact that the elicitation of the information required by a quantitative decision model is often not an easy task in practice. Another motivation is the genuine qualitativeness of human reasoning.

The handling of qualitative information in decision making is not a new question. The most famous decision rule of this kind is the maximin rule of Wald (**?**). It only presupposes that the arguments in X can be ranked in terms of merits by means of some utility function u mapping on any ordinal scale. Decisions are then ranked according to the merit of their worst arguments, following a pessimistic attitude. This approach captures the handling of negative affects. Purely positive decisions are sometimes separately handled in a symmetric way, namely on the basis of their best arguments. The case of ordinal ranking procedures using bipolar (both positive and negative) information has retained less attention. To the best of our knowledge, the only past work on this topic is in (**?**). They propose to merge all positive affects into a degree of satisfaction (using the max rule). If high, this degree does not play any role and the decision is made on the basis of the negative affects (using Wald's principle). If low, it is understood as a negative affect and merged with the other ones.

In the present paper, we follow a more systematic direction of research, characterising a set of procedures that are simultaneously ordinal and bipolar. Unsurprisingly, the reader will see that the corresponding decision rules are strongly related to possibility theory. This paper pursues an investigation started in (**?**) and proposes an improved decision rule and new representation theorems. Section will indeed present two basic qualitative and bipolar rules. We will see in Section how the basic properties of bipolar reasoning can be expressed axiomatically and going further, which axioms can capture the principle of qualitative bipolar reasoning. Efficient rules are studied in Section that are more decisive that the basic ones without going outside the qualitative requirement. The Section 2 is devoted to the background on monotonic set comparison.

Background

Comparing sets of more or less important elements is an old issue in uncertain reasoning, logics, measurement theory, etc. Let us first recall that, for any relation $\succeq$ on a power set 2^X, one can define its symmetric part ($A \sim B \iff A \succeq B$ and $B \succeq A$), its asymmetric part ($A \succ B \iff A \succeq B$ and $not(B \succeq A)$) and an incomparability relation: $A \asymp B \iff not(A \succeq B)$ and $not(B \succeq A)$. $\succeq$ is said to be *quasi-transitive* iff $\succ$ is transitive. $\succeq$ is a weak order iff it is complete and transitive. In the latter case, the comparison can obviously be can be captured by a monotonic set function or "capacity":

Definition 1 *A capacity on X is a mapping σ defined from 2^S to $[0,1]$ that is consistent ($\sigma(\emptyset) = 0$), non trivial ($\sigma(S) = 1$), and monotonic, i.e. such that: $\forall A, B \subseteq X, A \subseteq B \Rightarrow \sigma(A) \leq \sigma(B)$*

Capacities are meaningful in argument-based decision : if d is supported by a set of positive (resp. negative) arguments A ($\mathcal{C}(d) = A$), then this decision can be evaluated by means of $\sigma(A)$ — i.e. capacities suit the situations where all the elements of X are positive (resp. negative). The essence of capacities, i.e. the monotony principle saying that the larger the set, the higher its importance, is not tied to the use of a numerical measure nor to the use of a single scale of evaluation.

Consider for instance Dempster Shafer belief function – they evaluate the likelihood of events by an interval $[Bel(A), Pl(A)]$. $Pl(A)$ is a capacity and measures the plausibility of event A, while the capacity $Bel(A)$ its certainty. The comparison of events, given $A \succeq_{Bel,Pl} B \iff Bel(A) \geq PL(A)$ is thus an interval order, in the sense of the Ferrer relation, thus incomplete. It is nevertheless monotonic. (**?**; **?**) have extended the monotonicity principle to the relational framework, thus defining more general concept of *comparative capacity*:

Definition 2 *A relation $\succeq$ on a power set 2^X said to be positively monotonic (or "orderly") i.e. satisfies: $A \subseteq C, D \subseteq B, A \succeq B \Rightarrow C \succeq D$* [1].

A relation $\succeq$ on a power set 2^X is a comparative capacity iff it is reflexive, quasi-transitive, non-trivial ($X \succ \emptyset$) and positively monotonic

Contrary to numerical capacities, comparative capacities are not necessarily complete and transitive relations. For instance, the *discrimax order* ((**?**; see also (**?**)) relies on a rank function (possibility distribution) $\pi : X \mapsto [0,1]$. It is defined by: $A \succeq_{Discrimax} B$ iff $\Pi(A \setminus B) \geq \Pi(B \setminus A)$, where $\Pi(V) = max_{x \in V} \pi(x)$. This definition of $\succeq_{Discrimax}$ yields a complete but not fully transitive comparative capacity (indifference is not transitive). Another example is given by a family of probability distributions, say $\mathcal{F}$. The relation $A \succeq_{\mathcal{F}} B \iff \forall P \in \mathcal{F}, P(A) \geq P(B)$ yields a transitive but incomplete comparative capacity.

In the present paper, we aim at focusing on and at characterizing a particular class of bipolar decision making situations, namely the one that are qualitative rather than quantitative in essence. The next Section presents two basic rules that go this way. We will see in Section how the basic properties of bipolar capacities can be expressed axiomatically and going further, which axioms can capture the principle of qualitative reasonning. Efficient rules are studied in Section that are more decisive that the basic ones without going outside the qualitative requirement.

Two basic ordinal rules for comparing bipolar sets

For each decision d, let $\mathcal{C}(d)$ be the set of arguments relevant for d, including positive and negative ones. If any, argument irrelevant for d are outside $\mathcal{C}(d)$ For the sake of simplicity, we assume the positiveness (respectively the negativeness) of an argument is not a matter of degree. Hence we can suppose that X is divided into three disjoint subsets: X^+ is the set of positive arguments, X^- is the set of negative arguments, X^0 is the set of indifferent ones, if any. For a rich decision maker looking for a holiday accomodation, the price is not a relevant argument and belongs to X^0. But considering a house, having a garden is a positive argument (is in X^+) and that missing a garden is a negative argument (is in X^-). Obviously, only one of the two arguments belongs to $\mathcal{C}(d)$. Arguments that present both a positive and a negative aspect (for instance, being far from the main roads) can also be encompassed, through a similar duplication process : we will have two sub-arguments, one in X^- and one in X^+, that are either both present in $\mathcal{C}(d)$, or both absent of it.

So, any $A \subseteq X$ can thus also be partitioned in three disjoints subsets: let $A^+ = A \cap X^+$, $A^- = A \cap X^-$, $A^0 = A \cap X^0$ be respectively the positive, negative and indifferent arguments of A.

All the arguments obviously do not share the same degree of importance. In a pure qualitative, ordinal approach of decision making, let us suppose that the importance arguments is a transitive and complete notion, i.e. that it can be described on a totally ordered scale $L = [0_L, 1_L]$, e.g. by a function:

$$\pi : X \mapsto L = [0_L, 1_L]$$

$\pi(x) = 0_L$ means that the decision maker is indifferent to argument x ; 1_L is the highest level of attraction or repulsion (according to whether it applies to a positive or negative argument). π is supposed to be non trivial, i.e. at least one x receives a positive rating. Whenever $\pi(x) > \pi(y)$, the strength of x is considered at least one order of magnitude higher than the one of y, so that y is negligible in front of

[1] Remark that this implies the monotony of $\succeq$ implies the monotony of $\succ$, i.e. $A \subseteq C, D \subseteq B, A \succ B \Rightarrow C \succ D$

x. So the strength of A shall be measured by the following qualitative possibility measure:

$$\forall A \subseteq X, OM(A) = max_{x \in A} \pi(x)$$

$OM(A)$ reflects the order of magnitude of the sets of arguments A (irrespective of their signs) - hence the notation OM.

In order to ensure the indifferent arguments, i.e. the ones in X_0, do not affect the decision process, i.e. that $OM(A \cup \{x_0\}) = OM(A)$, we require that

$$\forall x_0 \in X^0 \iff \pi(x_0) = 0_L.$$

The Bipolar Qualitative Pareto Dominance

As said in the introduction, if all the arguments were negative, we could apply Wald's cautious principle, i.e. decide that (the object evaluated by) A is preferable to (the object evaluated by) B iff $OM(A) = OM(A^-) \leq OM(B) = OM(B^-)$. On the contrary, if all the arguments were positive, we could apply the optimistic rule, i.e. decide that A is preferable to B iff $OM(A) = OM(A^+) \geq OM(B) = OM(B^+)$. In the bipolar case, one could consider that each of the two scales defines a criterion, i.e. that the ranking of decisions does not rely on a single ordinal evaluation, like in the unipolar case, but on two ordinal evaluations, namely on the pair $(OM(A^+), OM(A^-))$. This yields the following Pareto-like rule (**?**), which does not assume commensurateness between the evaluation of positive and negative arguments:

$$A \succeq^{Pareto} B \iff \begin{array}{ll} & OM(A^+) \geq OM(B^+) \\ \text{and} & OM(A^-) \leq OM(B^-) \\ \text{where} & OM(V) = max_{x \in v} \pi(x) \end{array}$$

It is easy to see that $\succeq^{Pareto}$ is reflexive and transitive. Moreover:

- A is strictly preferred to B ($A \succ^{Pareto} B$) in two cases: either $OM(A^+) \geq OM(B^+)$ and $OM(A^-) < OM(B^-)$, or $OM(A^+) > OM(B^+)$ and $OM(A^-) \leq OM(B^-)$.
- A and B are indifferent when $OM(A^+) = OM(B^+)$ and $OM(A^-) = OM(B^-)$.
- There is a conflict when $OM(A^+) > OM(B^+)$ and $OM(A^-) > OM(B^-)$. Then A is not comparable with B: $\succeq^{Pareto}$ is a partial relation.

It is perhaps too partial: for instance, when $OM(A-) > OM(A^+)$, $\succeq^{Pareto}$ concludes that A is incomparable with $B = \emptyset$ and this even if the positiveness of A is negligible w.r.t its negativeness. In this case, one would rather say that getting A is bad and that getting nothing is preferable. Another drawback is observed when $OM(A^+) > OM(B^+)$ and $OM(A^-) = OM(B^-)$: the above definition enforces $A \succ^{Pareto} B$, and this even if $OM(A^+)$ is very weak w.r.t the order of magnitude of the negative arguments — in the latter case, a rational decider would examine the negative arguments in details before concluding.

The Bipolar Possibility Relation

The above decision rule does not account for the fact that the two evaluations that are used can *share a common scale*. The following new decision rule takes this commensurability into account. The principle at work here is simple: any argument against A (resp. against B) is an argument pro B (resp. pro A) and conversely, and the most supported decision is preferred:

$$A \succeq^{Poss} B \Leftrightarrow \begin{array}{l} max(OM(A^+), OM(B^-)) \\ \geq max(OM(B^+), OM(A^-)) \end{array}$$

In other terms, the rule focuses on the arguments of highest importance, deciding that A is at least as good as B iff, at that decision level, i.e. at level $OM(A \cup B)$ there are arguments in favor of A or arguments attacking B. Thus $A \succ^{Poss} B$ iff, at the highest level, there is at least a positive argument for A or an argument against B, but no negative argument against A and no positive argument pro B.

Importantly, $\succeq^{Poss}$ collapses to the max rule if $X = X^+ \cup X^0$ and to Wald's pessimistic rule if $X = X^- \cup X^0$, as $\succeq^{Pareto}$ also does. Like $\succeq^{Pareto}$ also, $\succeq^{Poss}$ satisfies the weak unanimity principle:

$$\begin{array}{ll} & OM(A^+) \geq OM(B^+) \\ \text{and} & OM(B^-) \geq OM(A^-) \end{array} \Rightarrow A \succeq^{Poss} B$$

But the converse implication is not valid. It may append that $A \sim^{Poss} B$ while $OM(A^+) > OM(B^+)$ and $OM(B^-) = OM(A^-)$

Finally, $\succeq^{Poss}$ is reflexive and quasi-transitive, and moreover complete (this is obviously not the case of $\succeq^{Pareto}$).

However, $\succeq^{Poss}$ and $\succeq^{Parero}$ are very rough rules that may be not decisive enough. In particular, if may happen that $A \subsetneq B$ while none of the rule is making a strict preference between A and B — the usual drowning effect of possibility theory reappears here.Variants will be presented in Section that overcome this difficulty. But the two rules have the advantage to capture the essence of ordinal decision making, as shown by the axiomatic study of the Section .

Remark

In a preliminary work (**?**), we have investigated a kind of dominance rule that is close to the Bipolar Possibility Relation. Like $\succeq^{Poss}$, it focuses on arguments of maximal strength, but applies a more restrictive dominance principle: A is at least as good as B iff, at level $OM(A \cup B)$ the existence of arguments in favor of B is counterbalanced by the existence of arguments in favor of A and the existence of arguments against A is cancelled by the existence of arguments against B. Formally:

Definition 3 $A \succeq^{DPoss}$ *iff :*

$$\begin{array}{lll} OM(B^+) = OM(A \cup B) & \Rightarrow & OM(A^+) = OM(B^+) \\ OM(A^-) = OM(A \cup B) & \Rightarrow & OM(B^-) = OM(A^-) \end{array}$$

$\succeq^{DPoss}$ is a reflexive and transitive rule but it is very often incomplete. For instance, any set containing a positive and a negative argument of the highest level, i.e. a conflicting set, is incomparable to any set of arguments of a lower level ($\succeq^{Poss}$ would rather conclude to an indifference). The rule is thus very interesting from a theoretic descriptive point of view, but such a large range of incomparability sounds difficult to use in practice.

Axioms for ordinal comparison on a bipolar scale

As usual in axiomatic characterizations, an abstract relation $\succeq$ on 2^X is considered and the natural properties that it should obey are formalised — here, the property that should be required for qualitative bipolar comparison relations. Let us begin by the general properties any bipolar procedure is succeptilble to obey

Axioms for monotonic bipolar set relations

The basic notion of bipolar reasoning over sets of arguments is the separation of X in good and bad arguments. The first axiom thus states that any argument is either positive or negative, i.e. better than nothing or worse than nothing:

Clarity of arguments $\forall x \in X, \{x\} \succeq \emptyset$ or $\emptyset \succeq \{x\}$

One can then trivially scale the elements of $\mathbb{X} = X \cup \{\emptyset\}$ and make a partition of them. Denoting $\succeq_{\mathbb{X}}$ the corresponding relation, we have:

$x \succeq_{\mathbb{X}} y \iff \{x\} \succeq \{y\}$ $\quad$ $X^+ = \{x, \{x\} \succ \emptyset\}$
$x \succeq_{\mathbb{X}} \emptyset \iff \{x\} \succeq \emptyset$ $\quad$ $X^- = \{x, \emptyset \succ \{x\}\}$
$\emptyset \succeq_{\mathbb{X}} x \iff \emptyset \succeq \{x\}$ $\quad$ $X^0 = \{x, \emptyset \sim \{x\}\}$

Now, arguments that are indifferent to the decision maker should obviously not affect the preference. This is the meaning of the next axiom:

Status quo consistency
If $\{x\} \sim \emptyset$ then $\forall A, B : A \succeq B \quad \Leftrightarrow \quad A \cup \{x\} \succeq B$
$\Leftrightarrow \quad A \succeq B \cup \{x\})$

The Status quo consistency axiom allows to forget about X_0.

Let us now discuss the property of monotonicity with respect to the addition of arguments. Monotonicity in the sense of Definition 2 can obviously not be obeyed as such in a bipolar scaling. Indeed, if B is a set of negative arguments, it generally happens that $A \succ A \cup B$. We rather need axioms of monotonicity *specific to* positive and negative arguments – basically, the one of bipolar capacities (**?**), expressed in a comparative way.

Positive monotonicity
$\forall C, C' \subseteq X^+, \forall A, B : A \succeq B \Rightarrow C \cup A \succeq B \setminus C'$

Negative monotonicity
$\forall C, C' \subseteq X^-, \forall A, B : A \succeq B \Rightarrow C \setminus A \succeq B \cup C'$

Now, the bipolar scale encodes all the relevant information, saying that only the positiveness and the negativeness of A and B are to be taken into account: if A is at least as good as B on both the positive and the negative sides, then A is at least as good as B. This is expressed by the axiom of weak unanimity.

Weak Unanimity
$\forall A, B, A^+ \succeq B^+$ and $A^- \succeq B^- \Rightarrow A \succeq B$

Notice that the axiom of weak unanimity can in some cases be reinforced by a second and more restrictive axiom, strong unanimity. It claims that only indifference on both sides results in indifference. We will see in next section that it is the essence of the Bipolar Qualitative Pareto Dominance rule.

Strong Unanimity $\forall A, B \neq \emptyset$:
$A^+ \succeq B^+$ and $A^- \succ B^- \Rightarrow A \succ B$
$A^+ \succ B^+$ and $A^- \succeq B^- \Rightarrow A \succ B$

Finally, adding an axiom of non triviality, we get the following generalisation of comparative capacities:

Non-Triviality: $X^+ \succ X^-$

Definition 4 *A relation on a power set* 2^X *is a* monotonic bipolar set relation *iff it is reflexive, quasi-transitive and satisfies the properties of Clarity of Arguments, Status Quo Consistency, Non-Triviality, Weak unanimity, Positive Monotonicity, and Negative Monotonicity*

Proposition 1 $\succeq^{Poss}$*is a monotonic bipolar set relations*

Proposition 2 $\succeq^{Pareto}$ *is a monotonic bipolar set relations*

In the present work, we are interested in relations that are only base on the strenght of the individual arguments they contain. It suppose that arguments should be well-ordered. This very natural property very natural one and can be viewed as a strong form of the Clarity axiom : a monotonic relation is said to be simply generated if it is entirely determined by a weak ordering of arguments in X. This implies first of all that $\succeq_{\mathbb{X}}$ is a weak order. A minimal condition of coherence with $\succeq_{\mathbb{X}}$ is that if an argument is replaced by a better one (resp. a worste one), the preference cannot be reversed. This can be view as a condition of monotonicity with respect to $\succeq_{\mathbb{X}}$

Monotonycity w.r.t. $\succeq_{\mathbb{X}}$ or ”$\mathbb{X}$-monotonicity”
$\forall A, B, x, x'$ such that $A \cap \{x, x'\} = \emptyset$ and $x \succeq_{\mathbb{X}} x'$
$A \cup \{x\} \succ B \quad \Rightarrow \quad A \cup \{x'\} \succ B$
$A \cup \{x\} \sim B \quad \Rightarrow \quad A \cup \{x'\} \succeq B$
$B \succ A \cup \{x'\} \quad \Rightarrow \quad B \succ A \cup \{x\}$
$B \sim A \cup \{x'\} \quad \Rightarrow \quad B \succeq A \cup \{x\}$

This axiom, that seems very natural, is richer that it seems. For instance, in implies some property of echangeability of equivalent arguments (provided that they are not already in the argumentation). This kind of property is often called ”anonymity” in social choice and decision theory.

In summary, we shall write the axiom of simple generation as follows:

Simple generation: $\succeq_{\mathbb{X}}$ is a weak order and $\succeq$ is monotonic with respect to $\mathbb{X}$-monotonicity” .

Proposition 3 $\succeq^{Poss}$ *is simply generated. It is moreover complete.*

Proposition 4 $\succeq^{Pareto}$ *is simply generated. It moreover satisfies strong unanimity.*

Qualitative bipolar relation

Definition 4 is actually very general and encompasses numerous models, be they qualitative (e.g. $\succeq^{Pareto}$ is obviously a monotonic bipolar set relations.) or not (e.g. cumulative prospect theory in its full generality). As we are interested in preference rules that derive from the principles of ordinal reasoning only we now focus on axioms that account for ordinality.

The ordinal comparison of sets was extensively used, especially in Artificial Intelligence (see for instance (**?**; **?**; **?**; **?**)). The basic concept of ordinal reasoning is Negligibility that presupposes that each level of importance should be interpreted as an order of magnitude, much higher than the next lower level. In the context of bipolarity, we propose to capture Negligibility by a pair of axioms. The first one enforces this property for positive sets, the second one, for negative sets.

NEG+ $\forall A, B, C : A \succ B$ and $A \succ C \Rightarrow A \succ B \cup C$

NEG-: $\forall A, B, C : B \succ A$ and $C \succ A \Rightarrow B \cup C \succ A$

The first axiom has been around in AI, directly under this form or through more demanding versions – the "union property" of non monotonic reasoning or Halpern's [1977] "Qualitativeness" axioms (see (**?**) for a discussion). The second axiom is effective in terms of negligibility for negative affects.

Ordinal reasoning generally comes along with a notion of Closeness. Since the addition of highest level negative (resp. positive) arguments can decrease (resp. increase) the preference to a set, closeness should also be expressed w.r.t positive and negative sets:

CLO= $\forall A, B, C : A \sim B$ and $A \sim D \Rightarrow A \sim B \cup C$

CLO+$\forall B, C : B \succeq C$ and $C \succeq \emptyset \subseteq X^+ \Rightarrow B \sim B \cup C$

CLO- $\forall B, C : B \preceq C$ and $C \preceq \emptyset \Rightarrow B \sim B \cup C$

Definition 5
A monotonic bipolar relation is said to be qualitative iff it satisfies NEG+, NEG-, CLO=, CLO+, CLO-.

Proposition 5 $\succeq^{Pareto}$ *and* $\succeq^{Poss}$ *are qualitative monotonic bipolar relations*

Conversely, consider a weak order $\succeq_{\mathbb{X}}$ on $\mathbb{X}$, encoding the order of magnitude the different arguments. Applying the principles of qualitative bipolar reasoning described by the previous axioms can lead to several different rules. But $\succeq^{Pareto}$ is the unique decision that follows from $\succeq_{\mathbb{X}}$, understood as an order of magnitude scale by applying only the principe of (i) qualitative bipolar decision making and (ii) weak and strong unanimity. Indeed, any other preference relation in the family refines it, i.e. obey the preference $A \succ^{Pareto} B$ when the Bipolar Qualitative Pareto Dominance rule allows to conclude. This direct result can be formalised as follows:

Definition 6 $\succeq'$ *refines* $\succeq$ *iff* $\forall A, B : A \succ B \Rightarrow A \succ' B$

Theorem 1 *For any* $\succeq_{\mathbb{X}}$ *be weak order on* $\mathbb{X}$, *let* $\mathcal{F}(\succeq_{\mathbb{X}}) = \{\trianglerighteq, \trianglerighteq_{\mathbb{X}} \equiv \succeq_{\mathbb{X}}\}$ *the set of monotonic bipolar rules generated by* $\geq_{\mathbb{X}}$. $\succeq^{Pareto}$ *is the least refined of the monotonic bipolar set relations in* $\mathcal{F}(\succeq_{\mathbb{X}}^{Pareto})$ *that are separable, transitive and qualitative and obey the principles of weak and strong unanimity.*

Theorem 3 shows that the Bipolar Qualitative Pareto Dominance rule is the least committed relation that follows from strong unanimity when embedded in a qualitative framework. On the other hand, Section has shown that this decision rule is too demanding, in the sense that it induces a strict preference that is counterintuitive in some situations. So, let us relax the unanimity postulate, requiring *weak unanimity* only.

Now, the characteristic of the of the $\succeq^{Poss}$ rule is that the principles of negligibility are not only applied within the positive subscale and within the negative subscale, but also apply when taking both positive and negative arguments into account. We need for instance to express that if A is so good that it can cope with rather negative set of elements B and also win the comparison with C, then $A \cup B$ is still better than C: $\forall A, B, C A \cup B \succ \emptyset$ and $A \succ C \Rightarrow A \cup B \succ C$ And similarly, if a globally negative A ($A \prec \emptyset$) is so bad that it is outperformed by C ($C \succ A$) and cannot be enhanced by B ($\emptyset \succ A \cup B$), then $C \succ A \cup B$, i.e. : $\forall A, B, C \emptyset \succ A \cup B$ and $C \succ A \Rightarrow C \succ A \cup B$. All these properties as well as the $NEG+$ and NEG^- axioms can be expressed by the following axiom of global negligibility:

NEG $\forall A, B, C, D$ pairwise disjoint sets, $A \succ B$ and $C \succ D \Rightarrow A \cup C \succ B \cup D$

This axiom is usually, ie in purely positive scales, considered as consequence of NEG+ and positive monotony. When a bi-scaled framework is to be taken into account, the NEG+ condition is not any more a sufficient condition for getting NEG. So, we lose a property that it is as the foundation of a pure order of magnitude reasonning.

Similarly, the three axioms of closeness can be in a more general property that is usual in a mono scale qualitative system but that need to be explicitely required when a bipolar scale is required:

CLO $\forall A, B, C, D$ pairwise disjoint sets, $A \succeq B$ and $C \succeq D \Rightarrow A \cup C \succ B \cup D$

These requirements of a pure qualitative scale is at the foundation of the $\succeq^{Poss}$ rule:

Theorem 2 *The following propositions are equivalent:*
- $\succeq$ *is a complete monotonic bipolar set relation on* 2^X, *is simply generated and satisfies NEG and CLO.*
- *there exists* $\pi : X \mapsto [0_L, 1_L]$ *such that* $\succeq \equiv \succeq^{Poss}$.

Alltrough $\succeq^{Pareto}$ does not appear as a good candidate for bipolar ordinal reasoning, it should be noticed that a similar theorem can be established for it. Here, the key property is the strong unanimity axiom that claims, in addititio to weak unanimity, that only indifference on both the positive and the negative side results in indifference:

Strong Unanimity $\forall A, B \neq \emptyset$
$$\begin{cases} A^+ \succeq B^+ & \text{and} \quad A^- \succ B^- \quad \Rightarrow \quad A \succ B \\ A^+ \succ B^+ & \text{and} \quad A^- \succeq B^- \quad \Rightarrow \quad A \succ B \end{cases}$$

The negligibility axioms cannot be required as such, because they induce a kind of commensurateness between the positive and the negative scale.

Theorem 3 *For any $\succeq_{\mathbb{X}}$ be weak order on $\mathbb{X}$, let $\mathcal{F}(\succeq_{\mathbb{X}}) = \{\trianglerighteq, \trianglerighteq_{\mathbb{X}} \equiv \succeq_{\mathbb{X}}\}$ the set of monotonic bipolar rules generated by $\geq_{\mathbb{X}}$. Any separable monotonic bipolar set relations in $\mathcal{F}(\succeq_{\mathbb{X}})$ that obey $NEG^{+}+$, $NEG^{-}-$ and strong unanimity is a refinement of $\succeq_{\mathbb{X}}^{Pareto}$.*

The efficiency principle

In summary $\succ^{Poss}$ encodes the a natural model of bipolar order of magnitude, and any other model follows the strict preference it proposes – but can be more decisive, hopefully. We have indeed seen that this rule (and it is also the case of $\succeq^{Pareto}$) suffers from a severe drowning effect, as usual in standard possibility theory. For instance, when B is included in A and even if all the proper elements of A are positive, the A is not necessarily strictly preferred to B. The proper extension of the principle of efficiency, that should be at work here, has one positive and one negative side:

Positive efficiency $B \subseteq A$ and $A \setminus B \succ \emptyset \Rightarrow A \succ B$
Negative efficiency $B \subseteq A$ and $A \setminus B \prec \emptyset \Rightarrow A \prec B$

$\succeq^{Poss}$ and $\succeq^{Pareto}$ also fail the classical condition of preferential independence, also called the principle of preadditivity, that encompasses the above conditions of efficiency (and also separability). This condition simply says that arguments present in both A and B should not influence the decision:

Preferential Independence: $\forall A, B, C$ such that $(A \cup B) \cap C = \emptyset : A \succeq B \iff A \cup C \succeq B \cup C$

Except in very special cases where all the arguments are of different levels of importance ($\succeq_{\mathbb{X}}$ is a total order), these axioms are incompatible with axioms of ordinality. It is already true in the pure positive case, i.e. when X^{-} is empty (**?**). But this impossibility result is not damning: the solution is to build relations that are in agreement with $\succeq^{Poss}$ (we shall give up $\succeq^{Pareto}$ because of its important drawbacks) and satisfy Preferential Independence. Such rules are presented in the next section.

Refining the bipolar possibility relation

To overcome the drowning effect, we can indeed propose comparison principles that refine $\succeq^{Poss}$, that is, relations $\succ$ compatible with $\succeq^{Poss}$ but more decisive, i.e. such that $A \succ^{Poss} B \Rightarrow A \succ B$. All relations presented here satisfy Preferential Independence and are thus efficient in both positively and negatively.

The "discri" rule just adds the principle of preferential independence to the ones proposed by $\succeq^{Poss}$, simply cancelling the elements that appear in both sets before applying the rule:

Definition 7 (Discri) $A \succeq^{Discri} B \iff A \setminus B \succeq^{Poss} B \setminus A$

$\succeq^{Discri}$ is complete but not transitive; its strict part, $\succ^{Discri}$ is obviously transitive. When $X = X^{+}$ (resp. $X = X^{-}$), sets of positive (resp. negative) arguments are to be compared; unsurprisingly, it is easy to check that in this case, $\succeq^{Discri}$ collapses to the discrimax (resp. discrimin) procedure.

$\succeq^{Discri}$ simply cancels any argument appearing in both A and B. One could moreover accept the cancellation of any positive (resp. negative) argument in A by another positive (resp. negative) argument in B that share the same order of magnitude. This yields the following two rules based on a levelwise comparison by cardinality. The arguments in A and B are scanned top down, until a level is reached such that the numbers of positive and negative arguments presented by the two alternatives are different; when it is the case, the set that present the lower number of negative arguments and the greater number of positive ones is preferred: a Pareto comparison of the two cardinality-based criteria is performed by $\succeq^{Bilexi}$.

Definition 8 (i-section) *For any level $i \in L$:*
$A_i = \{x \in A, \pi(x) = i\}$ is the i-section of A.
$A_i^{+} = A_i \cap X^{+}$ (resp. $A_i^{-} = A_i \cap X^{-}$) is its positive (resp. negative) i-section.

Definition 9 (BiLexi) $A \succeq^{Bilexi} B \iff |A_{lb}^{+}| \geq |B_{lb}^{+}|$ *and* $|A_{lb}^{-}| \leq |B_{lb}^{-}|)$
where $lb = Argmax\{i : |A_i^{+}| \neq |B_i^{+}|$ *or* $|A_i^{-}| \neq |B_{-}^{+}|\}$

It easy to show that $\succeq^{Bilexi}$ is reflexive, transitive, but not complete. Indeed, if at the decisive level (lb) one of the set wins on the positive side, and the other on the negative side, a conflict is revealed and the procedure concludes to an incomparability. This information is particulary interesting, and should not be confused with indifference: in case of incomparability, the decision maker has trouble with the situation and no alternative seems better than the other : (s)he does not know what to choose – for any choice, there is a reason to regret it. In case of indifference, both alternative are equally satisfactory, and the choice can be made without any regret. So, $\succeq^{Bilexi}$ concludes to an incomparability if and only if there is a conflict between the positive view and the negative view *at the decisive level*. From a descriptive point of view, this range of incomparability is a good point in favor of $\succeq^{Bilexi}$.

Now, if one looks for an even more decisive procedure, one could accept to lose this information about the conflict. A complete and transitive refinement of $\succeq^{Bilexi}$ will be obtained:

Definition 10 (Lexi)

$$A \succeq^{Lexi} B \iff \exists i \in L \text{ such that } \begin{cases} (\forall j > i, & |A_j^{+}| - |A_j^{-}| = |B_j^{+}| - |B_j^{-}|) \\ \text{and} & (|A_i^{+}| - |A_i^{-}| > |B_i^{+}| - |B_i^{-}|) \end{cases}$$

Finally, the projection of both $\succeq^{Bilexi}$ and $\succeq^{Lexi}$ on X^{+} (resp. X^{-}) is complete and transitive and amounts to the leximax (resp. leximin) preference relation (**?**). Each projection is thus representable by a qualitative capacity (see e.g. (**?**)). Thus:

Proposition 6 *There exists two capacities σ^+ and σ^- such that:*

$A \succeq^{Lexi} B \iff \sigma^+(A^+) - \sigma^-(A^-) \geq \sigma^+(B^+) - \sigma^-(B^-)$

$A \succeq^{Bilexi} B \iff \sigma^+(A^+) \succeq^{Bilexi} \sigma^-(B^+)$ *and* $\sigma^+(B^-) \succeq^{Bilexi} \sigma^-(A^-)$

The proposition is obvious using the classical encoding of the leximax (unipolar) procedure by a capacity, e.g. $\sigma^+(V) = \sigma^-(V) = \sum_{i \in L} |V_i| \cdot |X|^i$.

The three rules obviously define monotonic bipolar relation. Each of them refine $\succeq^{Poss}$ and satisfy Preferential Independence. They can be ranked from the least to the most decisive ($\succ^{Lexi}$), that is moreover complete and transitive.

Proposition 7 $A \succ^{Poss} B \Rightarrow A \succ^{Discri} B \Rightarrow A \succ^{Bilexi} B \Rightarrow A \succ^{Lexi} B$

Related works

Gigerenzer, G. Goldstein: chaque critere dit bon ou mauvais et a une force. On les classe qu plus fort au moins fort. Le premier quio dit bon pour a et mauvais pour b fait la decision.

Results in cognitive psychology have pointed out the importance of bipolar reasoning in human decisions. Psychologists have shown that the simultaneous presence of positive and negative affects prevents decisions from being simple to make (**?**) (see also (**?**; **?**)). Cumulative Prospect Theory (**?**) is the first attempt to account for this idea. CPT assumes reasons supporting a decision and reasons against it can be evaluated by means of two capacities σ^+ and σ^-, σ^+ measuring the importance of the group of positive affects, σ^- the importance of the group of negative affects. The higher σ^+, the more convincing the set of arguments and conversely the higher σ^-, the more deterring A. This approach moreover admits that it is possible to map these evaluations onto a so-called "net predisposition" expressed on a single scale. The net predisposition depends on the importance of each group and is given by:

$\forall A \subseteq X, CTP(A) = \sigma^+(A^+) - \sigma^-(A^-)$ where $A^+ = A \cap X^+, A^- = A \cap X^-$

Variants exist that measure utility by some function of σ^+ and σ^-.

On the other hand, since the comparison of net predisposition systematically provides a complete and transitive preference, it can fails to capture a large range of decision-making attitudes: the point is that, contrasting affects make decision difficult, so that the comparison of objects characterised by bipolar evaluations does not systematically yields a complete and transitive relation – but can imply some incomparabilities (see e.g. (**?**)). That is why bicapacities were generalised by the notion of bipolar capacity (**?**). The idea is to use two measures, a measure of positiveness (that increases with the addition of positive arguments and the deletion of negative arguments) and a measure of negativeness (that increases with the addition of negative arguments and the deletion of positive arguments), but without combining then. Then A is preferred to B iff it is the case with respect to both measures – i.e. according to the sole Pareto principle. This allows the representation of conflicting evaluations and can lead to a partial order. On the other hand, degenerated cases where the second measures is always 0 encompass the use of a aggregation of σ^+ and σ^-, as prescribed by CPT.

Our approach is clearly a qualitative counterpart to the above works. In the $\succeq^{Lexi}$ relation, the positive and negative sets of affects are evaluated separately by capacities σ^+ and a σ^- and the aggregated in agreement with net predisposition. The bilexi-rule does not make an aggregation of them, thus allowing the expression of conflicts. This concludes our argumentation in favor of $\succeq^{Lexi}$ and $\succeq^{Bilexi}$: they comply with the spirit of CTP as well at its practical advantages (transitivity and representability by a pair of functions), they are efficient and in accordance with but more decisive than OM reasoning.

This paper has adopted a prescriptive point of view in the sense that the rules were studied with respect to the properties that should obey a qualitative theory of bipolar decision making. Reciprocally, their descriptive power, i.e. their ability to represent the behavior of human decision maker have been tested — these experimental results, presented in (**?**) confirm $\succ^{Poss}$ as the basis of ordinal decision making and show the predominance $\succeq^{Lexi}$ as a decision rule for the decision makers that satisfy preferential independence. $\succeq^{Bilexi}$ seems to be the good model for the subset of decision maker for who conflicts leads to incomparability. Following (**?**) we think that the characterisation of $\succeq^{Bilexi}$ and $\succeq^{Lexi}$ should not be difficult and we leave it for further research.

Conclusion

This paper has focused on and characterised a particular class of bipolar decision making situations, namely the one that are qualitative rather than quantitative in essence. The proposed work is an extension of possibility theory to the handling of sets containing two-sorted elements considered as positive or negative. The results were couched in a terminology borrowing to argumentation and decision theories, and indeed we consider they can be relevant for both. Our framework is a qualitative application of Cumulative Prospect Theory and more recent proposals bipolar capacities. Its expressivity could be extended to elements whose positiveness and negativeness depend on the considered decision (using a duplication process of such x as x^+ and x^- and considering subsets containing one of them at most). The paper is also relevant in argumentation for the evaluation of sets of arguments in inference processes (**?**), and argument-based decisions (**?**). The next step in our research is naturally the extension to (qualitative) bipolar criteria whose satisfaction is a matter of degree (**?**). In the future, comparison between our decision rules and those adopted in the above works as well as aggregation processes in finite bipolar scales (**?**) is in order.

preuves

Lemma 1 $\succeq^{Poss}$ *is complete*

Proof. This follows from the simple fact that it is based on the comparison of two measures.

Lemma 2 $\succ^{Poss}$ *is transitive*

Proof. Suppose $A \succ^{Poss} B$ and $B \succ^{Poss} C$. Four cases are possible:

- $\exists a^+ \in A^+, \forall x \in A^- \cup B^+, \pi(a^+) > \pi(x)$ and $\exists b^+ \in B^+, \forall x \in B^- \cup C^+, \pi(b^+) > \pi(x)$. Then $\pi(a^+) > \pi(b^+)$ and thus $\forall x \in C^+, \pi(a^+) > \pi(x)$. So, $\exists a^+ \in A^+, \forall x \in A^- \cup C^+, \pi(a^+) > \pi(x) : OM(A^+ \cup C^-) > OM(A^- \cup C^+)$.
- $\exists b^- \in B^-, \forall x \in A^- \cup B^+, \pi(b^-) > \pi(x)$ and $\exists c^- \in C^-, \forall x \in B^- \cup C^+, \pi(c^-) > \pi(x)$. Then $\pi(c^-) > \pi(b^-)$ and thus $\forall x \cup A^-, \pi(c^-) > \pi(x)$. So, $\exists c^- \in C^-, \forall x \in A^- \cup C^+, \pi(c^-) > \pi(x) : OM(A^+ \cup C^-) > OM(A^- \cup C^+)$.
- $\exists a^+ \in A^+, \forall x \in A^- \cup B^+, \pi(a^+) > \pi(x)$ and $\exists c^- \in C^-, \forall x \in B^- \cup C^+, \pi(c^-) > \pi(x)$. So, the maximun of a^+ and c^- has a possibility degree strictly better than any $x \in A^-$ and that any $x \in C^+ : OM(A^+ \cup C^-) > OM(A^- \cup C^+)$.
- $\exists b^- \in B^-, \forall x \in A^- \cup B^+, \pi(b^-) > \pi(x)$ $\exists b^+ \in B^+, \forall x \in B^- \cup C^+, \pi(b^+) > \pi(x)$. Hence ce contradiction $\exists b^- \in B^-, \forall b^+ \in B^+, \pi(b^-) > \pi(b^+)$ and $\exists b^+ \in B^+, \forall b^- \in B^-, \pi(b^+) > \pi(b^-)$. This case never occurs.

So, in any case, $A \succ B$ and $B \succ C$ implies $A \succ C$: $\succ$ is transive, i.e. $\succeq$ quasi transitive.

Lemma 3 *:* $\sim^{Poss}$ *is not transitive*

Proof. Choose $a^+, a^-, b^+, b^-, c^+, c^-$ such that $\pi(b^+) = \pi(b^-) = \alpha > \pi(a^+) = \pi(c^-) = \beta > \pi(a^-) = \pi(c^+) = \gamma$, and build $A = \{a^-, a^+\}, B = \{b^-, b^+\}, C = \{c^-, c^+\}$.

So $OM(A^+ \cup B^-) = OM(C^+ \cup B^-) = OM(A^- \cup B^+) = OM(B^- \cup B^-) = \alpha$. We thus have $A \sim^{Poss} B$ and $B \sim^{Poss} C$, but $OM(A^+ \cup C^-) = \beta > \gamma = OM(A^- \cup C^+)$, hence $A \succeq^{Poss} C$.

The transitivity $A \succeq B, B \succeq C \implies A \succeq C$ can be proofed as soon as $OM(B^+) \neq OM(B^-)$, i.e. as soon as B is not suffering from an internal conflict.

Lemma 4 $\succeq^{Poss}$ *is simply generated*

Proof. When restricted to singleton, $\succeq^{Poss}$ is a complete preorder that ranks the positive arguments by decresing order of π, then the null arguments, then the negative argument by increasing value of π. This ranking defines is complete and transitive – $\succeq^{Poss}$ is simply generated

Lemma 5 $\succeq^{Poss}$ *is* $\mathbb{X}$*-monotonicity*

Proof. Let A, x, x' be such that $A \cap \{x, x'\} = \emptyset$ and$x' \succeq_{\mathbb{X}} x$.

If $x \in X^+$, then $x' \in\in X^+$ and $\pi(x') \geq \pi(x)$.

- if $A \cup \{x\} \succ^{poss} B$: then $OM(A^+ \cup \{x\} \cup B^-) > OM(A^- \cup B^+)$. Replacing x by x', i.e. $\pi(x)$ by $\pi(x')$, the first OM level increase, so we get $OM(A^+ \cup \{x'\} \cup B^-) > OM(A^- \cup B^+)$: if $A \cup \{x'\} \succ^{poss} B$.
- if $A \cup \{x\} \sim^{poss} B$: then $OM(A^+ \cup \{x\} \cup B^-) = OM(A^- \cup B^+)$. Replacing x by x', i.e. $\pi(x)$ by $\pi(x')$, the first OM level increases, so we get $OM(A^+ \cup \{x'\} \cup B^-) \geq OM(A^- \cup B^+)$: if $A \cup \{x'\} \succeq^{poss} B$.
- if $B \succ A \cup \{x'\}$, then $OM(B^+ \cup A^-) > OM(A^+ \cup \{x'\} \cup B^-)$. Replacing x' by x, i.e. $\pi(x')$ by $\pi(x)$, the secod OM level decrease, so we get $OM(B^+ \cup A^-) > OM(A^+ \cup \{x\} \cup B^-)$, i.e. $B \succ A \cup \{x\}$.
- if $B \sim A \cup \{x'\}$, then $OM(B^+ \cup A^-) = OM(A^+ \cup \{x'\} \cup B^-)$. Replacing x' by x, i.e. $\pi(x')$ by $\pi(x)$, the secod OM level decreases, so we get $OM(B^+ \cup A^-) \geq OM(A^+ \cup \{x\} \cup B^-)$, i.e. $B \succeq^{Poss} A \cup \{x\}$.

If $x' \in X^-$, then $x \in\in X^-$ and $\pi(x) \geq \pi(x')$. The same kind of proof in four cases can be drawn.

If $x' \in X^+$ and $x \in X^-$:

- if $A \cup \{x\} \succ^{poss} B$: A negative argument on the left is replaced by a positive one, so $OM(A^+ \cup B^- \cup \{x'\}) \geq OM(A^+ \cup B^-) > OM(A^- \cup B^+ \cup \{x\}) \geq OM(A^- \cup B^+)$: $A \cup \{x'\} \succ^{poss} B$.
- Similarly, if $A \cup \{x\} \sim^{poss} B$ we have: $OM(A^+ \cup B^- \cup \{x'\}) \geq OM(A^+ \cup B^-) = OM(A^- \cup B^+ \cup \{x\}) \geq OM(A^- \cup B^+)$: $A \cup \{x'\} \succeq^{poss} B$.
- if $B \succ A \cup \{x'\}$, a positive argument on the right is replaced by a negative one, $OM(B^+ \cup A^- \cup \{x\}) \geq OM(B^+ \cup A^-) > OM(A^+ \cup \{x'\} \cup B^-) \geq OM(A^+ \cup B^-)$: $B \succ A \cup \{x\}$.
- Similarly, if $B \succ A \cup \{x'\}$ we have : $OM(B^+ \cup A^- \cup \{x\}) \geq OM(B^+ \cup A^-) = OM(A^+ \cup \{x'\} \cup B^-) \geq OM(A^+ \cup B^-)$: $B \succeq A \cup \{x\}$.

Proof of Proposition ??. The quasi transitivity of $\succeq^{poss}$ is proved in the previous Lemma 1.

Positive and negative monotony, as well as status quo consistency follow from the monotony of OM, that is a possibility measure (i.e. $OM(U \cup V) \geq OM(V) \forall U, V$).

Non triviality of $\succeq^{poss}$ is obtained from the non triviality of π (there exists x such as $\pi(x) > 0$), that implies $OM((X^+)^+ \cup (X^-)^-) = OM(X^+ \cup X^-) > 0$ while $OM((X^+)^- \cup (X^-)^+) = OM(\emptyset) = 0$.

Clarity of argument is also trivial: if x is positive, then $OM(\{x\}^+ \cup (\emptyset)^-) = \pi(x) \geq OM(\{x\}^- \cup (\emptyset)^+) = OM(\emptyset) = 0$: $\{x\} \succeq^{Poss} \emptyset$. If x is negative, we get in the same way $\{x\} \preceq^{Poss} \emptyset$. If x has a degree 0, we get $OM(\{x\}^+ \cup (\emptyset)^-) = \pi(x) \geq OM(\{x\}^- \cup (\emptyset)^+) = OM(\emptyset) = 0$: $\{x\} \sim^{Poss} \emptyset$.

For proving weak unanimity, recall that $A^+ \succeq^{Poss} B^+$ implies $\exists a^+ \in A^+, \forall x \in A^+ \cup B^+, \pi(a^+) \geq \pi(x)$ and that $A^- \succeq^{Poss} B^-$ implies $\exists b^- \in B-, \forall x \in B^- \cup A^-, \pi(b^-) \geq \pi(x)$. So, $OM(A^+ \cup B^-) = \max(\pi(a^+), \pi(b^-)) \geq OM(A^- \cup B^+)$: $A \succeq^{Poss} B$

Proof that $\succeq^{pareto}$ is a monotonic bipolar set relations

Proof of Proposition 3. This proposition summarizes lemmas 1, 4, 5