An axiomatization of conditional possibilistic preference functionals

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Abstract. The aim of the paper is to extend the Savage like axiomatization of possibilistic preference functionals in qualitative decision theory to conditional acts, so as to make a step towards the dynamic decision setting. To this end, the de Finetti style approach to conditional possibility recently advocated by Coletti and Vantaggi is exploited, extending to conditional acts the basic axioms pertaining to conditional events.

1 Introduction

The natural counterparts to the expected utility criterion is the pair of possibilistic optimistic and pessimistic criteria, originally introduced by Yager [?] and Whalen [?] respectively. These criteria were axiomatized in the setting of Von-Neuman and Morgenstern theory, based on the comparison of possibilistic lotteries by Dubois et al. [?] and in the Savagean setting of acts under uncertainty by Dubois Prade and Sabbadin [?]. Later on, Giang and Shenoy [?] introduced a possibilistic criterion without pessimism nor optimism assumptions, using a bipolar qualitative scale concatenating the possibility and necessity scales. In this setting, each act is evaluated by a pair of qualitative values. The criterion is like the pessimistic one when all potential consequences are bad, and like the optimistic ones where all consequences are good. More recently Paul Weng [?] showed how to axiomatize this criterion in the Savage setting. All the above works propose a foundation to qualitative decision making in a static world. But the important issue of a qualitative decision theory when new input information can be received was left open.

In classical decision theory, this question turns out to be an easy one because of the sure thing principle. When the input information is obtained under the form of a true event A, the expected utility of acts comes down to restricting the acts to states of nature where this event is true, making the consequences outside A of all acts identical, regardless of what these common consequences are. Namely, if the preference relation indexed by the sure event is representable by an expected utility, then the same holds for the preference relation indexed by the event A, whenever the conditioning event A is not null. It comes down to changing the subjective probability into a conditional probability. This method can fail when the possible event A is null, that is, indifferent to the constant zero.

The issue of conditional qualitative criteria is more difficult, because in possibility theory the sure thing principle fails. As a consequence, the axiomatization of conditional possibilistic criteria must be reconsidered from scratch. It can be done either using a set of conditional preference relations on acts, or using a single preference relation on conditional acts. In the first approach, preference relations are indexed by an event that represents the information context in which the decision takes place. Additional axioms must be found in order to explain how preference relations indexed by different events can interplay. In the second approach one considers any act that takes place in a given information context. From an uncertainty-theoretic point of view it comes down to studying conditional set-functions not as a derived notion built from the unconditional ones, but as a primitive notion. This approach to uncertainty measures is the one adopted by de Finetti for probability theory, in order to allow for conditioning on hypothetical events with probability zero. This is the path followed in this paper. Recently Coletti and Vantaggi [**?**,**?**] introduced this approach in qualitative possibility theory, thus extending to the conditional setting the comparative possibility relation first proposed by Lewis [**?**], and retrieved by Dubois [**?**] as an ordinal account of Zadeh's possibility theory [**?**] in the spirit of comparative probability also originally proposed by de Finetti. The merit of qualitative conditional possibility after Coletti and Vantaggi is to provide an answer to conditioning on non-empty events of possibility zero, thus capturing a more general concept of conditioning (including some other proposals already studied in literature).

The aim of this paper is to bridge the gap between qualitative conditional possibility and the axiomatization of possibilistic preference functionals, thus paving the way toward possibilistic decision under uncertainty in a dynamic epistemic environment.

2 Decision-theoretic approach to possibility theory

A decision problem under uncertainty will be cast in the usual framework: we consider set S of states and a set $\mathcal{X}$ of potential consequences of decisions. States encode possible situations, states of affairs, etc. An act is viewed as a mapping f from the state space to the consequence set, namely, in each state $s \in S$, an act f produces a well-defined result $f(s) \in \mathcal{X}$. The decision maker must rank acts without knowing what is the current state of the world in a precise way. In qualitative decision theory, S is finite, and so is generally $\mathcal{X}$. n will denote the number of states in S. The consequences of an act can often be ranked in terms of their relative appeal: some consequences are judged better than others. This is often modeled by means of a numerical utility function u which assigns to each consequence $x \in \mathcal{X}$ a utility value $u(x) \in \boldsymbol{R}$.

The most widely found assumption is that there is a probability distribution p on S, and the most usual decision rule is based on the expected utility criterion.

When no information about the current state is available, the maximin criterion ranks acts according to its worst consequence:

$$W_u^-(f) = \min_{s \in S} u(f(s)). \tag{1}$$

Clearly this criterion has the major defect of being extremely pessimistic. Absolute qualitative approaches rely on extensions of Wald's criterion. The possibilistic qualitative criterion is based on a utility function u on $\mathcal{X}$ and a possibility distribution π on S [**?**], both mapping on the same totally ordered scale L. The ordinal value $\pi(s)$ represents the relative plausibility of state s. Here, L is equipped with its involutive order-reversing map $\mathbf{n}$; in particular $\mathbf{n}(1_L) = 0_L, \mathbf{n}(0_L) = 1_L$. So, $\mathbf{n}(\pi(s))$ represents the degree of potential surprise in case the state of the world is s [**?**]. In particular, $\mathbf{n}(\pi(s)) = 1_L$ for impossible states. A pessimistic criterion $W_{\pi,u}^-(f)$ is proposed [**?**,**?**] of the form :

$$W_{\pi,u}^-(f) = \min_{s \in S} \max(\mathbf{n}(\pi(s)), u(f(s))) \tag{2}$$

The value of $W_{\pi,u}^-(f)$ is small as soon as there exists a highly plausible state ($\mathbf{n}(\pi(s)) = 0_L$) with low utility value. This criterion is actually a prioritized extension of the Wald maximin criterion $W_u^-(f)$. The latter is recovered in case of total ignorance, ie. when $\pi(s) = 1_L$ for all $s \in S$. The decisions are again made according to the merits of acts in their worst consequences, now restricted to the most plausible states defined by a compromise between belief and utility expressed in the min-max expression.

The optimistic counterpart to this criterion [**?**,**?**] is:

$$W_{\pi,u}^+(f) = \max_{s \in S} \min(\pi(s), u(f(s))). \tag{3}$$

The optimistic and pessimistic possibilistic criteria are particular cases of a more general criterion based on the Sugeno integral (see [**?**]):

$$S_{\gamma,u}(f) = \max_{\lambda \in L} \min(\lambda, \gamma(F_\lambda)) \tag{4}$$

where $F_\lambda = \{s \in S, u(f(s)) \geq \lambda\}$, γ is a monotonic set function that reflects the decision-maker's attitude in front of uncertainty: $\gamma(A)$ is the degree of confidence in event A. The possibilistic criterion $W_{\pi,u}^+$ is obtained when γ is the possibility measure based on π ($\gamma(A) = \max_{s \in A} \pi(s)$), and $W_{\pi,u}^-$ is obtained when γ is the corresponding necessity measure ($\gamma(A) = \min_{s \notin A} \mathbf{n}(\pi(s))$ [3]).

We consider Sugeno integral and possibilistic criteria in the scope of Savage theory. Let us denote $\succeq$ a complete and transitive preference relation among acts of $\mathcal{X}^S$: $\succ$ will denote its strict part ($f \succ g \iff f \succeq g$ and $\neg(g \succeq f)$) and $\simeq$ will denote its symmetric part ($f \simeq g \iff f \succeq g$ and $g \succeq f$).

We denote fAh the act identical to f on a subset A and to h on its complementary: $\forall s, fAh(s) = f(s)$ if $s \in A, h(s)$ if $s \notin A$. The possibilistic criteria $W_{\pi,u}^+$ and $W_{\pi,u}^-$ satisfy a weak version of the sure-thing principle:

[3] Indeed, it is easy to show that $S_{\gamma,u}(f) = \max_{s \in S} \min(u(f(s)), \gamma(F_{u(f(s))}))$ is equal to $\min_{s \in S} \max(u(f(s)), \gamma(F_{\overline{u(f(s))}}))$, where $F_{\overline{\lambda}} = \{s \in S, u(f(s)) > \lambda\}$ [**?**].

Axiom WP2: $\forall A, \forall f, g, h, h', fAh \succ gAh \Rightarrow fAh' \succeq gAh'$.

Let us denote by $\succeq_P$ the utility ordering of consequences that derives from $\succeq$: $x \succeq_P y \iff f_x \succeq f_y$. where f_x (resp. f_y) is the constant act that concludes to consequence x (resp. y) for any state.

The rankings of acts obtained by a Sugeno integral satisfy the following weak version of Savage postulate P3:

Axiom WP3: $\forall A \subseteq S, \forall x, y \in \mathcal{X}, \forall f, x \succeq_P y$ implies $xAf \succeq yAf$.

But the converse may be false for events the plausibility of which is lower than the utility degree of x and y (the plausibility degree of A is in this case so negligible with respect to the utility of x and y that A is considered as null in this context).

The basic properties of Sugeno integrals exploit *disjunctive and conjunctive combinations* of acts. Let act $f \wedge g$ be the one always producing the worst consequences of f and g in each state, while $f \vee g$ always makes the best of them:

$$f \wedge g(s) = f(s) \text{ if } g(s) \succeq_P f(s) \text{ and } g(s) \text{ otherwise} \tag{5}$$

$$f \vee g(s) = f(s) \text{ if } f(s) \succeq_P g(s) \text{ and } g(s) \text{ otherwise} \tag{6}$$

They are union and intersection of fuzzy sets viewed as acts. Obviously, $S_{\gamma,u}(f \wedge g) \leq \min(S_{\gamma,u}(f), S_{\gamma,u}(g))$ and $S_{\gamma,u}(f \vee g) \geq \max(S_{\gamma,u}(f), S_{\gamma,u}(g))$ from weak Pareto monotonicity. These properties hold with equality whenever f or g is a constant act and are then characteristic of Sugeno integrals for monotonic aggregation operators [**?**]. Actually, these properties can be expressed by means of axioms, called restricted conjunctive and disjunctive dominance (RCD and RDD) on the preference structure $(\mathcal{X}^S, \succeq)$:

- Axiom RCD: if f is a constant act, $f \succ h$ and $g \succ h$ imply $f \wedge g \succ h$
- Axiom RDD: if f is a constant act, $h \succ f$ and $h \succ g$ imply $h \succ f \vee g$.

For instance, RCD means that limiting from above the potential utility values of an act g, that is better than another one h, to a constant value that is better than the utility of act h, still yields an act better than h. This is in contradiction with expected utility theory and strongly counterintuitive in the context of economic theory, with a continuous consequence set $\mathcal{X}$. However the range of validity of qualitative decision theory is precisely when both $\mathcal{X}$ and S are finite and steps in the finite value scale are far from each other.

This setting enables the axiomatization of Sugeno integrals in the style of Savage to be carried out. The following representation theorem holds:

Theorem 1 *[**?**]: A preference structure $(\mathcal{X}^S, \succeq)$ is a non-trivial weak order that satisfies WP3, RCD and RDD if and only if there exists a finite chain of preference levels L, an L-valued monotonic set-function γ, and an L-valued utility function u on $\mathcal{X}$, such that $f \succeq g$ if and only if $S_{\gamma,u}(f) \geq S_{\gamma,u}(g)$.*

The pessimistic criterion $W^-_{\pi,u}(f)$ can be axiomatized by strengthening axiom RCD into conjunctive dominance as follows [**?**]:

Axiom CD : $\forall f, g, h, f \succ h$ and $g \succ h$ imply $f \wedge g \succ h$.

Changing RDD into CD implies that the set-function γ is a necessity measure [**?**] and so, $S_{\gamma,u}(f) = W^{-}_{\pi,u}(f)$ for some possibility distribution π. Similarly, the criterion $W^{+}_{\pi,u}(f)$ can be axiomatized by strengthening axiom RDD into disjunctive dominance as follows:

Axiom DD : $\forall f, g, h, h \succ f$ and $h \succ g$ imply $h \succ f \vee g$.

Changing RCD into DD implies that the set-function γ is a possibility measure and so, $S_{\gamma,u}(f) = W^{+}_{\pi,u}(f)$ for some possibility distribution π. In order to figure out why axiom CD leads to a pessimistic criterion, let us notice here that CD can be equivalently replaced by the following property:

$$\text{(PESS)}\forall A \subseteq S, \forall f, g, fAg \succ g \text{ implies } g \succeq gAf. \tag{7}$$

Similarly, the following optimistic counterpart to (**??**) can serve as a substitute to axiom DD for the representation of criterion $W^{+}_{\pi,u}$:

$$\text{(OPT)}\forall A \subseteq S, \forall f, g, g \succ fAg \text{ implies } gAf \succeq g. \tag{8}$$

3 Qualitative conditional possibility

The notion of conditioning in possibility theory is a problem of long-standing interest. Starting from a triangular norm (t-norm) T various definitions of T-conditional possibility have been given [**?**]. In the following we use the axiomatic definition proposed in [**?**], restricted to the t-norm minimum:

Definition 1. *Let $S = \{s_1, ..., s_n\}$ be a state space and $\boldsymbol{E} = \boldsymbol{B} \times \boldsymbol{H}$ where $\boldsymbol{B}$ is a finite algebra of subsets of S, $\boldsymbol{H} \subseteq \boldsymbol{B} \setminus \{\emptyset\}$ an additive class of non-empty subsets of S (closed with respect to finite unions). A function $\Pi : \boldsymbol{E} \to [0,1]$ is a qualitative conditional possibility if it satisfies the following properties:*
1. $\Pi(E|H) = \Pi(E \wedge H|H)$, for every $E \in \boldsymbol{B}$ and $H \in \boldsymbol{H}$;
2. $\Pi(\cdot|H)$ is a possibility measure, for every $H \in \boldsymbol{H}$;
3. $\forall H, E \wedge H \in \boldsymbol{H}$, and $E, F \in \boldsymbol{B}, \Pi(E \wedge F|H) = \min(\Pi(E|H), \Pi(F|E \wedge H))$.

Condition *2* requires that, for every conditioning event $H \in \mathbf{H}$, the function $\Pi(\cdot|H)$ is a possibility, so it is normalized. A characterization of qualitative conditional possibilities in terms of a class of unconditional possibilities on the algebra $\mathbf{B}$ was given in [**?**]. An analogous result for T-conditional possibility, with T a strictly increasing t-norm, is in [**?**] and it is in the same line as the characterization theorem of conditional probabilities in de Finetti approach [**?**]. In both cases the conditional possibility $\Pi(\cdot|H)$ is not singled-out by the possibility of its conditioning event H, but its value is ruled by the values of other possibilities $\Pi(\cdot|E \wedge H)$, for suitable events E. It turns out that a conditional possibility cannot always be derived from just one "unconditional" possibility. The value $\Pi(E|H)$ follows directly from $\Pi(E \wedge H)$ and $\Pi(H)$ just in the case $\Pi(E \wedge H) < \Pi(H)$. Note that in such a case Definition **??** coincides with the one given by Dubois and Prade in [**?**], which is based on the minimum specificity principle and consists in taking for $\Pi(E|H)$ the greatest solution to the equation $\Pi(E \wedge H) = \min\{x, \Pi(H)\}$, that is $\Pi(E \wedge H)$ when $\Pi(E \wedge H) < \Pi(H)$ and 1_L otherwise. Definition **??** is more general than the latter.

For example, consider the following conditional possibility on $\mathbf{E} = \mathbf{B} \times \{H, \Omega\}$, with $S = \{s_1, s_2, s_3\}$ and $H = s_1 \vee s_2$:

$$\Pi(\{s_1\}) = \Pi(\{s_2\}) = 0.2; \Pi(\{s_1\}|H) = 0.6. \tag{9}$$

Note that the rules of possibility theory imply $\Pi(s_1 \vee s_2) = 0.2$ and $\pi(s_3) = \Pi(F) = 1$, where $s_3 \in F \in \mathbf{B}$. Similarly, the conditional constraint implies $\pi(s_2|H) = \Pi(H|H) = 1$. Let $\Pi_0 = \Pi_{|\Omega}$. It is a solution to both constraints in (**??**) but the equation $\Pi_0(s_1) = \min(x, \Pi_0(s_1 \vee s_2))$ does not define a unique conditional possibility. The solution to system (**??**) is a pair of unconditional possibilities (i.e. $\{\Pi_0, \Pi_1\}$, where $\pi_1(s_1) = 0.6; \pi_1(s_2) = 1; \pi_1(s_3) = 0$). Moreover, Π_1 is the unique solution to $\Pi(\{s_1\}|H) = 0.6$ on referential H.

Characterizations of ordinal relations $\preceq$ on a set of conditional events $\mathbf{E} = \mathbf{B} \times \mathbf{H}$ representable by qualitative conditional possibilities Π (i.e. for any $A|H, B|K \in \mathbf{E}$, $A|H \preceq B|K \leftrightarrow \Pi(A|H) \leq \Pi(B|K)$) have been provided in [**?**,**?**]. In the sequel we recall the main results.

Definition 2. *A binary relation $\preceq$ on conditional events $A|H \in \boldsymbol{E}$ is called comparative conditional possibility iff the following conditions hold:*

1. *$\preceq$ is a weak order;*
2. *for any $H, K \in \boldsymbol{H}$, $\emptyset|H \sim \emptyset|K \prec H|H \sim K|K$;*
3. *for any $A, B \in \boldsymbol{B}$ and $H, B \wedge H \in \boldsymbol{H}$, $A \wedge B|H \preceq A|B \wedge H$ and moreover if either $A \wedge B|H \prec B|H$ or $B|H \sim H|H$, then $A \wedge B|H \sim A|B \wedge H$;*
4. *for any $H \in \boldsymbol{H}$ and any $A, B, C \in \boldsymbol{B}, A|H \preceq B|H \Rightarrow (A \vee C)|H \preceq (B \vee C)|H$.*

Condition (3) requires that in the context of the new information "B" the degree of belief in an event A cannot be less than the degree of belief in $A \wedge B$ before supposing that B occurs. Moreover, if the new information B is less surprising than $A \wedge B$ in the context H, or even totally unsurprising, the occurrence of B cannot change the degree of belief in A in the context H. Condition (4) is essentially the one proposed by Dubois [**?**], just rewritten conditioned on the hypothesis H. Moreover, condition (4) is equivalent (see [**?**]), under transitivity, to $A|H \preceq B|K$ and $C|H \preceq D|K \Rightarrow (A \vee C)|H \preceq (B \vee D)|K$.

Theorem 1. *[**?**]: For a binary relation $\preceq$ on $\boldsymbol{E} = \boldsymbol{B} \times \boldsymbol{H}$ the following statements are equivalent:*

i. *$\preceq$ is a comparative conditional possibility;*
ii. *there exists a qualitative conditional possibility Π on $\boldsymbol{E}$ representing $\preceq$.*

Obviously, among the comparative conditional possibilities there are also the ordinal relations representable by conditional possibilities satisfying the minimum specificity principle, more precisely those satisfying a reinforcement of condition *3* of Definition **??**, that is

(sc) for every $A, B \in \mathbf{B}$ and $H, B \wedge H \in \mathbf{H}$, $(A \wedge B)|H \preceq A|(B \wedge H)$ and moreover if $A \wedge B \wedge H \neq \emptyset$ and $(A \wedge B)|H \sim B|H$, then $A|(B \wedge H) \sim H|H$.

4 Qualitative conditional possibilistic preference functional: optimistic case

Let S be a finite set of states, $\mathbf{B}$ be the power set of S and $\mathbf{H} \subseteq \mathbf{B} \setminus \{\emptyset\}$ be an additive class containing S.[4]

Given a set of consequences $\mathcal{X}$, a conditional act $f|H$ is formed by a pair: an act f and an event $H \in \mathbf{H}$. The event H in $f|H$ is not just representing a given fact, but it is an uncertain hypothetical event whose truth value may be unknown. It expresses the idea of choosing decision f in case H were true, not actually doing it when H occurs. It differs from an unconditional act of the form fHg even if the value of $f|H$ and fHg is equal to $f(s) \in \mathcal{X}$ for any state $s \in H$. Indeed, for $s \notin H$, the value of $f|H$ is *undetermined* (following the terminology of de Finetti).

Let x^* and x_* be the best and the worst consequences (according to a given preference) in $\mathcal{X}$. Moreover the event $E \in \mathbf{B}$ is in bijection with the binary act taking the best value x^* when E is true and the worst value x_* when E is false.

A qualitative conditional decision model consists of a conditional possibility $\Pi : \mathbf{B} \times \mathbf{H} \to L$, a utility function u on the consequences in $\mathcal{X}$ with $u(x^*) = 1_L$ and $u(x_*) = 0_L$. A *conditional* possibilistic optimistic criterion takes the form:

$$v^*(f|H) = \max_{s \in H} \{\min\{u(f(s)), \Pi(s|H)\}\} .$$

Note that the above model is such that, for any $H \in \mathbf{H}$,

$$v^*(x^*|H) = \max_{s \in H} \min\{u(x^*), \Pi(s|H)\} = \max_{s \in H} \Pi(s|H) = \Pi(H|H) = 1 = u(x^*)$$

and $v^*(x_*|H) = \max_{s \in H} \min\{u(x_*), \Pi(s|H)\} = u(x_*) = 0 = \Pi(\emptyset|H)$.

4.1 Axioms for the qualitative conditional model: optimistic case

Given a preference on the set of conditional acts $f|H$ with consequences on $\mathcal{X}$, we consider the following conditions:

1. $\preceq$ is a non-trivial weak order on $\mathcal{F} = \mathcal{X}^S \times \mathbf{H}$;
2. for any consequences $x, y \in \mathcal{X}$ such that $x >_p y$ and for any $H, K \in \mathbf{H}$ one has
$$y|H \sim y|K \text{ and } y|H \prec x|H;$$
3. (WP3) if x, y are consequences in $\mathcal{X}$ such that $x \geq_p y$, then $(yAh)|H \preceq (xAh)|H$ for any act h and any $A \in \mathbf{B}$ and $H \in \mathbf{H}$;
4. (OPT) for any $f|H, g|H$ and for any $A \in \mathbf{H}$
$$(fAg)|H \prec f|H \Rightarrow f|H \preceq (gAf)|H$$
5. (RCD) for any constant act f_x
$$f|H \prec g|H \text{ and } f|H \prec f_x|H \Rightarrow f|H \prec (g \wedge f_x)|H;$$

[4] this assumption could be dropped

6. for any $x, y \in \mathcal{X}$ such that $x >_p y$ and for any $A, B \in \mathbf{B}$ and $H, B \wedge H \in \mathbf{H}$,
$$(x(A \wedge B)y)|H \preceq (xAy)|B \wedge H$$
and moreover if $(x(A \wedge B)y)|H \prec (xBy)|H$ or $(xBy)|H \sim x|H$, then
$$(xA \wedge By)|H \sim (xAy)|B \wedge H.$$

Conditions 1,3,4,5 are trivial generalizations of axioms proposed in qualitative possibilistic decision theory. Conditions 2 and 6 compare conditional acts with different conditioning events and are generalizations of those proposed in Definition 2 [?] for comparative conditional possibility. Note that the approach reduces to axioms of qualitative possibilistic decision theory when fixing the conditioning event. Condition 2 is useful to compare constant acts with different conditioning events, stating that the merit of a constant act is not affected by the conditioning event. Note that condition 6 is actually a reinforcement of the axiom proposed in [?], by requiring its validity for all the conditional binary acts, i.e. conditional acts of the form $(xAy)|H$ having, when H is true, two consequences $x >_p y \in \mathcal{X}$, more precisely x when A is true and y when A is false. Actually, condition 6 involves all the pairs of constant acts $x >_p y$. The first part of the condition suggests the decision-maker always prefers a more precise context $(B \wedge H)$ for the act involving event A. Indeed, conditional act $(x(A \wedge B)y)|H$ is risky since a bad consequence obtains when $A \wedge B^c$ occurs, while this possibility is ruled out by act $(xAy)|B \wedge H$, in the context $B \wedge H$. The second part of condition 6 can be explained as follows: if improving consequence y into x on $A \wedge B^c$ makes the act $(x(A \wedge B)y)|H$ more attractive, assuming B is true in the context H makes act $x(A \wedge B)y$ indifferent to xAy. Moreover, the same conclusion is reached if, in context H, event B is considered so likely that act xBy is like the constant act f_x.

It is easy to see that the conditional possibilistic optimistic criterion satisfies these properties:

Proposition 1. *A conditional optimistic criterion induces a preference relation satisfying conditions* 1 *to* 6*.*

Proof. Condition 1 holds since v^* is valued on a totally ordered scale. The validity of conditions $3, 4, 5$ follows from [?]. Conditions $2, 6$ follow from [?].

Since $\preceq$ is a total preorder on $\mathcal{F}$, its restriction to constant acts induces the same type of relation $\geq_p$ on $\mathcal{X}$. Thus, among the consequences we can find the best and the worst acts, denoted by x^* and x_*, respectively. Now we can reconstruct the conditional possibilistic optimistic criterion using the following steps:

Lemma 1. *Let* $\preceq$ *be a preference relation on* $\mathcal{F}$ *satisfying conditions* $1, 3, 4$*. Then,* $(fAg)|H \prec f|H \Rightarrow f|H \preceq (hAf)|H$ *for any* h*.*

Proof. If $(fAg)|H \prec f|H$, then $(fAx_*)|H \prec f|H$.
Suppose there exists an act h such that $(hAf)|H \prec f|H$. Then $(x_*Af)|H \preceq (hAf)|H \prec f|H$, so a contradiction for condition 4 (OPT) arises.

The two results in the sequel are trivial generalizations of the ones given for the unconditional case in [**?**]

Lemma 2. *Let $\preceq$ be a preference relation on $\mathcal{F}$ satisfying conditions* $1, 3, 4$. *If $h = f \vee g$, then $h|H \sim f|H$ or $h|H \sim g|H$.*

Lemma 3. *Let $\preceq$ be a preference relation on $\mathcal{F}$ satisfying conditions* $1, 3, 4, 5$. *If $h = f \wedge f_x$, where f_x is a constant act with value x, then $h|H \sim f|H$ or $h|H \sim f_x|H$.*

The next step retrieves a comparative conditional possibility on events:

Theorem 2. *Let $\preceq$ be a preference relation on $\mathcal{F}$ satisfying conditions* 1 *to* 6. *Then the restriction of $\preceq$ on the acts of the form $x^*Ex_*|H$ with $E \in \boldsymbol{B}$ and $H \in \boldsymbol{H}$ is a comparative conditional possibility.*

Proof. We consider the bijection introduced in [**?**] between acts of the form x^*Ex_* and events E, where x^* and x_* are the best and the worst consequences (according to a given preference) in $\mathcal{X}$. It follows from condition 2 that, for any $H, K \in \mathbf{H}$, $\emptyset|H \sim \emptyset|K \prec H|H \sim K|K$. Condition 1 implies that the restriction of $\preceq$ is a non-trivial weak order on the set of conditional events $E|H \in \mathbf{B} \times \mathbf{H}$. From condition 6 it follows that $A \wedge B|H \preceq A|B \wedge H$, for any $A, B \in \mathbf{B}$ and $H, B \wedge H \in \mathbf{H}$. Moreover, when $A \wedge B|H \prec B|H$ or $B|H \sim H|H$, it follows $A \wedge B|H \sim A|B \wedge H$.
Taking condition 4 into play, and letting $f = x^*Bx_*$, $g = x^*Ax_*$ and $h = x^*A^cx_*$ one has $fAg = x^*(A \wedge B)x_*$, and $hAf = x^*(A^c \wedge B)x_*$, then condition 4 implies that if $(A \wedge B)|H \prec B|H$, then by Lemma 1 $A^c \wedge B|H \sim B|H$. Hence, $A \wedge B|H \sim B|H$ or $A^c \wedge B|H \sim B|H$, which is equivalent under monotonicity to $A|H \preceq B|H \Rightarrow A \vee C|H \preceq B \vee C|H$ (see [**?**]).

Corollary 1. *Let $\preceq$ be a preference relation on $\mathcal{F}$ satisfying conditions* 1 *to* 6. *Then the restriction of $\preceq$ on the acts of the form $x^*Ex_*|H$ with $E \in \boldsymbol{B}$ and $H \in \boldsymbol{H}$ is representable by a qualitative conditional possibility.*

Proof. From Theorem **??** it follows that the restriction of a preference relation on the conditional events, which satisfies condition 1 to 6, is a comparative conditional possibility, then the main result in [**?**] implies that it is representable by a qualitative conditional possibility.

Theorem 3. *Let S be a finite set of states, $\boldsymbol{B}$ be the power set of S and $\boldsymbol{H} \subseteq \boldsymbol{B} \setminus \{\emptyset\}$ an additive class of events such that $S \in \boldsymbol{H}$. Let $\preceq$ be a preference over $\mathcal{F} = \mathcal{X}^S \times \boldsymbol{H}$, which satisfies conditions* 1 *to* 6. *Then, there exists a finite totally ordered scale L, a utility function $u : \mathcal{X} \to L$, a qualitative conditional possibility $\Pi : \boldsymbol{B} \times \boldsymbol{H} \to L$, and a function $V : \mathcal{F} \to L$, which represents $\preceq$. Moreover V is of the form*

$$V(f|H) = \max_{s \in S} \{\min\{u(f(s)), \Pi(s|H)\}\}.$$

Proof. 1. *Building a utility scale* Since $\mathcal{F}$ is finite, from condition 1 it follows that there exists a function V that represents $\preceq$, taking values in a finite linear ordered scale L with smallest and the greatest values 0_L and 1_L, respectively. The value associated to the conditional act $f|H$ (and to its equivalent acts) is $V(f|H)$. Since $S \in \mathbf{H}$, take a constant act f_x and let $u(x) = V(f_x)$. Moreover, due to point-wise preference $u(x_*) = 0_L$ and $u(x^*) = 1_L$. By condition 2, since $f_x|H \sim f_x|K$, for any $H, K \in \mathbf{H}$ (and so $f_x \sim f_x|S$), it follows that, for any $H \in \mathbf{H}, V(f_x|H) = u(x)$ [5].
2. *Building a qualitative conditional possibility* The construction of a qualitative conditional possibility $\Pi(\cdot|\cdot)$ on $(x^*Ax_*)|H$ follows from Theorem **??** and Corollary **??**.
3. *Computation of the utilities of acts of the form $xEy|H$* Consider a conditional act of the form $(xEx_*)|H = (x^*Ex_*) \wedge f_x|H$, from Lemma **??** one has $(xEx_*)|H \sim (x^*Ex_*)|H$ or $(xEx_*)|H \sim f_x|H$, then, (see point 1 of the proof)

$$V(xEx_*|H) = V(x^*Ex_*|H) = \Pi(E|H) \text{ or } V(xEx_*|H) = V(f_x|H) = u(x).$$

Since $(xEx_*)|H \preceq (x^*Ex_*)|H$ and $(xEx_*)|H \preceq\ f_x|H$, then

$$V(xEx_*|H) = \min\{u(x), \Pi(E|H)\}.$$

A conditional binary act $(xEy)|H$, with $x \geq_P y$ (without loss of generality), can be written as $((xEx_*) \vee f_y|H$ and by Lemma **??** it follows $(xEy)|H \sim (xEx_*)|H$ or $(xEy)|H \sim f_y|H$, moreover $(xEx_*)|H \preceq (xEy)|H$ and $\mathbf{y}|H \preceq (xEy)|H$, hence

$$V(xEy|H) = \max\{V(xEx_*|H), V(f_y|H)\} = \max\{V(xEx_*|H), u(y)\}.$$

More generally, by decomposing any act through its value on states s, we get $f|H = \bigvee_{s\in H} f(s)\{s\}x_*|H$, then it follows

$$V(f|H) = \max_{s\in H}\{V(f(s)\{s\}x_*|H)\} = \max_{s\in S} \min\{\Pi(s|H), u(f(s))\}\}.$$

Note that we can also write $V(f|H) = \max_{s\in H} \min\{\Pi(s|H), u(f(s))\}\}$ since $\Pi(s|H) = 0$ if $s \notin H$.

4.2 Conditional possibilistic preference functional: the pessimistic case

A pessimistic qualitative possibilistic criterion presupposes a conditional necessity function $N : \mathbf{B} \times \mathbf{H} \to L$, a utility u on the consequences in $\mathcal{X}$ with $u(x^*) = 1_L$ and $u(x_*) = 0_L$. A *conditional* pessimistic criterion is of the form:

$$v_*(f|H) = \min_{s\in H} \max(u(f(s)), N(\{s\}^c|H)).$$

[5] If $S \notin \mathbf{H}$, consider $H^o = \bigvee_{H\in\mathbf{H}} H$ and put, for any constant act $f_x, u(x) = V(f_x|H^o)$. Thus, as in the previous case, $u(x_*) = 0_L$ and $u(x^*) = 1_L$ and again from condition 2 it follows $V(f_x|H) = u(x)$.

Note that the above functional is such that, for any $H \in \mathbf{H}$,

$$v_*(x^*|H) = \min_{s \in H} \max\{u(x^*), N(\{s\}^c|H)\} = u(x^*) = 1 = N(H|H)$$

and $v_*(x_*|H) = \min_{s \in H} \max\{u(x_*), N(\{s\}^c|H)\} = \min_{s \in H} N(\{s\}^c|H) = 0$.

Directly axiomatizing the pessimistic qualitative possibilistic criterion would require a drastic modification of condition 6 since the latter extends properties of conditional possibility orderings, not necessity orderings. It cannot be done here for lack of space. However, the necessity function can be expressed as $N(\{s\}^c|H) = \mathbf{n}(\pi(s|H))$ where $\mathbf{n}(\cdot)$ is the order reversing map in L. Then the pessimistic criterion can be expressed in terms of an expression close to the one of the optimistic criterion, since

$$\mathbf{n}(v_*(f|H)) = \max_{s \in H} \min(\mathbf{n}(u(f(s))), \pi(s|H))$$

which lays bare its meaning: $v_*(f|H)$ is all the higher as there is no plausible state with high disutility $\mathbf{n}(u(f(s)))$. So it maybe axiomatized by directly from pessimism axioms, preserving condition 6, and constructing a max-min disutility preferential $D(f \mid H) = \mathbf{n}(v_*(f|H))$ from a disutility function $\delta = \mathbf{n}(u)$ on $\mathcal{X}$ such that $\delta(x^*) = 0_L$ and $\delta(x_*) = 1_L$. The connection between a generalization of the specific condition introduced in [**?**] for comparative conditional necessities and the above model needs to be analyzed.

5 Conclusion

This paper takes a first step toward extending the scope of qualitative decision theory to conditional events, thus making it possible to update qualitative optimistic and pessimistic preference functionals. The rescaling function for the representation of uncertainty is rather simple since only the most plausible states allowed by the context induced by the new information are mapped to the top value of the scale. However conditioning on a null event may end up with a different possibiity distribution. Our results are only a first step, and several improvements could be envisaged

- Condition 6 is expressed in terms of binary acts, and is formally a copy of conditional possibility ordering axioms. It would be much more convincing to derive this condition from an axiom involving general acts.
- As the meaning of conditional acts $f|H$ may look difficult to grasp, it may sound more natural to axiomatize the conditional criteria in the setting of a preference relation indexed by the context, like $\prec_H$ (H not empty). For instance, $f \prec_H g$ may be another way of denoting $f|H \prec g|H$ [**?**]. But clearly, encoding the statement $f|A \prec g|B$ using relations of the form $\prec_H$ is not obvious, and the language of conditional acts is likely to be richer.

Among more advanced lines of further research that can be considered, the extension of the framework to more general set-functions, the application of these criteria to qualitative Markov decision processes and the study of dynamic consistency can be envisaged.

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