Cautious conjunctive merging of belief functions

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Abstract. When merging belief functions, Dempster rule of combination is justified only when information sources can be considered as independent. When this is not the case, one must find out a cautious merging rule that adds a minimal amount of information to the inputs. Such a rule is said to follow the principle of minimal commitment. Some conditions it should comply with are studied. A cautious merging rule based on maximizing expected cardinality of the resulting belief function is proposed. It recovers the minimum operation when specialized to possibility distributions. This form of the minimal commitment principle is discussed, in particular its discriminating power and its justification when some conflict is present between the belief functions.

Keywords: belief functions, least commitment, dependence.

1 Introduction

There exist many fusion rules in the theory of belief functions [13]. When several sources deliver information over a common frame of discernment, combining belief functions by Dempster's rule [4] is justified only when the sources can be assumed to be independent. When such an assumption is unrealistic and when the precise dependence structure between sources cannot be known, an alternative is to adopt a conservative approach to the merging of the belief functions (i.e. by adding no extra information nor assumption in the combination process). Adopting such a cautious attitude means that we apply the "least commitment principle", which states that one should never presuppose more beliefs than justified. This principle is basic in the frameworks of possibility theory, imprecise probability [15], and the Transferable Belief Model (TBM) [14]. It can be naturally exploited for cautious merging belief functions.

In this paper, we study general properties that a merging rule satisfying the least commitment principle should follow when the sources are logically consistent with one another. An idempotent cautious merging rule generalizing the minimum rule of possibility theory is proposed. Section 2 recalls some basics about belief functions. Section 3 recalls an approach to the conjunctive merging of belief functions proposed by Dubois and Yager in the early nineties and shows it provides a natural least committed idempotent merging rule for belief functions, where least commitment comes down to maximizing expected cardinality of the result. Finally, Section 4 discusses limitations of the expected cardinality criterion, raising interesting issues on the non-unicity of solutions, and discussing other rules proposed in the literature especially when some conflict is present between the sources.

2 Preliminaries

Let X be the finite space of cardinality $|X|$ with elements $X = x_1, \ldots, x_{|X|}$.

Definition 1. *A basic belief assignment (bba) [10] is a function m from the power set of X to $[0,1]$ s.t. $m(\emptyset) = 0$ and $\sum_{A \subseteq X} m(A) = 1$.*

Let $\mathcal{M}_X$ the set of bba's on $2^{|X|}$. A set A s.t. $m(A) > 0$ is called a focal set. The number $m(A) > 0$ is the mass of A. Given a bba m, belief, plausibility and commonality functions of an event $E \subseteq X$ are, respectively

$$bel(E) = \sum_{A \subseteq E} m(A) \ ; \ pl(E) = \sum_{A \cap E \neq \emptyset} m(A) = 1 - bel(A^c) \ ; \ q(E) = \sum_{E \subseteq A} m(A)$$

A belief function measures to what extent an event is directly supported by the available information, while a plausibility function measures the maximal amount of evidence that could support a given event. A commonality function measures the quantity of mass that may be re-allocated to a particular set from its supersets. The commonality function increases when bigger focal sets receive greater mass assignments, hence the greater the commonality degrees, the less informative is the belief function. A bba is said to be non-dogmatic if $m(X) > 0$ hence $q(A) > 0, \forall A \neq \emptyset$.

A bba m can also be interpreted as a probability family [15] $\mathcal{P}_m$ such that $Bel(A)$ and $Pl(A)$ are probability bounds: $\mathcal{P}_m = \{P | \forall A \subset X, \ Bel(A) \leq P(A) \leq Pl(A)\}$. In the sequel of the paper, we mainly focus on 2 special kinds of bbas : namely, possibility distributions and generalized p-boxes.

A possibility distribution [16] is a mapping $\pi : X \to [0,1]$ from which two dual measures (respectively the possibility and necessity measures) can be defined : $\Pi(A) = \sup_{x \in A} \pi(x)$ and $N(A) = 1 - \Pi(A^c)$. In terms of bba, a possibility distribution is equivalent to a bba whose focal sets are nested. The plausibility (Belief) measure then reduces to a Possibility (Necessity) measure.

A p-box [9] is a pair of cumulative distributions $[\underline{F}, \overline{F}]$ defining a probability family $\mathcal{P}_{[\underline{F},\overline{F}]} = \{P | \underline{F}(x) \leq F(x) \leq \overline{F}(x) \quad \forall x \in \Re\}$. A generalized p-box [6] is a generalization of a p-box, defined on an arbitrary (especially, finite) ordered space (whereas usual p-boxes are defined on the real line). If an order $\leq_R$ is defined on X, to any bba, a generalized p-box can be associated s.t. $\overline{F}(x)_R = Pl(\{x_i | x_i \leq_R x\})$ and $\underline{F}(x)_R = Bel(\{x_i | x_i \leq_R x\})$, but it retains only a part of the information contained in the bba, generally.

Dubois and Prade [7] defined three information orderings based on different notions related to belief functions :

– pl-ordering. if $pl_1(A) \leq pl_2(A)\ \forall A \subseteq X$, we write $m_1 \sqsubseteq_{pl} m_2$;
– q-ordering. if $q_1(A) \leq q_2(A)\ \forall A \subseteq X$, we write $m_1 \sqsubseteq_q m_2$;
– s-ordering. if m_1 is a specialization of m_2, we write $m_1 \sqsubseteq_s m_2$.

Informally, a bba m_2 is a specialization of a bba m_1 if every mass $m_1(A)$ can be reallocated to subsets of A in m_2 (i.e. the mass $m_1(A)$ "flows down" to subsets $B \subseteq A$ in m_2) so as to recover m_2. If m_2 is a specialization of m_1, it means that beliefs represented by the bba m_2 are more focused than those from the bba m_1. In other words, m_2 can be judged more informative than m_1. If we interpret bbas in terms of probability families, another means to compare them in terms of imprecision is to compare such families. We can say that m_1 is more precise than m_2 iff $\mathcal{P}_{m_1} \subset \mathcal{P}_{m_2}$. This is equivalent to the pl-ordering. More generally if we have $m_1 \sqsubseteq_x m_2$ (x corresponding to one of the three orderings), we say that m_2 is x-less committed than m_1. Dubois and Prade proved that $m_1 \sqsubseteq_s m_2$ imply $m_1 \sqsubseteq_q m_2$ and $m_1 \sqsubseteq_{pl} m_2$, but that the reverse is not true (hence, s-ordering is the strongest ordering of the three).

As these relations are partial orders, comparing bbas with respect to s, pl or q-ordering can be complex and often leads to incomparability (i.e. non unicity of the solution). A simpler tool for comparing bbas is to measure the non-commitment of a bba by its expected cardinality, which reads

$$I(m) = \sum_{A \subseteq X} m(A)|A|$$

where $|A|$ is the cardinality of A. Expected cardinality is an imprecision measure, and its value is the same as the cardinality of the fuzzy set equivalent to the contour function (i.e. $I(m) = \sum_{x_i \in X} pl(x_i)$). It is coherent with specialization ordering (and hence with the two others) since if m_1 is a specialization of m_2, then $I(m_1) \leq I(m_2)$. This definition is the one we will use in the sequel.

3 A least-committed merging rule

A bba built by merging two different bbas m_1, m_2 is supposed to be obtained by the following procedure, denoting $\mathcal{F}_i$ the set of focal sets of m_i:

1. A joint bba m is built on $X \times X$, having focal sets of the form $A \times B$ where $A \in \mathcal{F}_1, B \in \mathcal{F}_2$ and preserving m_1, m_2 as marginals. It means that $m_1(A) = \sum_{B \in \mathcal{F}_2} m(A, B)$ and likewise for m_2.
2. Each joint mass $m(A, B)$ should be allocated to the subset $A \cap B$ only, where A and B are focal sets of m_1 and m_2 respectively.

We call a merging rule satisfying these two conditions *conjunctive*[3], and denote $\mathcal{M}_X^{m_1 \cap m_2}$ the set of conjunctively merged bba's. The idea behind the conjunctive approach is to keep as much information as possible from the fusion process. However not every bba $m_\cap$ obtained by conjunctive merging is normalized (i.e.

[3] A disjunctive merging rule could be defined likewise, changing $\cap$ into $\cup$.

one may get $m(\emptyset) \neq 0$). It is clear that a merged bba $m_\cap$ on X in the above sense is a specialization of both m_1 and m_2.

In fact three situations may occur

- $\mathcal{M}_X^{m_1 \cap m_2}$ contains only normalized belief functions. It means that $\forall A \in \mathcal{F}_1, B \in \mathcal{F}_2, A \cap B \neq \emptyset$. Only in that case does the result of merging by Dempster rule of combination belong to $\mathcal{M}_X^{m_1 \cap m_2}$. The two bbas are said to be *logically consistent*.
- $\mathcal{M}_X^{m_1 \cap m_2}$ contains both subnormalized and normalized bbas. It means that $\exists A, B, A \cap B = \emptyset$ and that the marginal-preservation equations have solutions which allocate zero mass $m(A, B)$ to such $A \times B$.
- $\mathcal{M}_X^{m_1 \cap m_2}$ contains only subnormalized belief functions. A result from [3] indicates that this situation is equivalent to $\mathcal{P}_{m_1} \cap \mathcal{P}_{m_2} = \emptyset$. The two bbas are said to be conflicting.

A cautious merging rule is then one that selects a least committed bba in $\mathcal{M}_X^{m_1 \cap m_2}$ for any of the three orderings given above. In order to avoid incomparabilities, we define a least-committed bba in $\mathcal{M}_X^{m_1 \cap m_2}$ as one with maximal expected cardinality $I(m)$. A conjunctive merging rule is denoted $\oplus$, and a least-committed merging rule $\bigwedge$.

Now suppose $m_1 = m_2 = m$. The least committed specialisation of m is m itself. Hence the following natural requirement:

Idempotence The least-committed rule $\bigwedge$ should be idempotent.

The following property directly follows from this requirement:

Proposition 1. *Let m_1 be a specialization of m_2, then the result of the least committed rule $\bigwedge$ should be $m_1 \odot m_2 = \widehat{m}_{12} = m_1$.*

Although very important, this result concerns very peculiar cases and does not give us guidelines as to how general bbas should be combined to result in a least-committed bba (in the sense of expected cardinality). In [8], by using the concept of commensurate bbas, Dubois and Yager show that there are a lot of idempotent rules that combine two bbas, each of them giving different results. In the following, we slightly generalize the notion of bba and consider it as a relation between the power set of X and $[0, 1]$. In other words, a generalized bba may assign several weights to the same subset of X.

Definition 2. *Let m be a bba with focal sets $A_1, \ldots, A_n$ and associated weights $m^1, \ldots, m^n$. A split of m is a bba m' with focal sets $A'_1, \ldots, A'_{n'}$ and associated weights $m'^1, \ldots, m'^{n'}$ s.t. $\sum_{A'_i = A_i} m'^i = m^i$*

In other words, a split is a new bba where the original weight given to a focal set is separated in smaller weights given to the same focal set, with the sum of weights given to a specific focal set being constant. Two generalized bbas m_1,m_2 are said to be equivalent if $pl_1(E) = pl_2(E)$ and $bel_1(E) = bel_2(E)$ $\forall E \subseteq X$. If m_1 and m_2 are equivalent, it means that they are splits of the same regular bba [8]. In the following, a bba should be understood as a generalized one.

Definition 3. *Let m_1, m_2 be two bbas with respective focal sets $\{A_1,\ldots,A_n\}$, $\{B_1,\ldots,B_k\}$ and associated weights $\{m_1^1,\ldots,m_1^n\}$, $\{m_2^1,\ldots,m_2^k\}$. Then, m_1 and m_2 are said to be commensurate if $k = n$ and there is a permutation σ of $\{1,\ldots,n\}$ s.t. $m_1^j = m_2^{\sigma(i)}, \forall i = 1,\ldots,n$.*

Two bbas are commensurate if their distribution of weights over focal sets can be described by the same vector of numbers. In [8], Dubois and Yager propose an algorithm, given a prescribed ranking of focal sets on each side, that makes any two bbas commensurate by successive splitting. Based on this algorithm, they provide an idempotent rule $\bigoplus$ that allows to merge any two bbas. This merging rule is conjunctive and the result depends on the ranking of focal sets used in the commensuration algorithm, summarized as follows:

- Let m_1, m_2 be two bbas and $\{A_1,\ldots,A_n\}$, $\{B_1,\ldots,B_k\}$ the two sets of ordered focal sets with weights $\{m_1^1,\ldots,m_1^n\}$, $\{m_2^1,\ldots,m_2^k\}$
- By successive splitting of each bbas (m_1, m_2), build two generalised bbas $\{R_1^1,\ldots,R_1^l\}$ and $\{R_2^1,\ldots,R_2^l\}$ with weights $\{m_{R_1}^1,\ldots,m_{R_1}^l\}$, $\{m_{R_2}^1,\ldots,m_{R_2}^l\}$ s.t. $m_{R_1}^i = m_{R_2}^i$ and $\sum_{R_1^i=A_j} = m_1^j$, $\sum_{R_2^i=B_j} = m_2^j$.
- Algorithm results in two commensurate generalised bbas m_{R_1}, m_{R_2} that are respectively equivalent to the original bbas m_1, m_2.

Once this commensuration is done, the conjunctive rule $\bigoplus$ proposed by Dubois and Yager defines a merged bba $m_{12} \in \mathcal{M}_X^{m_1 \cap m_2}$ with focal sets $\{R^i_{1\bigoplus 2} = R_1^i \cap R_2^i, i = 1,\ldots,l\}$ and associated weights $\{m^i_{R_{1\bigoplus 2}} = m^i_{R_1} = m^i_{R_2}, i = 1\ldots,l\}$. The whole procedure is illustrated by the following example.

Example 1. Commensuration

m_1		m_2	
A_1	.5	B_1	.6
A_2	.3	B_2	.2
A_3	.2	B_3	.1
		B_4	.1

$\rightarrow$

l	m_{R^l}	R_1^l	R_2^l	$R^l_{1\bigoplus 2}$
1	.5	A_1	B_1	$A_1 \cap B_1$
2	.1	A_2	B_1	$A_2 \cap B_1$
3	.2	A_2	B_2	$A_2 \cap B_2$
4	.1	A_3	B_3	$A_3 \cap B_3$
5	.1	A_3	B_4	$A_3 \cap B_4$

From this example, it is easy to see that the final result crucially depends of the chosen rankings of the focal sets of m_1 and m_2. In fact, it can be shown that any conjunctively merged bba can be produced in this way.

Definition 4. *Two commensurate generalised bbas are said to be equi-commensurate if each of their focal sets has the same weight.*

Any two bbas m_1, m_2 can be made equi-commensurate. In our example, bbas can be made equi-commensurate by splitting the first line into five similar lines of weight 0.1 and the third line into two similar lines of weight 0.1. Every line then has weight 0.1, and applying Dubois and Yager's rule to these bbas yields a bba equivalent to the one obtained before equi-commensuration. Combining two equi-commensurate bbas $\{R_1^1,\ldots,R_1^l\}$, $\{R_2^1,\ldots,R_2^l\}$ by Dubois and Yager rule results in a bba s.t every focal element in $\{R^1_{1\bigoplus 2},\ldots,R^l_{1\bigoplus 2}\}$ has equal weight $m_{R_{1\bigoplus 2}}$ (0.1 in our example). The resulting bba is still in $\mathcal{M}_X^{m_1 \cap m_2}$.

Proposition 2. *Any merged bba in* $\mathcal{M}_X^{m_1 \cap m_2}$ *can be reached by means of Dubois and Yager rule using appropriate commensurate bbas equivalent to* m_1 *and* m_2 *and the two appropriate rankings of focal sets.*

Proof. We assume masses (of marginal and merged bbas) are rational numbers. Let $m \in \mathcal{M}_X^{m_1 \cap m_2}$ be the merged bba we want to reach by using Dubois and Yager's rule. Let $m(A_i, B_j)$ be the mass allocated to $A_i \cap B_j$ in m. It is of the form $k_{12}(A_i, B_j) \times 10^{-n}$ where k_{12}, n are integers. By successive splitting followed by a reordering of elements R_1^j, we can always reach m. For instance, let k_R be equal to the greatest common divisor of all values $k_{12}(A_i, B_j)$. Then, $k_{12}(A_i, B_j) = q_{ij} \times k_R$, for an integer q_{ij}. Then, it suffices to re-order elements R_1^k by a re-ordering σ s.t. for q_{ij} of them, $R_1^k = A_i$ and $R_2^{\sigma(k)} = B_j$. Then, by applying Dubois and Yager's rule, we obtain the result m. From a practical standpoint, restricting ourselves to rational numbers has no importance: rational numbers being dense in reals, this means that we can always get as close as we want to any merged bba.

For cautious merging, it is natural to look for appropriate rankings of focal sets so that the merged bba obtained via commensuration has maximal cardinality. The answer is : rankings should be extensions of the partial ordering induced by inclusion (i.e. $A_i < A_j$ if $A_i \subset A_j$). This is due to the following result:

Lemma 1. *Let* A, B, C, D *be four sets s.t.* $A \subseteq B$ *and* $C \subseteq D$*. Then, we have the following inequality*

$$|A \cap D| + |B \cap C| \leq |A \cap C| + |B \cap D| \tag{1}$$

Proof. From the assumption, the inequality $|(B \setminus A) \cap C| \leq |(B \setminus A) \cap D|$ holds. Then consider the following equivalent inequalities:

$$\begin{aligned} |(B \setminus A) \cap C| + |A \cap C| &\leq |A \cap C| + |(B \setminus A) \cap D| \\ |B \cap C| &\leq |A \cap C| + |(B \setminus A) \cap D| \\ |A \cap D| + |B \cap C| &\leq |A \cap C| + |A \cap D| + |(B \setminus A) \cap D| \\ |A \cap D| + |B \cap C| &\leq |A \cap C| + |B \cap D| \end{aligned}$$

hence the inequality (1) is true.

When using equi-commensurate bbas, masses in the formula of expected cardinality can be factorized, and expected cardinality then becomes: $I(m)_{R_{1 \oplus 2}} = m_{R_{1 \oplus 2}} \sum_{i=1}^{l} |R^i_{1 \oplus 2}| = m_{R_{1 \oplus 2}} \sum_{i=1}^{l} |R_1^i \cap R_2^i|$, where $m_{R_{1 \oplus 2}}$ is the smallest mass enabling equi-commensuration. We are now ready to prove the following proposition

Proposition 3. *If* $m \in \mathcal{M}_X^{m_1 \cap m_2}$ *is minimally committed for expected cardinality, there exists an idempotent conjunctive merging rule* $\bigwedge$ *constructing* m *by the commensuration method, s.t. focal sets are ranked on each side in agreement with the partial order of inclusion.*

Proof. Suppose $\widehat{m}_{12} \in \mathcal{M}_X^{m_1 \cap m_2}$ is minimally committed for expected cardinality. It can be obtained by commensuration. Let m_{R_1}, m_{R_2} be the two equi-commensurate bbas with n elements each derived from the two original bbas m_1, m_2. Suppose that the rankings used display four focal sets $R_1^i, R_1^j, R_2^i, R_2^j$, $i < j$, such that $R_1^i \supset R_1^j$ and $R_2^i \subseteq R_2^j$. By Lemma 1, $|R_1^j \cap R_2^j| + |R_1^i \cap R_2^i| \leq |R_1^j \cap R_2^i| + |R_1^i \cap R_2^j|$. Hence, if we permute focal sets R_1^i, R_1^j before applying Dubois and Yager's merging rule, we end up with a merged bba $m_{R'_{1\oplus 2}}$ s.t. $I(m_{R_{1\oplus 2}}) \leq I(m_{R'_{1\oplus 2}})$. Since any merged bba can be reached by splitting m_1,m_2 and by inducing the proper ranking of cocal sets of the resulting bbas m_{R_1}, m_{R_2}, any merged bba $\widehat{m}_{12} \in \mathcal{M}_X^{m_1 \cap m_2}$ maximizing expected cardinality can be reached by Dubois and Yager's rule, using rankings of focal sets in accordance with the inclusion ordering.

Ranking focal sets in accordance with inclusion is neither sufficient nor the only way of maximizing expected cardinality when merging two given bbas, as shown by the following examples.

Example 2. Let m_1,m_2 be two bbas of the space $X = x_1, x_2, x_3$. Let $m_1(A_1 = \{x_1, x_2\}) = 0.5$,$m_1(A_2 = \{x_1, x_2, x_3\}) = 0.5$ be the two focal sets of m_1 and $m_2(B_1 = \{x_1, x_2\}) = 0.2$,$m_2(B_2 = \{x_2\}) = 0.3$,$m_2(B_3 = \{x_1, x_2, x_3\}) = 0.5$ be the focal sets of m_2. The following table shows the result of Dubois and Yager's merging rule after commensuration:

1	m_{R^l}	R_1^l	R_2^l	$R^l_{1\oplus 2}$
1	.2	A_1	B_1	$A_1 \cap B_1 = \{x_1, x_2\}$
2	.3	A_1	B_2	$A_1 \cap B_2 = \{x_2\}$
3	.5	A_2	B_3	$A_2 \cap B_3 = \{x_1, x_2, x_3\}$

Although focal sets B_i are not ordered by inclusion ($B_1 \supset B_2$), the result maximizes expected cardinality (the result is m_2, which is a specialization of m_1). This shows that the technique based on proposition 3 is not necessary (nevertheless, the result is obtained by using order B_2, B_1, B_3).

Now, consider the same bba m_1 and another bba m_2 s.t. $m_2(B_1 = \{x_2\}) = 0.3$, $m_2(B_2 = \{x_2, x_3\}) = 0.3$, $m_2(B_3 = \{x_1, x_2\}) = 0.1$, $m_2(B_4 = \{x_1, x_2, x_3\}) = 0.3$. m_2 is no longer a specialization of m_1, and the order B_1, B_2, B_3, B_4 is one of the two possible extensions of the partial order induced by inclusion. The result of Dubois and Yager's rule gives us:

1	m_{R^l}	R_1^l	R_2^l	$R^l_{1\oplus 2}$
1	.2	A_1	B_1	$A_1 \cap B_1 = \{x_2\}$
2	.3	A_1	B_2	$A_1 \cap B_2 = \{x_2\}$
3	.1	A_2	B_2	$A_2 \cap B_2 = \{x_2, x_3\}$
4	.1	A_2	B_3	$A_2 \cap B_3 = \{x_1, x_2\}$
5	.3	A_2	B_4	$A_1 \cap B_4 = \{x_1, x_2, x_3\}$

and the expected cardinality of the merged bba is 1.8. If, instead of the order B_1, B_2, B_3, B_4, we choose the order B_1, B_3, B_2, B_4 (i.e. the other extension of the partial order induced by inclusion), applying Dubois and Yager's rule gives us a merged bba of expected cardinality 2.0, which is higher than the previous one. Hence, we see that proposition 3 is not sufficient in general to reach maximal cardinality. Thus, proposition 3 gives us guidelines for combining belief functions so as to maximise cardinality, but further conditions should be stated to select the proper total orderings of focal sets.

4 Beyond least-commitment based on expected cardinality

Least committed merging by expected cardinality maximisation is coherent with specialization since if an s-least committed bba exists, then it has maximal expected cardinality. But other notions of minimal commitment exist, that do not relate to expected cardinality. This section discusses arguments pro and con the use of this notion, first for logically consistent bbas and then for more general ones.

4.1 Retrieving the minimum rule of possibility theory

For the special case of possibility distributions, the order between focal sets induced by inclusion is complete. It means that, in this case, applying proposition 3 results in an unique consonant merged bba with contour function $\min(\pi_1, \pi_2)$, which corresponds to the usual minimum operator [8]. As the minimum is the most cautious conjunctive merging operator in possibility theory, it shows that our proposition is coherent with and thus justifies the possibilistic approach, as suggested by Smets[12]. One may also conjecture that merged bbas that maximize expected cardinality are also least-committed in the sense of the relative specificity of their contour functions (m_1 is less committed than m_2 in this sense if $pl_1(x) \geq pl_2(x) \; \forall x \in X$). Nevertheless, the minimum of two possibility distributions is not the only cardinality maximizer, as the next example shows:

Example 3. Consider the two following possibility distributions π_1,π_2, expressed as belief structures m_1, m_2

$\pi_1 = m_1$		$\pi_2 = m_2$	
Focal sets	Mass	Focal sets	Mass
$\{x_1, x_2, x_3\}$	0.5	$\{x_3, x_4, x_5\}$	0.5
$\{x_0, x_1, x_2, x_3, x_4\}$	0.5	$\{x_2, x_3, x_4, x_5, x_6\}$	0.5

The following merged bbas $C_1, C_2 \in \mathcal{M}_X^{m_1 \cap m_2}$ have the same contour function, hence (maximal) expected cardinality equal to 2.

$C_1 = \pi_{min}$		C_2	
Focal sets	Mass	Focal sets	Mass
$C_{11} = \{x_3\}$	0.5	$C_{21} = \{x_3, x_4\}$	0.5
$C_{12} = \{x_2, x_3, x_4\}$	0.5	$C_{22} = \{x_2, x_3\}$	0.5

This interesting example is discussed below.

4.2 Refining expected cardinality by the pl- or q-ordering

As maximizing expected cardinality is coherent with s-least commitment and can lead to non-uniqueness of the solution, discriminating different solutions can be done by using pl- or q-ordering. Choosing one or the other matters, since even for the simple example 3, we have $C_1 \sqsubset_{pl} C_2$ and $C_2 \sqsubset_q C_1$. Since $C_1 \sqsubset_{pl} C_2$ is equivalent to $\mathcal{P}_{C_1} \subset \mathcal{P}_{C_2}$ (e.g. the probability distribution $p(x_2) = 0.5, p(x_4) = 0.5$ is inside $\mathcal{P}_{C_2}$, and not in $\mathcal{P}_{C_1}$), choosing the *pl* ordering is coherent with a probabilistic interpretation of belief functions and shows the limitation of proposition 3. Note that in the example, the bba C_2 is the generalized p-box (with the order $x_1 <_R x_2 <_R \ldots <_R x_n$ on elements of X) corresponding to the possibility distribution C_1. It is not surprising that $\mathcal{P}_{C_1} \subset \mathcal{P}_{C_2}$, since the probability family induced by a possibility distribution is included in the family induced by its corresponding p-box [1].

Besides, choosing the $q-$ordering to discriminate solutions (which yields C_1 in example 3) seems more in accordance with proposition 3 (and thus with the particular case of possibility distributions). Moreover, as the commonality function increases when larger focal sets receive greater mass assignments, it could be argued that the $q-$ordering is more in accordance with the TBM approach. Smets [12] suggests without proof that in the case of merging possibility distributions, the minimum rule is least $q-$committed, like in the example.

4.3 Minimizing conflict

When two bbas are not logically consistent (i.e. there are focal elements A_i,B_j for which $A_i \cap B_j = \emptyset$), a conjunctively merged bba that maximizes expected cardinality may not, in general, minimize conflict (i.e. $m \in \mathcal{M}_X^{m_1 \cap m_2}$ s.t. $m(\emptyset)$ is minimal). This is illustrated by the following example:

Example 4. Consider the two following possibility distributions π_1,π_2, expressed as belief structures m_1, m_2

$\pi_1 = m_1$		$\pi_2 = m_2$	
Focal sets	Mass	Focal sets	Mass
$\{x_1, x_2\}$	0.5	$\{x_4\}$	0.5
$\{x_0, x_1, x_2, x_3, x_4\}$	0.5	$\{x_2, x_3, x_4, x_5, x_6\}$	0.5

And the following table shows the result of applying the minimum (thus maximising expected cardinality) and the unnormalized Dempster rule of combination

Min(π_1, π_2)		unnormalized Dempster's rule			
Focal sets	Mass	Focal sets	Mass	Focal sets	Mass
$\{x_2, x_3, x_4\}$	0.5	$\{x_2\}$	0.25	$\{x_2, x_3, x_4\}$	0.25
$\emptyset$	0.5	$\{x_4\}$	0.25	$\emptyset$	0.25

With Dempster rule, conflict value is 0.25 and expected cardinality is 1.25, while with the minimum, the conflict value is 0.5 and expected cardinality is 1.5.

Provided one considers that minimizing the conflict is as desirable as finding a least-committed way of merging the information, this can problematic. A possible alternative is then to find $m \in \mathcal{M}_X^{m_1 \cap m_2}$ that is least-committed among those for which $m(\emptyset)$ is minimal. This problem was studied by Cattaneo in [2]. Cattaneo proposes to find the merged bba $m \in \mathcal{M}_X^{m_1 \cap m_2}$ that maximizes the following function:

$$F(m) = m(\emptyset)f(0) + (1 - m(\emptyset)) \sum_{A \neq \emptyset} m(A) log_2(A) \quad (2)$$

with $f(0)$ a real number s.t. $f(0) < |X|$. In the above equation, $m(\emptyset)f(0)$ can be seen as a penalty given to the evaluation of the merged belief when conflict appears, while the second part of the right-hand side of equation (2) is equivalent to expected cardinality where $|A|$ is replaced by $log_2(|A|)$ (more generally, we can replace $|A|$ by any non-decreasing function $f(|A|)$ from N to R). A similar strategy (penalizing the appearance of conflict) could thus be adopted with expected cardinality (or with any function $f(|A|)$), nevertheless, it would not be without inconvenient:

- adding penalty to conflict is computationally less efficient than using expected cardinality alone, since proposition 3 does not hold.
- Cattaneo mentions that associativity and conflict minimization are incompatible, while our rule is at least associative in the case of possibility distributions (other cases still have to be explored).

Now, the claim that a cautious conjunctive rule should give a merged bba where the conflict is minimized is questionable. This is shown by our small example 4, where minimizing the conflict, by assigning zero mass to empty intersections while respecting the marginals, produces the bba $m(\{x_2\}) = 0.5, m(\{x_4\}) = 0.5$, which is the only probability distribution distribution in $\mathcal{P}_{m_1} \cap \mathcal{P}_{m_2}$. Indeed, this bba is the most precise possible result, and its informational content is clearly more adventurous than the bba corresponding to $\min(\pi_1, \pi_2)$.

4.4 Least commitment based on the weight function

Any non dogmatic belief function with bba m can be uniquely represented as the conjunctive combination of the form $m = \bigodot_{A \neq X} A^{w(A)}$ [11], where $w(A)$ is a positive weight, and $A^{w(A)}$ represents the (generalized) simple support function with bba μ such that $\mu(A) = 1 - w(A)$ and $\mu(X) = w(A)$, and $\bigodot$ denotes the unnormalized Dempster rule of combination. Note that if $w(A) \in [0, 1]$, μ is a simple support bba. Otherwise, it is not a bba. Denoeux [5] introduces another definition of least commitment calling a bba m_1 less w-committed than m_2 whenever $w_1(A) \leq w_2(A), \forall A \neq X$. Denoeux proposes to apply the following cautious rule to weight functions:

$$w_{12}(A) = \min(w_1(A), w_2(A)), \forall A \neq X.$$

and he shows that it produces the weight function of the least w-committed merged bba among those that are more w-committed than both marginals m_1 and m_2. If a bba is less w-committed than another one, then it is a specialisation thereof. Our conjunctive merging only requires the result to be more s-committed than m_1 and m_2. which is a weaker condition than to be more w-committed. Now, apply both rules to the following example (2 of [5]).

Example 5. :

Consider $X = \{a, b, c\}$, m_1 defined by $m_1(\{a, b\}) = 0.3$, $m_1(\{b, c\}) = 0.5$, $m_1(X) = 0.2$; m_2 defined by $m_2(\{b\}) = 0.3$, $m_2(\{b, c\}) = 0.4$, $m_2(X) = 0.3$. Results of both rules are given in the following table

Denoeux's rule (m^D)				Max. Exp. Card. rule (m^C)			
Focal Sets	Mass	Focal Sets	Mass	Focal Sets	Mass	Focal Sets	Mass
$\{b\}$	0.6	$\{b, c\}$	0.2	$\{b\}$	0.3	$\{X\}$	0.2
$\{a, b\}$	0.12	$\{X\}$	0.08	$\{b, c\}$	0.5		

In this example, our conjunctive cautious rule yields a merged bba m^C that is s-less committed (and hence has a greater expected cardinality) than m^D, the one obtained with Denoeux's rule. Nevertheless, the merged bba obtained by maximizing expected cardinality is not comparable in the sense of the w-ordering with any of the three other bbas (m_1, m_2, m^D), nor does it fulfil Denoeux's condition of being more w-committed than m_1 and m_2. The cautious w-merging of possibility distributions does not reduce to the minimum rule either. Thus, the two approaches are at odds. As it seems, using the w-ordering allows to easily find a unique least-committed element, at the expense of restricting the search to a subset of $\mathcal{M}_X^{m_1 \cap m_2}$ due to the use of the w-ordering (which can be questioned in the scope of a cautious approach).

In his paper, Denoeux generalizes both $\odot$ and his cautious rule with triangular norms. However, the set of non-dogmatic belief functions equipped with $\odot$ forms a group, as is the product of positive w-numbers. So the relevant setting for generalizing the product of weight functions seems to be the one of uninorms. But the minimum is not a uninorm on the positive real line. It is the greatest t-norm on $[0, 1]$, in particular, greater than product, and this property is in agreement with minimal commitment of contour functions. But the minimum rule no longer dominates the product on the positive real line, so that the bridge between Denoeux's idempotent and the idea of minimal commitment is not obvious beyond the w-ordering.

5 Conclusions

When our knowledge about the dependencies existing between multiple sources is poor, Dempster rule of combination cannot be applied. The merging of bbas should follow the principle of least-commitment, or said differently, we should adopt a cautious attitude. Nevertheless, the various definitions of least-commitment often lead to indecision (i.e. to non-unicity of the solution). In this paper, we have studied the maximisation of the expected cardinality of the merged bba and proposed an idempotent merging rule, based on the commensuration of bbas respecting the partial ordering induced by inclusion between focal sets. It encompasses the minimum rule on possibility distributions, thus justifying it in terms of least commitment. However more investigations are needed to make our proposition practically convenient and to articulate the expected cardinality criterion with other notions of least commitment, based on generalized forms of bba cardinality, on the comparison of contour functions, and other information orderings in the theory of belief functions.

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